ON THE LONG-TIME BEHAVIOUR OF SOLUTIONS OF THE p-LAPLACIAN PARABOLIC SYSTEM

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ABSTRACT. Convergence of global solutions to stationary solutions for a class of degenerate parabolic systems related to the p-Laplacian operator is proved. A similar result is obtained for variable exponent p. In the case of p constant, the convergence is proved to be $\mathcal{C}^1_{loc}$, in the variable exponent case — L^2 and $W^{1,p(x)}$-weak.

1. NOTATION AND CLAIM

Long time behaviour of degenerate parabolic equations have been widely studied, the p-Laplacian equation being a model example. In particular, Lieberman [4] proved that, for zero boundary data and in the scalar case ($m = 1$), the solutions of (1) are bounded in L^∞. Del Pino and Dolbeault established, for non-negative initial data and under some assumptions on p, not only the convergence to stationary solutions, but also an estimate on convergence rate [1]. We do not know, however, of any convergence result in the vectorial case.

In the paper we study global solutions of the system

$$u_t - \Delta_p u + \partial_2 f(x,u) = 0 \qquad u:\ [0,\infty)\times\Omega \to \mathbb{R}^m \tag{1}$$

in the space $w + W^{1,p}_0(\Omega)$, with initial condition $u(0,x) \equiv w(x)$, $p \geq 2$ and the following hypotheses on f, Ω, w:

The function $f(x,y)$ is a Carathéodory function such that $\partial_2 f(x,y)$ (the derivative w.r. to the second variable) exists a.e. in Ω. Moreover, $f(x,y)$ satisfies

- weak convexity: $\forall_{x\in\Omega}\ \langle \partial_2 f(x,y_1) - \partial_2 f(x,y_2), y_1 - y_2\rangle \geq 0$
- and growth conditions:

$$|f(x,y)| \leq C|y|^{Np/(N-p)}, \qquad |\partial_2 f(x,y)| \leq C|y|^{Np/(N-p)-1}.$$

The set Ω is a domain in $\mathbb{R}^N$ (i.e. it is open, bounded and connected), with $\partial\Omega \in \mathcal{C}^1$; the function w belongs to $W^{1,p}(\Omega)$, $w\big|_{\partial\Omega} \not\equiv 0$.

Remark. *From time to time we shall refer to somehow stronger hypotheses: that $\partial\Omega \in \mathcal{C}^{1,\tilde\alpha}$ and $w \in \mathcal{C}^{1,\tilde\alpha}(\Omega)$ for some $\tilde\alpha \in (0,1)$ (**strong hypotheses**).*

We consider the spacewise weak formulation of (1)

$$\langle u_t, \phi\rangle_{L^2(\Omega)} + M(u)(\phi) + \int_\Omega \langle \partial_2 f(x,u), \phi\rangle = 0 \qquad \forall_{\phi\in W^{1,p}_0(\Omega)}, \tag{2}$$

with $M(u)(\phi) = \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle$.

In what follows we shall use the following functionals:

$E_A(u) = 1/p \int_A |\nabla u|^p$, $E(u) = E_\Omega(u)$

and the variational functional related to (1)

$$\mathcal{E}(u) = \frac{1}{p} E(u) + \int\limits_\Omega f(x,u)$$

In order to ensure the existence of minima of $\mathcal{E}$, we suppose that the functional $\mathcal{E}(u) = \frac{1}{p}E(u) + \int_\Omega f(x,u)$ is coercive, that is

$$\lim_{\|u\|_{W^{1,p}(\Omega)} \to \infty} \mathcal{E}(u) = \infty$$

(see e.g. [3], Theorem 4.6). By convexity of $\mathcal{E}(\cdot)$ the minimum in $w + W_0^{1,p}$ is unique, and we shall denote it by $u_0(x)$. We shall write

$$\Omega_0 = \{x \in \Omega \ : \ |\nabla u_0(x)| = 0\}, \qquad \Omega_\epsilon = \{x \in \Omega \ : \ |\nabla u_0(x)| < \epsilon\}.$$

The main result of the paper is the following theorem:

Theorem. *The solution u and its gradient ∇u converge almost uniformly in Ω to u_0 and ∇u_0, respectively, i.e.*

$$\forall_{\Omega' \Subset \Omega} \quad |u(t,\cdot) - u_0(\cdot)|_{\mathcal{C}^1(\Omega')} \xrightarrow{t\to\infty} 0. \tag{3}$$

where u_0 is the (unique) minimum of the energy functional $\mathcal{E}(\cdot)$. If **strong hypotheses** *hold, the convergence is uniform in Ω.*

I shall use the following known facts on regularity of u and u_0:

- Regularity of u_0:

$$\forall_{\Omega' \Subset \Omega}\ \exists_{\alpha \in (0,1)} \quad u_0 \in \mathcal{C}^{1,\alpha}(\Omega')$$

  If **strong hypotheses** hold, we have that $u_0 \in \mathcal{C}^{1,\alpha}(\Omega)$ [5].
- Interior regularity of u:

$$\forall_{\Omega' \Subset \Omega}\ \exists_{\alpha \in (0,1)} \quad u(t,\cdot) \in \mathcal{C}^{1,\alpha}(\Omega')$$

  and the Hölder constant of ∇u is bounded independently of t for all $t \geq t_0 > 0$.

  In the case of **strong hypotheses** we obtain full regularity of $u(t,\cdot)$:

$$\exists_{\alpha \in (0,1)} \quad u(t,\cdot) \in \mathcal{C}^{1,\alpha}(\Omega)$$

  with Hölder constant of ∇u bounded independently of t for all $t \geq 0$ [2].

Remark. *The system* (2) *defines a gradient flow of $\mathcal{E}$ in $L^2(\Omega)$. In particular,*

$$\frac{d\mathcal{E}(u)}{dt} = M(u)(u_t) + \int\limits_\Omega \langle \partial_2 f(x,u), u_t \rangle = -\|u_t\|^2_{L^2(\Omega)} \leq 0, \tag{4}$$

so the energy $\mathcal{E}(u)$ decreases with $t \to \infty$.

2. Case of $f \equiv 0$

Let us start with a simpler, "toy" case of

$$u_t - \Delta_p u = 0 \tag{5}$$

In this case we can easily trace the main ideas of the general proof.

We have, by Young inequality

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)} &= \langle u_t, u-u_0\rangle_{L^2(\Omega)} = -M(u)(u-u_0) \\ &= -M(u)(u) + M(u)(u_0) \\ &= -E(u) + \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_0\rangle \\ &\le -pE(u) + (p-1)E(u) + E(u_0) \\ &= -(E(u)-E(u_0)) \le 0. \end{aligned} \tag{6}$$

Thus we see that $\|u-u_0\|^2_{L^2(\Omega)}$ is decreasing with $t \to \infty$, as u_0 is a minimizer for E. On the other hand, $\|u-u_0\|^2_{L^2(\Omega)}$ is bounded from below, so there must be a sequence t_i such that

$$\frac{d}{dt}\Big|_{t=t_i}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)} \xrightarrow{i\to\infty} 0 \tag{7}$$

To simplify the notation we shall write $u_i(\cdot)$ for $u(t_i,\cdot)$.

Next, let us notice that, by Bernoulli's inequality,

$$\begin{aligned} E(u)-E(u_0) &= \int_\Omega |\nabla u|^p - \int_\Omega |\nabla u_0|^p \\ &= \int_{\Omega_0} |\nabla u|^p + \int_{\Omega\setminus\Omega_0} \left[\left(|\nabla u_0|^2 + 2\langle \nabla u_0, \nabla(u-u_0)\rangle + |\nabla(u-u_0)|^2\right)^{p/2} - |\nabla u_0|^p\right] \\ &= E_{\Omega_0}(u-u_0) + \int_{\Omega\setminus\Omega_0} |\nabla u_0|^p \left[\left(1 + 2\frac{\langle \nabla u_0, \nabla(u-u_0)\rangle}{|\nabla u_0|^2} + \frac{|\nabla(u-u_0)|^2}{|\nabla u_0|^2}\right)^{p/2} - 1\right] \\ &\ge E_{\Omega_0}(u-u_0) + p\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}\langle \nabla u_0, \nabla(u-u_0)\rangle + \frac{p}{2}\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}|\nabla(u-u_0)|^2 \\ &= E_{\Omega_0}(u-u_0) + M(u_0)(u-u_0) + \frac{p}{2}\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}|\nabla(u-u_0)|^2. \end{aligned} \tag{8}$$

