On certain generalizations of the Gagliardo-Nirenberg inequality and their applications to capacitary estimates and isoperimetric inequalities

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Abstract

We derive the inequality

$$\int_{\mathbb{R}} M(|f^{'}(x)|h(f(x)))dx \leq C(M,h)\int_{\mathbb{R}} M\left(\sqrt{|f^{''}(x)\mathcal{T}_h(f(x))|}\cdot h(f(x))\right)dx,$$

with a constant $C(M,h)$ independent of f, where f belongs locally to the Sobolev space $W^{2,1}(\mathbb{R})$ and $f^{'}$ has compact support. Here M is an arbitrary N-function satisfying certain assumptions, h is a given function and $\mathcal{T}_h(\cdot)$ is its given transform independent of M. When $M(\lambda)=\lambda^p$ and $h\equiv 1$ we retrieve the well known inequality: $\int_{\mathbb{R}}|f^{'}(x)|^p dx \leq (\sqrt{p-1})^p\int_{\mathbb{R}}(\sqrt{|f^{''}(x)f(x)|})^p dx$. We apply our inequality to obtain some generalizations of capacitary estimates and isoperimetric inequalities due to Maz'ya.

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1 Introduction

The following interesting inequality is well known:

$$\int_{\mathbb{R}} |f^{'}|^p dx \leq \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f^{''}f|}\right)^p dx, \tag{1.1}$$

where $f \in C_0^\infty(\mathbb{R})$, $p \geq 2$. To prove it, one simply needs to integrate by parts and to note that $|f^{'}|^p = |f^{'}|^{p-2} f^{'} \cdot f^{'}$ (when $p = 2$ then the proof can be found in the classic book [6], Theorem 261, Section 7.9). On the other hand, another inspiring inequality can be found in the book by Maz'ya [10], Section 8:

$$\int_{\{x:f(x)>0\}} \left(\frac{|f^{'}|}{\sqrt{f}}\right)^p dx \leq \left(\frac{p-1}{|1-\frac{p}{2}|}\right)^{\frac{p}{2}} \int_{\mathbb{R}} \left(\sqrt{|f^{''}|}\right)^p dx, \tag{1.2}$$

where $f \in C_0^\infty(\mathbb{R})$ is nonnegative.
Motivated by both, one may ask if there is an interplay between (1.1) and (1.2), some general inequality yielding these inequalities as a special case. In [8] we gave an affirmative answer to that question, obtaining the inequality:

$$\int_{\mathbb{R}} |f^{'}(x)|^p h(f(x)) dx \leq \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f^{''}(x)\mathcal{T}_h(f(x))|}\right)^p h(f(x)) dx, \tag{1.3}$$

where f belongs locally to the Sobolev space $W^{2,1}(\mathbb{R})$ and $f^{'}$ has bounded support, $h(\cdot)$ is a given function and $\mathcal{T}_h(\cdot)$ is its certain transform, independent of p. When $h \equiv 1$ we retrieve (1.1), while if $h(\lambda) = \lambda^{-\frac{1}{2}}$ we retrieve (1.2). For example, in the family of those inequalities one finds the following:

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f^{'}|}{|f|^\theta}\right)^p dx \leq \left(\frac{p-1}{|1-\theta p|}\right)^{\frac{p}{2}} \int_{\{x:f(x)\neq 0\}} \left(\frac{\sqrt{|ff^{''}|}}{|f|^\theta}\right)^p dx, \tag{1.4}$$

linking (1.1) and (1.2), see [8], Proposition 6.1 for details.
It appears that inequality (1.3) can be applied to the regularity theory in nonlinear boundary value problems, see [8], Section 7.

In this paper we are inspired by the original motivation of inequality (1.2) due to Maz'ya. Namely, Maz'ya applied (1.2) as a key tool to obtain the capacitary inequality:

$$\int_{\Omega} \mathrm{cap}_p^+(\mathcal{N}_t, \Omega) t^{p-1} dt \leq C \int_{\Omega} |\nabla^{(2)} u(x)|^p dx, \tag{1.5}$$

where $\mathcal{N}_t = \{x \in \Omega : u(x) \geq t\}$,

$$\mathrm{cap}_p^+(E, \Omega) := \inf \left\{ \int_{\Omega} |\nabla^{(2)} u|^p dx : u \in C_0^\infty(\Omega), u \geq 0 \text{ on } \Omega, u \equiv 1 \text{ in a neighborhood of } E \right\},$$

whenever E is compactly included in Ω.
Let μ is a given Borel measure defined on open set Ω, N be the given N-function, N^* be the

Legendre transform of N and $L_N(\Omega,\mu)$ be an Orlicz space related to N (see Section 2). It is proven in [10], Theorem 8.3.1 (the original statement is given for $\mathbb{R}^n$, but the presented proof applies to the situation described below almost without changes) that the following statements (a) and (b) are equivalent:
(a) The embedding:

$$\| |u|^p \|_{L_N(\Omega,\mu)} \leq A \|\nabla^{(2)} u\|^p_{L^p(\Omega)} \tag{1.6}$$

holds for every nonnegative $u \in C_0^\infty(\Omega)$, with a u-independent finite constant A.
(b) The following isoperimetric inequality:

$$\mu(E)(N^*)^{-1}\left(\frac{1}{\mu(E)}\right) \leq B\mathrm{cap}_p^+(E,\Omega) \tag{1.7}$$

holds for every compact $E \subset \Omega$, such that $\mathrm{cap}_M^+(E,\Omega) > 0$.

Moreover, if A and B are the best constants in (1.6) and (1.7), respectively, then $B \leq A \leq pBC$, where C is the same as in (1.5).

One could ask about the validity of a more general embedding:

$$\|M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_\Omega M(|\nabla^{(2)} u|)dx, \tag{1.8}$$

where $u \in C_0^\infty(\Omega)$ is nonnegative, with some u-independent constant A, with a (possibly) general convex function M instead of λ^p. It appears that, under suitable assumptions on M, (1.8) is equivalent to the isoperimetric inequality:

$$\mu(E)(N^*)^{-1}\left(\frac{1}{\mu(E)}\right) \leq B\mathrm{cap}_M^+(E,\Omega), \tag{1.9}$$

holding over all compact sets $E \subset \Omega$ such that $\mathrm{cap}_M^+(E,\Omega) > 0$, where

$$\mathrm{cap}_M^+(E,\Omega) := \inf\left\{\int_\Omega M(|\nabla^{(2)} u|)dx : u \in C_0^\infty(\Omega), u \geq 0 \text{ on } \Omega, u \equiv 1 \text{ in a neighborhood of } E\right\}.$$

The precise statement is given in Section 4, Proposition 4.2.
Among these conditions on M, one requires the capacitary inequality:

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t,\Omega)dM(t) \leq \tilde{C}\int_\Omega M\left(|\nabla^{(2)} u|\right)dx, \tag{1.10}$$

holding for all nonnegative $u \in C_0^\infty(\Omega)$, with a constant $\tilde{C}$ not depending on u. As we show (see Proposition 4.1), inequality (1.10) follows from the following generalization of inequality (1.2):

$$\int_{\{x: f(x)>0\}} M\left(\frac{|f'|}{\sqrt{f}}\right)dx \leq \tilde{D}\int_{\mathbb{R}} M\left(\sqrt{|f''|}\right)dx, \tag{1.11}$$

holding for smooth, nonnegative, compactly supported functions f. With this in mind, we first derive the generalization of inequality (1.3) (see Proposition 3.1 for the precise statement):

$$\int_{\mathbb{R}} M(|f^{'}|h(f))dx \leq D \int_{\mathbb{R}} M\left(\sqrt{|f^{''}\mathcal{T}_h(f)|} \cdot h(f)\right)dx, \tag{1.12}$$

then we apply its special variant (1.11) to prove inequality (1.10) and consequently the equivalence of (1.8) and (1.9). The precise arguments are provided in Section 4. To the best of our knowledge such Orlicz generalizations of Mazya's capacitary estimates and isoperimetric inequalities are missing in the literature.

