51th Mathematical Olympiad in Poland

Problems of the first round, September – December 1999

1. Let $n \geq 3$ be a positive integer. Prove that the sum of the cubes of all natural numbers, coprime and less than n, is divisible by n.

2. In the acute-angled triangle ABC holds $\sphericalangle ACB = 2\sphericalangle ABC$. The point D lies on the side BC and satisfies $2\sphericalangle BAD = \sphericalangle ABC$. Prove that

$$\frac{1}{BD} = \frac{1}{AB} + \frac{1}{AC}\,.$$

3. The sum of positive numbers a, b, c is equal to 1. Prove that $a^2 + b^2 + c^2 + 2\sqrt{3abc} \leq 1$.

4. Each point of a circle is painted with one of three colours. Prove that there exist three points on the circle which have the same colour and are vertices of an isosceles triangle.

5. Find all pairs (a, b) of positive integers for which the numbers $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are cubes of positive integers.

6. A point X lies inside or on the boundary of the triangle ABC in which the angle $\sphericalangle C$ is a right angle. The points P, Q and R are the perpendicular projections of X onto the sides BC, CA and AB respectively. Prove that the equality $AR \cdot RB = BP \cdot PC + AQ \cdot QC$ holds if and only if the point X lies on the side AB.

7. Prove that for each positive integer n and each number $t \in (\frac{1}{2}, 1)$ there exist numbers $a, b \in (1999, 2000)$, such that

$$\frac{1}{2}a^n + \frac{1}{2}b^n < (ta + (1-t)b)^n.$$

8. The numbers $c(n,k)$ are defined for nonnegative integers $n \geq k$ in such a way, that the following equalities hold:

$$c(n,0) = c(n,n) = 1 \qquad \text{for all} \quad n \geq 0,$$
$$c(n{+}1,k) = 2^k c(n,k) + c(n,k{-}1) \qquad \text{for} \quad n \geq k \geq 1.$$

Prove that $c(n,k) = c(n,n{-}k)$ for $n \geq k \geq 0$.

9. Let m and n be given positive integers such that $mn | m^2 + n^2 + m$. Prove that m is a square of an integer.

10. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be pairwise perpendicular unit vectors in three dimensional space. Let ω be a plane containing the point O, and A', B', C' be the perpendicular projections of the points A, B, C onto the plane ω. Find the set of numbers $OA'^2 + OB'^2 + OC'^2$ for all planes ω.

11. Let m be a positive integer and M be a set containing $n^2{+}1$ positive integers and having the following property: in each $n+1$ numbers from the set M there are two numbers for which one is divisible by the other. Prove that the set M contains different numbers $a_1, \ldots, a_{n+1}$ for which $a_{i+1} | a_i$ for $i = 1, \ldots, n$.

12. In the acute-angled triangle ABC the points D, E, F lie on the sides BC, CA, AB respectively. The circumcircles of the triangles AEF, BFD, CDE meet in the point P. Prove that if

$$\frac{PD}{PE} = \frac{BD}{AE}\,, \qquad \frac{PE}{PF} = \frac{CE}{BF}\,, \qquad \frac{PF}{PD} = \frac{AF}{CD}\,,$$

then AD, BE, CF are the altitudes of the triangle ABC.