CENTER FOR CURRICULUM REDESIGN

PISA Mathematics in 2021

An analysis of the CENTER FOR CURRICULUM REDESIGN (CCR)

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<u>Important note:</u>

None of the recommendations in this document affect the distributions of scoring points, per the tables below from PISA 2015:

Table 1 Approximate distribution of score points by process category for PISA 2015

Process category	Percentage of score points
Formulating situations mathematically	Approximately 25
Employing mathematical concepts, facts, procedures and	Approximately 50
Interpreting, applying and evaluating mathematical outcomes	Approximately 25
TOTAL	**100**

Table 2 Approximate distribution of score points by content category for PISA 2012

Content category	Percentage of score points
Change and relationships	Approximately 25
Space and shape	Approximately 25
Quantity	Approximately 25
Uncertainty and data	Approximately 25
TOTAL	**100**

Table 3 Approximate distribution of score points by context category for PISA 2012

Context category	Percentage of score points
Personal	Approximately 25
Occupational	Approximately 25
Societal	Approximately 25
Scientific	Approximately 25
TOTAL	**100**

Introduction – the continued importance of Mathematics

Mathematics as a foundation for understanding the world, citizenship and economic growth

Education systems across the globe have been tuned to the demands of the Industrial Age, and are now struggling to prepare students for success in a rapidly transforming, present and future, Innovation Age. The last major changes to curriculum were effected in the late 1800s as a response to the sudden growth in societal and human capital needs. As the 21st century bears little resemblance to the 19th century, education curricula are overdue for emphasizing depth of understanding and versatility, to meet the needs of our global society.

Mathematics continues to be the foundation for:

- economic growth via Science, Technology, Engineering as the basis of innovation
- understanding the world and citizenship

To quote John Allen Paulos, mathematician[1] at Temple University, "*Gullible citizens are a demagogue's dream… almost every political issue has a quantitative aspect*". He has since 1988[2] advocated the need for educated citizenry and societies to deeply understand issues such as number sizing, coincidence, pseudoscience, etc. all of which are portended by Mathematics.

The PISA definition of mathematical literacy[3] is still accurately relevant:
"an individual's capacity to identify and understand the role that mathematics play in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen."

Science/Technology/Engineering/Math (STEM) is a critical element of the curriculum, particularly now as, worldwide, demand is outpacing supply, and STEM professions are perceived as a key driver of growth through innovation[4]. Mathematics is the foundation of STEM, and a critical literacy for developing innovators; as such, the situation requires urgent attention. And beyond STEM professions, we are witnessing significant innumeracy in a very large segment of the population[5], which has severe consequences for their ability to understand and solve the world's difficult problems and their own.

Current systems place the following emphasis on STEM as a proportion of total student learning time – approximately 30% in countries surveyed by the OECD[6]:

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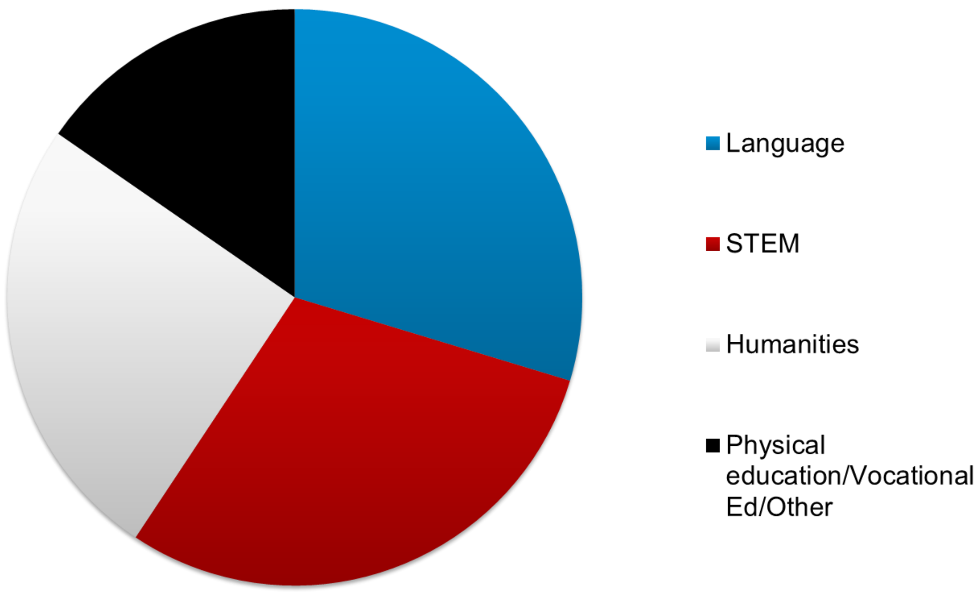
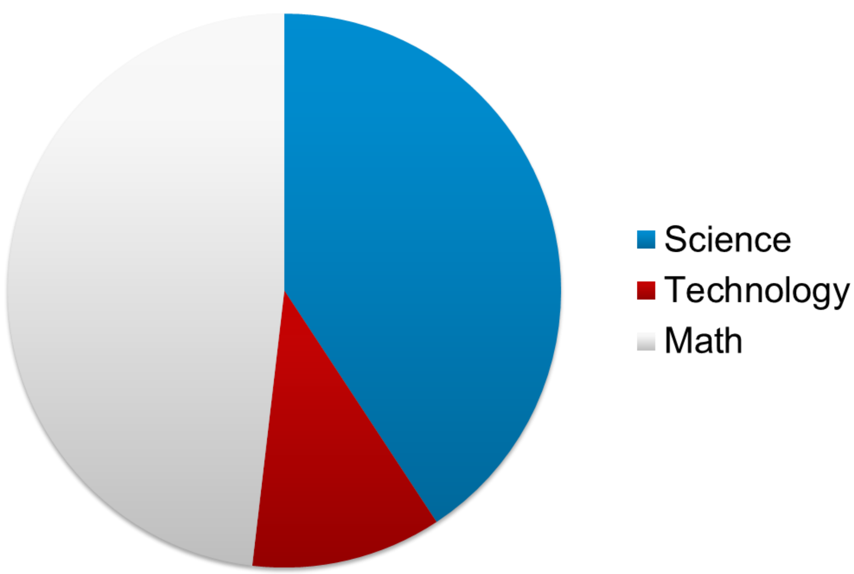
[1] Author, "*A Mathematician reads the newspaper*"
[2] Temple University, in his best-selling book "Innumeracy"
[3] OECD, PISA 2009, p.14