Let us mention that the Orlicz variant of inequality (1.1), i.e. the inequality

$$\int_{\mathbb{R}} M(|f^{'}|)dx \leq C \int_{\mathbb{R}} M\left(\sqrt{|f^{''}f|}\right)dx, \tag{1.13}$$

was obtained in [7] as a special case of the related inequality in n-dimensions. In particular, inequality (1.12) generalizing (1.3), links together inequalities (1.13) and (1.11), which are Orlicz extensions of inequalities (1.1) and (1.2).

Apart from the purely theoretical approach, we hope that inequality (1.12) can serve as a tool to derive apriori estimates. For example, it can possibly play a similar role in nonlinear eigenvalue problems as that played by $M(\lambda) = \lambda^p$, see [8].

We also hope to contribute to the investigation of the theory of Sobolev spaces, with particular emphasis on embedding theorems, see e.g. [1],[2],[10].

2 Notation and preliminaries

General notation. In general we assume that Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$ and we use the standard notation: $C_0^\infty(\Omega)$, $W^{m,p}(\Omega)$ and $W^{m,p}_{loc}(\Omega)$ for smooth compactly supported functions and global and local Sobolev functions defined on Ω, respectively. By $\nabla^{(2)}u(x)$ we denote the Hesse matrix of the function u at the point x. We will also be dealing with the special situation when $\Omega = I \subseteq \mathbb{R}$ is an interval (finite or not). If $A \subseteq \mathbb{R}$ and f is defined on A, by $f\chi_A$ we denote the extension of f by zero outside set A. More generally, if f is defined on A and $g : \mathbb{R} \to \mathbb{R}$ is zero outside A, by fg we denote the extension of fg by zero outside A. In the sequel $M : [0, \infty) \to [0, \infty)$ is continuous and locally absolutely continuous on $(0, \infty)$. By dM we denote the measure $M^{'}(t)dt$ defined on $[0, \infty)$.
We will be dealing with integrals of the form: $\int_{\mathbb{R}} M\left(|f^{'}|c(f)\right)dx$ and $\int_{\mathbb{R}} M\left(\sqrt{|f^{''}|}c(f)\right)dx$, where f is nonnegative, $M(0) = 0$, $c : (0, \infty) \to (0, \infty)$ is a continuous function and it might not be defined at zero. In all such cases we note that on the set $A = \{x : f(x) = 0\}$ we have $f^{'} = 0$ and $f^{''} = 0$ almost everywhere, so that functions $|f^{'}|c(f)$ and $\sqrt{|f^{''}|}c(f)$ are by our earlier definition equal zero almost everywhere on A. In particular such integrals are interpreted as $\int_{\{x:f(x)>0\}} M\left(|f^{'}|c(f)\right)dx$ and $\int_{\{x:f(x)>0\}} M\left(\sqrt{|f^{''}|}c(f)\right)dx$, respectively.

The special transform. The following definition will be crucial for our considerations.

Definition 2.1. *Let $h : (0,\infty) \to (0,\infty)$ be a given continuous function and let H be a locally absolutely continuous on $(0,\infty)$ primitive of h. We define the transform of h:*

$$\mathcal{T}_h(\lambda) := \frac{H(\lambda)}{h(\lambda)},$$

for all $\lambda \in (0,\infty)$.

N-functions. By an $N-$function we will call any function $M : [0,\infty) \to [0,\infty)$ which is convex and satisfies the conditions: $M(\lambda)/\lambda \to 0$ as $\lambda \to 0$ and $M(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$. In the sequel we use the following assumptions:

(M) $M : [0,\infty) \to [0,\infty)$ is a differentiable $N-$function, moreover, M satisfies the condition:

$$d_M \frac{M(\lambda)}{\lambda} \le M'(\lambda) \le D_M \frac{M(\lambda)}{\lambda} \quad \text{for every } \lambda > 0, \tag{2.1}$$

where $D_M \ge d_M \ge 2$.

(M1) $M : [0,\infty) \to [0,\infty)$ is a differentiable $N-$function and M satisfies inequality (2.1) with $D_M \ge d_M > 1$.

(h) $h : (0,\infty) \to (0,\infty)$ is locally Lipschitz and $H : (0,\infty) \to \mathbb{R}$ is its locally absolutely continuous primitive.

Remark 2.1. **(1)** The latter inequality in (2.1) implies that M satisfies Δ_2-condition: $M(2\lambda) \le CM(\lambda)$, with constant C independent of λ (see e.g. [9], Theorem 4.1).

(2) The condition $d_M > 1$ in (2.1) is equivalent to the Δ_2-condition for M^*, where $M^*(x) = \sup\{xy - M(y) : y > 0\}$ is the Legendre transform of f (see e.g. [9], Theorem 4.3). Moreover, for any $N-$function M, the left-hand side in (2.1) holds with $d_M = 1$.

(3) The condition $d_M \frac{M(\lambda)}{\lambda} \le M'(\lambda)$ is equivalent to the fact that $\frac{M(\lambda)}{\lambda^{d_M}}$ is nondecreasing. To verify this it suffices to compute $\left(M(\lambda)\lambda^{-d_M}\right)'$. Analogously, the condition $D_M \frac{M(\lambda)}{\lambda} \ge M'(\lambda)$ is equivalent to the fact that $\frac{M(\lambda)}{\lambda^{D_M}}$ is nonincreasing.

(4) If d_M and D_M are the best possible constants in (2.1), they obey the definition of Simonenko lower and upper index of M and are related to Boyd indices of $L^M(\mathbb{R}^n,\mu)$ (see [3], [11] for definitions, [4], [5], [12] for discussion on those and other indices of Orlicz spaces).

(5) If M is a differentiable $N-$function which satisfies **(M)** and $\frac{M(\lambda)}{\lambda^2}$ is nondecreasing then the function

$$M_1(\lambda) := \frac{M(|\lambda|)}{\lambda} \chi_{\{\lambda\ne 0\}}, \tag{2.2}$$

defined on $\mathbb{R}$, is locally Lipschitz. To verify this we note that when $\lambda > 0$, we have $0 \le M_1'(\lambda) = \frac{\lambda M'(\lambda) - M(\lambda)}{\lambda^2} \le (D_M - 1)\frac{M(\lambda)}{\lambda^2}$, so M_1' is bounded in every neighborhood of zero.

We will use the following lemmas.

