Gagliardo–Nirenberg inequalities in logarithmic spaces

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Abstract

We obtain interpolation inequalities for derivatives:

$$\int M_{q,\alpha}(|\nabla f(x)|)dx \le C\left[\int M_{p,\beta}(\Phi_1(x,|f|,|\nabla^{(2)}f|))dx + \int M_{r,\gamma}(\Phi_2(x,|f|,|\nabla^{(2)}f|))dx\right],$$

and their counterparts expressed in Orlicz norms:

$$\|\nabla f\|^2_{(q,\alpha)} \le C\|\Phi_1(x,|f|,|\nabla^{(2)}f|)\|_{(p,\beta)}\,\|\Phi_2(x,|f|,|\nabla^{(2)}f|)\|_{(r,\gamma)},$$

where $\|\cdot\|_{(s,\kappa)}$ is the Orlicz norm relative to the function $M_{s,\kappa}(t) = t^s(\ln(2+t))^\kappa$. The parameters $p,q,r,\alpha,\beta,\gamma$, as well as the Carathéodory functions Φ_1, Φ_2 are supposed to satisfy certain consistency conditions. Some of the classical Gagliardo-Nirenberg inequalities follow as a special case. Gagliardo-Nirenberg inequalities in logarithmic spaces with higher order gradients are also considered.

MSC (2000): Primary 26D10, Secondary 46E35.

1 Introduction and statement of results

The purpose of this paper is to obtain variants of the Gagliardo-Nirenberg interpolation inequalities for derivatives:

$$\|\nabla^{(k)}f\|_{L^q} \le C\|f\|^{1-k/m}_{L^p}\|\nabla^{(m)}f\|^{k/m}_{L^r}, \tag{1.1}$$

(where $f \in W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $p,q,r \in [1,\infty]$, $\frac{1}{q} = (1-\frac{k}{m})\frac{1}{p} + \frac{k}{m}\frac{1}{r}$, $0 < k < m$ and k, m are positive integers), expressed in logarithmic-type Orlicz spaces instead of L^p, L^q and L^r in (1.1).

Inequalities of the form (1.1) have been extensively investigated and have evolved in many directions (see [5, 6, 8, 10, 11, 19, 21, 24, 27, 29, 30, 33, 34, 35, 36, 37] and their references), but their generalizations to Orlicz spaces are nearly missing in the literature. In 1996 Bang [1], see also [2, 3, 4], proved variants of (1.1) for a one-variable function, within the same Orlicz space L^M.

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The authors have recently obtained inequalities of the form

$$\int M(|\nabla f|)dx \leq C(\int H(|f|)dx + \int J(|\nabla^{(2)} f|)dx), \tag{1.2}$$

$$\|\nabla f\|^2_{(M)} \leq C\|f\|_{(H)}\|\nabla^{(2)} f\|_{(J)} \tag{1.3}$$

dealing with functions of n variables and Orlicz spaces L^M, L^H and L^J defined by possibly distinct Young functions M, H, J which satisfy certain compatibility conditions (see [26]). In this work we adapt this abstract approach to Young functions $M_{s,\kappa}(t) = t^s(\ln(2+t))^\kappa$, with related Orlicz norms denoted by $\|\cdot\|_{(s,\kappa)}$.

The parameters we deal in Theorems 1.1, 1.2 and 1.3 stated below will be subject to the following two consistency conditions:

(A) $\beta, \gamma \in \mathbb{R}$, $p, r > 1$, ($q > 2, \alpha \in \mathbb{R}$ or $q = 2, \alpha \geq 0$) and $\left(\frac{2}{q} = \frac{1}{p} + \frac{1}{r}, \frac{2\alpha}{q} \leq \frac{\beta}{p} + \frac{\gamma}{r}\right)$,

(B) $\beta, \gamma \in \mathbb{R}$, $\alpha < 0$, $p, r > 1$, $q = 2$, $\frac{1}{p} + \frac{1}{r} = 1$, $\beta(r-1) + \gamma \geq 0$.

One of our results is the following logarithmic variant of the Gagliardo-Nirenberg inequality.

Theorem 1.1 *Suppose that $p, q, r, \alpha, \beta, \gamma$ are given real numbers such that Condition* **(A)** *or* **(B)** *is satisfied. Then for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support one has:*

$$\int |\nabla f|^q(\ln(2+|\nabla f|))^\alpha dx \leq C\left[\int |f|^p(\ln(2+|f|))^\beta dx + \int |\nabla^{(2)} f|^r(\ln(2+|\nabla^{(2)} f|))^\gamma dx\right], \tag{1.4}$$

and also

$$\|\nabla f\|^2_{(q,\alpha)} \leq C\|f\|_{(p,\beta)}\| \, \|\nabla^{(2)} f\|_{(r,\gamma)}, \tag{1.5}$$

with the constant C independent of f.

In the particular case when we take $\alpha = \beta = \gamma = 0$, we obtain the classical Gagliardo-Nirenberg inequality (1.1) restricted here to $q \geq 2$, while for $p = q = r \geq 2$, $\alpha = \beta = \gamma$ (negative values of α permitted only for $q > 2$) and a scalar function f, we retrieve Bang's result from [1]. Observe that q is in this case the harmonic mean of p and r, and when $p = q = r$ and **(A)** holds then α does not exceed the arithmetic mean of β and γ.

The special cases of (1.4), when one of the parameters α or β or γ equals to zero, follow from our previous work [25], where we dealt with variants of (1.4) within logarithmic spaces $L^s(\ln(\mu + L^a))^\alpha$, with $\mu \in \{1, 2\}$, under the restriction that one of the spaces considered: for f, $|\nabla f|$ or $|\nabla^{(2)} f|$ was the homogeneous space L^s (see Remark 4.3).

We will prove a more general variant of Theorem 1.1 which reads as follows.