[4] http://www.oecd.org/sti/oecd-science-technology-and-industry-outlook-19991428.htm
[5] The Wall Street Journal, January 7, 2012 "Americans Stumble on Math of Big Issues" http://online.wsj.com/article/SB10001424052970203471004577144632919979666.html
[6] OECD "Information at a glance 2014" http://www.oecd.org/edu/eag.htm



And digging deeper, Mathematics represents approximately 45% of the STEM total, or roughly 11% of the total instruction time (and for OECD countries only, approximately 15%, representing a spending of $235B annually![7])

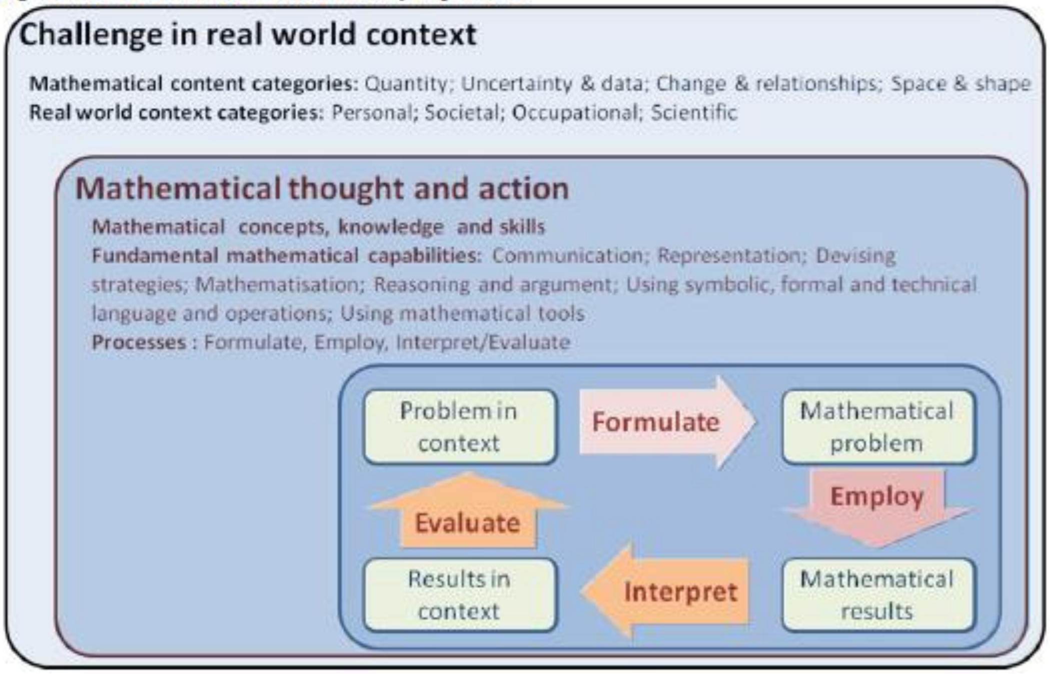


[7] Calculation by Andreas Schleicher, OECD Director for the Directorate of Education and Skills, in a private communication on 7/13/2015

Rationale for improving PISA for 2021

First, we shall be reminded of the very valid and relevant construct of the PISA 2015 Mathematics framework, which will be referenced throughout this document:

Figure 1 A model of mathematical literacy in practice



Mathematical reasoning required to understand the world

An important dimension of mathematical literacy, as defined in the PISA Mathematics Framework, is the individual's capacity to reason mathematically.

Mathematical reasoning stands behind the solution of any mathematical problem, by the very nature of mathematics. The solution of some problems requires producing a coherent sequence of arguments. But even when the solution reduces to a simple calculation that we know how to do "by heart", the reflection on the plausibility of the solution involves mathematical reasoning. A correctly performed mathematical reasoning brings a prize, uniquely reserved for mathematics: we arrive at an *irrefutable eternal truth*.

The ability to make sound logical conclusions from the available facts is crucial in life, as it can be applied far beyond mathematical contexts. Whenever we want to justify precisely our opinions based on facts, we use logical constructions practiced in the math class: one conclusion implies another and a well-chosen counterexample can turn a false opponent's argument into ashes.

The ability to use mathematical reasoning, while presenting our opinions or analyzing opinions of others, is crucial for making "well-founded judgments and decisions needed by constructive, engaged and reflective citizens" (PISA 2015 Draft Mathematics Framework, page 5). Therefore teachers of mathematics should pay special attention to develop this ability. This task is much harder than just teaching the students to memorize routine ways of dealing with standard math problems. To achieve success, students should be prepared, without the teacher's direct support, to construct chains of arguments. When so practiced, this skill becomes habit, which encourages students, not only to present their opinions (e.g., by clicking "likes" on Facebook), but also to justify and defend them[8].

Changes in global employability requirements

Over several decades, the needs in mathematics have evolved toward new branches and topics, reflected in the OECD survey[9] of industry in the table below; simply put, modern industry requires different mathematics, above and beyond traditional branches such as arithmetic, geometry, and algebra. What is relevant today is not what was relevant 15 years ago or more.

Themes	Responses
	Knowledge areas
Complexity	Complex systems
Uncertainty	Statistics & probability
Multiple scales	Complex systems
Simulations & modeling	Computational mathematics (algorithmic)
Data & information	Statistics & probability
	Skills areas
Multidisciplinarity	Collaboration
Transfer of knowledge	Communication

Evidently, the need for surveyors and woodworkers - needing more trigonometry - in modern economies has gone down, supplanted by need to understand data. Hal Varian, Google Chief Economist was quoted[10] as saying: *"I keep saying the sexy job in the next ten years will be*

[8] Pr. Harold Fawcett "*The nature of proof*", NCTM 1938/1995
[9] OECD Global Science Forum Report on Mathematics in Industry - July 2008
[10] McKinsey Quarterly, Jan 2009

statisticians. People think I'm joking but who would've guessed that computer engineers would've been the sexy job of the 1990s."