On the other hand, u_0 is a minimizer for E, so $M(u_0)(\cdot) \equiv 0$, and we get

$$E(u)-E(u_0) \ge E_{\Omega_0}(u-u_0) + \frac{p}{2}\int_{\Omega\setminus\Omega_0} |\nabla u_0|^{p-2}|\nabla(u-u_0)|^2. \tag{9}$$

At the moment we know by (7) that $E(u_i) \xrightarrow{i\to\infty} E(u_0)$. We have, however, proved that $E(u(t,\cdot))$ is non-increasing, therefore

$$E(u(t,\cdot)) \xrightarrow{t\to\infty} E(u_0(\cdot)). \tag{10}$$

Let us denote the right hand side of (9) by $V(u-u_0)$. By (10) we have $V(u(t,\cdot)-u_0(\cdot)) \xrightarrow{t\to\infty} 0$. In the next step we shall prove the following lemma:

Lemma 1.

$$V(u(t,\cdot)-u_0(\cdot)) \xrightarrow{t\to\infty} 0 \qquad \Rightarrow \qquad \forall_{\Omega'\Subset\Omega\setminus P} \quad \sup_{\Omega'} |\nabla(u-u_0)| \xrightarrow{t\to\infty} 0$$

Proof. Let us suppose otherwise. After passing to a subsequence $\{t_i\}\to\infty$, we have, for a fixed Ω' and some $b>0$,

$$\sup_{\Omega'} |\nabla(u_i-u_0)| > b.$$

Let us suppose that, for every i, $|\nabla(u_i(\xi_i)-u_0(\xi_i))| > b$. Again, by passing to a subsequence, we may suppose that $\xi_i \xrightarrow{i\to\infty} \xi_\infty \in \overline{\Omega'}$.

The functions u_i-u_0 are all in $\mathcal{C}^{1,\alpha}$, with Hölder constant of the gradient bounded by some G, independently of i. Thus, there exists $\rho=\rho(\alpha,G,b,N)$ such that $\forall_i\ |\nabla(u_i-u_0)| > \frac{1}{2}b$ on $B_\rho(\xi_i)\cap\Omega'$. For i sufficiently large we have also $|\nabla(u_i-u_0)| > \frac{1}{2}b$ on $S := B_{\rho/2}(\xi_\infty)\cap\Omega'$.

We may always suppose (possibly after enlarging Ω') that $\overline{\Omega'} = \overline{\mathrm{int}\Omega'}$. Moreover, we may take ϵ small enough to have $\mu(\Omega_\epsilon\setminus\Omega_0) < \frac{1}{2}\mu(S)$. Then either $S\cap\Omega_0$, or $S\cap\Omega\setminus\Omega_\epsilon$ is of non-zero Lebesgue measure. If $\mu(S\cap\Omega_0)>0$, then $E_{\Omega_0}(u_i-u_0)$ cannot tend to zero (the integrand is bounded from below on the set, independently of i), and if $\mu(S\cap\Omega\setminus\Omega_\epsilon)>0$, the second term in $V(u_i-u_0)$ cannot tend to zero. □