Theorem 1.2 *Suppose that $p, q, r, \alpha, \beta, \gamma$ are given real numbers satisfying* **(A)** *or* **(B)** *and let $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory functions (i.e. measurable with respect to $x \in \mathbb{R}^n$ and continuous with respect to $(\lambda_1, \lambda_2) \in \mathbb{R}^2$), such that $\Phi_1(x, \lambda_1, \lambda_2)\Phi_2(x, \lambda_1, \lambda_2) = \lambda_1\lambda_2$ a.e. Take any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support. Then, denoting $w_1(x) = \Phi_1(x, |f|, |\nabla^{(2)} f|)$,*

$w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|)$, *we have:*

$$\int M_{q,\alpha}(|\nabla f(x)|)dx \leq C \left[\int M_{p,\beta}(w_1(x))dx + \int M_{r,\gamma}(w_2(x))dx \right], \tag{1.6}$$

and also

$$\|\nabla f\|^2_{(q,\alpha)} \leq C \|w_1\|_{(p,\beta)} \, \|w_2\|_{(r,\gamma)}, \tag{1.7}$$

both inequalities holding with a constant C independent of f.

For a particular choice of $\Phi_1(x, \lambda_1, \lambda_2) = \omega(x)\lambda_1^{\theta_1}\lambda_2^{\theta_2}$, $\Phi_2(x, \lambda_1, \lambda_2) = \frac{1}{\omega(x)}\lambda_1^{1-\theta_1}\lambda_2^{1-\theta_2}$, where $\omega : \mathbb{R}^n \to (0, \infty)$ is a measurable, a.e. positive function, we obtain the following theorem.

Theorem 1.3 *Suppose that $p, q, r, \alpha, \beta, \gamma$ are given real numbers such that Condition* **(A)** *or* **(B)** *is satisfied, let $(\theta_1, \theta_2) \in [0, 1]^2 \setminus \{(0, 0), (1, 1)\}$ and let ω be an arbitrary positive a.e., measurable function. Then for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support one has:*

$$\int M_{q,\alpha}(|\nabla f|)dx \leq C \left[\int M_{p,\beta}(|f|^{\theta_1}|\nabla^{(2)} f|^{\theta_2}\omega)dx + \int M_{r,\gamma}(|f|^{1-\theta_1}|\nabla^{(2)} f|^{1-\theta_2}\omega^{-1})dx \right], \tag{1.8}$$

and also

$$\|\nabla f\|^2_{(q,\alpha)} \leq C \| |f|^{\theta_1}|\nabla^{(2)} f|^{\theta_2}\omega\|_{(p,\beta)} \| \, \| |f|^{1-\theta_1}|\nabla^{(2)} f|^{1-\theta_2}\omega^{-1}\|_{(r,\gamma)}, \tag{1.9}$$

both inequalities holding with a constant C independent of f, (θ_1, θ_2) and ω.

Observe that Theorem 1.1 is a particular case of Theorem 1.3 (it corresponds to $\theta_1 = 1, \theta_2 = 0$ and $\omega \equiv 1$), but Theorem 1.3 (and so also Theorem 1.2) is more general.

Yet another choice of parameters: $\theta_1 = \theta_2 = 1/2$, $p = q = r$, $\alpha = \beta = \gamma$ and $\omega \equiv 1$ in Theorem 1.3, yields the following result.

Theorem 1.4 *Suppose that either $q > 2, \alpha \in \mathbb{R}$ or $q = 2, \alpha \geq 0$. Then for an arbitrary smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support we have*

$$\int M_{q,\alpha}(|\nabla f|)dx \leq C \int M_{q,\alpha}(\sqrt{|f| \, |\nabla^{(2)} f|})dx, \tag{1.10}$$

and also

$$\|\nabla f\|_{(q,\alpha)} \leq C \|\sqrt{|f| \, |\nabla^{(2)} f|}\|_{(q,\alpha)}, \tag{1.11}$$

both inequalities holding with a constant C independent of f.

For completeness we write down the statement of Theorem 1.3 in homogeneous spaces ($\alpha = \beta = \gamma = 0$).

Corollary 1.1 *If p, q, r are given real numbers such that $(q \geq 2, p, r > 1)$ and $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$, then for arbitrary $(\theta_1, \theta_2) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$, an arbitrary $f \in C_0^\infty(\mathbb{R}^n)$ and any positive a.e., measurable function ω we have:*

$$(\int |\nabla f|^q dx)^{2/q} \leq C(\int (|f|^{\theta_1} |\nabla^{(2)} f|^{\theta_2} \omega)^p dx)^{1/p} (\int (|f|^{(1-\theta_1)} |\nabla^{(2)} f|^{1-\theta_2} \omega^{-1})^r dx)^{1/r}, \tag{1.12}$$

with a constant C independent of f, (θ_1, θ_2) and ω.

We also point out two special cases of Corollary 1.1.

Corollary 1.2 ($\theta_1 = \theta_2 = \frac{1}{2}, \omega \equiv 1, p = q = r$) *If $q \geq 2$ and $f \in C_0^\infty(\mathbb{R}^n)$, we have*

$$\int |\nabla f|^q dx \leq C \int (|f|\, |\nabla^{(2)} f|)^{q/2} dx,$$

with a constant C independent of f.

Corollary 1.3 ($\theta_2 = 0$) *If p, q, r are given real numbers such that $(q \geq 2, p, r > 1)$ and $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$, then for an arbitrary $\theta \in [0,1]$, any $f \in C_0^\infty(\mathbb{R}^n)$ and an arbitrary measure $\mu(dx) = \omega(x)dx$ with a positive weight ω, we have*

$$(\int |\nabla f|^q dx)^{2/q} \leq C(\int (|f|^{\theta p} d\mu)^{1/p} (\int (|f|^{(1-\theta)} |\nabla^{(2)} f|)^r \omega^{-\frac{r}{p}} dx)^{1/r}, \tag{1.13}$$

with a constant C independent of f, θ and ω.

Note that on the right hand side of (1.13) we can manage the terms $\int |f|^s d\mu$ with $s = \theta p$ being smaller than 1 and an arbitrary weighted measure $\mu(dx) = \omega(x)dx$, with a positive weight ω. In such a case $\|f\|_{L^s_\mu} = (\int |f|^s d\mu)^{1/s}$ is no longer a norm.