And if one analyzes the needs of professions in general, per the table "Usage of mathematics by various professions" given in Appendix 1, it becomes clear that emphasis is required beyond traditional branches.

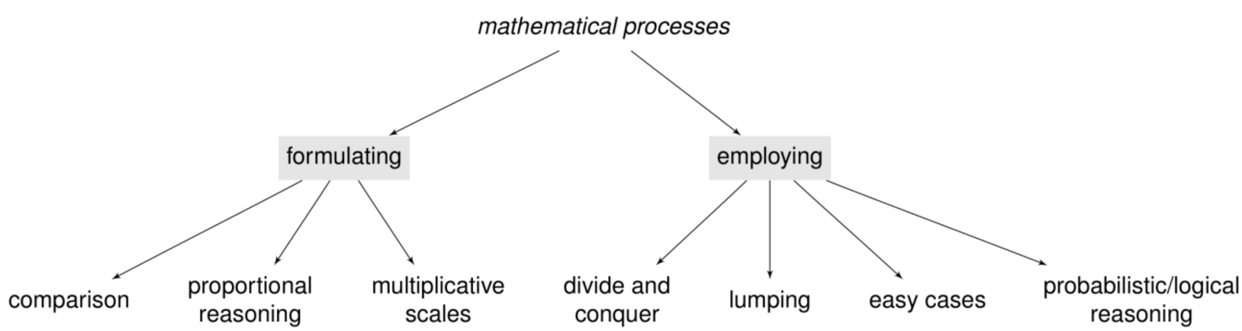
Explicit reasoning and processes

Seven reasoning tools

The PISA test on mathematics measures, among other things, students' ability to employ mathematical reasoning while solving problems. The items used for this purpose are usually hard for the participating students; many are not able to produce the expected chain of arguments.

Nevertheless, many students who failed still did attempt to solve those items, as they usually offer an interesting challenge, in contrast to items that require for example just a simple calculation. Because of that, and also because of the importance of mathematical reasoning for mathematical literacy, it is important in the future PISA tests to evaluate those attempts[11], to give credit to those education systems which put their students at least on the right track. This can be achieved by a careful selection of items which will be constructed so that they can be solved by the right reasoning approach and that will offer a partial credit for such an approach.

Therefore we suggest enriching the descriptions of the mathematical processes (formulate, employ, interpret), identified in the PISA Mathematics Framework as the main components of the modeling cycle, with seven reasoning tools[12] which are most commonly used in search of the right approach:



A successful application of any of these tools identified in the student's solution will measure his/her readiness to approach problems demanding mathematical reasoning[13]. In this way, the measurement of students' ability to reason mathematically will become more precise.

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[11] Commentary from Alberta Education: "We're assuming that these would be constructed response items. Coding for the various types of responses and different approaches to solving problems would help jurisdictions identify areas of strength and opportunities for growth."

[12] Source: Sanjoy Mahajan, in *NumberSight* (unpublished)

[13] Commentary from Alberta Education: "Coding to identify which approaches are being used would be helpful for jurisdictions. If marketed properly, this information could also be helpful for classroom teachers."

The PISA 2015 draft Mathematics framework states: "Items in the 2015 PISA mathematics survey will be assigned to one of three mathematical processes:

- **Formulating** situations mathematically;
- **Employing** mathematical concepts, facts, procedures, and reasoning; and
- **Interpreting**, applying and evaluating mathematical outcomes."

While **formulating** a problem mathematically, one has often to deal with quantitative data. A quantity has no meaning alone; its meaning comes from connections to other quantities. To identify those connections, we have three basic tools:

- comparison
- proportional reasoning
- applying multiplicative scales

The ability to use these tools is a very important aspect of the Formulate process:

Comparison: No quantity is meaningful until it is compared to another, related quantity. For example, a monetary rate of $1 trillion dollars per year means little by itself; formulated as a fraction of a government budget or a GDP, it acquires meaning. Another example: reading a bare number in a newspaper article, such as "1 million fans came into the streets to celebrate their sports team" (as appears regularly in Boston newspapers) conveys little information. Only when the 1 million is compared to the total population of Boston and the inner suburbs, also approximately 1 million, does the number acquire meaning (as being highly implausible).

Proportional reasoning: Comparison naturally produces ratios. Understanding how ratio changes are related (e.g. doubling lengths quadruples areas) – the essence of proportional reasoning – is essential in formulating mathematical descriptions of real-world problems. Proportional reasoning provides a simple approach to the *Pizzas* question in the PISA framework.

Multiplicative scales (including exponential growth): The next tool is counting with ratios as the unit – e.g. by how many factors of 2 (or 10) is the Sun larger in mass than the Earth? Counting ratios leads to representation of quantities on logarithmic and exponential scales. Logarithmic scales organize the huge dynamic range of lengths, times, energies in the Universe. In the social world, exponential-growth models help to model the growth of populations, economies, epidemic spread or resource use.

Employing mathematical concepts and facts in a mathematical model can be difficult also because of the model's complexity. A student can be overwhelmed by the many meanings of mathematical objects appearing in the model. Therefore the ability to simplify the model is a very important aspect of the Employ process. There are four basic tools[14] for simplification which help us formulate the complexity of the world in comprehensible ways:

- divide and conquer,

[14] Commentary from Alberta Education: "For each of these, coding to identify effective approaches versus ineffective/inefficient approaches provide useful information."

- lumping,
- consider first easy cases,
- Probabilistic and logical reasoning.

Divide and conquer. We split hard problems into manageable pieces. This tool is valuable from the earliest ages. For example, in adding 24 + 38, students can divide 38 into 6 and 32, and then add 24+6=30 and 30+32=62. For PISA-age students, a rich class of problems that use divide-and-conquer reasoning are so called Fermi problems. For example: "At what rate (passengers per hour) can a busy train track carry passengers?" One divides the hard estimate into simpler estimates: how many trains per hour, how many cars on each train, and how many passengers in each train. Fermi problems also require comfort with imprecision, an ability developed by the next reasoning tool of lumping.

