PROPERTIES OF THE HOLMES SPACE.

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Abstract. The following properties of the Holmes space H are established:
(i) H has the Metric Approximation Property (MAP).
(ii) The w^*-closure of the set of extreme points of the unit ball B_{H^*} of the dual space H^* is the whole ball B_{H^*}.

A family of compact subsets $X \subset U$ of the Urysohn space is described such that the Lipschitz-free space $\mathcal{F}(X)$ has a finite-dimensional decomposition and is not complemented in H.

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1. Introduction

Let (X,d) be a metric space and $m_0 \in X$. $\mathrm{Lip}_0(X)$ will stand for the space of all real-valued Lipschitz functions which vanish at m_0 with the norm

$$(1.1) \qquad ||f||_{\mathrm{Lip}} = \sup\left\{\frac{|f(u)-f(v)|}{d(u,v)} : \ u,v\in X, \ u\neq v\right\}.$$

The space of all Lipschitz functions on a metric space X will be denoted by $\mathrm{Lip}(X)$. Clearly $\|.\|_{\mathrm{Lip}}$ is only a pseudonorm on this space. The Lipschitz-free space over X, denoted by $\mathcal{F}(X)$, is the canonical predual of $\mathrm{Lip}_0(X)$, i.e. the norm closed linear subspace of $\mathrm{Lip}_0(X)^*$ spanned by the evaluation functionals $\delta(u)$ with $u \in X$. For properties of Lipschits-free spaces see [W] and [GK]. The case $X = U$ the Urysohn metric space is of a special interest. Recall that the Urysohn space is a separable complete metric space with the following extension property

(E) *For any two finite metric spaces $A \subset B$ and any isometry $\phi : A \to U$, there is an isometric extension $\psi : B \to U$.*

Property (E) characterizes the space U (up to isometry) in the class of all separable complete metric spaces (see [U]).

Fix a point $m_0 \in U$ and denote $H = \mathcal{F}(U)$. Holmes proved [H] the following remarkable property of H. Assume that $\psi : U \to Y$ is an isometry of U into a Banach space Y and E is the norm closure of the linear span of $\psi(U)$ in Y. Then E is isometric to H. In [H] Holmes asks whether the space H has a basis? In section 2 we prove a weaker property of H, namely that it has the metric approximation property (MAP).

Recall that a separable Banach space Y has MAP if there is a sequence $\{V_n\}$ of finite-rank operators in Y with $||V_n|| = 1, \ n = 1,2,...$, such that $\lim_n V_n x = x$, for any $x \in Y$.

Since the Urysohn space U contains an isometric copy of any separable metric space, it follows from [GK] that the Holmes space H in universal in the class of all separable Banach spaces, i.e. for any separable Banach space Y there is a subspace $Z \subset H$ that is linearly isometric to Y. We recall two examples of universal (in the class of all separable Banach spaces) Banach spaces. They are $C[0,1]$ and the Gurariy space G (see [G]). The Holmes space is isomorphic to neither $C[0,1]$ nor G. Indeed, both spaces $C[0,1]$ and G are Lindenstrauss spaces, i.e. their duals are $L_1(\mu)$-spaces (see [LL]); it is known (see e.g. [PW, III.C.14]) that any $L_1(\mu)$-space is weakly sequentially complete and $H^* = \mathrm{Lip}_0(U)$ is not. However, the spaces H and G have a common property, namely

$$(1.2) \qquad w^*\text{-cl}\,\mathrm{ext}B_{H^*} = B_{H^*}, \quad w^*\text{-cl}\,\mathrm{ext}B_{G^*} = B_{G^*}.$$

For the space G it was observed by A. Pełczyński, see [LL1]. For H we prove it in section 3. Note that the property (1.2) uniquely defines G in the class of all separable Lindenstrauss spaces (see [Lu1, Lu2]). In section 3 we show that for the space H the property (1.2) is not characteristic in the class of all separable $\mathcal{F}(X)$-spaces.

