A non-reflexive Banach space with all contractions mean ergodic

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Dedicated to the memory of Aryeh Dvoretzky

Abstract

We construct on any quasi-reflexive of order 1 separable real Banach space an equivalent norm, such that all contractions on the space and all contractions on its dual are mean ergodic, thus answering negatively a question of Louis Sucheston.

1 Introduction

A linear operator T on a (real or complex) Banach space X is called *mean ergodic* if the limit

$$E(T)x := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k x \quad \text{exists} \quad \forall x \in X.$$

Mean ergodicity was shown by Von-Neumann (1931) for unitary operators on Hilbert spaces, by Riesz (1937) for contractions on Hilbert spaces, and (independently) by Lorch (1939), Kakutani (1938) and Yosida (1938) for power-bounded operators on reflexive spaces. We refer the reader to [9] for proofs, discussion and references. A natural question is whether reflexivity is necessary. In [6] the authors proved that if every *power-bounded* operator on a Banach space with a basis is mean ergodic, then the space must be reflexive; it is still not known if the same holds without the existence of a basis. For additional references related to this question see [6].

Sucheston [18] posed the following question: *If every contraction in a Banach space is mean ergodic, must the space be reflexive?* In this paper we construct an example which gives a negative answer to Sucheston's question.

2 Operators on quasi-reflexive spaces of order 1

Definition A Banach space X is *quasi-reflexive of order 1* if $\dim X^{**}/X = 1$, where we always consider $X \subset X^{**}$ via the canonical isometric embedding. The first example of such a space (over $\mathbb{R}$, separable with a basis), and one of the important ones, is the James space [7] (for more of its properties see [4]).

Throughout this note we consider Banach spaces over $\mathbb{R}$.

For the sake of completeness we include the following result of [13, §2.2].

Proposition 1. *Let X be quasi-reflexive of order 1. Then there exists a linear, multiplicative functional $q : L(X) \to \mathbb{R}$ of norm 1 such that $\ker q$ is exactly the space of weakly compact operators from X into X.*

Proof. For $T : X \to X$ let us consider $T^{**} : X^{**} \to X^{**}$. Since $T^{**}(X) \subset X$ we see that it induces an operator $\widetilde{T^{**}} : X^{**}/X \to X^{**}/X$. It is easy to check that $\|\widetilde{T^{**}}\| \leq \|T^{**}\| = \|T\|$. Since X^{**}/X is one-dimensional, $\widetilde{T^{**}}$ is a multiplication by a number which we denote by $q(T)$. It is clear that the map q is the desired norm 1 linear multiplicative functional. The kernel of q consists of those operators $T : X \to X$ such that $T^{**}(X^{**}) \subset X$, but this is exactly the set of all weakly compact operators. □

Now for $T : X \to X$ we write $T = q(T)I + (T - q(T)I)$. Since $q\,(T - q(T)I) = 0$, by the proposition $W := T - q(T)I$ is weakly compact.

If $T : X \to X$ is power-bounded we get

$$c \geq \|T^n\| \geq |q(T^n)| = |q(T)|^n$$

for $n = 1, 2, \ldots$ so $|q(T)| \leq 1$. Let us summarize this discussion with

Corollary 1. *Every linear operator T on X quasi-reflexive of order 1 has a unique decomposition as $T = \lambda I + W$ where $\lambda \in \mathbb{R}$ and W is weakly compact. If T is power-bounded, then $|\lambda| \leq 1$.*

Proposition 2. *Let X be quasi-reflexive of order 1. Suppose $T = \lambda I + W : X \to X$ is power-bounded, with $\lambda \neq 1$. Then both T and T^* are mean-ergodic.*

Proof. For the space of fixed points of T, denoted by Fix (T), we have that

$$\text{Fix}\ (T) = \{x \in X \ :\ \lambda x + W(x) = x\} = \{x \in X \ :\ W(x) = (1 - \lambda)x\}$$

is an eigenspace corresponding to a non-zero eigenvalue of a weakly compact operator, so it is a reflexive space.

