Lusin type theorem with quasiconvex hulls

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We obtain Lusin type theorem showing that after extracting an open set of an arbitrary small measure one can apply some variant of convex integration theory using quasiconvex hulls of sets instead of lam-convex hulls (called by Gromov P-convex hulls in the more general setting) or rank-one convex hulls in the approach by Müller and Šverák.

1 Introduction

In this paper we deal with Partial Differential Inclusion:

$$Du \in K \subseteq \mathbf{M}^{m\times n}, \tag{1}$$

where Du is the distributional gradient of the Lipshitz mapping $u : \mathbf{R}^n \supseteq \Omega \to \mathbf{R}^m$, Ω is a bounded domain, K is the compact set and $\mathbf{M}^{m\times n}$ is the space of matrices having m rows and n columns. The scalar one-dimensional case of (1), $m = n = 1$, is well understood by now, see e.g. [1]. The celebrated results by Nash and Kupier [29, 17] were dealing with the multidimensional variants of the PDI's like (1) to obtain the existence of the non-trivial C^1 isometric immersions. Since this time the theory of the PDI's has evolved in several directions. Gromov [13] developed general theory called convex integration, based on the techniques by Nash and Kupier [13], see also later book [31]. Nash and Kupier approaches deal only with C^1 solutions, while Gromov had also results for Lipshitz mappings. It is important to consider Lipshitz mappings as well, as for example they explain some problems in the analysis of crystal microstructure, see e.g. [4, 5, 8]. There are two approaches in this direction. The first one, based on the Baire cathegory method, is due to Dacorogna and Marcellini [9, 10]. It has its rudiments in the previous papers [6, 11, 12]. The second one is based on Gromov's ideas extended further by Müller and Šverák [25, 26] (see also [2, 23, 24, 27, 28, 33, 14, 33, 15], their references and independent earlier paper [30]). It results in constructions of singular Lipshitz solutions of elliptic and parabolic PDE's.

The idea of convex convex integration is based on the successive construction of solutions to the PDI:

$$Du_k \in U^k, \quad u_k = g \text{ on } \partial\Omega \tag{2}$$

where open sets U^k better and better approximate set K. The key point there is to construct such a decreasing family of sets $\{U^k\}_{k\in\mathbf{N}}$ approximating K that the existence of the solution of (2) with U^k implies the existence of solution of (2) with U^{k+1}. Then one proves that finally the sequence $\{Du_k\}_{k\in\mathbf{N}}$ converges to the solution of (1) with the same boundary data as all the u_k's. If this machinery works one only needs to prove that the solution of (2) with $k = 1$ exists.

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This leads to the concept of in-approximation of K which can be introduced in several ways. Gromov used in the simplest version the in-approximation based on the lam-convexification proces. Namely, by the lam-convex hull (called by Gromov P-convex hull in the more general context) of a set $K \subseteq \mathbf{R}^{m\times n}$ we mean the smallest set denoted by K^{lc} with the property that if $A, B \in K^{lc}$ and $A - B$ is a rank-one matrix then K^{lc} contains all the segment $[A, B]$.

Müller and Šverák were dealing with rank-one convex hulls of sets, called functionally rank-one convex by Matoušek and Plecháč [19]. Namely, if K is compact then its rank-one convex hull is defined by

$$K^{rc} = \{x : f(x) \leq \sup_{y\in K} f(y), \text{ whenever } f : \mathbf{M}^{m\times n} \to \mathbf{R} \text{ is rank-one convex}\}$$

and by rank-one convex function we mean such a one which is convex along all the directions of rank-one matrices in $\mathbf{M}^{m\times n}$. For U being open the hull U^{rc} is defined as the (set theoretic) sum of rank-one- convex hulls of all its compact subsets.

The key point in the approach by Gromov, Müller and Šverák was that the existence of solution of

$$Du \in V \quad u = g \text{ on } \partial\Omega, \tag{3}$$

where V is either U^{lc} in the constructions by Gromov or U^{rc} in the approach by Müller and Šverák, implies the existence of solution of (3) with U instead of V. Sets $\{U^k\}$ in the in-approximation of K have the property that U^k is contained in $(U^{k+1})^{lc}$ or $(U^{k+1})^{rc}$. Therefore one can inductively construct the solutions of (2).

