The 20-th Austrian–Polish Mathematics Competition
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1. Lines ℓ_1 and ℓ_2 intersect in point P. Circles S_1 and S_2 are externally tangent in P; the line ℓ_1 is their common tangent in this point. Similarly, circles T_1 and T_2 are externally tangent in point P and ℓ_2 is their common tangent in this point. Circles S_1 and T_1 intersect in points P and A. Similarly, circles S_1 and T_2 intersect in points P and B, circles S_2 and T_2 intersect in points P and C and circles S_2 and T_1 intersect in points P and D. Prove that the points A, B, C and D are concyclic if and only if the lines ℓ_1 and ℓ_2 are perpendicular.

2. Given is an $n \times m$ chessboard. Each field is assign a pair of coordinates (x, y) where $1 \leq x \leq m$ and $1 \leq y \leq n$. Let p and q be positive integers. A pawn standing on the field (x, y) may be moved to the field (x', y') if and only if
$$|x - x'| = p \qquad \text{and} \qquad |y - y'| = q \,.$$
There is a pawn on each field. We want to move each pawn at the same time such that after the move there is still only one pawn on each field. In how many ways can this be done?

3. There are 97 numbers written on the blackboard: 48, 24, 16, ..., 48/97, that is, rational numbers $48/k$ for $k = 1, 2, \ldots, 97$. In each step we choose two numbers a and b from the blackboard, remove them and write $2ab - a - b + 1$ instead. After 96 steps there will remain exactly one number on the blackboard. Determine the set of these numbers.

4. In a convex quadrilateral $ABCD$ the sides AB and CD are parallel. The diagonals AC and BD intersect in point E. F is the point of intersection of altitudes of triangle EBC, and G is the point of intersection of altitudes of triangle EAD. Prove that the midpoint of GF lies on the line passing through E and perpendicular to AB.

5. Let p_1, p_2, p_3, p_4 be four distinct primes. Prove that there is no polynomial $Q(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that
$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3 \,.$$

6. Prove that there does not exist function $f \colon \mathbf{Z} \to \mathbf{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x, y.

   Note: $\mathbf{Z}$ denotes the set of integers.

7. *a*) Prove that for any real numbers p, q the following inequality holds:
$$p^2 + q^2 + 1 > p(q + 1).$$

b) Determine the biggest real number b such that for all real numbers p, q the following inequality holds:
$$p^2 + q^2 + 1 \geq bp(q + 1).$$

c) Determine the biggest real number c such that for all integer numbers p, q the following inequality holds:
$$p^2 + q^2 + 1 \geq cp(q + 1).$$

8. Let n be a positive integer and let M be an n-element set. Determine the biggest positive integer k with the following property: *There exists a k-element family K consisting of 3-element subsets of the set M, such that each two sets belonging to K have nonempty intersection.*

9. Given is a parallelepiped P. Let V_P be its volume, S_P the area of its surface and L_P the sum of the lengths of its edges. For a real number $t \geq 0$ let P_t be the solid consisting of all points X whose distance from P is not bigger than t. Prove that the volume of the solid P_t is given by the formula
$$V(P_t) = V_P + S_P\,t + \frac{\pi}{4}L_P\,t^2 + \frac{4\pi}{3}t^3\,.$$

Explanation: A distance from point X to parallelepiped P is not bigger than t if there exists point Y belonging to parallelepiped P such that $|XY| \leq t$.