Analogous arguments show that

$$\text{Fix}\ (T^{**}) = \{x^{**} \in X^{**} \ :\ W^{**}(x^{**}) = (1 - \lambda)x^{**}\}\,.$$

Since W is weakly compact, we have $W^{**}(X^{**}) \subset X$, which implies Fix $(T^{**}) =$ Fix (T). It is known that Fix (T^*) always separates Fix (T) (e.g. use [9, Theorem 2.1.3, p. 73]), so in our case we get that Fix (T^*) separates Fix (T^{**}). By Sine's criterion [17], [9, p. 74], we obtain that T^* is mean-ergodic. But analogously Fix (T^{**}) always separates Fix (T^*), so in our case Fix (T) separates Fix (T^*), and by Sine's criterion T is mean-ergodic. □

Remark. It was shown in [6, Theorem 5] that for T power-bounded on a quasi-reflexive space of order 1 we always have T or T^* mean ergodic, and if the space has a basis there exists T power-bounded which is not mean ergodic; by the previous proposition this T is of the form $I+W$ with W weakly compact.

3 Renorming spaces with separable second dual

In this section X is a Banach space over $\mathbb{R}$. We use the standard notations

$$S_X = S_{(X,\|\cdot\|)} = \{x \in X : \|x\| = 1\|\} \quad \text{and} \quad B_X = B_{(X,\|\cdot\|)} = \{x \in X : \|x\| \le 1\}$$

for the *unit sphere* and the *closed unit ball* of X, respectively.

Proposition 3. *Let X be a non-reflexive Banach space with X^{**} separable. Then there exist an equivalent norm $|||\cdot|||$ on X, a functional $f_0 \in S_{(X^{**},|||\cdot|||)} \setminus X$, and a functional $F_0 \in S_{(X^{***},|||\cdot|||)} \cap X^{\perp}$ such that*

(i) $F_0(f_0) = 1$ and $F_0(g) < 1$ for any $g \in B_{(X^{**},|||\cdot|||)}$, $g \ne f_0$.

(ii) If $H \in S_{(X^{***},|||\cdot|||)}$ and $H(f_0) = 1$,

then $H \in X^{\perp}$. Moreover, if there exists a Banach space Y such that $X = Y^*$ (isometrically), then the norm $|||\cdot|||$ can be taken as a dual norm.

Proof. We prove first the case that $X = Y^*$; we will then indicate how to modify the proof for the non-dual case.

As usual we assume that $Y \subset Y^{**} = X^*$ and $X = Y^* \subset Y^{***} = X^{**}$, under the canonical isometric embeddings.

Claim 1. *For any $y^{\perp} \in S_{Y^{\perp}} \subset Y^{***}$, we have $d(y^{\perp}, Y^*) \ge 1/2$. Proof of Claim.* For $y^* \in Y^*$ we have $\|y^{\perp} - y^*\| \ge \sup_{\|y\|\le 1} |(y^{\perp} - y^*)(y)| = \|y^*\|$. Hence $\|y^{\perp} - y^*\| \ge \max\{\|y^{\perp}\| - \|y^*\|, \|y^*\|\} \ge \frac{1}{2}\|y^{\perp}\| = \frac{1}{2}$, which proves the claim.

We construct a new norm $|||\cdot|||$ in 2 steps. Pick a functional $f \in S_{Y^{\perp}}$. Clearly, $f \in S_{X^{**}} \setminus X$, and by Claim 1 $d(f, X) \ge 1/2$. By the Hahn-Banach theorem there is $F \in S_{X^{***}} \cap X^{\perp}$ such that $F(f) = d(f, X) \ge 1/2$. Next take a sequence $\{x_n\} \subset S_X$ such that $w^* - \lim x_n = f$ (in the w^*-topology of X^{**}). Put

$$W = \operatorname{cl}\operatorname{co}\{B_X \cup \{\pm 3x_n\}_{n=1}^{\infty}\}, \qquad W^{**} = w^* - \operatorname{cl}W.$$