Lemma 2.1. *Suppose that M is an N-function such that $\frac{M(\lambda)}{\lambda^2}$ is nondecreasing. Then for all $a, b > 0$ we have*

$$\frac{M(a)}{a^2} b^2 \leq M(a) + M(b). \tag{2.3}$$

Proof. If $a \geq b$ then $\frac{M(a)}{a^2} b^2 \leq \frac{M(a)}{a^2} a^2 = M(a)$. If $b > a$ then $\frac{M(a)}{a^2} \leq \frac{M(b)}{b^2}$ and $\frac{M(a)}{a^2} b^2 \leq \frac{M(b)}{b^2} b^2 = M(b)$. □

Lemma 2.2 ([9])**.** *Suppose that M is an N-function satisfying* **(M)***. Then for every $r > 0$ and $\lambda > 0$*

$$\min(\lambda^{d_M}, \lambda^{D_M}) M(r) \leq M(\lambda r) \leq \max(\lambda^{d_M}, \lambda^{D_M}) M(r).$$

Orlicz spaces. Let M be an $N-$function. By $L_M(\Omega, \mu)$ we denote the space of all real, μ-measurable functions, for which

$$\|u\|_{L_M(\Omega,\mu)} = \sup \left\{ \left| \int_\Omega uv d\mu \right| : \int_\Omega M^*(v) d\mu \leq 1 \right\} < \infty,$$

where μ is an arbitrary measure. It is known that $L_M(\Omega, \mu)$ is a Banach space with the norm $\| \cdot \|_{L_M(\Omega,\mu)}$, and we have

$$\|\chi_E\|_{L_M(\Omega,\mu)} = \mu(E) M^{*-1}\left(\frac{1}{\mu(E)}\right), \tag{2.4}$$

$$\int_\Omega M(|u(x)|) d\mu(x) = \int_0^\infty M'(t) \mu(\mathcal{N}_t) dt = \int_0^\infty \mu(\mathcal{N}_t) dM(t), \tag{2.5}$$

where $\mathcal{N}_t = \{x \in \Omega : u(x) \geq t\}$. Proofs of (2.4) and (2.5) can be found in [10], Sections 2.3.2 and 1.2.3.

Capacities. We will be using the following notion of capacity.

Definition 2.2. *Let $E, F \subset \Omega$ and suppose that E is compactly included in F. Assume further that M is an N-function. We define the capacity of E with respect to F as follows:*

$$\mathrm{cap}_M^+(E, F) := \inf \left\{ \int_\Omega M\left(|\nabla^{(2)} u(x)| \right) dx : \ u \in \mathcal{G} \right\},$$

where

$$\mathcal{G} := \{u \in C_0^\infty(F), \ u \geq 0 \text{ on } F, \ u \equiv 1 \text{ in a neighborhood of } E\}.$$

We recall some well known properties of the capacities ([10]).

Proposition 2.1. *Let $E_1 \subset E_2 \subset F_1 \subset F_2$ be subsets of Ω, such that E_1 and E_2 are compactly included in F_1. Then we have*

1. $\operatorname{cap}_M^+(E_1,F_1) \le \operatorname{cap}_M^+(E_2,F_1)$,
2. $\operatorname{cap}_M^+(E_1,F_1) \ge \operatorname{cap}_M^+(E_1,F_2)$.

The useful lemma. In the sequel we will use the following simple observation.

Lemma 2.3 ([10]). *If $0 < R \le +\infty$, $f : [-R,R] \to [\alpha,\beta]$ is absolutely continuous and $L : [\alpha,\beta] \to \mathbb{R}$ is Lipshitz, then the function $(L \circ f)(x) := L(f(x))$ is absolutely continuous on $[-R,R]$.*

3 Interpolation inequality

Our goal is to obtain the following result.

Proposition 3.1. *Suppose that M satisfies* **(M)**, *$h : (0,\infty) \to (0,\infty)$ satisfies* **(h)**,

$$C(y) = \inf \left\{ C(\delta, y) := \frac{\delta \max\{(\frac{1}{\delta})^{\frac{D_M}{2}}, (\frac{1}{\delta})^{\frac{d_M}{2}}\}}{y-\delta} : \ 0 < \delta < y \right\}, \text{ where } y > 0, \tag{3.1}$$

and moreover, we have either (i), (ii) or (iii), where

(i)

- **(a)** $|h^{'}H| \le Eh^2$ *and* $(D_M - 1)E < 1$,
- **(b)** *the function h is either nonincreasing or locally bounded in a neighborhood of zero,*
- **(c)** *for every $a > 0$, the function $(0,\infty) \ni \lambda \mapsto M\left(\sqrt{a\mathcal{T}_h(\lambda)}h(\lambda)\right)$ is either nonincreasing or locally bounded in a neighborhood of zero,*
- **(d)** $y := \frac{1}{D_M-1} - E$;

(ii)

- **(a)** $|h^{'}H| \ge eh^2$, $e(d_M - 1) > 1$,
- **(b)** *h is nonincreasing,*
- **(c)** *for every $a > 0$, the function $(0,\infty) \ni \lambda \mapsto M\left(\sqrt{a\mathcal{T}_h(\lambda)}h(\lambda)\right)$ is either nonincreasing or locally bounded in a neighborhood of zero,*
- **(d)** $y := \frac{e(d_M-1)-1}{D_M-1}$;

(iii)

- **(a)** $|h^{'}H| \ge eh^2$,
- **(b)** *h is nondecreasing,*
- **(c)** *for every $a > 0$, the function $(0,\infty) \ni \lambda \mapsto M\left(\sqrt{a\mathcal{T}_h(\lambda)}h(\lambda)\right)$ is either nonincreasing or locally bounded in a neighborhood of zero,*
- **(d)** $y := \frac{(d_M-1)e+1}{D_M-1}$.

Then for every nonnegative $f \in W^{2,1}(\mathbb{R})$ such that f' is compactly supported, we have

$$\int_{\mathbb{R}} M(|f'(x)|h(f(x)))dx \le C(y) \int_{\mathbb{R}} M\left(\sqrt{|f''(x)\mathcal{T}_h(f(x))|} \cdot h(f(x))\right) dx.$$

Proof. Let $f \in W^{2,1}_{loc}(\mathbb{R})$ be a given nonnegative function and

$$f_\epsilon \ := \ f + \epsilon.$$

Denote further

$$I(\epsilon) := \int_{\mathbb{R}} M(|f'|h(f_\epsilon))dx, \qquad J(\epsilon) := \int_{\mathbb{R}} M\left(\sqrt{|f''\mathcal{T}_h(f_\epsilon)|} \cdot h(f_\epsilon)\right) dx.$$

Then we have

$$I(\epsilon) = \int_{\mathbb{R}} \frac{M(|f_\epsilon'|h(f_\epsilon))}{f_\epsilon' h(f_\epsilon)} \cdot f_\epsilon' h(f_\epsilon) \chi_{\{x: f_\epsilon'(x) \neq 0\}} dx.$$

As $d_m \ge 2$, the function $M_1(\lambda)$ is locally Lipschitz (see Remark 2.1, parts (3), (5), assumption **(M)**). Moreover, $f_\epsilon' h(f_\epsilon)$ belongs to $W^{1,1}(\mathbb{R})$ and is compactly supported and bounded, in particular $f_\epsilon' h(f_\epsilon)$ is absolutely continuous on $\mathbb{R}$. Therefore $M_1(|f_\epsilon'|h(f_\epsilon))$ is absolutely continuous on $\mathbb{R}$ and is compactly supported (see Lemma 2.3). By similar arguments the function $H(f_\epsilon)$ is locally absolutely continuous on $\mathbb{R}$ and we have $(H(f_\epsilon))' = f_\epsilon' h(f_\epsilon)$ (in sense of distributions and almost everywhere). This allows us to integrate by parts in the expression above to get:

$$I(\epsilon) = \int_{\mathbb{R}} \left(M_1(f_\epsilon' h(f_\epsilon))\right) \cdot (H(f_\epsilon))' dx = -\int_{\mathbb{R}} \left(M_1(f_\epsilon' h(f_\epsilon))\right)' \cdot H(f_\epsilon) dx.$$