Lumping. By rounding numbers or by approximating complicated shapes with simple shapes, we discard less important details in order to understand the essential parts of a problem. We thereby formulate complicated problems as manageable ones. For example, the *Climbing Mount Fuji* question asked students to find the step length of a person who walked 22,500 steps up Mount Fuji covering 9 km. A simple lumping calculation gives a plausible estimate: roughly 10,000 m in roughly 20,000 steps means about 0.5 m per step.

Easy cases. When a problem is still too hard, we look at its special cases, ones that we can understand. For example, in evaluating a mathematical conjecture, we first test it with n=0 and n=1. Or, in the personal and societal contexts, mortgages can, at one extreme, be like annuities, where interest rate × loan period >> 1; or, at the other extreme, they can like installment loans, where interest rate × loan period << 1. In each extreme the payment is easy to calculate, and the calculation is easier to understand than it is in the general case.

Probabilistic and logical reasoning. The complete *Employ* process requires also probabilistic or logical reasoning. We represent our incomplete knowledge through probability – in particular, through the probabilities of hypotheses. Collecting data and evidence changes knowledge and thus changes probabilities, according to Bayes theorem, which connects evidence and belief. Probabilistic reasoning includes logical reasoning: in the easy (extreme) case where probabilities are either 0 (false) or 1 (true), probabilistic reasoning simplifies to logical reasoning.

Knowledge relevance

Increasing focus on important existing content areas
Incorporating new important/relevant areas

The Freudenthal Institute states, when describing "workplace mathematics"[15]:

> "Most important in many occupations are:
>
> > Number, quantity, measure
> > Data handling and uncertainty
>
> Followed by
>
> > Space and shape
> > Relations, change, formulas"

The UK's Royal Society describes[16] the following requirements for mathematics in the workplace:

- Mathematical modeling (e.g. energy requirement of a water company; cost of sandwich; etc.)
- Use of software, and coping with problems (e.g. oil extraction; dispersion of sewage; etc.)
- Costing (allocation, dispute management) (e.g. contract cleaning of hospital; management of railway; etc.)
- Performance and ratios (e.g. insurance ratios; glycemic index; etc.)
- Risk (e.g. clinical governance; insurance; etc.)
- Quality/SPC control (e.g. furniture; machine downtime; deviation of rails; etc.)

More generally, the US National Science Foundation has stated[17]:

- "more emphasis on estimation, mental maths…
- "less emphasis on paper/pencil execution…"
- "content in… algebra, geometry, pre-calculus and trigonometry needs to be… streamlined to make room for important new topics."
- "discrete Mathematics, statistics/probabilities and computer science must be introduced".

Based on all the discussions of this paper until this point, the quest to reflect areas of relevant mathematics knowledge, of importance to a wide range of personal, societal and occupational[18] needs (while respecting the PISA taxonomy and focus) has led to the following recommendations:

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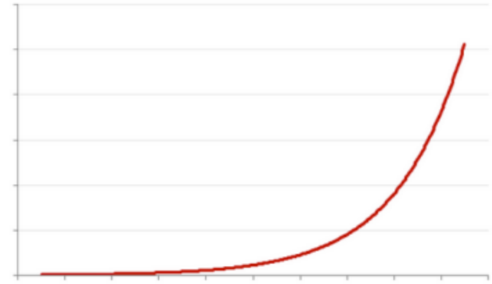
[15] Arthur Bakker as presented at the CCR's Stockholm conference: http://curriculumredesign.org/wp-content/uploads/bakker-21st-2013-04-24final-small-Compatibility-Mode.pdf
[16] ACME 2011 "Mathematics in the workplace and higher education"
[17] National Science Foundation: "The Mathematical Sciences Curriculum K-12: What is still fundamental and what is not" (1982)
[18] See Appendix: "Usage of Mathematics by various professions" by CCR

Item type	Weight (unchanged)	Summary - Areas of emphasis and increased focus
Shape & Space	25%	Unpredictable shapes; Proportionality
Change & Relationships	25%	Exponential; Proportionality; Algorithmic Mathematics
Uncertainty & Data	25%	Bayesian/Conditional probability; Discrete Maths [combinatorics; game theory; complex systems]
Quantity	25%	Number Sense; Estimation

PISA Math Literacy Areas	Explanations of areas of emphasis and increased focus; what's expected from students
Shape & Space	**Unpredictable Shapes –** Irregular, yet realistic shapes: Real world situations do not always directly conform to expectations or the regular geometric models studied. Still, often the need arises to find the area, volume, circumference, length, or size of their angles for such shapes. Students need to be able to apply and adapt their knowledge of standard geometric models and other mathematical tools flexibly to arrive at acceptable solutions to such problems. This involves breaking up the irregular shapes into smaller regular shapes for which solutions are possible and then assembling them back together to solve the original problem or at least provide a reasonable estimate thereof. Metaphorically this is like sewing the fabric covering each sub-shape into an overall canopy that approximates the irregular shape. **Proportionality –** Recognizing that the constant of proportionality is a ratio often expressed as a scale on a diagram, blueprint, or map. Using this insight leads to applications in building scale models, interpreting schematics, and using maps. Proportionality appears also in the simplest relations between physical quantities.
Change & Relationships	**Exponential –** This type of non-linear function models many diverse aspects of the real world, from the decay of radioactive isotopes to the population growth of bacteria/virus (e.g. Ebola) to the growth of a bank account from compounding interest.