Although in this paper we deal mostly with derivatives of order $0, 1$ and 2, some generalisations to higher order derivatives are also possible. This results in Theorem 4.3, where some cases of Theorem 1.1 are generalised to higher order derivatives. We also obtain stronger variants of inequalities (1.4), (1.6) and (1.8) (Theorem 4.1). As we deal with the logarithmic functions, we were able to get nonlinear variants of inequalities (1.6), i.e. such inequalities between Young functionals: $I_1 = \int M_{q,\alpha}(|\nabla f|)dx$, $I_0 = \int M_{p,\beta}(w_1)dx$ and $I_2 = \int M_{r,\gamma}(w_2)dx$, with w_1 and w_2 introduced in Theorem 1.2, where I_1 is estimated from the above by a nonlinear expression involving I_0 and I_2. The precise statement is given in Theorem 4.2.

In the proof of Theorem 1.3 we adapt abstract techniques described in [26]. These techniques specialized to logarithmic Orlicz spaces require an additional and independent analysis (see also Remark 4.3). The results obtained (Theorems: 1.1,1.2, 1.3 and 1.4) are in general new, while results within homogeneous spaces (Corollaries 1.1–1.3) are also covered by the abstract approach presented in [26]. On the other hand, the additional results presented in Section 4 (Theorems 4.1, 4.2 and 4.3) are based on the special structure of logarithmic Orlicz spaces and they have no abstract counterparts previously treated in [26]. The importance of logarithmic Orlicz spaces in various disciplines of analysis and PDE's (e.g. [7], [9], [12], Section 4.3, [13], [14], [15], [16], [17],

[18], [20], [22], [23], [31], [38], Theorems 11.7 and Corollary 15.4, and references therein), to our opinion justifies separate investigation of the logarithmic-type Gagliardo-Nirenberg inequalities.

Notation. Throughout the paper, the symbol $\nabla^{(k)}f$ stands for the k–th gradient of the mapping $f:\mathbb{R}^n\to\mathbb{R}$ i.e. the vector $(D^\alpha f)_{|\alpha|=k}$. If A is a vector or a matrix, by $|A|$ we denote its Euclidean norm induced by the standard scalar product $\langle\cdot,\cdot\rangle$, while A^t stands for its transposition.

By q^* we will denote the Hölder conjugate to a real number $1<q<\infty$ and by C — a general constant whose value can change even within the same line. When the domain of integration is not specified, it is meant to be the whole of $\mathbb{R}^n$. If F is a Young function, by F^* we denote its Legendre transform, defined by $F^*(t)=\sup_{s\ge 0}[st-F(s)]$.

Suppose $M,N:[0,\infty)\to[0,\infty)$ are two given functions. When $N(\lambda)\le CM(k\lambda)$ for $\lambda\ge\lambda_0$ (for $0\le\lambda\le\lambda_0$; for $\lambda\ge 0$) with constants C,k independent of x, then we say that M dominates N at infinity (near zero; globally). This relation is denoted by $M\succ N$. We say that M is equivalent to N (symb. $M\sim N$) when $M\succ N$ and $N\succ M$. It is not hard to convince oneself (see e.g. Theorems 2.1 and 3.1 of [28]) that this domination is reversed by taking the Legendre transform of Young functions: $M\succ N$ (at infinity, near zero, globally) implies $N^*\succ M^*$ (at infinity, near zero, globally). Note that if M satisfies the Δ_2-condition then $M\succ N$ if and only if $N(\lambda)\le CM(\lambda)$ with some constant C independent of λ.

When the domain of integration is not specified, it is meant to be the whole of $\mathbb{R}^n$.

2 Preliminaries

We will be dealing with the following functions:

$$M_{q,\alpha}(t):=t^q(\ln(2+t))^\alpha \ \text{ where } \ q>1,\alpha\in\mathbb{R}. \tag{2.1}$$

Within this range of parameters q,α all functions $M_{q,\alpha}$ are Young functions (i.e. convex, $M_{q,\alpha}(0)=0$, $\lim_{t\to 0+}\frac{M_{q,\alpha}(t)}{t}=0$, $\lim_{t\to\infty}\frac{M_{q,\alpha}(t)}{t}=\infty$). Therefore the set

$$L_{(q,\alpha)}=\{f:\mathbb{R}^n\to\mathbb{R} \text{ measurable s.t. for some } K>0\int M_{q,\alpha}(\frac{|f(x)|}{K})dx<\infty\}$$

becomes a Banach space when equipped with the Luxemburg norm:

$$\|f\|_{(q,\alpha)}:=\inf\{K>0:\int M_{q,\alpha}(\tfrac{|f(x)|}{K})dx\le 1\}.$$

This is the Orlicz space defined by $M_{q,\alpha}$. Note that for $\alpha=0$ it coincides with the usual L^q space. The functions $M_{q,\alpha}$ satisfy the Δ_2-condition, i.e. $M_{q,\alpha}(2t)\le CM_{q,\alpha}(t)$, with a constant $C=C(q,\alpha)$ independent of $t\ge 0$. It is known that

$$\int M_{q,\alpha}(\tfrac{|f(x)|}{\|f\|_{(q,\alpha)}})dx=1 \quad \text{and} \quad \|f\|_{(q,\alpha)}\le\textstyle\int M_{q,\alpha}(|f(x)|)dx+1. \tag{2.2}$$

For details we refer the reader to the book [28], Chapter 1.

For later use observe that

$$M_{q,\alpha}\circ M_{\mu,\kappa}\sim M_{q\mu,q\kappa+\alpha}. \tag{2.3}$$

Finally, let us prove a lemma.

Lemma 2.1 *Suppose that* $\mu > 1, \kappa \in \mathbb{R}$ *and* $\tilde{\kappa} \geq \kappa_1 = -\kappa(\mu^* - 1)$. *Then there exists a constant* $C > 0$ *such that for all* $u, v \geq 0$ *one has*

$$uv \leq M_{\mu,\kappa}(u) + CM_{\mu^*,\tilde{\kappa}}(v). \tag{2.4}$$

Proof. It is immediate: as $M_{\mu,\kappa}(u) \sim u^{\mu}(\ln u)^{\kappa}$ for u large, then $M^*_{\mu,\kappa}(v) \sim M_{\mu^*,\kappa_1}(v)$, for large values of v (see [28], Theorem 7.1). On the other hand, for u small we have $M_{\mu,\kappa}(u) \sim u^{\mu}$, thus $M^*_{\mu,\kappa}(v) \sim v^{\mu^*} \sim M_{\mu^*,\kappa_1}$, for v small. Therefore $M^*_{\mu,\kappa} \sim M_{\mu^*,\kappa_1}$ globally.