From now the proof follows under assumptions (i), (ii) and (iii) considered separately.
(i):
Note that

$$\begin{aligned} 0 \le M_1'(\lambda) &= \frac{|\lambda| M'(|\lambda|) - M(|\lambda|)}{\lambda^2} \le (D_M - 1)\frac{M(|\lambda|)}{\lambda^2}, \qquad (3.2)\\ (M_1(\lambda_{f_\epsilon}))' &= M_1'(\lambda_{f_\epsilon})\left\{f'' h(f_\epsilon) + (f')^2 h'(f_\epsilon)\right\}, \text{ where}\\ \lambda_{f_\epsilon} &:= f' h(f_\epsilon). \end{aligned}$$

Consequently

$$\begin{aligned} I(\epsilon) &\le \left|\int_{\mathbb{R}} \left(M_1(f_\epsilon' h(f_\epsilon))\right)' \cdot H(f_\epsilon) dx\right| \le (D_M - 1)\int_{\mathbb{R}} \frac{M(|\lambda_{f_\epsilon}|)}{\lambda_{f_\epsilon}^2} \left|f'' h(f_\epsilon) H(f_\epsilon)\right| dx \\ &+ (D_M - 1)\int_{\mathbb{R}} \frac{M(|\lambda_{f_\epsilon}|)}{\lambda_{f_\epsilon}^2} (f')^2 \left|h'(f_\epsilon) H(f_\epsilon)\right| dx =: A + B. \qquad (3.3) \end{aligned}$$

We apply Lemma 2.1 to estimate

$$\begin{aligned} A &= (D_M-1)\delta \int_{\mathbb{R}\cap\{x: f'(x)\neq 0\}} \left(\frac{M(|f'|h(f_\epsilon))}{(f'h(f_\epsilon))^2} \right) \cdot \left(\frac{|f''h(f_\epsilon)H(f_\epsilon)|}{\delta} \right) dx \leq \\ &\leq (D_M-1)\delta \int_{\mathbb{R}} M(|f'|h(f_\epsilon))dx + (D_M-1)\delta \int_{\mathbb{R}} M\left(\sqrt{\frac{|f''H(f_\epsilon)|h(f_\epsilon)}{\delta}} \right) dx. \end{aligned}$$

The first integral equals $(D_M-1)\delta I(\epsilon)$. To estimate the second one we note that

$$|f''H(f_\epsilon)h(f_\epsilon)| = |f''\mathcal{T}_h(f_\epsilon)|h^2(f_\epsilon)$$

and that according to Lemma 2.2: $M(\frac{a}{\sqrt{\delta}}) \leq \max\{(\frac{1}{\delta})^{\frac{D_M}{2}}, (\frac{1}{\delta})^{\frac{d_M}{2}}\}M(a)$. This implies:

$$\begin{aligned} A &\leq (D_M-1)\delta I(\epsilon) + (D_M-1)\delta \cdot \max\left\{ \left(\frac{1}{\delta}\right)^{\frac{D_M}{2}}, \left(\frac{1}{\delta}\right)^{\frac{d_M}{2}} \right\} \int_{\mathbb{R}} M\left(\sqrt{|f''\mathcal{T}_h(f_\epsilon)|} \cdot h(f_\epsilon) \right) dx \\ &= (D_M-1)\delta I(\epsilon) + (D_M-1)\delta \cdot \max\left\{ \left(\frac{1}{\delta}\right)^{\frac{D_M}{2}}, \left(\frac{1}{\delta}\right)^{\frac{d_M}{2}} \right\} J(\epsilon). \end{aligned} \tag{3.4}$$

On the other hand, we have

$$B \leq (D_M-1)E \int_{\mathbb{R}} \frac{M(|f'|h(f_\epsilon))}{(f'h(f_\epsilon))^2} \left(f'h(f_\epsilon) \right)^2 dx = (D_M-1)EI(\epsilon). \tag{3.5}$$

Combining estimations (3.4) and (3.5) we obtain

$$I(\epsilon) \leq (D_M-1)(\delta+E)I(\epsilon) + (D_M-1)\delta \cdot \max\left\{ \left(\frac{1}{\delta}\right)^{\frac{D_M}{2}}, \left(\frac{1}{\delta}\right)^{\frac{d_M}{2}} \right\} J(\epsilon).$$

Consequently, when $(D_M-1)(\delta+E) < 1$, we have $\delta < y \leq 1$ and

$$I(\epsilon) \leq C(\delta, y)J(\epsilon), \tag{3.6}$$

where in our case $C(\delta, y) := \frac{\delta^\alpha}{y-\delta}$, $\alpha = 1 - \frac{D_M}{2} \leq 0$, $y = \frac{1}{D_M-1} - E > 0$. The minimization of $C(\delta, y)$ with respect to $\delta < y$ gives the inequality with the constant $C(y)$, achieved at $\delta_y = \frac{y\left(\frac{D_M}{2}-1\right)}{\frac{D_M}{2}} \in [0, y)$.

Now we will let ϵ converge to zero in the inequality $I(\epsilon) \leq CJ(\epsilon)$. Observe that

$$\begin{aligned} I(\epsilon) &= \int_{\mathbb{R}} M(|f'|h(f_\epsilon))dx = \int_{\mathbb{R}\cap\{x:|f(x)|<\bar{\delta}\}} M(|f'|h(f_\epsilon))dx + \int_{\mathbb{R}\cap\{x:|f(x)|\geq\bar{\delta}\}} M(|f'|h(f_\epsilon))dx \\ &=: C(\epsilon) + D(\epsilon). \end{aligned}$$

When $\epsilon < \bar{\delta}$ for $|f(x)| \geq \bar{\delta}$, we have $\bar{\delta} \leq f_\epsilon(x) \leq f(x) + \bar{\delta} \leq \|f\|_\infty + \bar{\delta}$. Therefore

$$M(|f'|h(f_\epsilon)) \leq \sup\{M(\|f'\|_\infty \cdot h(\lambda)) : \lambda \in [\bar{\delta}, \|f\|_\infty + \bar{\delta}]\} < \infty$$

and by Lebegue's Dominated Convergence Theorem we obtain

$$D(\epsilon) \to D = \int_{\mathbb{R}\cap\{x:|f(x)|\geq\bar{\delta}\}} M(|f^{'}|h(f))dx.$$

On the other hand, when $\epsilon < \bar{\delta}$, for the sufficiently small $\bar{\delta}$ and $|f(x)| < \bar{\delta}$, we can assume that $\epsilon \mapsto h(f_\epsilon(x))$ either increases when ϵ converges to zero or it is bounded by a constant independent of x. Therefore by Lebegue's Monotonic Convergence Theorem or by Lebegue's Dominated Convergence Theorem we obtain:

$$C(\epsilon) \to C = \int_{\mathbb{R}\cap\{x:|f(x)|<\bar{\delta}\}} M(|f^{'}|h(f))dx.$$

This implies $I(\epsilon) \to \int_{\mathbb{R}} M(|f^{'}|h(f))dx$. Similar arguments applied to the right hand side in (3.6) give:

$$J(\epsilon) \to J := \int_{\mathbb{R}} M\left(\sqrt{|f^{''}\mathcal{T}_h(f)|}\cdot h(f)\right)dx.$$

This finishes the proof of part (i).