In the case of **strong hypotheses** we may choose, of course, $\Omega'=\Omega$, which yelds uniform convergence of ∇u to ∇u_0 in Ω. This, in turn, implies $W^{1,p}$ and then, via Sobolev embedding theorem, L^k convergence for all $k\le np/(n-p)$, in particular for $k=2$:

$$\lim_{t\to\infty} \|u(t,\cdot)-u_0(\cdot)\|_{L^2(\Omega)} \to 0.$$

If only weak hypotheses hold, we must proceed differently. Notice that $\{u(t,\cdot)\}$ form a bounded set in $w+W_0^{1,p}(\Omega)$ — thus, by Alaoglu and Rellich-Kondrashov theorems, one can find a sequence $\{u(t_i,\cdot)\}$ with $t_i\to\infty$ (as before, I denote $u(t_i,\cdot)$ by $u_i(\cdot)$), such that

$$\begin{aligned} u_i &\rightharpoonup u_\infty \text{ weakly in } w+W_0^{1,p}(\Omega)\\ u_i &\to u_\infty \text{ strongly in } L^2(\Omega). \end{aligned}$$

By weak convergence in $w+W_0^{1,p}(\Omega)$,

$$M(u_\infty)(u_\infty-u_i) \xrightarrow{i\to\infty} 0, \tag{11}$$

but, using Young inequality as in (6) and (10),

$$M(u_\infty)(u_\infty-u_i) \ge E(u_\infty)-E(u_i) \xrightarrow{i\to\infty} E(u_\infty)-E(u_0) \ge 0.$$

If $u_\infty \not\equiv u_0$, then, by uniqueness of minimum of E, $E(u_\infty - E(u_0) > 0$, which contradicts the convergence in (11).

This shows that $u(t_i,\cdot) \xrightarrow[L^2(\Omega)]{i\to\infty} u_0(\cdot)$, but the monotonicity of $\|u-u_0\|_{L^2(\Omega)}$ implies that this convergence holds, in fact, for $t \to \infty$:

$$\|u-u_0\|_{L^2(\Omega)} \xrightarrow{t\to\infty} 0.$$

We may also conclude that $u(t,\cdot)$ converges weakly to $u_0(\cdot)$ in $w+W_0^{1,p}(\Omega)$. Let us suppose that such a convergence does not hold. There exist thus a (weak-topology) neighbourhood U of u_0 and a sequence $t_i \to \infty$ such that $\{u(t_i,\cdot)\}_{i\in\mathbb{N}} \cap U = \emptyset$. On the other hand the sequence $\{u(t_i,\cdot)\}$ is bounded in $w+W_0^{1,p}(\Omega)$, we may thus choose a subsequence weakly convergent to some u_∞ — but the above reasoning shows that this u_∞ is equal to u_0, which is a contradiction.

Having established L^2-convergence of u to u_0, we may now repeat the trick from lemma 1: for a fixed $\Omega' \Subset \Omega$, the functions $u(t,\cdot)-u_0(\cdot)$ are Lipschitz, with the Lipschitz constant bounded independently of t ($\nabla(u-u_0)$ converges uniformly to 0 in Ω', thus it is bounded independently of t). If $u-u_0$ does not converge uniformly to 0 in Ω', we may choose a sequence u_i-u_0 such that

$$\forall_i \sup_{\Omega'} |u_i(\cdot)-u_0(\cdot)| \geq b > 0,$$

and thus a set $A \subset \Omega'$ of positive measure, on which

$$\forall_{i>i_0} \forall_{x\in A} \ |u_i(x)-u_0(x)| > b/2,$$

which contradicts the already proved L^2-convergence of u to u_0.

This concludes the proof of theorem 1 for $f \equiv 0$.

3. Non-homogeneous case

In this section we shall prove theorem 1:

Theorem 1. *Let $u(t,x)$ be a global solution of (2) in $w+W_0^{1,p}$, with f satisfying the assumptions from Section 1. Then*

$$\forall_{\Omega'\Subset\Omega} \quad \|u(t,\cdot)-u_0(\cdot)\|_{C^1(\Omega')} \xrightarrow{t\to\infty} 0,$$

i.e. u and ∇u are, for $t\to\infty$, almost-uniformly convergent to u_0 and ∇u_0 in Ω, where u_0 is the (unique) minimizer of the energy functional $\mathcal{E}$. If **strong hypotheses** *hold, the convergence is uniform in Ω.*