When $\tilde{\kappa} \geq \kappa_1$, then $M_{\mu^*,\tilde{\kappa}}$ dominates M_{μ^*,κ_1} globally, and so, for $u, v \geq 0$

$$uv \leq M_{\mu,\kappa}(u) + M^*_{\mu,\kappa}(v) \leq M_{\mu,\kappa}(u) + CM_{\mu^*\kappa_1}(v) \leq M_{\mu,\kappa}(u) + CM_{\mu^*,\tilde{\kappa}}(v),$$

with certain constant $C > 0$. $\square$

3 Proofs of Theorems 1.1–1.4

As indicated in Section 1, what only needs to be shown is Theorem 1.2. The remaining results: Theorems 1.1, 1.3, 1.4 (together with Corollaries 1.1–1.3) follow as consequences.

Proof of Theorem 1.2. The proof is carried out in several steps.
STEP 1. We show that

$$I := \int M_{q,\alpha}(|\nabla f|)dx \leq C \int M_{q-2,\alpha}(|\nabla f|)|f|\,|\nabla^{(2)} f|\,dx, \tag{3.1}$$

with a constant C not depending on f (with a slight abuse of notation: the number $q - 2$ can be smaller than 1 here, but the formula (2.1) defining $M_{q-2,\alpha}$ remains valid).

Proof of this inequality is basically taken from [25]; we sketch it here to make the paper self-contained.

As $M_{q,\alpha}(|\lambda|) = M_{q-2,\alpha}(\lambda)\langle\lambda, \lambda\rangle$, where $\lambda = (\lambda_1, ..., \lambda_n)$, after integrating by parts we obtain

$$I = -\int \operatorname{div}\left(S(\nabla f(x))\right) f(x)dx, \tag{3.2}$$

where $S = (S_1, ..., S_n)$ and $S_i(\lambda) = M_{q-2,\alpha}(|\lambda|)\lambda_i$ (since $q \geq 2$ this integration by parts is allowed according to the Nikodym ACL Characterization Theorem, see [32], Th.2, Sec. 1.1.3).

In particular $S_i(\nabla f) = M_{q-2,\alpha}(|\nabla f|)\frac{\partial f}{\partial x_i}$, and so

$$\operatorname{div} S(\nabla f) = \frac{M'_{q-2,\alpha}(|\nabla f|)}{|\nabla f|}[\nabla f]^t[\nabla^{(2)} f][\nabla f] + M_{q-2,\alpha}(|\nabla f|)\Delta f.$$

Elementarily we check that $M'_{q-2,\alpha}(t) \sim M_{q-2,\alpha}(t)t^{-1}$ on the positive half-line. Moreover $|v^t A v| \leq |A|\,|v|^2$ and $|\operatorname{tr} A| \leq \sqrt{n}|A|$ (so that $|\Delta f| \leq \sqrt{n}|\nabla^{(2)} f|$). This gives

$$|\operatorname{div} S(\nabla f)| \leq CM_{q-2,\alpha}(|\nabla f|)|\nabla^{(2)} f|,$$

and together with (3.2) completes the proof of (3.1).

STEP 2. Now assume that **(A)** holds. We show that in this case we have, for all $u, v, w \geq 0$:

$$M_{q-2,\alpha}(u)vw \leq M_{q,\alpha}(u) + C\left[M_{p,\beta}(v) + M_{r,\gamma}(w)\right]. \tag{3.3}$$

To see this, first observe that

$$M_{q-2,\alpha}(s)t^2 \leq M_{q,\alpha}(s) + M_{q,\alpha}(t). \tag{3.4}$$

It is immediate: when $t \leq s$, then $M_{q-2,\alpha}(s)t^2 = M_{q,\alpha}(s)\left(\frac{t}{s}\right)^2 \leq M_{q,\alpha}(s)$. Since $M_{q-2,\alpha}$ is increasing, then for $s \leq t$ one has $M_{q-2,\alpha}(s)t^2 \leq M_{q-2,\alpha}(t)t^2 = M_{q,\alpha}(t)$.

Next, take $\mu = \frac{2p}{q}$, $\kappa = \frac{\beta-\alpha}{q}$, $\tilde{\kappa} = \frac{\gamma-\alpha}{q}$. Under current restriction on the parameters, it is not hard to convince oneself that $\tilde{\kappa} \geq \kappa_1 = -\kappa(\mu^* - 1)$.

Therefore the assumptions of Lemma 2.1 are satisfied and (2.4) can be applied, resulting in the following series of inequalities:

$$\begin{aligned} M_{q-2,\alpha}(u)vw &\leq M_{q,\alpha}(u) + CM_{q,\alpha}(\sqrt{vw}) \\ &\leq M_{q,\alpha}(u) + CM_{q,\alpha}\left(M_{\mu,\kappa}(\sqrt{v}) + M_{\mu^*,\tilde{\kappa}}(\sqrt{w})\right) \\ &\leq M_{q,\alpha}(u) + C\left[M_{q,\alpha} \circ M_{\mu,\kappa}(\sqrt{v}) + M_{q,\alpha} \circ M_{\mu^*,\tilde{\kappa}}(\sqrt{w})\right] \end{aligned}$$

(the last line inequality follows from the fact that for an arbitrary nondecreasing function F satisfying the Δ_2-condition one has $F(a+b) \leq F(2\max(a,b)) \leq F(2a)+F(2b) \leq C(F(a)+F(b))$). Using now the property (2.3) we see that $M_{q,\alpha} \circ M_{\mu,\kappa}(\sqrt{v}) \sim M_{p,\beta}(v)$ and $M_{q,\alpha} \circ M_{\mu^*,\tilde{\kappa}}(\sqrt{w}) \sim M_{r,\gamma}(w)$, so that (3.3) follows.

