53-rd Mathematical Olympiad in Poland
Second Round, February 22–23, 2002

First Day

1. Prove that all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$\forall\, x \in \mathbb{R} \quad f(x) = f(2x) = f(1-x)$$
are periodic.

2. In a convex quadrangle $ABCD$ the following equalities
$$\sphericalangle ADB = 2\sphericalangle ACB \quad \text{and} \quad \sphericalangle BDC = 2\sphericalangle BAC$$
hold. Prove that $AD = CD$.

3. A positive integer n is given. In an association consisting of n members work 6 commissions. Each commission contains at least $n/4$ persons. Prove that there exist two commissions containing at least $n/30$ persons in common.

Second Day

4. Find all prime numbers $p \le q \le r$ such that all the numbers
$$pq + r\,, \quad pq + r^2\,, \quad qr + p\,, \quad qr + p^2\,, \quad rp + q\,, \quad rp + q^2$$
are prime.

5. Triangle ABC with $\sphericalangle BAC = 90^\circ$ is the base of the pyramid $ABCD$. Moreover it holds
$$AD = BD \quad \text{and} \quad AB = CD\,.$$
Prove that $\sphericalangle ACD \ge 30^\circ$.

6. Find all positive integers n such that for all real numbers $x_1, x_2 \ldots x_n,\ y_1, y_2 \ldots y_n$ the following inequality
$$x_1x_2 \ldots x_n + y_1y_2 \ldots y_n \le \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_1^2} \cdot \ldots \cdot \sqrt{x_n^2 + y_n^2}$$
holds.