Proof. In this case $u_0(\cdot)$ is a minimizer for $\mathcal{E}$, therefore

$$M(u_0)(\phi) + \int_\Omega \langle \partial_2 f(x,u_0), \phi\rangle = 0 \qquad \forall_{\phi\in W_0^{1,p}(\Omega)}. \tag{12}$$

Let us now calculate a counterpart of (6) in our case:

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)} &= \langle u_t, u-u_0\rangle_{L^2(\Omega)} \\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle \\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle + M(u_0)(u-u_0) \\
&\quad + \int_\Omega \langle \partial_2 f(x,u_0), u-u_0\rangle \\
&= -E(u) + M(u)(u_0) + M(u_0)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u) - \partial_2 f(x,u_0), u-u_0\rangle \\
&\le -E(u) + M(u)(u_0) + M(u_0)(u-u_0).
\end{aligned} \tag{13}$$

We shall continue this calculation in two ways. In the first one, we use Hölder inequality, obtaining

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)} &\le -pE(u) + M(u)(u_0) + M(u_0)(u-u_0) \\
&= -pE(u) + \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_0\rangle + \int_\Omega |\nabla u_0|^{p-2}\langle \nabla u_0, \nabla u\rangle - pE(u_0) \\
&\le -pE(u) + pE(u)^{(p-1)/p}E(u_0)^{1/p} + pE(u)^{1/p}E(u_0)^{(p-1)/p} \\
&= -p\left(E(u)^{1/p} - E(u_0)^{1/p}\right)\left(E(u)^{(p-1)/p} - E(u_0)^{(p-1)/p}\right) \\
&\le 0.
\end{aligned} \tag{14}$$

Notice that in this case we do not necessarily have $E(u) > E(u_0)$, because u_0 is a minimizer for $\mathcal{E}$, not for E.

The second continuation of (13) uses Young's inequality:

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)} &\le -pE(u) + M(u)(u_0) + M(u_0)(u-u_0) \\
&= -pE(u) + \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla u_0\rangle + M(u_0)(u-u_0) \\
&\le -pE(u) + \int_\Omega |\nabla u|^{p-1}|\nabla u_0| + M(u_0)(u-u_0) \\
&\le -pE(u) + \int_\Omega \left(\frac{p-1}{p}|\nabla u|^p + \frac{1}{p}|\nabla u_0|^p\right) + M(u_0)(u-u_0) \\
&= -(E(u) - E(u_0) - M(u_0)(u-u_0)).
\end{aligned} \tag{15}$$

Using Young inequality once again, this time on $M(u_0)(u-u_0)$, shows that the above quantity is still non-positive.

Just as in the case of $f \equiv 0$, there exists a sequence t_i such that $\frac{d}{dt}\big|_{t=t_i} \|u-u_0\|^2_{L^2(\Omega)} \xrightarrow{i\to\infty} 0$. We write $u_i(\cdot) := u(t_i,\cdot)$.

The inequality (14) implies that $E(u_i) \xrightarrow{i\to\infty} E(u_0)$, and (15), together with (8), that $V(u_i-u_0) \xrightarrow{i\to\infty} 0$. Lemma 1 shows then that ∇u_i is almost-uniformly (uniformly with **strong hypotheses**) convergent to ∇u_0 in Ω.

We mimic the proof given in the previous section: if **strong hypotheses** hold, uniform convergence of ∇u_i to ∇u_0 implies

$$\begin{aligned} u_i &\to u_0 \text{ strongly in } w + W_0^{1,p}(\Omega) \\ &\Downarrow \\ u_i &\to u_0 \text{ strongly in } L^k(\Omega) \text{ for } k \le np/(n-p). \end{aligned}$$

Next, we need to pass from convergence of the sequence u_i to the convergence for all t. First, we need to establish the convergence of $\mathcal{E}(u)$ to $\mathcal{E}(u_0)$ with $t\to\infty$. By monotonicity of $\mathcal{E}(u(t,\cdot))$ it is enough to prove it for the sequence t_i.