Since $f \in Y^{\perp}$ it follows that $w^* - \lim x_n = 0$ in w^*-topology defined on X by its predual Y. Since B_X is w^*-compact (in the w^*-topology defined on X by the predual Y), it easily follows that W is w^*-compact (indeed, since $w^* - \lim x_n = 0$, it follows that $A = \operatorname{cl}\operatorname{co}\{\pm 3x_n\}$ is w^*-closed, and hence $W = \operatorname{co}\{A \cup B_X\}$ is w^*-compact, by [3, Lemma V.2.5]). By Milman's theorem [12] (see [16, Prop. 1.5]),

$$\operatorname{ext}W^{**} \subset \operatorname{ext}B_{X^{**}} \cup \{\pm 3x_n\}_{n=1}^{\infty} \cup \{\pm 3f\}. \tag{1}$$

Since X^{**} is separable, W^{**} is a weak-* compact convex set with $\operatorname{ext}W^{**}$ separable, so by [8],[5] (see also [16, p. 26]), we have that $W^{**} = \operatorname{cl}\operatorname{co}\{\operatorname{ext}W^{**}\}$ – the norm closed convex hull (this can be deduced also from the Bessaga-Pełczyński theorem [2]). It follows that

$$\sup F(W^{**}) = \sup F(\operatorname{ext}W^{**}).$$

Since $\|F\| = 1$ and $F \in X^\perp$, by (1) we have $\sup F(W^{**}) = F(3f) > 1$, and, moreover, $F(g) < F(3f)$ for any $g \in W^{**},\ g \neq 3f$.

Let $\{u_i\}_{i=1}^\infty \subset B_X$ be a sequence dense in B_X. Put

$$f_0 = 3f, \quad t_n = 3x_n, \quad F_0 = \frac{1}{F(3f)}F, \quad K = \mathrm{cl}\,\mathrm{co}\{\pm 2^{-i}u_i\}_{i=1}^\infty,$$

$$T_n = \{\lambda t_n + x :\ x \in K,\ \lambda^2 + ||x||^2 \le 1\}, \quad n = 1, 2, ...,$$
$$T_0 = \{\lambda f_0 + x :\ x \in K,\ \lambda^2 + ||x||^2 \le 1\},$$

Since K is symmetric, each T_n is symmetric, and we define

$$V = \mathrm{cl}\,\mathrm{co}\{W \cup \cup_{n=1}^\infty T_n\}, \quad V^{**} = w^* - \mathrm{cl}V.$$

Claim 2. *V is w^*-closed in the w^*-topology on X defined by the predual Y.*

Proof of Claim. Put $A = w^* - \mathrm{cl}\,\mathrm{co} \cup_{n=1}^\infty T_n$. An easy consideration shows that $w^* - \mathrm{cl} \cup_{n=1}^\infty T_n = \cup_{n=1}^\infty T_n \cup K$ (recall that $w^* - \lim t_n = 0$). Since A is a weak-* compact convex set (in $X = Y^*$), by Milman's theorem $\mathrm{ext}A \subset w^* - \mathrm{cl} \cup_{n=1}^\infty T_n = \cup_{n=1}^\infty T_n \cup K$, so by [8],[5] $A = \mathrm{cl}\,\mathrm{co}\{\cup_{n=1}^\infty T_n \cup K\}$. Since $K \subset B_X \subset W$, it follows that $A \subset V$. Finally, since also W is w^*-compact and convex, we obtain (using [3, Lemma V.2.5])

$$V \subset w^* - \mathrm{cl}\,\mathrm{co}\{W \cup A\} = \mathrm{co}\{W \cup A\} \subset V,$$

hence $V = w^* - \mathrm{cl}\,\mathrm{co}\{W \cup A\}$ is w^*-closed, which finishes the proof of Claim 2.