As rank-one convex hull of a set can be essentially larger than its lam- convex hull (see e. g. [15], Theorems 1-3), the technique of convex integration dealing with rc- in approximations is more powerful.

It is natural to ask what other convexifications $\hat{U}$ of U different than lam- and rc- convexifications could be used to built the solutions of (1) by the successive improvements of (2). We want this convexifications to satisfy $\hat{U} \supseteq U^{rc}$, so also $\hat{U} \supseteq U^{lc}$.

We are now in the position to explain our point of view. Let us recall the notion of quasiconvexity introduced by Morrey [20, 21]. Namely, the function $f : \mathbf{M}^{m\times n} \to \mathbf{R}$ is called quasiconvex if f satisfies the quasiconvexity condition

$$⨍_Q f(A + \nabla\phi(x))dx \geq f(A),$$

for every $A \in \mathbf{M}^{m\times n}$, every cube $Q \subseteq \mathbf{R}^n$ and arbitrary $\phi \in C_0^\infty(Q, \mathbf{R}^m)$.
Quasiconvex functions characterize all energy functionals $I_f(u) = \int_\Omega f(\nabla u)dx$ that are lower semi-continuous with respect to the sequential weak $*$ convergence of gradients in $L^\infty(\Omega)$, see also e.g. [3, 7, 16, 32] and their references for some selected results on quasiconvexity.

The quasiconvex hull of a compact set K is defined by

$$K^{qc} = \{x : f(x) \leq \sup_{y\in K} f(y), \text{ whenever } f : \mathbf{M}^{m\times n} \to \mathbf{R} \text{ is quasiconvex}\}.$$

Analogously as before one defines the quasiconvex hull of an open set. It is know [20, 21] that every quasiconvex function is rank-one convex (while the celebrated result by Šverák [32] shows the in the case $m \geq 3, n \geq 2$ the reverse implication is not true). Therefore we have $V^{qc} \supseteq V^{rc}$ for any V.

It was questioned by Müller and Šverák (see Chapter 5 of [24]) that perhaps it is possible to apply the machinery of convex integration by the successive improvements of (2) dealing with qc-in-approximations of K. This problem still remains open.

Here we show that if K admits qc-in approximation $\{U^k\}_{k\in\mathbf{N}}$ and the solution of the problem

$$Du \in U^1, \quad u = g \text{ on } \partial\Omega$$

exists then one can find an open set $U \subseteq \Omega$ of an arbitrary small measure and a Lipshitz mapping v such that $Dv \in K$ everywhere on $\Omega \setminus U$. Moreover, v satisfies the same boundary data as u and it can be chosen arbitrary close to u in the supremum norm.

2 The result

Notation. By $\mathbf{M}^{m\times n}$ we denote the space of matrices with m rows and n columns. By $W^{1,p}(\Omega, \mathbf{R}^m)$ we denote Sobolev spaces defined on an arbitrary open bounded set Ω, and by $W_0^{1,p}(\Omega, \mathbf{R}^m)$ we mean the completion of $C_0^\infty(\Omega, \mathbf{R}^m)$ in $W^{1,p}(\Omega, \mathbf{R}^m)$. We say that $u = v$ on $\partial\Omega$ where $u, v \in W^{1,p}(\Omega, \mathbf{R}^m)$ if $u - v \in W_0^{1,p}(\Omega, \mathbf{R}^m)$. If P is the subset of an Euclidean space E and $\delta > 0$, by $(P)_\delta$ we denote the set $\{p \in E : \mathrm{dist}(p, P) \le \delta\}$, where $\mathrm{dist}(p, P) := \inf_{x\in P}|x - p|$ and $\|P\|_\infty := \sup\{|F| : F \in P\}$.

We will deal with piecewise affine functions defined below.

Definition 2.1 Let $\Omega \subseteq \mathbf{R}^n$ be a bounded domain. The continuous function $u : \Omega \to \mathbf{R}^m$ is called piecewise affine if we have: $\Omega = \cup_i\Omega_i \cup A$ where Ω_i's are open, $|A| = 0$ and f is affine on each Ω_i.