(ii):
In this case instead of (3.3) we use the precise equation:

$$I(\epsilon) = -\int_{\mathbb{R}} M_1^{'}(f^{'}h(f_\epsilon))f^{''}h(f_\epsilon)H(f_\epsilon)dx + \int_{\mathbb{R}} M_1^{'}(f^{'}h(f_\epsilon))(f^{'})^2(-1)h^{'}(f_\epsilon)H(f_\epsilon)dx. \tag{3.7}$$

Moreover, as $(-1)h^{'} = |h^{'}|$, the integrand in the second term above equals:

$$M_1^{'}(f^{'}h(f_\epsilon))(f^{'})^2|h^{'}(f_\epsilon)|H(f_\epsilon) = \left(|\lambda_{f_\epsilon}|M^{'}(|\lambda_{f_\epsilon}|) - M(|\lambda_{f_\epsilon}|)\right)\frac{|h^{'}|(f_\epsilon)H(f_\epsilon)}{h^2(f_\epsilon)},$$

under notation (3.2). Therefore we can equivalently write:

$$\begin{aligned}\mathcal{L} := \int_{\mathbb{R}}\left\{\left(|\lambda_{f_\epsilon}|M^{'}(|\lambda_{f_\epsilon}|) - M(|\lambda_{f_\epsilon}|)\right)\frac{|h^{'}(f_\epsilon)|H(f_\epsilon)}{h^2(f_\epsilon)} - M(|\lambda_{f_\epsilon}|)\right\}dx = \\ = \int_{\mathbb{R}} M_1^{'}(f^{'}h(f_\epsilon))f^{''}h(f_\epsilon)H(f_\epsilon) \leq A,\end{aligned} \tag{3.8}$$

where A is the same as in (3.3). On the other hand, by our assumption **(M)** and by (iia), we have

$$\begin{aligned}|\lambda_f|M^{'}(|\lambda_{f_\epsilon}|) - M(|\lambda_{f_\epsilon}|) &\geq (d_M-1)M(|\lambda_{f_\epsilon}|),\\ \frac{|h^{'}(s)H(s)|}{h^2(s)} &\geq e \text{ for every } s.\end{aligned}$$

This gives

$$\mathcal{L} \geq [e(d_M-1)-1]\int_{\mathbb{R}} M(|\lambda_{f_\epsilon}|)dx = [e(d_M-1)-1]I(\epsilon).$$

This combined with (3.8) and (3.4) gives:

$$[e(d_M-1)-1]I(\epsilon) \le (D_M-1)\delta I(\epsilon) + (D_M-1)\delta \max\left\{ \left(\frac{1}{\delta}\right)^{\frac{D_M}{2}}, \left(\frac{1}{\delta}\right)^{\frac{d_M}{2}} \right\} J(\epsilon).$$

Now it suffices to rearrange to get (3.6), where: $\delta < y = \frac{e(d_M-1)-1}{D_M-1}$ and $y > 0$. This assumption does not necessarily force the condition $\delta < 1$. Minimization with respect to $y > \delta > 0$ gives inequality (3.6) with constant $C(y)$. Finally, we let ϵ converge to zero and complete the the proof in the same way as we have finished the proof of part (ii).

(iii):
Now $h' \ge 0$ and so we can modify (3.7) to get

$$\begin{aligned} \mathcal{L} := \int_{\mathbb{R}} \left\{ \left(|\lambda_{f_\epsilon}| M'(|\lambda_f|) - M(|\lambda_{f_\epsilon}|) \right) \frac{|h'(f_\epsilon) H(f_\epsilon)|}{h^2(f_\epsilon)} + M(|\lambda_{f_\epsilon}|) \right\} dx \\ = |\int_{\mathbb{R}} M_1'(\lambda_{f_\epsilon}) f'' h(f_\epsilon) H(f_\epsilon) dx| \le A, \end{aligned} \tag{3.9}$$

under the same notation. This, our assumptions and (3.4) imply:

$$((d_M-1)e+1)I(\epsilon) \quad \le \quad \mathcal{L} \le (D_M-1)\delta I(\epsilon) + (D_M-1)\delta \max\left\{ \left(\frac{1}{\delta}\right)^{\frac{D_M}{2}}, \left(\frac{1}{\delta}\right)^{\frac{d_M}{2}} \right\} J(\epsilon).$$

This implies (3.6) with $y = \frac{(d_M-1)e+1}{D_M-1}$. Minimization of constants, then the final step, letting ϵ converge to zero and final conclusion follows by almost the same arguments. □

Remark 3.1. As presented in the proof, in case of the assumption (i) in Proposition 3.1 we compute that

$$C := \left\{ \frac{1}{D_M-1} - E \right\}^{-\frac{D_M}{2}} \left(\frac{D_M}{2} - 1 \right)^{1-\frac{D_M}{2}} \left\{ \frac{D_M}{2} \right\}^{\frac{D_M}{2}}$$

(where we interpret 0^0 as 1). We omit the presentation of other constants as they are rather complicated.

Corollary 3.1. *Let $\alpha \in \mathbb{R} \setminus \{-1, 0\}$ and $e(\alpha) := \frac{|\alpha|}{|\alpha+1|} > \frac{1}{d_M-1}$ in case $-1 < \alpha < 0$. Then there exists a constant $C = C(\alpha) > 0$ such that for every nonnegative $f \in W^{2,1}(\mathbb{R})$, such that f' is compactly supported, we have*

$$\int_{\mathbb{R}} M(|f'| f^\alpha) dx \le C(\alpha) \int_{\mathbb{R}} M\left(\sqrt{|\alpha+1|^{-1} |f'' f|} \cdot f^\alpha \right) dx. \tag{3.10}$$

Moreover, $C(\alpha) \le C(y_\alpha)$, where $C(y)$ is defined by (3.1), $y_\alpha = \frac{e(\alpha)(d_M-1)+1}{D_M-1}$ in case $\alpha > 0$, $x_\alpha = \frac{e(\alpha)(d_M-1)-1}{D_M-1}$ in case when $\alpha < 0$.

Proof. For $\alpha > 0$ we easily verify condition (iii), while if $\alpha < -1$ we have condition (ii), as then $e = E = \frac{|\alpha|}{|\alpha+1|} > 1 > \frac{1}{d_M - 1}$. If $-1 < \alpha < 0$ the condition $e = \frac{|\alpha|}{|\alpha+1|} > \frac{1}{d_M-1}$ must be assumed and then again we can use condition (ii). □

Remark 3.2. In case of $\alpha > 0$, under the assumption $E(\alpha) := \frac{\alpha}{\alpha+1} < \frac{1}{D_M-1}$ estimation (3.10) follows also from assumption (i) in Proposition 3.1 (obtained with a different constant).