In section 4 we initiate the investigation of the following problem. Let $X \subset U$ be a compact subset of U. Under which conditions on X (or on $\mathcal{F}(X)$) the space

$\mathcal{F}(X)$ is complemented in H? We give a general method of construction of compact metric spaces X such that the space $\mathcal{F}(X)$ has a finite-dimensional decomposition and $\mathcal{F}(X)$ is not complemented in H.

We use a standard Geometry of Banach Spaces notation (see [JL]). For instance B_L stands for the unit ball of a normed space L. Let us note also that for a Lipschitz map ψ from one metric space into another we will denote its smallest Lipschitz constant by the $\|\psi\|_{\mathrm{Lip}}$. This should not lead to any misunderstanding.

2. The metric approximation property

In this section we prove

Theorem 2.1. *The space H has the metric approximation property (MAP).*

We collect below the facts we will need for the proof of Theorem 2.1.

Fact 1.[BP, Proposition 1.1] *Let M be a finite metric space and $u_0 \in M$. Then there is an isometric embedding $h : M \to l_\infty^n$, $n = |M| - 1$, with $h(u_0) = 0$.*

Fact 2.[W] *Let X be a metric space and $A \subset X$. Then $\mathcal{F}(A) \subset \mathcal{F}(X)$.*

Fact 3.[W] *Let $\psi : X \to X$ be a Lipschitz map from a metric space X into X. Assume that $u_0 \in X$ and $\psi(u_0) = u_0$. Then there is a linear operator $T : \mathcal{F}(X) \to \mathcal{F}(X)$ with $||T|| = \|\psi\|_{\mathrm{Lip}}$, and such that $T|_X = \psi$.*

Fact 4.[U] *Let M be a separable metric space and $L \subset M$ be a finite subset of M. Assume that $\xi : L \to U$ is an isometry. Then there is an isometric extension $\tilde{\xi} : M \to U$ of ξ.*

Fact 5.[BL, Proposition 2.4] *The space $l_\infty(\Gamma)$ is an absolute 1-Lipschitz retract.*

Fact 6.[GK, Proposition 5.1] *Let E be a finite-dimensional Banach space. Then for any $\varepsilon > 0$ and $R > 0$ there is a Lipschitz map $\phi : E \to \mathcal{F}(E)$ with finite-dimensional range and such that*

$$\phi(0) = 0, \ \ ||\phi||_{\mathrm{Lip}} < 1 + \varepsilon, \quad ||\phi(x) - \delta(x)|| < \varepsilon, \ x \in RB_E.$$

Proof of Theorem 2.1. We construct a sequence of finite rank linear operators $V_n : H \to H$ such that $||V_n|| = 1, \ n = 1, 2, ...,$ and $\lim_n V_n x = x$, for any $x \in H$.

Let $\{u_i\}_{i=0}^\infty \subset U$ be a dense countable subset of U. Fix $n \in \mathbb{N}$. By using Facts 1 and 4 we find a subset $E_n \subset U$ isometric l_∞^n such that $\{u_i\}_{i=0}^n \subset E_n$. By Fact 5 there is a 1-Lipschitz retraction $r_n : U \to E_n$. Next apply Fact 6 for $\varepsilon = 1/n, \ R = \max\{d(u_0, u_i) : \ \ i = 1, ..., n\}$, and find a Lipschitz map $\phi_n : E_n \to \mathcal{F}(E_n)$ with finite-dimensional range and such that

$$\phi_n(0) = 0, \ \ ||\phi_n||_{\mathrm{Lip}} < 1 + 1/n, \quad ||\phi_n(u_i) - \delta(u_i)|| < 1/n, \ i = 1, ..., n.$$

Put $t_n = \phi_n \circ r_n$. Then by Facts 3 and 2 there is a linear operator $T_n : H \to H$, $||T_n|| < 1 + 1/n$, and such that $T|_U = \phi_n$. Clearly, T_n has a finite-dimensional range and $||T_n(\delta(u_i)) - \delta(u_i)|| < 1/n, \ \ i = 1, ..., n$. Therefore $\lim_n T_n(\delta(u_i)) =$

$\delta(u_i),\ i = 0, 1, ...$ Since $\sup_n ||T_n|| < \infty$ and $\text{span}\{u_i\}_{i=1}^{\infty}$ is dense in H, it follows that $\lim_n T_n x = x$, for any $x \in H$. Put $V_n = ||T_n||^{-1} T_n,\ n = 1, 2, ...$, and finish the proof.