The approximate and quasiconvex hulls: K^{app} and K^{qc} of set K are defined as follows (see e. g. [22], Section 4.4).

Definition 2.2 Let $K \subseteq \mathbf{M}^{m\times n}$ be the compact subset and $\Omega \subseteq \mathbf{R}^n$ be a bounded domain.

1) We say that $F \in K^{app}$ if there exists the sequence $u^\nu : \Omega \to \mathbf{R}^m$ bounded in $W^{1,\infty}(\Omega, \mathbf{R}^m)$ such that

$$\mathrm{dist}(Du^\nu, K) \to 0 \text{ a. e.}$$
$$u^\nu = Fx \text{ on } \partial\Omega$$

2)
$$K^{qc} := \{F \in \mathbf{M}^{m\times n} : f(F) \le \sup_K f \text{ for every quasiconvex } f : \mathbf{M}^{m\times n} \to \mathbf{R}\}.$$

The following fact is known (see e.g. Theorem 4.10, part i) in [22]).

Theorem 2.1 *If $K \subseteq \mathbf{M}^{m\times n}$ is the compact subset then $K^{app} = K^{qc}$.*

In particular, as the definition of K^{qc} is independent on Ω, we see that the set K^{app} is also independent on Ω.

As the consequence of the above facts we obtain the following result. It may be known to the specialists, but for completeness of our arguments we present its proof.

Proposition 2.1 *Let $K \subseteq \mathbf{M}^{m\times n}$ be the compact subset and $\Omega \subseteq \mathbf{R}^n$ be the bounded domain. Then $F \in K^{qc}$ if and only if for every $\delta > 0$ there exists the sequence $u^\nu : \Omega \to \mathbf{R}^m$ of piecewise affine mappings, such that*

$$\begin{aligned} \mathrm{dist}(Du^\nu, K) &\to 0 \text{ a. e.} && (4)\\ \|Du^\nu\|_\infty &\le C\|K\|_\infty, && (5)\\ u^\nu &= Fx \text{ on } \partial\Omega, && (6)\\ \|u^\nu - Fx\|_\infty &< \delta \text{ in } \Omega, && (7) \end{aligned}$$

and the constant C in (5) depends on m and n only.

Proof. The implication "$\Leftarrow$" follows from Theorem 2.1, so the only part "$\Rightarrow$" remains nontrivial. The proof is obtained by steps: 1. we show that there exists the sequence of functions $\{u^\nu\}$ (not necessarily piecewise affine) which satisfies (4), (5), (6) and (7); 2. we show that (4), (5), (6) and (7) holds true with the sequence of affine mappings.

Proof of Step 1. Suppose that $F \in K^{qc}$. Then, according to Theorem 2.1 there exists the sequence $\{u^\nu\}$ which is bounded in $W^{1,\infty}(\Omega, \mathbf{R}^m)$ with the property (4) and (6). The fact that we may additionally assume that $\{u^\nu\}$ satisfies (5) follows from Zhang's Lemma (see Lemma 3.1 in [34] or Lemma 4.21, part (ii) in [22]) and Rellich Compactness Theorem (see e.g. Section 1.4.6 in [18]). To show that we may additionally assume that $\{u^\nu\}$ satisfies (7) we use the following rescaling argument. We may assume without loss of generality that $F = 0$ (consider $\tilde{u}^\nu = u^\nu - Fx$). Let us cover Ω up to a set of measure 0 by disjoint copies of Ω: $\Omega_i = a_i + r_i\Omega$ that are contained in Ω and such that $r_i < \epsilon$ for every i, and define

$$\omega^\nu(x) := \begin{cases} r_i u^\nu(\frac{x-a_i}{r_i}) & \text{in} \quad \Omega_i \\ 0 & \text{in} \quad \Omega \setminus \cup\Omega_i \end{cases}$$