	 **Proportionality (see Space & Shape also) –** the slope or rate of change in a bivariate relation may be either positive, as in a directly proportional relation, or negative, as in an inverse proportion. Recognizing the difference between these two types of proportional relations is essential to making decisions about medicinal dosages, the amount of paint needed for a project, or the time needed to complete a project as a function of the number of individuals available. **Algorithmic Mathematics –** Some quantitative problems are not so readily addressed because of the complexity of the required mathematics. An alternate approach is to address these problems iteratively by using simpler mathematical expressions, equations, and functions in a sequential fashion through simulation. This involves discerning from a real world situation the pertinent variables and how they are related, and using these in a step-wise fashion to derive approximate solutions. A current example of this is algorithmic trading, which is used to place trades on the stock market.
Uncertainty & Data	**Bayesian Probability/Conditional Probability –** Students need to recognize the concepts of conditional probability and independence in everyday language and everyday situations. For example, understanding that the chances of having lung cancer if you are a smoker, and the chances of being a smoker if you have lung cancer, are not necessarily the same. **Discrete Mathematics:** **Combinatorics and probability:** This involves situations in which outcomes are expressed as discrete alternatives or increments. As a result it involves students having an elementary understanding of probability and real life combinatorial situations. This does not include students' understanding the formal rules for combinations, permutations, and decision trees, but that they can handle simple combinatorial situations mostly by common sense. This is useful in studying and describing objects and problems in branches of computer science, such as computer algorithms, programming languages, cryptography, automated theorem proving, and software development. Conversely, computer implementations are significant in applying such ideas to real-world problems, such as in operations research. **Game theory:** in particular deals with situations where success depends on

	the choices of others, which makes choosing the best course of action more complex (e.g. political systems and voting). **Complex systems**: occur naturally in the real world and often appear to be chaotic as well as inherently non-linear. Students need to understand complexity not as chaos that is non-interpretable but as a challenge that can be addressed by examining the consequences of the behavior of such a system in an iterative fashion through the use of feedback loops (e.g. global warming).
Quantity	**Number Sense:** It is important for students to develop a sense of numbers relative to the human scale of daily life. This is especially the case for very large or very small numbers which are difficult to make sense of unless placed in some sense relative to familiar quantities. Such a number sense is also difficult for students with various geometric quantities such as volume and angles. Judging solutions based on what is generally known about relative magnitude of quantities and number relations is also important. **Estimation:** Fast, order-of-magnitude judgments are formed through the application of various estimation strategies such as rounding and using simplified operations that may be easily accomplished mentally, without pencil-and-paper or a calculator. These estimates help to judge the accuracy of complex computation done either by hand, calculator[19], or computer.

Linkage to explicit reasoning and processes: Given the orientation of the above content focus areas, the reasoning/process skills described previously are closely aligned and become an integral part of them, providing the tools needed to address these content areas, especially in real-world contexts.

[19] Commentary from Alberta Education: "To fully test estimation skills, a non-calculator portion of the exam is recommended."

Embedding learning competencies (Skills, Character, Meta-Learning)

Note: for Critical Thinking (skill), see section above on Explicit Reasoning.

1) **Skills, and Meta-Learning:** Creativity (non-canonical answers) and Metacognition**:** The CCR has defined the following progression in Mathematics creativity and metacognition:
    - Solve exercises and problems using standard solutions
    - Solve exercises and problems using non-standard solutions (creative stretch)
    - Find new real-world problems, and solve using both standard and non-standard solutions (creative stretch)
    - Create new problems, and solve using both standard and non-standard solutions (creative stretch)
    - Create new classes of problems (metacognitive stretch) and explore solvability

Measuring Creativity in mathematical thinking

A recognized goal for mathematics students is to have them flexibly apply their mathematical knowledge, skills, and tools to situations and challenges encountered in various aspects of the real world. This notion is an explicit part of the PISA definition of mathematics literacy. The flexible application of knowledge to novel situations and problems often includes recognizing alternative approaches, especially the non-standard ones. This is recognized as the "holy grail" of education/learning. This idea of the flexible application of mathematics knowledge is related to the notion of creatively approaching and resolving a mathematical challenge.

In light of the challenge educators face in achieving the "holy grail" of having students flexibly apply (transfer) their knowledge from the known to the novel, it remains a substantial challenge to measure such phenomena. Traditional multiple-choice items that assess student knowledge around standard solutions using basic algorithms and concepts are not likely to tap into flexible or creative thinking. Items to do this are likely to require students to explain how they think about one or more approaches to a mathematical challenge or, perhaps, to evaluate several different approaches including recognizing that there might be a non-standard solution as well. This type of item has been successfully used to measure the knowledge teachers have with respect to the teaching of mathematics[20]. The following item prototypes provide an illustration of what such items might look like in attempting to assess students' creative approach to a mathematical situation.

In the example below focused on Creativity, the student is asked to evaluate four solutions provided by exemplar students. Allan's solution is most likely considered the standard solution but that of Cristine is also correct. Students must evaluate each solution against the criteria

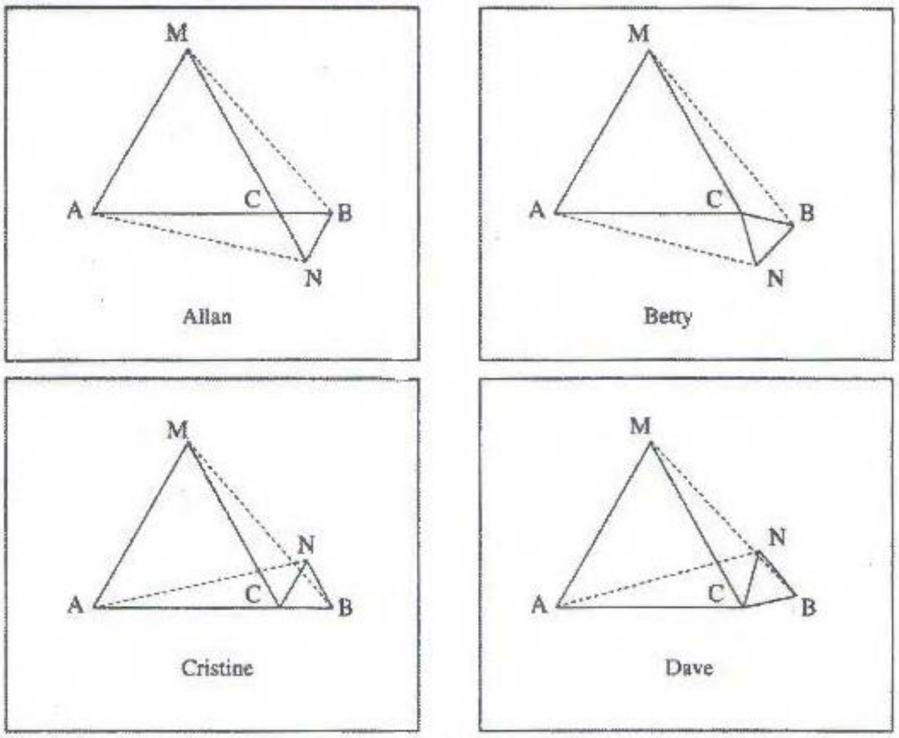
[20] Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? American Educator, 29(3), 14-22. And Schmidt, W. H., Blömeke, S., & Tatto, M. T. (2011). Teacher Education Matters: A Study of Middle School Mathematics teacher Preparation in Six Countries. New York, NY: Teachers College Press.

provided. This requires them to evaluate non-standard solutions rather than simply providing a solution of their own. Consequently, they must apply their knowledge in evaluating each solution provided in order to generate a response of their own. Only one box may be checked correctly (option F).