3. Extreme points of the dual ball B_{H^*}.

The main result of this section is the following

Theorem 3.1. *Let H be the Holmes space. Then*

$$w^*\text{-cl ext} B_{H^*} = B_{H^*}. \tag{3.3}$$

A proof of Theorem 3.1 is a combination of the following auxiliary results and the defining property of the Urysohn space U.

Lemma 3.2. *Let M be a finite metric space, $m_0 \in M$, and $L = \text{Lip}_0(M)$. TFAE:*
(1) $f \in \text{ext} B_L$.
(2) $\|f\|_{\text{Lip}} = 1$ and for any $t_0 \in M$ there is a chain $\{t_i\}_{i=0}^n \subset M$ with $t_n = m_0$ and such that

$$\frac{|f(t_{i+1}) - f(t_i)|}{d(t_{i+1}, t_i)} = 1, \quad i = 0, 1, ..., n-1.$$

Proof. (1)⇒(2). Fix $f \in \text{ext} B_L$ and $t_0 \in M$. For any $t \in M$ denote

$$A(t) = \{s \in M : \frac{|f(t) - f(s)|}{d(t, s)} = 1\},$$

and consider the following tree

$$T = \{t_0\} \cup A(t_0) \cup \bigcup_{s \in A(t_0)} A(s) \cup \bigcup_{u \in \cup_{s \in A(t_0)} A(s)} A(u) \cup \ldots$$

It is enough to prove that $m_0 \in T$. Assume to the contrary that $m_0 \notin T$. Clearly, for any $t \in T$ and for any $v \in M \setminus T$ we have

$$\frac{|f(t) - f(v)|}{d(t, v)} < 1,$$

and hence there is a $\delta > 0$ such that

$$\frac{|f(t) \pm \delta - f(v)|}{d(t, v)} < 1, \quad t \in T, \quad v \in M \setminus T,$$

(recall that M is finite).
Define a function h on M as follows

$$h(t) = \delta, \ \ t \in T, \quad h(t) = 0, \ \ t \in M \setminus T.$$

By our assumption $h(m_0) = 0$, and hence $h \in L$. It is not difficult to see that $||f \pm h||_{\text{Lip}} = 1$, contradicting $f \in \text{ext} B_L$. The proof of (1)⇒(2) is complete.
(2)⇒(1). Assume that for some $h \in L$ we have $||f \pm h|| = 1$, and prove that $h = 0$. Fix $v \in M$, $v \neq m_0$, and by (2) find a chain $\{t_i\}_{i=0}^n \subset M$ with $t_0 = m_0$ and $t_n = v$ and such that

$$\frac{|f(t_{i+1}) - f(t_i)|}{d(t_{i+1}, t_i)} = 1, \quad i = 0, 1, ..., n-1.$$

Since $f(t_0) = h(t_0) = 0$, we easily get $h(t_1) = 0$. Next we pass to the pair t_1, t_2, and by using

$$\frac{|f(t_2) - f(t_1)|}{d(t_2, t_1)} = 1, \quad \frac{|f(t_2) \pm h(t_2) - f(t_1)|}{d(t_2, t_1)} \leq 1,$$

we get $h(t_2) = 0$, and so on. Finally we get $h(t_n) == h(v) = 0$. Since $v \in M$ is an arbitrary point, it follows that $h = 0$, and the proof of the lemma is complete.