Remark 3.3. Of our special interest is the case $\alpha = -\frac{1}{2}$, $d_M > 2$. Then $e = E = 1$, so the condition $e > \frac{1}{d_M-1}$ is satisfied. In that case we have $y = \frac{d_M-2}{D_M-1} < 1$, so that $C(\delta, y) := \frac{\delta^\kappa}{y-\delta}$, $\kappa = 1 - \frac{D_M}{2} < 0$. Its minimum is achieved at $\delta_y = \frac{y\left(\frac{D_M}{2}-1\right)}{\frac{D_M}{2}}$. As now $\sqrt{\mathcal{T}_h(f)}h(f) \equiv \sqrt{2}$, we obtain the inequality

$$\int_{\mathbb{R}} M\left(\frac{|f'|}{\sqrt{f}}\right) dx \leq \tilde{C} \int_{\mathbb{R}} M\left(\sqrt{2|f''|}\right) dx, \tag{3.11}$$

satisfied for every nonnegative function $f \in W^{2,1}(\mathbb{R})$, such that f' is compactly supported with $\tilde{C} = \left(\frac{d_M-2}{D_M-1}\right)^{-\frac{D_M}{2}} (\frac{D_M}{2} - 1)^{1-\frac{D_M}{2}} \left(\frac{D_M}{2}\right)^{\frac{D_M}{2}}$.

When $M(\lambda) = \lambda^{2p}$, we have $d_M = D_M = 2p$ and we obtain the inequality

$$\int_{\mathbb{R}} \frac{|f'|^{2p}}{f^p} dx \leq C_p \int_{\mathbb{R}} |f''|^p dx,$$

$C_p = \left\{ \left(\frac{2(p-1)}{2p-1}\right)^{-p} (p-1)^{1-p} p^p \right\} \cdot 2^p = \left(\frac{2p-1}{p-1}\right)^p \cdot \left(p^p (p-1)^{1-p}\right)$. Mazya, [10], Lemma 1, Section 8.2.1, obtained the same inequality with constant $B_p = \left(\frac{2p-1}{p-1}\right)^p$, which is better than our. Note that $\left(p^p(p-1)^{1-p}\right) = \left(1 + \frac{1}{p-1}\right)^{p-1} \cdot p \leq ep$. We obtain the estimation with a bigger constant because in our general proof we cannot use the property $M(ab) = M(a)M(b)$, which is the case of $M(\lambda) = \lambda^{2p}$ but does not hold in general.

Remark 3.4. Let $M(\lambda) = \lambda^p$, $p \geq 2$, $h : (0, \infty) \to (0, \infty)$ be as in Proposition 3.1 (in particular $d_M = D_M = p$) and let $\tilde{h}(\lambda) = (h(\lambda))^p$. Proposition 3.1 implies the inequality

$$\int_{\mathbb{R}} |f'|^p \tilde{h}(f) dx \leq C \int_{\mathbb{R}} \sqrt{|f'' \mathcal{T}_h(f)|^p} \tilde{h}(f) dx, \tag{3.12}$$

where $f \in W^{2,1}(\mathbb{R})$ is nonnegative, f' is compactly supported and C is a constant dependent on $\tilde{h}$ and p. Inequalities like (3.12) were obtained earlier in [8], Propositions 4.1, 4.2 and 4.3, under more general assumptions on $\tilde{h}$ and f, with the constant $C = \left(\sqrt{p-1}\right)^p$ independent of $\tilde{h}$.

4 Capacitary estimates and izoperimetric inequalities

As a direct application of Proposition 3.1 we obtain the following capacitary estimate. When $M(\lambda) = \lambda^p$ this result was obtained by Maz'ya in [10], Section 8. To our best knowledge the presented below Orlicz variant of Mazya's results is missing in the literature.

Proposition 4.1. *Let M be an N-function satisfying* **(M1)**. *Assume further that M satisfies the condition*

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty. \tag{4.1}$$

Then for all nonnegative $u \in C_0^\infty(\Omega)$, we have

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega) dM(t) \le \tilde{C}\int_\Omega M\left(|\nabla^{(2)}u(x)|\right) dx, \tag{4.2}$$

for some positive constant $\tilde{C}$ not depending on u, where $\mathcal{N}_t := \{x \in \Omega : |u(x)| \ge t\}$.

Remark 4.1. The proof of Proposition 4.1 is based on similar arguments as that given in [10], Section 8, dealing with the p-homogeneous case $M(\lambda) = \lambda^p$, $p > 1$. In our more general case we assume additionally (4.1). Obviously this assumption is satisfied with $L = 1$ in the p-homogeneous case. Moreover, according to Lemma 2.2 we have: $M(1) \le \frac{M(\lambda)}{\lambda^{D_M}} \le \lambda^{d_M - D_M}M(1)$ for $\lambda \le 1$ and $M(1) \le \frac{M(\lambda)}{\lambda^{d_M}} \le \lambda^{D_M - d_M}M(1)$ for $\lambda \ge 1$, in particular $L \ge M(1)$.

The proof will be based on the following lemma.

Lemma 4.1. *Let M be an N-function satisfying condition* **(M1)**, *$n \in \mathbb{N}$ and $u \in C_0^\infty(\mathbb{R}^n)$, $u \ge 0$. Then we have*

$$\int_{\{x:u(x)>0\}} M\left(\frac{|\nabla u(x)|^2}{u(x)}\right) dx \le C\int_{\mathbb{R}^n} M(|\nabla^{(2)}u(x)|)dx,$$

where $C = n^{D_M}2^{D_M}\left(\frac{d_M-1}{D_M-\frac{1}{2}}\right)^{D_M}(D_M-1)^{1-D_M}D_M^{D_M}$.

Proof. As $M(\sum_{i=1}^n a_i) \le \frac{1}{n}\sum_{i=1}^n M(na_i)$ and $|\frac{\partial^2 u}{\partial x_i^2}| \le |\nabla^{(2)}u|$, the proof reduces to the case $n = 1$ (when we substitute nu instead of u). Consider function $\tilde{M}(\lambda) := M(\lambda^2)$. Then $\tilde{M}'(\lambda) = 2\lambda M'(\lambda^2)$ and we easly check that $\tilde{M}$ is an N-function satisfying **(M)**, where $d_{\tilde{M}} = 2d_M$, $D_{\tilde{M}} = 2D_M$. Therefore by inequality (3.11) in Remark 3.3 applied to $\tilde{M}$, we have

$$\begin{aligned}\int_{\{x:u(x)>0\}} M\left(\frac{|u'|^2}{u}\right) dx &= \int_{\{x:u(x)>0\}} \tilde{M}\left(\frac{|u'|}{u^{\frac{1}{2}}}\right) dx \le \tilde{C}\int_{\mathbb{R}} \tilde{M}\left(\sqrt{2|u''|}\right) dx \\ &= \tilde{C}\int_{\mathbb{R}} M\left(2|u''|\right) dx \le 2^{D_M}\tilde{C}\int_{\mathbb{R}} M\left(|u''|\right) dx,\end{aligned}$$

where $\tilde{C} = \left(\frac{d_M-1}{D_M-\frac{1}{2}}\right)^{D_M}(D_M-1)^{1-D_M}D_M^{D_M}$. The last inequality follows from Lemma 2.2. This finishes the proof. □

