17-th Austrian-Polish Mathematics Competition
Poland, June 29 – July 1, 1994

1. The function $f\colon \mathbf{R} \to \mathbf{R}$ satisfies for all $x \in \mathbf{R}$ the conditions $f(x+19) \le f(x)+19$ and $f(x+94) \ge f(x)+94$. Prove that $f(x+1) = f(x)+1$ for all $x \in \mathbf{R}$.

2. The sequence (a_n) is given by the conditions
$$a_0 = \frac{1}{2} \qquad \text{and} \qquad a_{n+1} = \frac{2a_n}{1+a_n^2} \qquad \text{for} \quad n \ge 0,$$
and the sequence (c_n) is defined by the formulas
$$c_0 = 4 \qquad \text{and} \qquad c_{n+1} = c_n^2 - 2c_n + 2 \qquad \text{for} \quad n \ge 0.$$
Prove that for all $n \ge 1$
$$a_n = \frac{2c_0c_1\ldots c_{n-1}}{c_n}\,.$$

3. A rectangular building consists of two rows of 15 square rooms (situated like the cells in two neighbouring rows of a chessboard). In each room there are three doors which lead to one, two or all the three neighbouring rooms. (Doors leading outside the building are not counted.) The doors are distributed in such a way that one can pass from any other room to any other one without leaving the building. How many distributions of the doors (in the walls between the 30 rooms) can be found so as to satisfy the given conditions?

4. Let $n \ge 2$ be a fixed natural number and let P_0 be a fixed vertex of a regular $(n{+}1)$-gon. The remaining vertices denote by $P_1, P_2, \ldots, P_n$, in any order. To each side of the $(n{+}1)$-gon assing a natural number in the following way: if the endpoints of the side are P_i and P_j, the assigned number is equal to $|i-j|$. Let S be the sum of the all $n+1$ numbers assigned to the sides. (Of course, S depends on the way the vertices are assigned.)

(a) What is – for a fixed n – the smallest possible value of S?

(b) How many distinct assignments of the vertices are there so that S attains the smallest value?

5. Solve in integers the following equation
$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz\,.$$

6. Let $n > 1$ be an odd positive integer. We assume that the integers $x_1, x_2, \ldots, x_n \ge 0$ satisfy the following system of equations:
$$\begin{aligned}
&(x_2-x_1)^2 + 2(x_2+x_1) + 1 = n^2\\
&(x_3-x_2)^2 + 2(x_3+x_2) + 1 = n^2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&(x_1-x_n)^2 + 2(x_1+x_n) + 1 = n^2\,.
\end{aligned}$$
Prove that either $x_1 = x_n$ or there exits j $(1 \le j \le n-1)$, such that $x_j = x_{j+1}$.

7. Determine all two-digit (in decimal representation) positive integers $n = (ab)_{10} = 10a + b$ $(a \ge 1)$ with the property that for all integers x the difference $x^a - x^b$ is divisible by n.

8. Consider the following fuctional equation $f(x,y) = af(x,z) + bf(y,z)$ with real contstants a, b. For each pair of the reals a, b give the general form of the functions $f\colon \mathbf{R}^2 \to \mathbf{R}$ satisfying the above equation for all $x, y, z \in \mathbf{R}$.

9. In the plane there are given four distinct points A, B, C, D lying (in that order) on the line g, at distances $AB = a$, $BC = b$, $CD = c$.

(a) Construct – if it's possible – a point P, not lying on the line g, such that the angles $\angle APB$, $\angle BPC$, $\angle CPD$ are all equal.

(b) Prove that such a point P exists, if and only if the inequality $(a+b)(b+c) < 4ac$ holds.