Lemma 3.3. *Let M be a finite metric space, $m_0 \in M$, and $L = \mathrm{Lip}_0(M)$. Assume that $f \in B_L$. Then there is a metric space M_1 containing M with $M_1 \backslash M$ a singleton, and such that there is $g \in \mathrm{ext}B_{\mathrm{Lip}_0(M_1)}$ with $g|_M = f$.*

Proof. Let $M = \{m_i\}_{i=0}^n,\ d_{ij} = d(m_i, m_j),\ a_i = f(m_i),\ i, j = 0, ..., n$. We define the distances $d_i = d(m_i, m_{n+1}),\ i = 0, ..., n$, and $a = g(m_{n+1})$, such that g will be an extreme point of $B_{\mathrm{Lip}_0(M_1)}$, where $M_1 = \{m_i\}_{i=0}^{n+1}$. To this end it is enough to fulfill the following conditions:
(i) $d_{ij} \leq d_i + d_j,\ \ i, j = 0, ..., n.$
(ii) $|d_i - d_j| \leq d_{ij},\ \ i, j = 0, ..., n.$
(iii) $|a - a_i| = d_i,\ i = 0, 1, ..., n.$
Note that (i)-(ii) guarantee that M_1 is a metric space, while (iii) guarantees that $g \in \mathrm{ext}B_{\mathrm{Lip}_0(M_1)}$, (see Lemma 3.2).
If $a \geq \max\{|a_i| :\ i = 1, ..., n\}$, then from (iii) we get $d_i = a - a_i,\ i = 0, 1, ..., n.$ Therefore (ii) is equivalent to $|a_i - a_j| \leq d_{ij},\ i, j = 0, ..., n$, which is satisfied since $||f||_{\mathrm{Lip}} \leq 1$. If moreover we take

$$a \geq \max\{d_{ij} :\ i, j = 0, ..., n\} + \max\{|a_i| :\ i = 1, ..., n\}$$

we fulfill (i). The proof is complete.

Lemma 3.4. *Let $A \subset B$ be two metric spaces and $m_0 \in A$. Then for any $f \in \mathrm{ext}B_{\mathrm{Lip}_0(A)}$ there is $g \in \mathrm{ext}B_{\mathrm{Lip}_0(B)}$ with $g|_A = f$.*

Proof. It is known that $F(A) \subset F(B)$ and $F^*(A) = \mathrm{Lip}_0(A),\ F^*(B) = \mathrm{Lip}_0(B)$. By the Krein-Milman theorem we have $\mathrm{ext}B_{\mathrm{Lip}_0(A)} \subset \mathrm{ext}B_{\mathrm{Lip}_0(B)}|_{F(A)} = \mathrm{ext}B_{\mathrm{Lip}_0(B)}|_A$ which finishes the proof.

Corollary 3.5. *Let M be a finite metric space, $m_0 \in M$, and $L = \mathrm{Lip}_0(M)$. Assume that $N \subset B_L$ is a finite set. Then there is a finite metric space $\hat{M}$ containing M such that for any $f \in N$ there is $g \in \mathrm{ext}B_{\mathrm{Lip}_0(\hat{M})}$ with $g|_M = f$.*

Proof. Let $N = \{f_j\}_{j=1}^s$. First we apply Lemma 3.3 to f_1 to get metric space $M_1 \supset M$ and $\hat{f}_1$ an extreme point in the ball of $\mathrm{Lip}_0(M_1)$ such that $\hat{f}_1|M = f_1$. Next using Theorem 1.5.6 from [W] we extend f_2 to the function $\tilde{f}_2$ on M_1 with preservation of the Lipschitz norm and apply to $\tilde{f}_2$ Lemma 3.3 to get a metric space $M_2 \supset M_1$ and $\hat{f}_2$ an extreme point in the ball of $\mathrm{Lip}_0(M_2)$ such that $\hat{f}_2|M_1 = \tilde{f}_1$. We continue in this manner to get an increasing sequence of metric spaces $M \subset M_1 \subset M_2 \subset \cdots \subset M_s = \hat{M}$ and functions $\hat{f}_j$ which are extreme points of the unit ball of

$\mathrm{Lip}_0(M_j)$ and $\hat{f}_j|M = f_j$. Lemma 3.4 applied to each pair $M_j \subset \hat{M}$ and function $\hat{f}_j$ gives the claim.