Your class was given the following problem

Let C be a point on the segment AB. Equilateral triangles ACM and BCN are drawn. Is it true that AN=BM?

Four students drew the following figures:



1) Which student's figure(s) are consistent with the given conditions?

*Check **one** box.*

A) Allan's only .... ☐
B) Betty's only .... ☐
C) Cristine's only .... ☐
D) Dave's only .... ☐
E) Allan's and Betty's only .... ☐
F) Allan's and Cristine's only .... ☐
G) Betty's and Dave's only .... ☐
H) All four figures .... ☐

Some[21] have defined creativity as being able to make innovative decisions from existing information which in the PISA case would include quantitative information. An example is included below as a prototype:

> The Defense Budget Item[22]:
> In a certain country, the defense budget was $30 million for 1980. The total budget for that year was $500 million. The following year, the defense budget was $35 million, whereas the total budget was $605 million. Inflation during the period between the two budgets was 10 percent.
> (a) Someone is invited to hold a lecture for a pacifist society. They want to explain that the defense budget has decreased this year. Explain how to do this.
> (b) Someone is invited to lecture to a military academy. They want to claim that the defense budget has increased this year. Explain how to do this.

Metacognition: In the example below, the student is asked to evaluate the solutions to the problem generated by three exemplar students. To successfully evaluate the responses, students must understand three concepts: definition of natural number, definition of the square of a number, and how to determine a probability. Once the student has identified which response is appropriate (Monica's), the student in asked to select two explanations for the incorrect solutions the other students provided. This requires the student to reflect on the reasoning each student has provided and identify the underlying reasoning error that led to an inappropriate response to the question. This type of item requires the students to reflect on different solutions and to generate hypothesis capturing the thinking represented in the different solutions – a metacognitive task.

[21] Holmes, N. G., Wieman, C. E., & Bonn, D. A. (2015). Teaching Critical Thinking. *Proceedings of the National Academy of Sciences*. Retrieved from: http://www.pnas.org/content/early/2015/08/12/1505329112 doi:10.1073/pnas.1505329112

[22] de Lange, J. (2003). Mathematics for literacy. In B. L. Madison & L. A. Steen (Eds.), Quantitative literacy: Why numeracy matters for schools and colleges (pp. 75-89). Princeton, NJ: The National Council on Education and the Disciplines.

Students are given the following problem.

John chooses an arbitrary natural number, squares it, and takes the last digit of the square. What is the probability that this digit is 1?

Following are the responses of three students:

Lisa

There are 10 digits altogether. Each of them has an equal chance to be the last digit. Therefore the answer is 1/10=10%.

Monica

The last digit of a square depends only on the square of the last digit of the number chosen. The last digits of squares of the first 10 natural numbers are 1, 4, 9, 6, 5, 6, 9, 4, 1, 0. Since there are two 1's in this sequence, the answer is 2/10=20%.

Silvia

This probability cannot be determined, since there are infinitely many natural numbers and we cannot check all possibilities.

Which of the three responses do you consider the *most* appropriate?

Check one box.

- Lisa .... ☐
- Monica .... ☐
- Silvia .... ☐

Which of the following ***best*** describes the problems that the other two students have?

*Check up to **2** boxes.*

1. does not understand what happens when you square. .... ☐
2. does not understand what a natural number is. .... ☐
3. does not take into consideration that the last digit is already determined to be 1. .... ☐
4. finds the problem too difficult and gives up. .... ☐
5. does not consider the appropriate natural numbers. .... ☐
6. does not understand concepts of probability. .... ☐

2) **Character** (Resilience/perseverance) – see section below on Log Data

Innovative Tools

The most significant methodological recent change in PISA is the transition from a paper-based to a computer-based assessment (CBA) that was implemented in 2015 in most of the participating countries. This innovation allows the assessment to improve its reliability and more fundamentally to keep up-to-date with the digitalisation of society. In the assessment of mathematical literacy, the advent of CBA opens new possibilities to measure the students' ability to reason mathematically.

Computer-***based*** Mathematics:

Note: this is not computer-assisted instruction! Nor is it Computer Science or Programming[23]:

At its heart, mathematics is a problem solving activity. According to the PISA 2015 Draft Mathematics Framework, it progresses through an *iterative* cycle of problem definition – translation into mathematics – computation of results – interpretation. Mathematics education and assessment focuses almost exclusively on moving along this cycle by using paper-based procedures. The increasing availability of computers shifts the emphasis and hence creates an opportunity for PISA to introduce new assessments that will measure the broader skills of mathematics.

Building interactive challenges and scenarios within a creative coding environment enables the full cycle of skills to be assessed in a controlled fashion. Cloud-based delivery ensures wide-ranging access to the assessments and ensures that students can receive unique and realistic data, which was hardly possible in the test administered in the paper version. Also collaborative skills can be fully exercised in a realistic and demanding fashion.

For example, imagine students being assessed on their ability to abstract into code using short closed problems that can be automatically assessed. In addition, it will be possible to assess the outcomes associated with more open problems, following the full problem solving cycle, end to end, via an extended task with broader scope.