Since V is a bounded closed convex symmetric subset of X containing B_X, it is easy to show that $|||x||| := \inf\{t > 0 : \frac{1}{t}x \in V\}$ defines an equivalent norm on X with $B_{(X,|||\cdot|||)} = V$. From Claim 2 we get that $|||\cdot|||$ is a dual norm (see, e.g., [19]). Next, by Milman's theorem (in the weak-* topology of X^{**})

$$\mathrm{ext}V^{**} \subset W^{**} \cup \cup_{n=0}^\infty T_n,$$

and since V^{**} is weak-* compact convex, it is the norm-closed convex hull of $\mathrm{ext}V^{**}$ ([8],[5]), and by the choice of F_0 (in particular, $F_0 \in X^\perp$), we have

$$\sup F_0(V^{**}) = \sup F_0(\mathrm{ext}V^{**}) = \sup F_0(W^{**}) = F_0(f_0) = 1.$$

Moreover, f_0 is the only point in $W^{**} \cup \cup_{n=0}^\infty T_n$ where F_0 attains the value 1. Indeed, for W^{**} it was already mentioned above, while for the set $\cup_{n=0}^\infty T_n$ it easily follows from $F_0 \in X^\perp$. Finally assume that $H \in X^{***}$ satisfies $H(f_0) = 1 = \max H(V^{**})$; we prove that $H \in X^\perp$. Fix $y \in K$ and put

$$D = \{af_0 + by :\ a^2 + b^2 \le 1\}.$$

We show that $D \subset T_0$: For $af_0 + by \in D$ we have $a^2 + b^2 \le 1$. Since $K \subset B_X$ is absolutely convex, $y \in K$ and $|b| \le 1$ imply that $x = by \in K$, and $\|y\| \le 1$ yields $a^2 + ||x||^2 \le a^2 + b^2 \le 1$. Therefore $af_0 + by \in T_0 \subset V^{**}$. Thus we have

$$1 = H(f_0) = \max H(D).$$

Let $H(y) = \gamma$. We then have $(\sqrt{1+\gamma^2})^{-1}(f_0 + \gamma y) \in D$ and

$$H\Big(\frac{1}{\sqrt{1+\gamma^2}} f_0 + \frac{\gamma}{\sqrt{1+\gamma^2}} y\Big) = \frac{1}{\sqrt{1+\gamma^2}} + \frac{\gamma^2}{\sqrt{1+\gamma^2}} = \sqrt{1+\gamma^2} \leq 1$$

so we must have $H(y) = 0$. Since $y \in K$ is arbitrary and $\mathrm{cl\,span} K = X$, it follows that $H \in X^\perp$.

When X is not a dual space, we skip Claim 1 and start directly by taking $f \in S_{X^{**}} \setminus X$ with $d(f, X) > 1/3$, and then procced with the same proof, ignoring (the now unnecessary) Claim 2. □

Remark. Clearly, the condition X^{**} is separable may be weakened. For instance, the same proof works, using deeper results, if X^* is separable and does not contain ℓ_1 (use the remark at the end of [15], instead of [8],[5] or [2], for expressing W^{**} and V^{**} as the norm-closed convex hull of their respective extreme points).

Corollary 2. *Let X be a separable quasi-reflexive of order 1 Banach space. Then there exist an equivalent norm $|||\cdot|||$ on X, a functional $f_0 \in S_{(X^{**},|||\cdot|||)} \setminus X$, and a functional $F_0 \in S_{(X^{***},|||\cdot|||)} \cap X^\perp$, such that*

(i) $F_0(f_0) = 1$ and $F_0(g) < 1$ for any $g \in B_{(X^{**},|||\cdot|||)}$, $g \neq f_0$.

(ii) F_0 is the only functional in $S_{(X^{***},|||\cdot|||)}$ with $F_0(f_0) = 1$. Moreover, if there exists a Banach space Y such that $X = Y^*$ (isometrically), then the norm $|||\cdot|||$ can be taken as a dual norm.

Proof. For part (ii), note that $X^{**} = X \oplus [f_0]$ by quasi-reflexivity of order 1. □

4 Mean ergodicity of contractions

In this section we construct non-reflexive separable Banach spaces such that every contraction is mean ergodic (abbreviated ME in the sequel).

We start with a general lemma on mean ergodicity.