STEP 3. THE CONCLUSION UNDER THE CONDITION **(A)**.
Applying (3.1) we get

$$I \leq \frac{1}{2}\int M_{q-2,\alpha}(|\nabla f|) \cdot \left(2C|f||\nabla^{(2)} f|\right) dx. \tag{3.5}$$

Since, by definition of w_1 and w_2, $|f(x)||\nabla^{(2)} f(x)| = w_1(x)w_2(x)$, applying (3.3) and using the Δ_2-condition we get that I is not bigger than

$$\frac{1}{2}I + C\int M_{p,\beta}(w_1(x))dx + C\int M_{r,\gamma}(w_2(x))dx,$$

(with C possibly different than in (3.5)) which after rearranging yields (1.6).

In order to prove (1.7), fix $t_1, t_2 > 0$ and write the inequality (3.5) for $\widetilde{f} = \frac{f}{t_1 t_2}$. We get that

$$I \leq \frac{1}{2}\int M_{q-2,\alpha}(|\nabla \widetilde{f}|)\left(2C\tilde{w}_1\tilde{w}_2\right) dx,$$

where we put $\tilde{w}_i = \frac{w_i}{t_i^2}$ (it is so because of the relation $|\tilde{f}||\nabla^{(2)}\tilde{f}| = \tilde{w}_1\tilde{w}_2$).

Using (3.3) and repeating the subsequent steps with f, w_1 and w_2 replaced by $\tilde{f}, \tilde{w}_1$ and $\tilde{w}_2$ we obtain

$$\int M_{q,\alpha}(|\nabla\widetilde{f}|)dx \leq C\left(\int M_{p,\beta}(\widetilde{w}_1)dx + \int M_{r,\gamma}(\widetilde{w}_2)dx\right),$$

with constant C independent of f and t_1, t_2. Now choose $t_1^2 = \|w_1\|_{(q,\beta)}$, $t_2^2 = \|w_2\|_{(r,\gamma)}$. As the condition $t_i = 0$ implies $w_1 w_2 = 0$, which by the inequality (3.1) forces $f \equiv 0$ (as f is compactly supported and smooth), without loss of generality we can assume that $t_1, t_2 > 0$. Moreover, we have $\int M_{p,\beta}(\tilde{w}_1)dx = \int M_{p,\beta}(\frac{w_1}{\|w_1\|_{(p,\beta)}})dx = 1$, and similarly

$\int M_{r,\gamma}(\tilde{w}_2)dx = \int M_{r,\gamma}(\frac{w_2}{\|w_1\|_{(r,\gamma)}})dx = 1$. We end up with $\int M_{q,\alpha}(|\nabla \widetilde{f}|)dx \le C$. This together with (2.2) gives $\|\nabla \widetilde{f}\|_{(q,\alpha)} \le C + 1$, so that

$$\|\nabla f\|^2_{(q,\alpha)} \le (C+1)\|w_1\|_{(p,\beta)}\,\|w_2\|_{(r,\gamma)}.$$

Step 4. The conclusion under the Condition **(B)**.
First, apply (3.1), but instead of using (3.3) observe that for $q = 2$ and $\alpha < 0$ the function $M_{q-2,\alpha}$ is bounded. Therefore (using the same notation as above)

$$I \le C \int |f|\,|\nabla^{(2)} f| dx = C \int w_1\, w_2\, dx.$$

The conditions imposed on the parameters β and γ imply that (see Lemma 2.1) $w_1 w_2 \le M_{p,\beta}(w_1) + C M_{q,\gamma}(w_2)$, and consequently

$$I \le C \left(\int M_{p,\beta}(w_1)\, dx + \int M_{r,\gamma}(w_2)\, dx \right),$$

which proves (1.6) in this case. The proof of (1.7) goes now along the same lines as in Step 3 and so we skip it. ☐

4 Extensions and remarks

We start with the following result which shows that inequality (1.6) in Theorem 1.2 and its special variants: inequalities (1.4) and (1.8), can be transformed into a stronger form, where one of the summands can be made arbitrarily small. We obtain:

Theorem 4.1 *Suppose that $p, q, r, \alpha, \beta, \gamma$ are given real numbers satisfying* **(A)** *or* **(B)** *and let $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory functions such that $\Phi_1(x, \lambda_1, \lambda_2)\Phi_2(x, \lambda_1, \lambda_2) = \lambda_1 \lambda_2$ a. e. Take any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support and denote:*

$$w_1(x) \;=\; \Phi_1(x, |f|, |\nabla^{(2)} f|),\;\; w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|),$$

$$h_{s,\kappa}(\delta) = \begin{cases} M_{s,\kappa}(\delta) & \text{for } \kappa \ge 0 \\ \delta^s \ln(2 + \frac{1}{\delta})^{-\kappa} & \text{for } \kappa < 0. \end{cases} \tag{4.1}$$

Then there exists a constant $C = C(\beta, \gamma)$ such that for an arbitrary $\delta > 0$ we have:

$$\int M_{q,\alpha}(|\nabla f(x)|)dx \le C \left(h_{p,\beta}(\delta) \int M_{p,\beta}(w_1(x))dx + h_{r,\gamma}(\delta^{-1}) \int M_{r,\gamma}(w_2(x))dx \right). \tag{4.2}$$

In particular, for an arbitrary $\epsilon > 0$ there exists a constant C_ϵ, depending on ϵ, p, r, β and γ such that

$$\int M_{q,\alpha}(|\nabla f(x)|)dx \leq \epsilon \int M_{p,\beta}(w_1(x))dx + C_\epsilon \int M_{r,\gamma}(w_2(x))dx, \tag{4.3}$$

$$\int M_{q,\alpha}(|\nabla f(x)|)dx \leq C_\epsilon \int M_{p,\beta}(w_1(x))dx + \epsilon \int M_{r,\gamma}(w_2(x))dx. \tag{4.4}$$

Proof. Take an arbitrary $\delta > 0$ and apply (1.6) with $\tilde{w_1} = \delta w_1$ and $\tilde{w_2} = w_2/\delta$ replacing w_1 and w_2. Then it suffices to prove that for $s > 1, \kappa \in \mathbb{R}$ we have

$$M_{s,\kappa}(\delta\lambda) \leq C h_{s,\kappa}(\delta) M_{s,\kappa}(\lambda), \text{ for } \delta, \lambda \geq 0, \tag{4.5}$$

with C depending on κ only. To obtain (4.5), at first we note that

$$\ln(2+\delta\lambda) \leq C \ln(2+\delta)\ln(2+\lambda), \tag{4.6}$$

with C independent of δ and λ.