Examples of shorter tasks:

Title	Description of Problem	Outcomes include
Pattern find	From a list of words, design a filtering rule that would filter out those that contain the letters 'p i s a', or those that meet other conditions.	Defining the question. Pattern matching

[23] In CCR's opinion, Programming, or Computer Science more generally, is highly relevant for education in this century, but it is not merely a branch or project in Mathematics thus belongs elsewhere in the curricula, on its own footing which it deserves. Of course, it could be put to good use in learning Mathematics.

Describe the data set	Compute statistics and display visuals that describe a data set	Communicating and collaborating. Standard deviation; Histogram
Polygon stars	Produce a function that will draw a 2D star of a given size with a given number of points.	Abstracting to maths concepts. Angle

Examples of extended tasks:

Title	Description of Problem	Outcomes include
How old am I?	Use the current time and your birth date to compute your age in days, hours, minutes, and seconds.	Managing a computation. Generalizing models.
Do referees favour home teams?	Using real data from football/rugby/basketball, analyse whether the home team gains any advantage from supporters influencing the referee's awards of penalties/free kicks/yellow cards.	Communicating and collaborating. Interpreting results. Critiquing and verifying. Statistics & probability.
Where should the hubs be built?	To efficiently implement a national delivery network, where should distribution hubs be located?	Planning and managing computations. Graph theory.

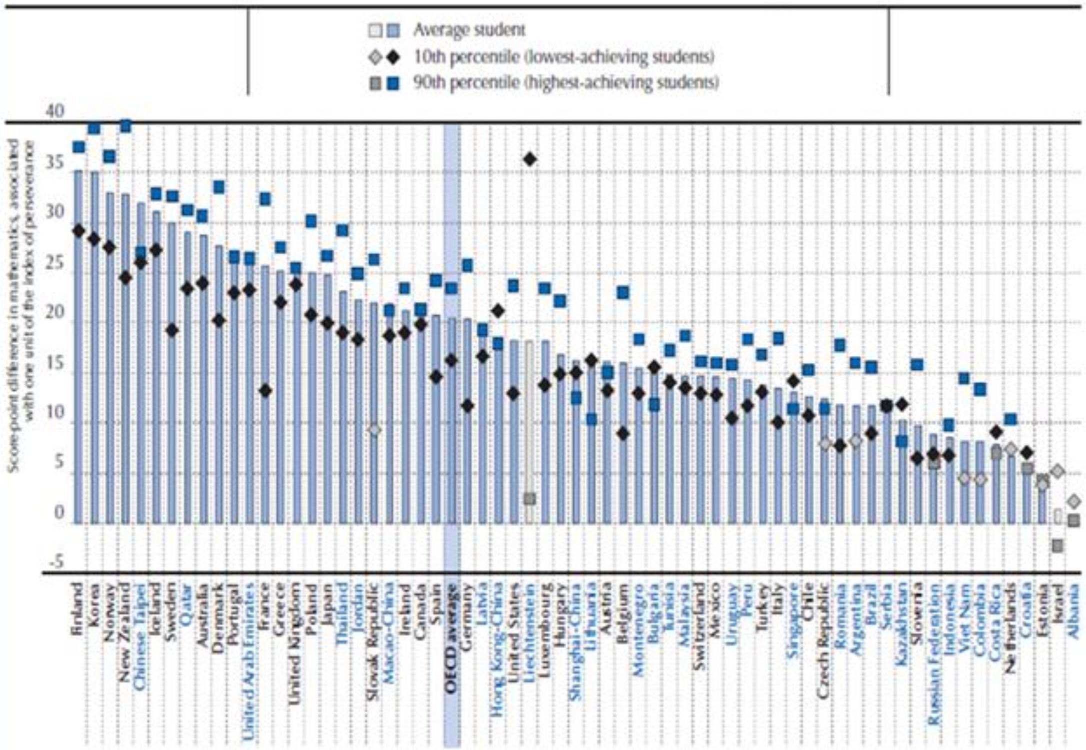
Use of log data:

Analysing the steps and actions that students take during an assessment can help understand how students try to reach a solution: applying mathematical reasoning, following repetitive learned procedures, by trial-and-error, or simply guessing, in multiple-choice questions. For this purpose, mathematics PISA items should be designed so that students' ability to reason mathematically can be effectively measured. For instance, we should develop items where the solution can be reached through a long, repetitive process or using a shorter route that involves deep mathematical understanding. The goal is ultimately to complement the wrong/correct dichotomy with information on students' ability to apply mathematical reasoning[24].

Another example of the use of log data is to measure persistence/resilience, which is critical to success in mathematics:

[24] Commentary from Alberta Education: "The interpretation of this data needs to be carefully considered; for instance, if the time stamp is a short duration of time, this could mean the student guessed as opposed to reason."

Relationship between perseverance and mathematics performance



Note: Differences that are statistically significant at the 5% level ($p < 0.05$) are marked in a darker tone.
Countries and economies are ranked in descending order of the average score-point difference in mathematics associated with one unit of the index of perseverance.
Source: OECD, PISA 2012 Database, Table III.3.1e.
StatLink http://dx.doi.org/10.1787/888932963825

Adaptive Testing:

Adaptive testing in digital assessments can also help PISA move in a similar direction. Questions that require deep mathematical understanding are often criticised for their inability to discriminate sufficiently among students. Frequently only a tiny fraction of students answer these questions correctly (e.g. Revolving Door Q2 in PISA 2012) which means that students could be “wasting” their time in items that barely help to tell apart the low from the top performers. A solution would be to develop items also of low and medium difficulty that require or could be solved using deep mathematical reasoning but this is easier said than done. An easier solution might be to use adaptive testing so that the best students are asked to use more mathematical reasoning as they progress in the test. Adaptive testing therefore is a tool that can help measure creativity, mathematical reasoning, and critical thinking, at the same time that it provides a more accurate view on the knowledge and skills of the top performing 15-year-olds, as well as low-performing students.

Student and teacher questionnaires[25]:

To strengthen even further the measurement of mathematical reasoning, it is also important to look into the possibilities that the student and teacher questionnaires offer. The questions on learning strategies, teaching strategies, and exposure to mathematics content already provide valuable information. However, students might be asked explicitly which strategies they use to solve mathematical problems, for instance by showing different ways of solving a mathematical problem and choosing the one they would most likely have used were they faced with a similar question. Possible methods could include using their experience of similar problems, trial-and-error, spatial reasoning, algebraic reasoning, etc.

