The 24th Austrian–Polish Mathematics Competition
Austria, June 2001

1. Determine the number of positive integers a for which there exist nonnegative integers $x_0, x_1, \ldots, x_{2001}$ satisfying

$$a^{x_0} = \sum_{k=1}^{2001} a^{x_k}\,.$$

2. Let n be a positive integer greater than 2. Solve in nonnegative real numbers the following system of equations

$$x_k + x_{k+1} = x_{k+2}^2\,, \quad k = 1, 2, \ldots, n\,,$$

where $x_{n+1} = x_1$ and $x_{n+2} = x_2$.

3. Real numbers a, b, c are lengths of the sides of a triangle. Prove that

$$2 < \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - \frac{a^3+b^3+c^3}{abc} \le 3\,.$$

4. Prove that if a, b, c, d are lengths of the successive sides of a quadrangle (not necessarily convex) with the area equal to S, then the following inequality holds

$$S \le \frac{1}{2}(ac + bd)\,.$$

For which quadrangles does the inequality become equality?

5. The fields of the 8×8 chessboard are numbered from 1 to 64 in the following manner: For $i = 1, 2, \ldots, 63$ the field numbered by $i+1$ can be reched from the field numbered by i by one move of the knight. Let us choose positive real numbers $x_1, x_2, \ldots, x_{64}$. For each white field numbered by i define the number $y_i = 1 + x_i^2 - \sqrt[3]{x_{i-1}^2 x_{i+1}}$ and for each black field numbered by j define the number $y_j = 1 + x_j^2 - \sqrt[3]{x_{j-1} x_{j+1}^2}$ where $x_0 = x_{64}$ and $x_1 = x_{65}$. Prove that

$$\sum_{i=1}^{64} y_i \ge 48\,.$$

6. Let k be a fixed positive integer. Consider the sequence defined by

$$a_0 = 1\,, \quad a_{n+1} = a_n + [\sqrt[k]{a_n}] \quad n = 0, 1, \ldots$$

where $[x]$ denotes the greatest integer less than or equal to x. For each k find the set A_k containing all integer values of the sequence $(\sqrt[k]{a_n})_{n \ge 0}$.

7. Consider the set A containing all positive integers whose decimal expansion contains no 0, and whose sum $S(N)$ of the digits divides N.
a) Prove that there exist infinitely many elements in A whose decimal expansion contains each digit the same number of times as each other digit.
b) Explain that for each positive integer k there exists an element in A having exactly k digits.

8. The prism with the regular octagonal base and with all edges of the length equal to 1 is given. The points $M_1, M_2, \ldots, M_{10}$ are the midpoints of all the faces of the prism. For the point P from the inside of the prism denote by P_i the intersection point (not equal to M_i) of the line M_iP with the surface of the prism. Assume that the point P is so chosen that all associated with P points P_i do not belong to any edge of the prism and on each face lies exactly one point P_i. Prove that

$$\sum_{i=1}^{10} \frac{M_i P}{M_i P_i} = 5\,.$$

9. Let $n > 10$ be a positive integer and let A be a set containing $2n$ elements. The family $\{A_i : i = 1, 2, \ldots, m\}$ of subsets of the set A is called suitable if:

- for each $i = 1, 2, \ldots, m$ the set A_i contains exactly n elements,
- for all $i \neq j \neq k \neq i$ the set $A_i \cap A_j \cap A_k$ contains at most one element.

For each n determine the length of a maximal suitable family.

10. The sequence $a_1, a_2, \ldots, a_{2010}$ has the following properties:

- each sum of the 20 successive values of the sequence is nonnegative,
- $|a_i a_{i+1}| \leq 1 \qquad$ for $i = 1, 2, \ldots, 2009\,.$

Determine the maximal value of the expression $\sum_{i=1}^{2010} a_i\,.$