Then ω^ν's are continuous and satisfy (4), (5), (6). They also satisfy (7) if we take ϵ small enough, which follows from the estimation:

$$\|\omega^\nu\|_\infty \leq r_i\|u^\nu\|_\infty \leq C^{'}\epsilon\|Du^\nu\|_\infty \leq C\epsilon\|K\|_\infty = \delta,$$

where we take $\epsilon = \frac{\delta}{C\|K\|_\infty}$ (we may assume that $\|K\|_\infty \neq 0$, as otherwise the inequality is trivial) and constants $C^{'}, C$ depend on $m, n, \mathrm{diam}\Omega$ and do not depend on the sequence. We have used Poincare's inequality in the version: $\|u\|_\infty \leq C\|Du\|_\infty$, where $C = C(m, n, \mathrm{diam}\Omega)$, $u \in W_0^{1,\infty}(\Omega, \mathbf{R}^m)$. It is an easy consequence of density argument combined with the Sobolev's integral formulae (see e.g. Theorem 1.1.10/2 in [18]):

$$u = \sum_{i=1}^{n} K_i * D_i u,$$

where $u \in C_0^\infty(\mathbf{R}^n)$, $K_i = \frac{1}{nv_n}\frac{-x_i}{|x|^n}$ and v_n denotes the volume of the unit ball. Its application was possible as all the u^ν's vanish on $\partial\Omega$ and Ω is bounded.

Proof of Step 2. Now let us show that the functions $\{u^\nu\}$ can be taken piecewise affine. This is obtained in the standard way by introducing the sufficiently fine triangulation of the set Ω by subsets Ω_i and improving $\{u^\nu\}$ to be affine on every Ω_i, so that (4), (5) and (7) are satisfied (with possibly different C in (5) and 2δ instead of δ in (7)). If the diameters of Ω_i converge to zero when $\min_{x\in\Omega_i}\mathrm{dist}(x, \partial\Omega) \to 0$, then the improved functions must satisfy the same boundary conditions as the original ones, which are given by (6). The proof of the proposition is complete. □

For an open set U we define

$$U^{qc} := \cup_{K\subseteq U, K-\mathrm{compact}} K^{qc}$$

The following definition is the generalization of the well known definition of in–approximation due to Gromov (see e.g. [13]).

Definition 2.3 Assume that K and U_i (where $i \in \mathbf{N}$) are subsets in $\mathbf{M}^{m\times n}$, given for $i \in \mathbf{N}$, K is compact and the U_i's are open. We say that the sequence $\{U_i\}_{i\in\mathbf{N}}$ is an qc–in–approximation of K (or that K admits the qc-in–approximation by $\{U_i\}_{i\in\mathbf{N}}$) if $U_i \subseteq U_{i+1}^{qc}$ and for every i we have $\sup_{x\in U_i}\mathrm{dist}(x, K) \to 0$ as $i \to \infty$.

Our main result reads as follows.

THEOREM 2.2 *Assume that $\Omega \subseteq \mathbf{R}^n$ is a bounded domain and the compact set $K \subseteq \mathbf{M}^{m\times n}$ admits qc–in–approximation by $\{U_i\}_{i\in\mathbf{N}}$, where $U_i \subseteq \mathbf{M}^{m\times n}$ are open subsets. Suppose further that there is the solution $u \in W^{1,\infty}(\Omega, \mathbf{R}^m)$ of differential inclusion:*

$$Du(x) \in U_1 \text{ a. e. in } \Omega. \tag{8}$$

Then for every $\epsilon, \delta > 0$ there exists the closed set $F \subseteq \Omega$ and the mapping $v \in W^{1,\infty}(\Omega, \mathbf{R}^m)$ such that

$$\begin{aligned} |\Omega \setminus F| &< \epsilon && (9)\\ Dv(x) &\in K \text{ for every } x \in F, && (10)\\ \|Dv\|_\infty &\leq C\|U_1\|_\infty, && (11)\\ v &= u \text{ on } \partial\Omega, \\ \|u-v\|_\infty &\leq \delta \end{aligned}$$

The proof will be proceeded by the sequence of lemmas.