Proof of Proposition 4.1. Throughout the proof we will denote

$$\begin{aligned}\tau_j &:= \max(2^{d_M j}, 2^{D_M j}), \quad a_j := \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \mathcal{N}_{2^{-j-1}}), \quad b_j := M(2^{-j+1}), \\ d_j &:= \int_{\mathcal{N}_{2^{-j-1}}\setminus\mathcal{N}_{2^{-j}}} M(|\nabla^{(2)}f(u)|)dx, \quad \mathcal{L} := \int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega)dM(t),\end{aligned}$$

where $j \in \mathbb{Z}$ are integer numbers. $>$From now the proof follows by steps.
STEP 1. We show that

$$\mathcal{L} \leq \sum_{j=-\infty}^{+\infty} a_j b_j.$$

To prove this we note that

$$\mathcal{L} = \sum_{j=-\infty}^{+\infty} \int_{2^{-j}}^{2^{-j+1}} \mathrm{cap}_M^+(\mathcal{N}_t, \Omega) dM(t).$$

Proposition 2.1 implies that the function $t \to \mathrm{cap}_M^+(\mathcal{N}_t, \Omega)$ is nonincreasing and $\mathrm{cap}_M^+(\mathcal{N}_t, \Omega) \leq \mathrm{cap}_M^+(\mathcal{N}_t, \mathcal{N}_s)$ for $t > s$. Therefore, we have

$$\mathcal{L} \leq \sum_{j=-\infty}^{+\infty} \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \mathcal{N}_{2^{-j-1}})(M(2^{-j+1}) - M(2^{-j})) \quad \leq \textstyle\sum_{j=-\infty}^{+\infty} a_j b_j. \tag{4.3}$$

This ends the proof of Step 1. STEP 2. We show that

$$\sum_{j=-\infty}^{+\infty} a_j b_j \leq C_1 \int_\Omega M(|\nabla^{(2)} f(u)|) dx,$$

for $C_1 := \max\{4^{D_M} \sup_{\lambda \leq 4} \frac{M(\lambda)}{\lambda^{D_M}}, 4^{d_M} \sup_{\lambda > 4} \frac{M(\lambda)}{\lambda^{d_M}}\} < \infty$ and f to be specified below. To do so, we shall use the smooth truncation procedure. Let $\alpha \in C^\infty[0,1]$ be a nondecreasing function, which is equal to zero in a neighborhood of $t = 0$ and $\alpha \equiv 1$ in a neighborhood of $t = 1$. Suppose further that $f \in C^\infty(0, +\infty)$ is defined on each interval $[2^{-j-1}, 2^{-j}]$ as follows:

$$f(\lambda) = 2^{-j-1} + \alpha\left(\frac{\lambda - 2^{-j-1}}{2^{-j} - 2^{-j-1}}\right)(2^{-j} - 2^{-j-1}).$$

Then the function

$$v(x) := \begin{cases} \frac{f(u) - 2^{-j-1}}{2^{-j} - 2^{-j-1}} = \alpha\left(\frac{u - 2^{-j-1}}{2^{-j} - 2^{-j-1}}\right) = \alpha(2^{j+1}u - 1) & \text{for} \quad x \in \mathcal{N}_{2^{-j-1}} \setminus \mathcal{N}_{2^{-j}}, \\ 1 & \text{for} \quad x \in \mathcal{N}_{2^{-j}}, \\ 0 & \text{for} \quad x \notin \mathcal{N}_{2^{-j-1}}. \end{cases}$$

is admissible in the definition of $\mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \mathcal{N}_{2^{-j-1}})$, moreover $\nabla^{(2)} v = \nabla^{(2)} f(u) \cdot 2^{j+1}$. Therefore we have

$$\begin{aligned} a_j = \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \mathcal{N}_{2^{-j-1}}) \leq \int_{\mathcal{N}_{2^{-j-1}} \setminus \mathcal{N}_{2^{-j}}} M(|\nabla^{(2)} f(u)| \cdot 2^{j+1}) dx \quad &\leq \\ \tau_{j+1} \int_{\mathcal{N}_{2^{-j-1}} \setminus \mathcal{N}_{2^{-j}}} M(|\nabla^{(2)} f(u)|) dx \quad &= \quad \tau_{j+1} d_j, \end{aligned}$$

where the last inequality follows from Lemma 2.2. Thus we have

$$\sum_{j=-\infty}^{+\infty} a_j b_j \leq \sum_{j=-\infty}^{+\infty} \tau_{j+1} d_j b_j \leq \sup_{j \in \mathbb{Z}} (\tau_{j+1} b_j) \cdot \sum_{j=-\infty}^{+\infty} d_j = \sup_{j \in \mathbb{Z}} (\tau_{j+1} b_j) \int_\Omega M(|\nabla^{(2)} f(u)|) dx.$$

An easy computation gives that $\sup_{j\in\mathbb{Z}}(\tau_{j+1}b_j) \le C_1$. This finishes the proof of Step 2.
Step 3. We show that

$$\int_\Omega M(|\nabla^{(2)} f(u)|)dx \le C_2 \int_\Omega M(|\nabla^{(2)} u|)dx,$$

where $C_2 = 2^{D_M-1}(2^{D_M}\|\alpha^{''}\|_\infty^{D_M} C + \|\alpha^{'}\|_\infty^{D_M})$, $C = n^{D_M} 2^{D_M} \left(\frac{d_M-1}{D_M-\frac{1}{2}}\right)^{D_M} (D_M-1)^{1-D_M} D_M^{D_M}$.
By the very definition of the function f we have $\|f^{'}\|_\infty \le \|\alpha^{'}\|_\infty$ and
$|f^{''}(\lambda)| = 2^{j+1}|\alpha^{''}(2^{j+1}\lambda - 1)| \le \frac{2\|\alpha^{''}\|_\infty}{\lambda}$ (as $\alpha^{'}$ is supported in $[0,1]$). We have:

$$\int_\Omega M(|\nabla^{(2)} f(u)|)dx = \int_\Omega M\left(\left|\nabla u \otimes \nabla u \cdot f^{''}(u) + \nabla^{(2)} u \cdot f^{'}(u)\right|\right)dx,$$

where the symbol $a \otimes b$ denotes the tensor product of vectors $a, b \in \mathbb{R}^n$, i. e. the $n \times n$ matrix with $a_i b_j$ on place (i,j). By the monotonicity and the convexity of M (observe that $M(a+b) \le \frac{1}{2}(M(2a) + M(2b))$), this is no bigger than:

$$\frac{1}{2}\int_\Omega M\left(4\|\alpha^{''}\|_\infty \frac{|\nabla u|^2}{u}\right)dx + \frac{1}{2}\int_\Omega M\left(2\|\alpha^{'}\|_\infty |\nabla^{(2)} u|\right)dx \le$$
$$B_1 \int_\Omega M\left(\left|\frac{(\nabla u)^2}{u}\right|\right)dx + B_2 \int_\Omega M\left(|\nabla^{(2)} u|\right)dx,$$

where $B_1 = 2^{2D_M-1}\|\alpha^{''}\|_\infty^{D_M}$, $B_2 = 2^{D_M-1}\|\alpha^{'}\|_\infty^{D_M}$. To calculate B_1 and B_2 we used the fact that $1 \le \|\alpha^{'}\|_\infty$ and $1 \le \|\alpha^{''}\|_\infty$. This follows from the following estimations: $1 = |\alpha(1) - \alpha(0)| \le \|\alpha^{'}\|_\infty$ and $1 \le \|\alpha^{'}\|_\infty = |\alpha^{'}(\lambda_0) - \alpha^{'}(0)| = \frac{|\alpha^{'}(\lambda_0) - \alpha^{'}(0)|}{\lambda_0}\lambda_0 \le \|\alpha^{''}\|_\infty$, where we chose $\lambda_0 \in (0,1)$, such that $\alpha^{'}(\lambda_0) = \|\alpha^{'}\|_\infty$. According to 4.1 we deduce that

$$B_1 \int_\Omega M\left(\left|\frac{(\nabla u)^2}{u}\right|\right)dx \le B_1 C \int_\Omega M\left(|\nabla^{(2)} u|\right)dx$$

and this finishes the proof of Step 3. Inequality (4.2) holds with $\tilde{C} = C_1 C_2$. □