Proof of Theorem 3.1. Let $\{m_i\}_{i=0}^{\infty}$ be a dense subset of U. By using Corollary 3.5 and the property (E) of U (see Introduction) we construct an increasing sequence $\{M_n\}$ of finite subsets of U and a sequence $\{N_n\}$ of finite $1/n$-nets in B_{L_n} ($L_n = \mathrm{Lip}_0(M_n)$) with the following properties:
(i) For any n we have $\{m_i\}_{i=0}^{n} \subset M_n$.
(ii) For any n and for any $f \in N_n$ there is $g \in \mathrm{ext} B_{\mathrm{Lip}_0(M_{n+1})}$ with $g|_{M_n} = f$.

Fix $h \in \mathrm{Lip}_0(U)$, $||f|| \leq 1$, and put $h_n = f|_{M_n}$, $n = 1, 2, \ldots$. Clearly $h_n \in B_{L_n}$ and hence there is $f_n \in N_n$ with $||h_n - f_n|| \leq 1/n$, $n = 1, 2, \ldots$ By (ii) for any n there is $g_n \in \mathrm{ext} B_{\mathrm{Lip}_0(M_{n+1})}$ with $g_n|_{M_n} = f_n$. By the Krein-Milman theorem there is $t_n \in \mathrm{ext} B_{\mathrm{Lip}_0(U)}$ with $t_n|_{M_{n+1}} = g_n$, $n = 1, 2, \ldots$. Recall that the w^*-convergence in $B_{H^*} = \mathrm{Lip}_0(U)$ is just the point-wise convergence on U. It easily follows that $w^*\text{-}\lim t_n = h$. The proof is complete.

Remark. There is a countable discrete metric space D such that the space $\mathcal{F}(D)$ has property (3.3), i.e. $w^*\text{-cl}\ \mathrm{ext} B_{\mathrm{Lip}_0(D)} = B_{\mathrm{Lip}_0(D)}$. The proof runs along the lines of the proof of Theorem 3.1. The difference (actually, a simplification) is that we use only Corollary 3.5 for construction of a metric space $D = \cup_{n=1}^{\infty} M_n$ without using the Urysohn space. Note that it follows from the proof of Lemma 3.3 that D is discrete. On the other hand note also that if $\mathbb{N}$ is the set of integers and $m_0 = 0$ then the sequence $(a_n)_{n \in \mathbb{N}} \in \mathrm{ext} B_{\mathrm{Lip}_0(\mathbb{N})}$ if and only if $|a_n - a_{n+1}| = 1$ for all $n \in \mathbb{N}$. One easily sees that this set is w^*–closed. For a separable Banach space X the property $w^*\text{-cl}\ \mathrm{ext} B_{X^*} = B_{X^*}$ shows that in some sense there are many extreme points in B_{X^*}. It holds in particular if every point of the unit sphere is extreme. This however cannot happen for $\mathrm{Lip}_0(M)$ if M has more than two points.

4. Non-complemented subspaces of the Holmes space

We start with the proposition that shows that the complementability of $\mathcal{F}(K)$ with K metric compact, does not depend on how a compact space K is embedded into U.

Proposition 4.1. *Let K_1 and K_2 be two isometric compact subsets of U. Then $\mathcal{F}(K_1)$ is complemented in H if and only if $\mathcal{F}(K_2)$ is complemented in H.*

Proof. Let $\phi : K_1 \to K_2$ be an isometry. By [Hu] there is an isometry $\psi : U \to U$ of U onto U with $\psi|_{K_1} = \phi$. By [GK] there is a linear isometry $T_\psi : H \to H$ of H onto H with $T_\psi|_U = \psi$. If $P : H \to \mathcal{F}(K_1)$ is a linear bounded projection on $\mathcal{F}(K_1)$ then it is easy to check that $Q = T_\psi^{-1} P T_\psi$ is a linear bounded projection on $\mathcal{F}(K_2)$. The proof is complete.

In view of Proposition 4.1 the following problem seems to be interesting.
Problem. Characterize those compact metric spaces K for which $\mathcal{F}(K)$ is complemented in H.