LEMMA 2.1 *Let $U \subseteq \mathbf{M}^{m\times n}$ be an open bounded set, $u : \Omega \to \mathbf{R}^m$ be the piecewise affine mapping such that $Du \in U^{qc}$ for almost every $x \in \Omega$. Then for every $\epsilon, \delta > 0$ there is piecewise affine mapping $v : \Omega \to \mathbf{R}^m$ and an open set R such that*

$$\begin{aligned} |\Omega \setminus R| &< \epsilon, \\ Dv &\in U \text{ on } R \\ \|Dv\|_\infty &\leq C\|U\|_\infty, \\ u &= v \text{ on } \partial\Omega, \\ \|u-v\|_\infty &\leq \delta, \end{aligned} \tag{12}$$

and the constant C in (12) depends on n and m only.

Proof of Lemma 2.1: As u is piecewise affine, we have $\Omega = \cup_i \Omega_i \cup A$, where $|A| = 0$, Ω_i's are open and u is affine on each Ω_i, so that: $u = F_i x + C_i$ where $F_i \in U^{qc}$ on Ω_i. Since $U^{qc} = \cup_{P\subseteq U, P-\text{compact}} P^{qc}$, we have $F_i \in P_i^{qc}$ for some compact set P_i, and we can choose $\delta_i > 0$ such that $(P_i)_{\delta_i} \subseteq U$. Let $\epsilon_i > 0$ be taken arbitrary. According to Proposition 2.1, we find the piecewise affine function v_i and mesasurable set

$$\tilde{\Omega}_i := \{x \in \Omega_i : \text{dist}(Dv_i(x), P_i) > \delta_i\} \tag{13}$$

such that

$$\begin{aligned} \text{dist}(Dv_i, P_i) \leq \delta_i \quad &\text{on} \quad \Omega_i \setminus \tilde{\Omega}_i \text{ where } |\tilde{\Omega}_i| < \epsilon_i, \\ \|Dv_i\|_{\infty,\Omega_i} &\leq C\|P_i\|_\infty \leq C\|U\|_\infty, \\ v_i &= F_i x + C_i \text{ on } \partial\Omega_i, \\ \|v_i - u\|_{\infty,\Omega_i} &\leq \delta \end{aligned}$$

In particular, on the set $\Omega \setminus \tilde{\Omega}_i$ we have $Dv_i \in (P_i)_{\delta_i} \subseteq U$. Let us choose v_i subordinated to the choice of $\epsilon_i := \epsilon/2^i$ and let $v(x)$ be equal to $v_i(x)$ on each Ω_i. Then v verifies the assertions of the lemma. It remains to show that one can assume that the set $R := \{x \in \Omega : Dv \in U\}$ is open. To see that we use the following argument. As v is piecewise affine, we have $\Omega = \cup V_k \cup A$, where $|A| = 0$, $Dv = G_k$ on each V_k and the V_k's are open. Take $I := \{k : G_k \in U\}$ and define

$$R := \cup_{k\in I} V_k.$$

Then R is open and by construction we have $\Omega \setminus R \subseteq \cup_{i=1}^\infty \tilde{\Omega}_i$, where $\tilde{\Omega}_i$'s are the same as in (13). In particular $|\Omega \setminus R| \leq \sum_{i=1}^\infty \frac{\epsilon}{2^i} = \epsilon$. This ends the proof of the lemma. □

LEMMA 2.2 *Assume that $\Omega \subseteq \mathbf{R}^n$ is a bounded domain and that the compact set $K \subseteq \mathbf{M}^{m\times n}$ admits qc-in-approximation by open sets $\{U_i\}_{i\in\mathbf{N}}$. Let $u : \Omega \to \mathbf{R}^m$ be the piecewise affine mapping such that*

$$Du \in U_1^{qc},$$

for almost every $x \in \Omega$, and let $\{\epsilon_k\}_{k\in\mathbf{N}}$ and $\{\delta_k\}_{k\in\mathbf{N}}$ be two given sequences of positive numbers. Then there exist

1. *a decreasing sequence $\{R_k\}_{k\in\mathbf{N}}$ of open subsets of Ω such that $|\Omega \setminus R_1| < \epsilon_1$ and $|R_k \setminus R_{k+1}| < \epsilon_k$ for every $k \in \mathbf{N}$,*

2. *the sequence of piecewise affine lipshitz mappings $u_k : \Omega \to \mathbf{R}^m$ such that:*