Enhancing the Measurement of Opportunity to Learn in PISA 2021

For the first time, PISA 2012 included several questions to measure students' opportunity to learn (OTL) mathematics concepts and skills in their schooling. OTL is a common sense construct that the time a student spends in learning something (including no such exposure) is related to what that student learns (although care has to be exercised in not confusing duration for ability); a notion that is fundamental to schools and schooling. The PISA measurement of mathematics literacy – the application of mathematics knowledge to real world, everyday situations – rather than the mathematics students study most typically in schools, however, raises the question as to what sort of relationship might exist between what is studied in schools and such a literacy measure.

Several different types of OTL items were included in the PISA 2012 student questionnaire. One question listed 13 mathematics concepts commonly taught in school and asked students to indicate how familiar they were with each one. An aggregate index constructed from this item demonstrated a strong and robust relationship with all of the PISA mathematics literacy scores both at the individual student level and at the school level. Furthermore, this was the only OTL measure that was strongly related in this manner to the various scores and subscores for every participating country[26].

The other type of item which was found to be significantly related to PISA performance asked students how frequently they encountered different types of problems in their classroom instruction and in the tests they took. A continuum of category types were chosen from standard algorithmic types of items such as solving a simple linear equation to typical textbook word problems and to more complex problems involving applications of mathematics both to mathematics problems themselves as well as real-world situations. These were also found to be significantly related to PISA performance but in a more complex non-linear way.

[25] Commentary from Alberta Education: "Jurisdictions should be made aware of the usefulness of the data in the questionnaires. Note too that students' perceptions may vary from teachers 'perceptions when it comes to teaching strategies and learning strategies.

[26] Schmidt, W. H., Zoido, P., & Cogan, L. S. (2013). Schooling Matters: Opportunity to Learn in PISA 2012. *OECD Education Working Papers, No. 95*. Retrieved from: http://dx.doi.org/10.1787/5k3v0hldmchl-en

That work was only a beginning and suffered from limited space on the questionnaire but even so was a powerful predictor of performance related to schooling. Building on the limited but highly successful PISA 2012 OTL measures, we propose that the goals for OTL in PISA 2021 are twofold. One goal is to expand and bring greater focus to the measurement both of the specific mathematics concepts studied in schools as well as to students' experiences in class with problems based on real-world situations. This includes carefully selecting the mathematics concepts from a range of topics studied towards the end of identifying with greater precision which concepts may be linked to various aspects of mathematics literacy. It also involves including a greater range of categories of items along the proposed continuum from standard algorithmic problems to problems with complex real-word situations. The second goal is to be mindful of the newly proposed focal areas in measuring mathematics literacy and to identify and select concepts and problems which might especially be related to performance in those focal areas.

Additionally, the PISA 2012 Field Test included more OTL measures based on actual PISA literacy items than were included in the final survey in order to define the range of OTL experiences students had with problem types. These items performed well in the Field Trial. Revisions and refinement to the way previously used assessment items or modifications of current assessment items are included in OTL measurement could yield valuable insight into the type of school experiences that relate to how well students do on PISA. This type of OTL may well prove valuable in providing context and insight into the new proposed mathematics focal content areas for PISA 2021.

Concluding note on Worked examples

The team fully acknowledges the difficulty of constructing valid and reliable test items of the type this paper proposes. Worked examples will be developed during 2016, to assist all PISA jurisdictions in adapting to the recommendations herewith.

The CCR looks forward to the suggestions and recommendations of the PISA community.

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CENTER FOR CURRICULUM REDESIGN

Making Education *More* Relevant

Appendix 1:
Usage of Mathematics by various professions

© 2014 Charles Fadel **Occupation (below)**	**Algebra**	**Applied Maths**	**Calculus**	**Discrete Mathematics**	**Foundations**	**Geometry**	**Numbers & Operations**	**Statistics & Probability**	**Topology & Recreational**
Taxonomy & Ontology: Wolfram Research →	**Matrices, Operations, Vectors etc**	**Complex systems, Control, Game theory, etc**	**Analysis, Transforms, Polynomials, etc**	**Automata, Graphs, Computational maths etc**	**Sets, Logic etc**	**Curves, Dimensions, Transformations, Trigonometry, etc**	**Arithmetic operations, Fractions, Sequences, etc**	**Distributions, Analysis, Estimation, etc**	**Knots, Figures, Folding, Spaces, etc**
Agriculture						X	X	X	
Architecture		X				X	X	X	X
Astronomy/Cosmology	X	X	X	X		X	X	X	X
Biology, Botany, Zoology		X		X			X	X	
Biotechnology, Genetics	X	X	X	X		X	X	X	X
Business		X					X	X	
Cinematography/Photography						X	X		X
Civil engineering	X	X	X	X		X	X	X	X
Communication		X					X	X	
Computer science	X	X	X	X	X	X	X	X	X
Craftsmanship						X	X		X
Dance						X	X		X
Design						X	X		X
Drawing						X	X		X
Economics & Finance	X	X	X	X		X	X	X	
Education	X	X	X			X	X	X	
Electrical engineering	X	X	X	X		X	X	X	
Environmental science	X	X	X	X		X	X	X	
Ethics							X		
Geography/Geology	X	X	X	X		X	X	X	X
Health							X	X	
History/Archeology	X	X		X			X	X	
Journalism	X	X					X	X	
Languages/Linguistics	X	X		X			X	X	
Law		X					X	X	
Materials Science/Nanotechnology	X	X	X	X		X	X	X	X
Mechanical engineering, Robotics	X	X	X	X		X	X	X	X
Medicine/Pharmacy/Veterinary		X					X	X	
Music	X						X	X	
Painting						X	X		
Philosophy		X			X		X	X	
Physics	X	X	X	X	X	X	X	X	X
Poetry/Prose							X		
Psychology/Sociology/Anthropology	X	X		X			X	X	
Sculpture						X	X		X
Sewing/Knitting/Tapestry						X	X		X
Spirituality/Religions							X		
Theater/Acting							X		X