If $K \subset U$ is a Lipschitz retract then it is easy to see that $\mathcal{F}(K)$ is complemented in H. Clearly, this condition is not a necessary one (take K finite). In the rest of this section we give a general construction of compact metric spaces X for which the space $\mathcal{F}(X)$ has a finite-dimensional decomposition and is not complemented in H.

Therefore the main result of this section is the following

Theorem 4.2. *There exists a compact metric space X such that $\mathcal{F}(X)$ has a finite dimensional decomposition and for any isometric embedding of X into the Uryshon space U, $\mathcal{F}(X)$ is not complemented in $\mathcal{F}(U)$.*

To prove the theorem we need several auxiliary results.

Proposition 4.3. *Given $A > 0$ there is a pair of finite-dimensional Banach spaces $X \subset Y$ such that for any linear extension $E : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$, we have $||E|| \geq A$.*

Proof. It is well-known that there is a pair of finite-dimensional Banach spaces $X \subset Y$ such that for any projection $Q : Y \to X$ we have $||Q|| \geq A$, see e.g. [PW, III.B.16] We prove that this pair works. Assume that $E : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a linear extension operator. Denote $r_1 : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ the restriction operator. Clearly, $r_1E = Id_{\mathrm{Lip}_0(X)}$. Also it is clear that $X^* \subset \mathrm{Lip}_0(X)$ and $Y^* \subset \mathrm{Lip}_0(Y)$. By [BL], Proposition 7.5, there are projections $P : \mathrm{Lip}_0(Y) \to Y^*$ and $P_0 : \mathrm{Lip}_0(X) \to X^*$ with $||P|| = ||P_0|| = 1$, and such that if $r_2 : Y^* \to X^*$ is the restriction map, then $r_2P = P_0r_1$. Put $s = PE|_{X^*}$. Then s is a linear extension of "linear functionals" from X to Y. Hence r_2 is a projection with norm $A \leq ||r_2|| \leq ||s|| \leq ||E||$. The proof is complete.

Proposition 4.4. *Let $X \subset Y$ are two separable metric spaces with fixed point $x_0 \in X$. Assume that there is a constant $A > 0$ such that for any pair of finite sets $x_0 \in M \subset N \subset Y$ such that $N \cap X = M$ there exists a linear extension operator $E_{MN} : \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(N)$ with $\|E_{MN}\| \leq A$. Then there exists a linear extension operator $E : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ with $\|E\| \leq A$.*

Proof: Let us fix a sequence of finite sets $M_n \subset N_n$ as in the assumptions such that $M_n \subset M_{n+1}$, $N_n \subset N_{n+1}$, $M_\infty =: \bigcup_{n=1}^\infty M_n$ is dense in X and $N_\infty =: \bigcup_{n=1}^\infty N_n$ is dense in Y. Let us fix extensions given in the assumptions and denote $E_n =: E_{M_n,N_n}$. Now let LIM be a fixed Banach limit on $\mathbb{N}$. For $f \in \mathrm{Lip}_0(X)$ and $z \in N_\infty$ we define

$$E(f)(z) =: \underset{n}{LIM}\, E_n(f|M_n)(z).$$

Note that $z \in N_n$ only for n greater then some K_z so $E_n(f|M_n)(z)$ is formally defined also only for $n \geq K_z$. This however does not influence the value of the LIM (we can formally put $E_n(f|M_n)(z) = 0$ for $n < K_z$). Clearly E is a linear map from functions on X to the functions on N_∞ and for $x \in M_\infty$ and any f we have $E(f)(x) = f(x)$. To estimate the norm of E note that for $z_1, z_2 \in N_\infty$ we have

$$\begin{aligned} |E(f)(z_1) - E(f)(z_2)| &= |\,LIM\left(E_n(f|M_n)(z_1) - E_n(f|M_n)(z_2)\right)| \\ &\leq LIM\,\|E_n(f|M_n)\|_{\mathrm{Lip}} \leq \sup \|E_n(f|M_n)\| \leq A\|f\|_{\mathrm{Lip}}. \end{aligned}$$

which gives that the norm of E as an operator from $\mathrm{Lip}_0(X)$ into $\mathrm{Lip}_0(N_\infty)$ is at most A. Since M_∞ is dense in X, E is actually an extension operator. The proof is complete.