This is easy: suppose that $\delta \leq \lambda$, then the left hand side in the inequality above is not bigger than $\ln(2+\lambda^2) \sim \ln(2+\lambda)$. Also, $\ln(2+\delta) \geq \ln 2 > 0$, which completes the proof of (4.6).

Now the inequality (4.5) follows immediately from (4.6) when $\kappa \geq 0$, while for negative κ we have:

$$\begin{aligned} M_{s,\kappa}(\delta\lambda) &= \delta^s \left(\frac{\ln(2+\delta\lambda)}{\ln(2+\lambda)}\right)^{\kappa} M_{s,\kappa}(\lambda) \leq \delta^s \left(\sup_{\lambda>0} \frac{\ln(2+\lambda)}{\ln(2+\delta\lambda)}\right)^{-\kappa} M_{s,\kappa}(\lambda) \\ &\leq \delta^s \left(\sup_{\lambda>0} \frac{\ln(2+\lambda\delta^{-1})}{\ln(2+\lambda)}\right)^{-\kappa} M_{s,\kappa}(\lambda) \leq C_\kappa \delta^s (\ln(2+\delta^{-1}))^{-\kappa} M_{s,\kappa}(\lambda), \end{aligned}$$

where for the last inequality we have used (4.6).

This gives (4.2). To derive (4.3) and (4.4) we observe that $\lim_{\delta\to 0} h_{s,\kappa}(\delta) = 0$, so we can find δ such that $Ch_{p,\beta}(\delta) = \epsilon$ (for inequality (4.3)) and such that $Ch_{r,\gamma}(\delta^{-1}) = \epsilon$ (for inequality (4.4)). □

Now we will derive multiplicative variants of inequality (1.6) in Theorem 1.2, involving not Orlicz norms, but Orlicz functionals. Consequently, inequalities (1.4) and (1.8) will also have multiplicative counterparts involving Orlicz functionals.

The result presented below is restricted to the case $\beta, \gamma \geq 0$. When $\beta < 0$ or $\gamma < 0$, then a similar statement will hold, but the third and fourth factors in (4.7) will be different.

Theorem 4.2 *Suppose that $p, q, r, \alpha, \beta, \gamma$ are given real numbers satisfying* **(A)** *or* **(B)**, *$\beta, \gamma \geq 0$, and let $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ be Carathéodory functions such that $\Phi_1(x,\lambda_1,\lambda_2)\Phi_2(x,\lambda_1,\lambda_2) = \lambda_1\lambda_2$ a.e. Take any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support and denote:*

$$w_1(x) = \Phi_1(x, |f|, |\nabla^{(2)} f|), \quad w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|).$$

Then there exists a constant $C = C(p, r, \beta, \gamma) > 0$ such that:

$$\begin{aligned} \left(\int M_{q,\alpha}(|\nabla f(x)|)dx\right)^{\frac{2}{q}} &\leq C \left(\int M_{p,\beta}(w_1(x))dx\right)^{\frac{1}{p}} \left(\int M_{r,\gamma}(w_2(x))dx\right)^{\frac{1}{r}} \cdot \\ &\cdot \left(\ln(2 + \frac{\int M_{p,\beta}(w_1(x))dx}{\int M_{r,\gamma}(w_2(x))dx})\right)^{\frac{\gamma}{r}} \left(\ln(2 + \frac{\int M_{r,\gamma}(w_2(x))dx}{\int M_{p,\beta}(w_1(x))dx})\right)^{\frac{\beta}{p}}. \end{aligned} \tag{4.7}$$

Proof. Let us denote:

$$a := \int M_{q,\alpha}(|\nabla f(x)|)dx, \;\; b := C\int M_{p,\beta}(w_1(x))dx, \;\; c := C\int M_{r,\gamma}(w_2(x))dx,$$

where C is the constant from (4.2). Then (4.2) reads

$$a \leq M_{p,\beta}(\delta)b + M_{r,\gamma}(\delta^{-1})c, \tag{4.8}$$

where $\delta > 0$ can be taken arbitrary.

Now we minimize the right hand side of (4.8) with respect to $\delta > 0$. Observe that $M'_{s,\kappa}(\lambda) \sim M_{s,\kappa}(\lambda)/\lambda$, and so the minimum of the right hand side of (4.8) with respect to $\delta > 0$ is achieved at a point δ_0 for which the following is true:

$$C_1\frac{c}{b} \leq R(\delta_0) \leq C_2\frac{c}{b}, \;\; \text{where} \;\; R(\lambda) := \frac{M_{p,\beta}(\lambda)}{M_{r,\gamma}(\lambda^{-1})}, \tag{4.9}$$

with constants C_1, C_2 independent of c and b.

As $R(\lambda) \sim \frac{\lambda^{p+r}}{|\ln\lambda|^{\gamma}} = \frac{1}{(1/\lambda)^{p+r}(\ln(1/\lambda))^{\gamma}}$ for λ close to 0, and $R(\lambda) \sim \lambda^{p+r}(\ln\lambda)^{\beta}$ for λ large, and $(\lambda|\ln\lambda|)^{-1} \sim \frac{\lambda}{|\ln\lambda|}$ for both small and large values of λ (here ϕ^{-1} denotes the function inverse to ϕ), we verify that the function inverse to R satisfies:

$$R^{-1}(\lambda) \sim \left(\lambda\frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}}\right)^{\frac{1}{p+r}}. \tag{4.10}$$

Using (4.6) and (4.9) we establish that

$$\widetilde{C_1}R^{-1}\left(\frac{c}{b}\right) \leq \delta_0 \leq \widetilde{C_2}R^{-1}\left(\frac{c}{b}\right), \tag{4.11}$$

with $\tilde{C_1}, \tilde{C_2}$ not dependent on b and c. Moreover, we have

$$M_{p,\beta}\circ R^{-1}(\lambda) \sim \lambda^{\frac{p}{p+r}}(\ln(2+\lambda^{-1}))^{\frac{\gamma p}{p+r}}(\ln(2+\lambda))^{\frac{\beta r}{p+r}}. \tag{4.12}$$