Remark 4.2. The estimation of $\mathcal{L}$ in (4.3) can be also performed in a slightly different way. Namely,

$$\begin{aligned}
\mathcal{L} \le \sum_{j=-\infty}^{+\infty} \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \Omega)(M(2^{-j+1}) - M(2^{-j})) \quad &\le \quad \sum_{j=-\infty}^{+\infty} \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \Omega) 2^{-j} \sup_{t\in[2^{-j}, 2^{-j+1}]} M^{'}(t) \le \\
\sum_{j=-\infty}^{+\infty} \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \Omega) 2^{-j} \sup_{t\in[2^{-j}, 2^{-j+1}]} D_M \frac{M(t)}{t} \quad &\le \quad D_M \sum_{j=-\infty}^{+\infty} \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \Omega) 2^{-j} \frac{M(2^{-j+1})}{2^{-j+1}} \le \\
\frac{D_M}{2} \sum_{j=-\infty}^{+\infty} \mathrm{cap}_M^+(\mathcal{N}_{2^{-j}}, \mathcal{N}_{2^{-j-1}}) M(2^{-j+1}) \quad &= \quad \frac{D_M}{2} \sum_{j=-\infty}^{+\infty} a_j b_j,
\end{aligned}$$

where we apply the mean value theorem and the fact that $\frac{M(\lambda)}{\lambda}$ is nondecreasing. This gives us a better estimation in case when $1 < D_M < 2$ - here $\tilde{C} = \frac{D_M}{2} C_1 C_2$.

Remark 4.3. Obviously our constant $\tilde{C}$ cannot be optimal. Following the proof of Proposition 4.1 one obtains the estimation: $\tilde{C} \leq A_M n^{D_M} + B_M$, where A_M, B_M are ceratin constants independent of the dimension.

Remark 4.4. We are now to discuss conditions (4.1) and **(M1)**.

1. As already noticed in Remark 4.1, the function $M(\lambda) = \lambda^p$ satisfies assumptions (4.1) and **(M1)**, whenever $p > 1$.
2. Let $p > q > 1$. The function

$$M(\lambda) := \begin{cases} \lambda^p & \text{for} \quad 0 \leq \lambda < (\frac{q}{p})^{\frac{1}{p-q}} \\ \lambda^q + \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \text{for} \quad (\frac{q}{p})^{\frac{1}{p-q}} \leq \lambda \end{cases}$$

   is an N-function different than p-homogeneous and it satisfies condition **(M1)** with $d_M = q$ and $D_M = p$. Moreover, we have $\limsup_{\lambda\to 0^+} \frac{M(\lambda)}{\lambda^{D_M}} = \limsup_{\lambda\to 0^+} \frac{\lambda^p}{\lambda^p} = 1$ and $\limsup_{\lambda\to+\infty} \frac{M(\lambda)}{\lambda^{d_M}} = \limsup_{\lambda\to+\infty} \frac{\lambda^q}{\lambda^q} = 1$. Therefore M satisfies also (4.1).
3. When $p > q > 1$, the function $M(\lambda) := \lambda^p + \lambda^q$ satisfies **(M1)** with $d_M = q$ and $D_M = p$, but it does not satisfy assumption (4.1).

We are now to present the last result of this section. Its proof is taken from [10], Section 8.3, after the minor modification. We present it for reader's convenience.

Proposition 4.2. *Let Ω be an open subset of $\mathbb{R}^n$, equipped with the Borel measure μ and let M and N be N-functions such that:*
(a) *M satisfies conditions* **(M1)** *and (4.1),*
(b) *inequality*

$$\| M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_\Omega M(|\nabla^{(2)} u(x)|)dx$$

holds with best constant A independent on u, whenever $u \in C_0^\infty(\Omega)$ is nonnegative,
(c) $B := \sup \left\{ \frac{\mu(E) N^{*-1}(\frac{1}{\mu(E)})}{\mathrm{cap}_M^+(E,\Omega)} : \ E \subset \Omega,\ E - compact,\ \mathrm{cap}_M^+(E,\Omega) > 0 \right\}.$

Then $M(1)B \leq A \leq B\tilde{C}$, where $\tilde{C}$ is the constant from Proposition 4.1.

Proof. We first show that $A \leq B\tilde{C}$. Using the notation from (2.5):

$\mathcal{N}_t = \{x \in \Omega : |u(x)| \geq 0\}$ and the very definition of norm in $L_N(\Omega, \mu)$, we obtain:

$$\begin{aligned}
\| M(|u|)\|_{L_N(\Omega,\mu)} &= \sup \left\{ \int_\Omega M(|u|) v d\mu : \int_\Omega N^*(v) d\mu \leq 1 \right\} \\
&= \sup \left\{ \int_0^\infty M'(t)(v \cdot \mu)(\mathcal{N}_t) dt : \int_\Omega N^*(v) d\mu \leq 1 \right\} \\
&= \sup \left\{ \int_0^\infty \left(\int_{\mathcal{N}_t} v d\mu \right) dM(t) : \int_\Omega N^*(v) d\mu \leq 1 \right\} \\
&\leq \int_0^\infty \sup \left\{ \int_\Omega \chi_{\mathcal{N}_t} v d\mu : \int_\Omega N^*(v) d\mu \leq 1 \right\} dM(t) \\
&= \int_0^\infty \|\chi_{\mathcal{N}_t}\|_{L_N(\Omega,\mu)} dM(t).
\end{aligned}$$

Therefore, by (2.4):

$$\| M(|u|)\|_{L_N(\mu)} \leq \int_0^\infty \mu(\mathcal{N}_t) N^{*-1} \left(\frac{1}{\mu(\mathcal{N}_t)} \right) dM(t).$$

Definition of B and Proposition 4.1 imply that

$$\| M(|u|)\|_{L_N(\mu)} \leq B \int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega) dM(t) \leq B\tilde{C} \int_\Omega M(|\nabla^{(2)} u|) dx.$$

Hence $A \leq B\tilde{C}$.
Now we prove that $M(1)B \leq A$. Let the function $u \in C_0^\infty(\Omega)$ be nonnegative on Ω and equal to 1 in a neighborhood of a compact $E \subset \Omega$. Then by the definition of A:

$$M(1)\|\chi_E\|_{L_N(\Omega,\mu)} = \|M(\chi_E)\|_{L_N(\Omega,\mu)} \leq \|M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_\Omega M(|\nabla^{(2)} u|) dx.$$

Taking the infimum on the right-hand side over such functions u we obtain

$$M(1)\mu(E) N^{*-1} \left(\frac{1}{\mu(E)} \right) \leq A \mathrm{cap}_M^+(E, \Omega).$$

This finishes the proof. □

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