Of the two terms in $\mathcal{E}(u_i)$ we already have that $E(u_i) \xrightarrow{i\to\infty} E(u_0)$. What is left to prove is that

$$\int_\Omega [f(x,u(t_i,x)) - f(x,u_0(x))] \xrightarrow{i\to\infty} 0.$$

By convexity of f we have, for some $\theta\in(0,1)$ and $u_\theta = \theta u_i + (1-\theta)u_0$,

$$\begin{aligned} \int_\Omega [f(x,u_i)-f(x,u_0)] &= \int_\Omega \langle \partial_2 f(x,u_\theta), u_i-u_0\rangle \\ &= \frac{1}{\theta}\int_\Omega \langle \partial_2 f(x,u_\theta), u_\theta-u_0\rangle \ge \frac{1}{\theta}\int_\Omega \langle \partial_2 f(x,u_0), u_\theta-u_0\rangle \\ &= \int_\Omega \langle \partial_2 f(x,u_0), u_i-u_0\rangle = -M(u_0)(u_i-u_0) \\ &\ge E(u_0)-E(u_i). \end{aligned} \tag{16}$$

Similarly

$$\begin{aligned} \int_\Omega [f(x,u_i)-f(x,u_0)] &= \int_\Omega \langle \partial_2 f(x,u_\theta), u_i-u_0\rangle \\ &= \frac{1}{1-\theta}\int_\Omega \langle \partial_2 f(x,u_\theta), u_i-u_\theta\rangle \le \frac{1}{1-\theta}\int_\Omega \langle \partial_2 f(x,u_i), u_i-u_\theta\rangle \\ &= \int_\Omega \langle \partial_2 f(x,u_i), u_i-u_0\rangle = -M(u_i)(u_i-u_0) - \left\langle u_t\big|_{t=t_i}, u_i-u_0\right\rangle_{L^2(\Omega)} \\ &\le E(u_0)-E(u_i) - \frac{d}{dt}\Big|_{t=t_i}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)}. \end{aligned} \tag{17}$$

As we see, both the lower (16) and upper (17) estimate for

$$\int_\Omega [f(x,u_i)) - f(x,u_0)]$$

tend, for this particular sequence t_i, to 0. This concludes the proof of energy convergence.

Next, notice that by (16) and (8)

$$\begin{aligned}\mathcal{E}(u) - \mathcal{E}(u_0) &= E(u) - E(u_0) + \int_\Omega [f(x,u) - f(x,u_0)] \\ &\geq E(u) - E(u_0) - M(u_0)(u - u_0) \\ &\geq V(u - u_0) \geq 0,\end{aligned}$$

and, applying lemma 1 we obtain almost uniform (uniform for **strong hypotheses**) convergence of ∇u to ∇u_0.

In the case of **strong hypotheses** we already have $L^2(\Omega)$-convergence of u_i to u, which, together with monotonicity of $\|u - u_0\|_{L^2(\Omega)}$, gives us that

$$u \to u_0 \text{ in } L^2(\Omega).$$

If only weak hypotheses hold, we proceed as in the previous section. By (14) (which implies boundedness of $\{u(t,\cdot)\}_{t>0}$ in $W^{1,p}(\Omega)$) we can choose a sequence $\{t_i\}$, $t_i \to \infty$, such that

$$\begin{aligned} u_i &\rightharpoonup u_\infty \text{ weakly in } w + W_0^{1,p}(\Omega) \\ u_i &\to u_\infty \text{ strongly in } L^2(\Omega); \end{aligned}$$

as before, $u_i(\cdot) = u(t_i,\cdot)$. By this weak convergence, convexity of f (see (17)), and Young inequality (like in (15))

$$\begin{aligned}\mathcal{E}(u_\infty) - \mathcal{E}(u_i) &= E(u_\infty) - E(u_i) + \int_\Omega [f(x,u_\infty) - f(x,u_i)] \\ &\leq M(u_\infty)(u_\infty - u_i) + \int_\Omega \langle \partial_2 f(x,u_\infty), u_\infty - u_i\rangle \xrightarrow{i\to\infty} 0\end{aligned} \tag{18}$$

On the other hand $\mathcal{E}(u_\infty) - \mathcal{E}(u_i) \xrightarrow{i\to\infty} \mathcal{E}(u_\infty) - \mathcal{E}(u_0)$.