On the other hand, according to (4.11), and using the fact that $M_{p,\beta}$ satisfies the Δ_2-condition,

$$M_{p,\beta}(\delta_0)b \leq M_{p,\beta}(\tilde{C}_2R^{-1}(\frac{c}{b}))b \leq C_3(M_{p,\beta}\circ R^{-1}(\frac{c}{b}))b := \mathcal{A}, \tag{4.13}$$

and by (4.9),

$$M_{r,\gamma}(\delta_0^{-1})c \leq C_1^{-1}M_{p,\beta}(\delta_0)b \leq C_1^{-1}\mathcal{A}. \tag{4.14}$$

Now we apply (4.8) with $\delta = \delta_0$, and also (4.13), (4.14) and (4.12). This gives

$$a \leq M_{p,\beta}(\delta_0)b + M_{r,\gamma}(\delta_0^{-1})c \leq C_4\mathcal{A} \leq C_5\, b^{\frac{r}{p+r}}c^{\frac{p}{p+r}}(\ln(2+\tfrac{b}{c}))^{\frac{\gamma r}{p+r}}(\ln(2+\tfrac{c}{b}))^{\frac{\beta r}{p+r}},$$

with C_5 independent of b and c, and completes the proof of (4.7). □

Remark 4.1 Note that for $\beta = \gamma = 0$ (4.7) is exactly the Gagliardo–Nirenberg's inequality restricted to $q \geq 2$.

The results of Theorem 1.1 can be iterated to higher derivatives. In particular we obtain the following theorem:

Theorem 4.3 *Suppose that $k, m \in \mathbb{Z}_+$, $0 < k < m$ and $p, q, r, \alpha, \beta, \gamma$ are given real numbers such that:*

$$\frac{1}{q} = (1 - \frac{k}{m})\frac{1}{p} + \frac{k}{m}\frac{1}{r},\ p, r > 2 \ \text{ and } \ \frac{\alpha}{q} \le (1 - \frac{k}{m})\frac{\beta}{p} + \frac{k}{m}\frac{\gamma}{r}. \tag{4.15}$$

Then for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support one has:

$$\begin{aligned} \int M_{q,\alpha}(|\nabla^{(k)} f(x)|)dx &\le C\left(\int M_{p,\beta}(|f(x)|)dx + \int M_{r,\gamma}(|\nabla^{(m)} f(x)|)dx\right), && (4.16)\\ \|\nabla^{(k)} f(x)\|_{(q,\alpha)} &\le C\|f\|_{(p,\beta)}^{1-\frac{k}{m}} \|\nabla^{(m)} f\|_{(r,\gamma)}^{\frac{k}{m}}, && (4.17)\end{aligned}$$

with a constant C independent of f.

Proof. We give the proof of (4.16) only, leaving (4.17) to the reader. As $M_{q,\tilde{\alpha}} \le M_{q,\alpha}$ whenever $\tilde{\alpha} \le \alpha$, it suffices to prove Theorem under the condition:

$$\frac{1}{q} = (1 - \frac{k}{m})\frac{1}{p} + \frac{k}{m}\frac{1}{r},\ p, r > 2 \text{ and } \frac{\alpha}{q} = (1 - \frac{k}{m})\frac{\beta}{p} + \frac{k}{m}\frac{\gamma}{r}. \tag{4.18}$$

For simplicity we will use the following notation. Let $D = \{\begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathbb{R} \setminus \{0\}, y \in \mathbb{R}\}$, $h : D \to D$ and $G_s : \mathbb{R}_2 \times \mathbb{R}_2 \to \mathbb{R}_2$ defined for $s \in [0,1]$, be given by the expression

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/x \\ y/x \end{pmatrix}, \quad G_s(\overline{\lambda}_1, \overline{\lambda}_2) = s\overline{\lambda}_1 + (1-s)\overline{\lambda}_2, \tag{4.19}$$

where $\overline{\lambda}_1, \overline{\lambda}_2 \in \mathbb{R}_2$.

Then conditions (4.18) read

$$h\begin{pmatrix} q \\ \alpha \end{pmatrix} = G_{\frac{k}{m}}\left(h\begin{pmatrix} p \\ \beta \end{pmatrix}, h\begin{pmatrix} r \\ \gamma \end{pmatrix}\right), p, r > 2, \gamma, \beta \in \mathbb{R}. \tag{4.20}$$

We proceed by induction on $m \ge 2$ and prove that for $k \in \{1, ..., m-1\}$, all $k, m, q, p, r, \alpha, \beta, \gamma$ satisfying (4.20) and arbitrary $\epsilon > 0$ there exists a constant $C_\epsilon = C(\epsilon, k, m, p, r, \gamma, \beta) > 0$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$

$$I_{q,\alpha}(|\nabla^{(k)} f|) \le \epsilon I_{p,\beta}(|f|) + C_\epsilon I_{r,\gamma}(|\nabla^{(m)} f|), \tag{4.21}$$

where $I_{s,\kappa}(g) = \int M_{s,\kappa}(|g|)dx$.

When $m = 2, k = 1$, then (4.21) is just (4.3) and there is nothing to prove. Suppose then that (4.21) holds for all $m \in \{2, \ldots, M\}$ and all $0 < k < m$, provided that the parameters $k, m, q, p, r, \alpha, \beta, \gamma$ satisfy (4.20). Now we take $m = M + 1$, $0 \le k \le M + 1$ and denote:

$$\overline{\lambda}_k := \begin{pmatrix} q_k \\ \alpha_k \end{pmatrix} = h^{-1} \circ G_{k/(M+1)}\left(h\begin{pmatrix} p \\ \beta \end{pmatrix}, h\begin{pmatrix} r \\ \gamma \end{pmatrix}\right). \tag{4.22}$$

In particular $\overline{\lambda}_0 = \begin{pmatrix} q_0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} p \\ \beta \end{pmatrix}$ and $\overline{\lambda}_{M+1} = \begin{pmatrix} q_{M+1} \\ \alpha_{M+1} \end{pmatrix} = \begin{pmatrix} r \\ \gamma \end{pmatrix}$. To abbreviate, we will write $I_k = I_{q_k,\alpha_k}(|\nabla^{(k)} f|)$. In this notation, the induction step reduces to the proof of

$$I_k \leq \epsilon I_0 + C_\epsilon I_{M+1}, \tag{4.23}$$

with $C_\epsilon = C(\epsilon, k, M, p, r, \beta, \gamma)$ and for all $k \in \{1, \ldots, M\}$.