- **a)** *$Du_k \in U_k$, for almost every $x \in R_k$,*
- **b)** $\sup_k \|Du_k\|_\infty \le C\|U_1\|_\infty$, *with C depending on m and n only,*
- **c)** *$u_{k+1} = u_k$ a. e. on $\Omega \setminus R_k$, for every $k \in \mathbf{N}$,*
- **d)** *$u_k = u$ on $\partial\Omega$, for every $k \in \mathbf{N}$,*
- **e)** *$\|u_1 - u\|_\infty < \delta_1$ and $\|u_{k+1} - u_k\|_\infty < \delta_{k+1}$ for every $k \in \mathbf{N}$.*

Proof. We inductively apply Lemma 2.1. In the first step we use Lemma 2.1 on Ω with $U := U_1$, $\epsilon := \epsilon_1$ and $\delta := \delta_1$, and find the piecewise affine mapping $u_1 := v$ and an open set R_1 such that:

$$\begin{array}{rcl} Du_1 & \in & U_1 \subseteq U_2^{qc} \text{ a. e. on } R_1 \text{ and } |\Omega \setminus R_1| < \epsilon_1, \\ \|Du_1\|_\infty & \le & C\|U_1\|_\infty \text{ on } \Omega \text{ with } C = C(m,n), \\ u_1 & = & u \text{ on } \partial\Omega \\ \|u - u_1\|_\infty & \le & \delta_1. \end{array}$$

Suppose now that we have already constructed pairs $\{u_k, R_k\}$ with the desired properties for $l = 1, \dots, k$. To construct u_{k+1} first we apply Lemma 2.1 on $\Omega = R_k$, with $u = u_k$, $U = U_{k+1}$ (so that $Du_k \in U_{k+1}^{qc}$ on R_k), $\delta = \delta_{k+1}$ and $\epsilon = \epsilon_{k+1}$ and construct v such that:

$$\begin{array}{rcl} Dv & \in & U_{k+1} \subseteq U_{k+2}^{qc} \text{ on } R_{k+1} \text{ and } |R_k \setminus R_{k+1}| < \epsilon_{k+1}, \\ \|Dv\|_\infty & \le & C(m,n)\|U_{k+1}\|_\infty, \text{ on } R_k \\ v & = & u_k \text{ on } \partial R_k, \\ \|u_k - v\|_{\infty,R_k} & < & \delta_{k+1}. \end{array}$$

Then we extend v to the whole of Ω by expression

$$u_{k+1}(x) = \begin{cases} u_k(x) & \text{for} \quad x \in \Omega \setminus R_k \\ v & \text{for} \quad x \in R_k. \end{cases}$$

□

Now we are in the position to prove Theorem 2.2.

Proof of Theorem 2.2. The proof is obtained by the slight modification of general techniques due to Gromov, Müller and Šverák, see e.g. [22], Theorem 5.3. At first we note that we may assume that the solution u of the inclusion (8) is piecewise affine. This is arranged by the same arguments as that in the proof of Step 2 in Proposition 2.1. Then the proof is obtained by the successive corrections of the solution u of (8) within piecewise affine mappings on sets Ω_i where u is affine. From now the arguments are similar to that given in [22], the proof of Theorem 5.3.

Let

$$\Omega_i = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 2^{-i}\},$$

and $\rho \in C_0^\infty(\mathbf{R}^n)$ be the usual modifying kernel on $\mathbf{R}^n$, i. e. $\mathrm{supp}\rho \subseteq B(1)$, $\int \rho dx = 1$ and $\rho_\epsilon = \epsilon^{-n}\rho(x/\epsilon)$. Take an arbitrary $\epsilon, \delta > 0$. We may assume that $\epsilon, \delta < 1$. Then we take $\epsilon_1 = \frac{\epsilon}{4}$ and choose $\epsilon_i \in (0, 2^{-(i+1)}\epsilon)$ for $i = 2, 3, \dots$ to satisfy:

$$\|\rho_{\epsilon_i} * Du_i - Du_i\|_{L^1(\Omega_i)} < 2^{-i}. \tag{14}$$

We define the sequence $\{\delta_i\}$ by putting $\delta_1 = \frac{\delta}{4}$ and $\delta_{i+1} = \delta_i \epsilon_i$, and use Lemma 2.2 with $\{\epsilon_i\}$ and $\{\delta_i\}$. In particular $\sum_i \delta_i \leq \delta/2$, and we observe from part e) in Lemma 2.2 that the constructed sequence $\{u_k\}$ is the Cauchy sequence in $L^\infty(\Omega)$. Hence, and using property e) in Lemma 2.2 there exists the function $u_\infty \in L^\infty(\Omega)$ such that

$$u_k \to u_\infty \text{ in } L^\infty(\Omega) \text{ and } \|u - u_\infty\| \leq \delta.$$

On the other hand, using part b) in Lemma 2.2, we see that $\sup_k \|Du_k\|_\infty < \infty$, so we may assume (after eventually extracting the subsequence) that

$$Du_k \overset{*}{\rightharpoonup} Du_\infty \text{ in } L^\infty(\Omega).$$

Then property d) in Lemma 2.2 implies that

$$u_\infty = u \text{ on } \partial\Omega.$$

We will show that

$$Du_\infty \in K \text{ a. e. in } R := \cap_k R_k. \tag{15}$$

Note that R is measurable and

$$|\Omega \setminus R| \leq |\Omega \setminus R_1| + \sum_{k\geq 1} |R_k \setminus R_{k+1}| < \sum_{k \geq 1} \epsilon_k < \frac{\epsilon}{2}.$$

To prove (15) at first we see that

$$\begin{aligned} \|\rho_{\epsilon_k} * (Du_k - Du_\infty)\|_{L^1(\Omega_k)} &= \|D\rho_{\epsilon_k} * (u_k - u_\infty)\|_{L^1(\Omega_k)} \leq \frac{C}{\epsilon_k}\|u_k - u_\infty\|_{\infty, R_k} \leq & (16)\\ &\leq \frac{C}{\epsilon_k}\sum_{l\geq k}\|u_l - u_{l+1}\|_\infty \leq \frac{C}{\epsilon_k}\sum_{l \geq k+1}\delta_l < C^{'}\delta_k, & (17)\end{aligned}$$

with constants $C, C^{'}$ independent on k and

$$\|Du_k - Du_\infty\|_{L^1(\Omega)} \leq \|Du_k - Du_\infty\|_{L^1(\Omega\setminus\Omega_k)} + \|Du_k - Du_\infty\|_{L^1(\Omega_k)}.$$

The first term above, according to part b) of Lemma 2.2 is no bigger than $2C\|U_1\|_\infty|\Omega \setminus \Omega_k|$, so can be arbitrary small if we take k big enough. The second term can be estimated by

$$\|\rho_{\epsilon_k} * (Du_k - Du_\infty)\|_{L^1(\Omega_k)} + \|Du_k - \rho_{\epsilon_k} * Du_k\|_{L^1(\Omega_k)} + \|Du_\infty - \rho_{\epsilon_k} * Du_\infty\|_{L^1(\Omega_k)}.$$

Using (14) and (17) we observe that first two expressions converge to 0 as $k \to \infty$. To deal with the third expression we note that

$$\|Du_\infty - \rho_{\epsilon_k} * Du_\infty\|_{L^1(\Omega_k)} \leq \|w - \rho_{\epsilon_k} * w\|_{L^1(\Omega)},$$

where w is an extension of Du_∞ by 0 outside Ω. Therefore the last expression also converges to 0 as $k \to \infty$. This gives $\|Du_k - Du_\infty\|_{L^1(\Omega)} \to 0$. After extracting the subsequence we may assume that $Du_k \to Du_\infty$ almost everywhere. This together with the fact that $Du_k \in U_k$ a. e. on R and $U_k \to K$ as $k \to \infty$ shows that $Du_\infty \in K$ a. e. on R. Finally, we observe that the set R is of G_δ-type, so it is not necessarily closed. As R is measurable, we can find the closed subset $F \subseteq R$ such that $|R \setminus F| < \frac{\epsilon}{2}$ and $Du_\infty \in K$ everywhere on F. Then (15) holds true everywhere with R substituted by F, so that set F does the job. □

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