If $u_\infty \not\equiv u_0$, then, by uniqueness of minimum for $\mathcal{E}$, $\mathcal{E}(u_\infty) - \mathcal{E}(u_0) > 0$, which gives a contradiction. Therefore $u_\infty \equiv u_0$ in $w + W_0^{1,p}(\Omega)$.

The fact that $u(t_i,\cdot)$ converges strongly in $L^2(\Omega)$ to $u_0(\cdot)$ together with monotonicity of $\|u - u_0\|$ yields

$$u(t,\cdot) \xrightarrow[L^2(\Omega)]{t\to\infty} u_0(\cdot). \tag{19}$$

This, together with the fact that $u(t,\cdot)$ are Lipschitz with Lipschitz constant time-independent, gives us the almost uniform (uniform for **strong hypotheses**) convergence of $u(t,\cdot)$ to $u_0(\cdot)$ (see previous section). This ends the proof of theorem 1. □

Remark. *The same argument as in the previous section gives us also the weak convergence*

$$u(t,\cdot) \xrightarrow{t\to\infty} u_0(\cdot) \text{ weakly in } w + W_0^{1,p}(\Omega).$$

4. Variable exponent case

In this section we shall study the convergence of a counterpart of (1) for a measurable, bounded exponent function $p(x)$, $p_1 > p(x) \geq 2$:

$$\langle u_t, \phi \rangle_{L^2(\Omega)} + M(u)(\phi) + \int_\Omega \langle \partial_2 f(x,u), \phi \rangle = 0 \qquad \forall_{\phi \in W_0^{1,p(x)}(\Omega)} \tag{20}$$

This time

$$E(u) = \int_\Omega |\nabla u|^{p(x)}, M(u)(\phi) = \int_\Omega p(x) |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle, \tag{21}$$

$$\mathcal{E}(u) = E(u) + \int_\Omega f(x,u). \tag{22}$$

The main difference between the variable exponent case and the preceding ones is that no partial regularity result for time-dependent solutions of (20) is known. Therefore the methods used for $p = const$ cannot yield any pointwise convergence. However, this regularity result (widely believed to be true, at least under sufficient continuity assumptions on $p(x)$) is the only missing detail, and were it proved, one could apply the same technique as for p constant, obtaining the same convergence result.

From now on:

$$\begin{aligned} |f(x,y)| \leq C|y|^{p(x)^*} & \qquad p(x)^* = p(x)N/(N-p(x)) \\ \langle \partial_2 f(x,y_1) - \partial_2 f(x,y_2), y_1 - y_2 \rangle \geq 0 & \qquad \text{for a.e. } x \end{aligned}$$

and, as before, we suppose that $\mathcal{E}(\cdot)$ is coercive.

Theorem 2. *With the above assumptions on $p(x)$ and $f(x,y)$ a global solution $u(t,x)$ of (20) in $w + W_0^{1,p(x)}(\Omega)$ converges to a stationary solution $u_0(x)$ strongly in $L^2(\Omega)$ and weakly in $W^{1,p(x)}(\Omega)$.*

We repeat, with slight alterations, the calculation (13), using Young inequality (notice that only pointwise inequalities — i.e. not the Hölder inequality — can be safely used in the $p(x)$ case):

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|u-u_0\|^2_{L^2(\Omega)} &= \langle u_t, u-u_0\rangle_{L^2(\Omega)} \\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle \\
&= -M(u)(u-u_0) - \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle + M(u_0)(u-u_0) \\
&\quad + \int_\Omega \langle \partial_2 f(x,u_0), u-u_0\rangle \\
&= -M(u)(u-u_0) + M(u_0)(u) - \int_\Omega p(x)|\nabla u_0|^{p(x)} \\
&\quad - \int_\Omega \langle \partial_2 f(x,u) - \partial_2 f(x,u_0), u-u_0\rangle \\
&\le -M(u)(u-u_0) + \int_\Omega p(x)|\nabla u_0|^{p(x)-1}|\nabla u| - \int_\Omega p(x)|\nabla u_0|^{p(x)} \\
&\le -M(u)(u-u_0) + \int_\Omega (p(x)-1)|\nabla u_0|^{p(x)} + \int_\Omega |\nabla u|^{p(x)} - \int_\Omega p(x)|\nabla u_0|^{p(x)} \\
&= E(u) - E(u_0) - M(u)(u-u_0).
\end{aligned}
\tag{23}
$$