Linear extensions are (as is well known) closely related to projections. We will need the following obvious observation: *Let $X \subset Y$ be metric spaces and let $\mathcal{F}(X)$ be naturally embeded into $\mathcal{F}(Y)$. If P is a projection from $\mathcal{F}(Y)$ onto $\mathcal{F}(X)$ then $P^* : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a linear extension operator.*

Now we discuss a general construction which may be considered as a "direct sum" of Lipschitz spaces. Let $\{X_n, \rho_n\}_{n=1}^{\infty}$ be a sequence of metric spaces (which we consider to be disjoint) each of finite diameter. Dilating if necessary we assume that the diameter of X_n is at most 2^{-n}. On $\hat{X} =: \bigcup_{n=1}^{\infty} X_n$ we define a metric by the formula

$$\rho(y_1, y_2) = \begin{cases} \rho_n(y_1, y_2) & \text{if } \exists n \ : \ y_1, y_2 \in X_n \\ \left|\frac{1}{n} - \frac{1}{m}\right| & \text{if } y_1 \in X_n, y_2 \in X_m \text{ with } n \neq m \end{cases} \tag{4.4}$$

We will assume that the distinguished point in $\hat{X}$ is in X_1. We fix (arbitrary) points $z_{n+1} \in X_{n+1}$ and define a projection in $\mathrm{Lip}_0(\hat{X})$ as

$$P_n(f)(z) = \begin{cases} f(z) & \text{if } z \in X_k \text{ with } k \leq n \\ f(z_{n+1}) & \text{if } z \in X_k \text{ with } k > n \end{cases} \tag{4.5}$$

It is clear that P_n is a sequence of commuting, norm one projections. One can check that $\ker P_n \cap P_{n+1}(\mathrm{Lip}_0(\hat{X}))$ is the set of all functions which are zero on $\bigcup_{k=1}^{n} X_k$, are zero on z_{n+1} and are constant on $\bigcup_{k=n+2}^{\infty} X_k$. Clearly this space is uniformly in n isomorphic to $\mathrm{Lip}(X_{n+1})$. The norm closure of $\bigcup_{n=1}^{\infty} P_n(\mathrm{Lip}_0(\hat{X}))$ in $\mathrm{Lip}_0(\hat{X})$ equals $\{f \in \mathrm{Lip}_0(\hat{X}) \ : \ \lim_{m\to\infty} \|f|\bigcup_{n\geq m} X_n\| = 0\}$.

Lemma 4.5. *There exists a sequence of commuting, norm one projections Q_n on $\mathcal{F}(\hat{X})$ such that $Q_n^* = P_n$ and Q_n is pointwise in norm convergent to $Id_{\mathcal{F}(\hat{X})}$.*

Proof: We know that $\mathcal{F}(\hat{X})$ is naturally a subspace of $\mathrm{Lip}_0(\hat{X})^*$ spanned in norm by functionals $\delta(z)$ of point evaluations with $z \in \hat{X}$. Thus it suffices to check that for $z \in \hat{X}$ we have $P_n^*(\delta(z)) \in \hat{X}$ and that $P_n^*(\delta(z)) = z$ for n big enough. All this follows from (4.5), the fact that for $f \in \mathrm{Lip}_0(\hat{X})$ and $y \in \hat{X}$ we have $P_n^*(\delta(y))(f) = P_n(f)(y)$. The proof is complete.