To get it, at first we check that $q_i > 2$ for $i \in \{0, \ldots, M+1\}$, and moreover, for all s, l, t such that $0 \leq s < l < t \leq M+1$ we have

$$h(\overline{\lambda}_l) = G_{\frac{l-s}{t-s}}(h(\overline{\lambda}_s), h(\overline{\lambda}_t)).$$

By the inductive assumption, this implies that (4.21) holds true with parameters: $q = q_l, \alpha = \alpha_l, p = p_s, \beta = \alpha_s, r = q_t, \gamma = \alpha_t, k = l - s, m = t - s$, provided $0 < t - s \leq M$. An application of (4.21) to all $g = D^\alpha f$ with $|\alpha| = s$, with such range of parameters,

together with the inequality:

$$M_{q_l,\alpha_l}(|\nabla^{(l)} f|) \leq C \sum_{\alpha, |\alpha| = s} M_{q_l,\alpha_l}(|\nabla^{(l-s)} D^\alpha f|),$$

with C independent of f, implies that once $0 \leq s < l < t \leq M+1$ and $t - s \leq M$, then we have

$$I_l \leq \epsilon I_s + C_\epsilon I_t, \tag{4.24}$$

with $C_\epsilon = C(\epsilon, s, t, l, p, r, \alpha, \beta)$ This gives for all $0 < k < M$,

$$I_k \leq \delta I_0 + C_\delta I_M \leq \delta I_0 + C_\delta(\epsilon I_k + C_\epsilon I_{M+1}),$$

for every $\epsilon, \delta > 0$. Choosing $\epsilon = \epsilon_\delta$ such that $C_\delta \epsilon = \frac{1}{2}$ and rearranging we obtain (4.23) with all $0 < k < M$. To get (4.23) with $k = M$ we use the inequalities:

$$I_M \leq \epsilon I_{M-1} + C_\epsilon I_{M+1} \quad \text{and} \quad I_{M-1} \leq \delta I_0 + C_\delta I_M.$$

They imply

$$I_M \leq \epsilon \delta I_0 + \epsilon C_\delta I_M + C_\epsilon I_{M+1},$$

for every $\epsilon, \delta > 0$. Take $\epsilon \leq \epsilon_\delta$, where ϵ_δ satisfies $\epsilon_\delta C_\delta = \frac{1}{2}$. After rearranging we obtain

$$I_M \leq 2\epsilon\delta I_0 + 2C_\epsilon I_{M+1},$$

which completes the induction argument and concludes the proof of the Theorem. □

Remark 4.2 In [26] we have shown that when M is a Young function satisfying the Δ_2-condition with $M'(t)/t$ bounded next to zero and F is an arbitrary Young function, then for every $f \in C_0^\infty(\mathbb{R}^n)$ we have:

$$\begin{aligned} \int M(|\nabla f|)dx &\leq C(\int M(F(\sqrt{|f|}))dx + \int M(F^*(\sqrt{|\nabla^{(2)} f|}))dx), \\ \|\nabla f\|^2_{(M)} &\leq C\|f\|_{(H)} \|\nabla^{(2)} f\|_{(J)}, \end{aligned}$$

where $H(t) = M(F(\sqrt{t}))$, $J(t) = M(F^*(\sqrt{t}))$, and the constant C is independent of f. Analogous results remain true with arbitrary Carathéodory functions $\Phi_1, \Phi_2 : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ such that $\Phi_1(x, \lambda_1, \lambda_2)\Phi_2(x, \lambda_1, \lambda_2) = \lambda_1\lambda_2$ and $w_1(x) = \Phi_1(x, |f|, |\nabla^{(2)} f|)$ and $w_2(x) = \Phi_2(x, |f|, |\nabla^{(2)} f|)$ replacing $|f(x)|$ and $|\nabla^{(2)} f(x)|$. In present paper we show that in the particular case of logarithmic-type functions: $M(t) = M_{q,\alpha}(t)$ and $F(t) = M_{\mu,\kappa}(t)$, with parameters μ and κ suitably chosen, we end up with (1.4)–(1.13), illustrating the abstract approach of [26].

Remark 4.3 In our previous work [25] we have dealt with the following logarithmic inequalities:

$$\int |\nabla f|^q (\ln(\mu + |\nabla f|^a))^\alpha dx \le C\Big((\int |f|^p (\ln(\mu + |f|^b))^\beta dx)^{1/p^*} \|\nabla^{(2)} f\|_r + \|\nabla^{(2)} f\|_r^r \Big), \tag{4.25}$$

$$\int |\nabla f|^q (\ln(\mu + |\nabla f|^a))^\alpha dx \le C\left((\int |\nabla^{(2)} f|^r (\ln(\mu + |\nabla^{(2)} f|^b))^\gamma dx)^{1/p^*} \|f\|_p + \|f\|_p^p \right), \tag{4.26}$$

$$\int |\nabla f|^q dx \le C\left(\int |f|^p (\ln(\mu + |f|^a))^\beta dx + \int |\nabla^{(2)} f|^r (\ln(\mu + |\nabla^{(2)} f|^b))^\gamma dx \right),$$

where $\mu \in \{1, 2\}$. In the particular case when $\mu = 2$, $a = b = 1$, by the classical Young inequality ($xy \le \frac{x^p}{p} + \frac{y^{p^*}}{p^*}$, $p > 1$) applied to (4.25) and (4.26) we see that they both imply (1.4) for β or γ equal to 0. The last inequality in this series with $a = b = 1$ and $\mu = 2$ is just (1.4) for $\alpha = 0$. Note that (4.25) and (4.26) for $a = b = 1$ and $\mu = 2$ are in general stronger than the special case of (1.4) when β or γ equals zero. It appears that the ranges of parameters in inequalities (4.25) and (4.26) under the restrictions $a = b = 1$ and $\mu = 2$ obtained in [25] and that in (1.4) in this paper are coherent.

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