Applying Young inequality once more, this time to the second term in $M(u)(u-u_0) = M(u)(u) - M(u)(u_0)$, proves that the right hand side of (23) is still non-positive. We get thus a sequence $t_i \to \infty$ such that

$$
\frac{d}{dt}\Big|_{t=t_i} \|u-u_0\|_{L^2(\Omega)},
$$

in particular

$$
E(u_i) - E(u_0) - M(u_i)(u_i-u_0) \xrightarrow{i\to\infty} 0,
$$

where, as before, $u_i(\cdot) = u(t_i,\cdot)$.

This allows us to prove the energy convergence, by estimates similar to those in (17)

$$
\begin{aligned}
\mathcal{E}(u) - \mathcal{E}(u_0) &= E(u) - E(u_0) + \int_\Omega f(x,u) - \int_\Omega f(x,u_0) \\
&\le E(u) - E(u_0) + \int_\Omega \langle \partial_2 f(x,u), u-u_0\rangle \\
&= E(u) - E(u_0) - M(u)(u-u_0) - \langle u_t, u-u_0\rangle_{L^2(\Omega)} \\
&= E(u) - E(u_0) - M(u)(u-u_0) - \frac{d}{dt}\|u-u_0\|_{L^2(\Omega)}.
\end{aligned}
\tag{24}
$$

For the sequence $t = t_i$ the right hand side of (24) tends to 0 with $i \to \infty$, and $\mathcal{E}(u) - \mathcal{E}(u_0) \geq 0$. However, as in the previous sections, the energy $\mathcal{E}(u)$ is decreasing with t:

$$\frac{d}{dt}\mathcal{E}(u) = -M(u)(u_t) - \int_\Omega \langle \partial_2 f(x,u), u_t \rangle = -\|u_t\|^2_{L^2(\Omega)}.$$

therefore we have $\mathcal{E}(u) \xrightarrow{t\to\infty} \mathcal{E}(u_0)$.

We proceed as in the previous section. By coercivity of $\mathcal{E}$ and the convergence of $\mathcal{E}$ proved above, the set $\{u(t)\}_{t>0}$ is bounded in $w + W_0^{1,p(x)}$, and thus in $W_{1,2}(\Omega)$. By Alaoglu and Rellich-Kondrashov theorems we may choose a sequence $t_i \to \infty$ $(u_i(\cdot) = u(t_i, \cdot))$ such that

$$\begin{aligned} u_i &\rightharpoonup u_\infty \text{ weakly in } w + W_0^{1,p(x)} \\ u_i &\to u_\infty \text{ strongly in } L^2(\Omega). \end{aligned}$$

Using exactly the same calculation as in (18) I show that $u_\infty = u_0$. Monotonicity of $\|u - u_0\|_{L^2(\Omega)}$ ensures that the L^2-convergence holds for all t, while the argument from section 2 ($\{u(t,\cdot)\}_{t>0}$ is precompact and every $W^{1,p}$-weakly convergent sequence with $t \to \infty$ converges to u_0) — that we have $w + W_0^{1,p(x)}$-weak convergence.

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References

[1] Manuel Del Pino, Jean Dolbeault, *Asymptotic behavior of nonlinear diffusions.* Math. Res. Lett. 10, No.4, 551-557 (2003).

[2] Emmanuele DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag 1993.

[3] Enrico Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing 2003.

[4] Gary M. Lieberman, *Study of global solutions of parabolic equations via* a priori *estimates III. Equations of p-Laplacian type*, Singularities and Differential Equations, Banach Center Publ. Vol. 33 (1996).

[5] Karin Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. 138 (1977), pp. 219-240.

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