Corollary 4.6. *Assume that the metric spaces X_n are finite. Then the space $\mathcal{F}(\hat{X})$ has a finite dimensional decomposition.*

Proposition 4.7. *There exist two compact metric spaces $X \subset Y$ such that both $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ have finite dimensional decompositions and $\mathcal{F}(X)$ is not complemented in $\mathcal{F}(Y)$.*

Proof: From Propositions 4.3 and 4.4 we find a sequence of pairs of finite metric spaces $X_n \subset Y_n$ such that every linear extension from $\mathrm{Lip}_0(X_n)$ to $\mathrm{Lip}_0(Y_n)$ has norm at least n. Clearly the same is true for extensions from $\mathrm{Lip}(X_n)$ to $\mathrm{Lip}(Y_n)$. Now we build spaces $\hat{X}$ and $\hat{Y}$ by the procedure described above. Clearly $\hat{X} \subset \hat{Y}$, so $\mathcal{F}(\hat{X}) \subset \mathcal{F}(\hat{Y})$ and by Corollary 4.6 both those spaces have finite dimensional decomposition. If there would be a projection from $\mathcal{F}(\hat{Y})$ onto $\mathcal{F}(\hat{X})$ with norm

$\leq A$, we would have a bounded linear extension from $\mathrm{Lip}_0(\hat{X})$ to $\mathrm{Lip}_0(\hat{Y})$. Now for $n > 1$ we identify isometrically the space $\mathrm{Lip}(X_n)$ with the subspace of $\mathrm{Lip}_0(\hat{X})$ as a space of functions supported on X_n. Considering our linear extension only on this this subspace and next restricting to Y_n we get a a linear extension operator from $\mathrm{Lip}(X_n)$ to $\mathrm{Lip}(Y_n)$ with norm at most A. But this contradicts our choice. Our metric spaces are not compact but their completions are compact. This proves the Proposition.

Proof of Theorem 4.2. The desired X is the space X from Proposition 4.7. By the properties of Uryshon space (see [H] Corollary in Part I) any isometric embedding of X into U can be extended to an isometric embedding of Y into U, so we have $X \subset Y \subset U$, so also $\mathcal{F}(X) \subset \mathcal{F}(Y) \subset \mathcal{F}(U)$. This shows that complementation of $\mathcal{F}(X)$ in $\mathcal{F}(U)$ implies complementation of $\mathcal{F}(X)$ in $\mathcal{F}(Y)$ which is impossible by Proposition 4.7. The proof is complete.

References

[BL] Y. Benyamini and J. Lindenstrauss, **Geometric nonlinear functional analysis**. Vol. 1. American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.

[BP] C. Bessaga and A. Pełczyński, **Selected topics in infinite-dimensional topology**. Monografie Matematyczne, Tom 58. [Mathematical Monographs, Vol. 58] PWN—Polish Scientific Publishers, Warsaw, 1975.

[GK] G. Godefroy and N.J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. 159 (1), 2003, 121–141.

[H] M.R. Holmes, *The universal separable metric space of Urysohn and isometric embeddings thereof in Banach spaces.* Fund. Math. 140 (1992), no. 3, 199–223.

[G] V. I. Gurariy, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces.* (Russian) Sibirsk. Mat. Z. 7 (1966) 1002–1013.

[Hu] Huhunaišvili, G. E. *On a property of Uryson's universal metric space.* (Russian) Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 607–610.

[JL] W.B. Johnson and J. Lindenstrauss, *Basic concepts in geometry of Banach spaces*, in Handbook of the Geometry of Banach Spaces, v.1, Elsevier, 2001.

[LL] A. J. Lazar and J. Lindenstrauss, *On Banach spaces whose duals are L_1 spaces.* Israel J. Math. 4 1966 205–207.

[LL1] A.J. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1 spaces and their representing matrices.* Acta Math. 126 (1971), 165–193.

[Lu1] W. Lusky, *On separable Lindenstrauss spaces*, J. Func. Anal. 26 (1977), 103–120.

[Lu2] W. Lusky, *The Gurariy spaces are unique*, Arch. Math. 27, no. 2 (1976), 627–635.

[W] N. Weaver, **Lipschitz Algebras** World Scientific, 1999.

[PW] P. Wojtaszczyk, **Banach spaces for analysts**, Cambridge Studies in Advanced Mathematics 25, Cambridge University Press, Cambridge 1991

[U] P. S. Uryson, *Sur un espace metrique universel*, Bull. Sci. Math. (2) 51 (1927), 43–64,

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