Lemma 1. *Let R be a mean ergodic power-bounded operator on a Banach space Y, with ergodic projection $Ey = \lim \frac{1}{n}\sum_{k=1}^n R^k y$. Then $E^*(Y^*) = Fix\ (R^*)$.*

Proof. Since $RE = ER = E = E^2$, we have $R^*E^* = E^*$, which yields $E^*(Y^*) \subset \mathrm{Fix}\ (R^*)$. For the converse, let $y^* \in \mathrm{Fix}\ (R^*)$. Then for $y \in Y$ we have

$$E^*y^*(y) = y^*(Ey) = y^*(\lim_n \frac{1}{n}\sum_{k=1}^n R^k y) = y^*(y).$$

□

Throughout this section, we assume that X is a separable Banach space quasi-reflexive of order 1, endowed with the norm given by Corollary 2, and we use the notations of Corollary 2. We will show that every contraction on X is mean ergodic.

Lemma 2. *Let Q be a norm 1 projection from $(X^{**}, |||\cdot|||)$ onto the one-dimensional subspace spanned by f_0. Then $Qx^{**} = F_0(x^{**})f_0$.*

Proof. Clearly $Qx^{**} = \alpha(x^{**})f_0$, with α linear (by linearity of Q), and $|||\alpha||| = 1$ since $|\alpha(x^{**})| = |||Qx^{**}||| \le \|x^{**}|||$ with equality on $Q(X^{**}) \neq \{0\}$. Now

$$\alpha(f_0)f_0 = Qf_0 = Q^2 f_0 = \alpha^2(f_0)f_0$$

so $\alpha(f_0) = 1$ since $Q \neq 0$. By Corollary 2(ii), $\alpha = F_0$. □

Lemma 3. *Let $T = I+W$ be a contraction in $(X, |||\cdot|||)$ with W weakly compact. Then $f_0 \in Fix\ (T^{**})$.*

Proof. Since W is weakly compact, it follows that $W^{**}f_0 \in X$, and hence $F_0(W^{**}f_0) = 0$. Hence

$$1 \ge |||T^{**}f_0||| = |||f_0 + W^{**}f_0||| \ge F_0(f_0) + F_0(W^{**}f_0) = 1 + F_0(W^{**}f_0) = 1.$$

Therefore $|||f_0 + W^{**}f_0||| = 1$, and we conclude that F_0 attains its norm on $f_0 + W^{**}f_0$. Corollary 2(i) yields that $W^{**}f_0 = 0$, i.e. $f_0 \in \text{Fix}\ (T^{**})$. □

Theorem 1. *Every separable Banach space which is quasi-reflexive of order 1 has an equivalent norm in which any contraction is mean ergodic.*

Proof. Endow X with the norm $|||\cdot|||$ obtained in Corollary 2. Let $T : X \to X$ be a contraction. By Proposition 2 we have to prove only the case $T = I + W$ where W is weakly compact (to which Lemma 3 is applicable). By [6], T or T^* (or both) is ME, so without loss of generality we may assume that T^* is ME. Besides, we can assume that Fix $(T^*) \neq \{0\}$ (otherwise $X = \overline{(I-T)X}$ so T is ME and we are done).

Define

$$Px^* = \lim \frac{1}{n}\sum_{i=1}^{n} T^{*k}x^*, \qquad x^* \in X^*.$$

Clearly, P is a projection onto Fix (T^*), and since Fix $(T^*) \neq \{0\}$ and $|||T||| \le 1$, it follows that $|||P||| = 1$ (i.e. P is not 0). Since T^* is ME we have the following ergodic decomposition

$$X^* = P(X^*) \oplus \overline{(I-T^*)X^*} = F(T^*) \oplus \overline{W^*X^*}.$$

By Lemma 3 $f_0 \in \text{Fix}\ (T^{**})$, so Fix $(T^{**}) = \text{Fix}\ (T) \oplus [f_0]$. Define

$$Qx^{**} = F_0(P^*x^{**})f_0.$$

Then $Qf_0 = f_0$ and $|||Q||| = 1$, $Q^2 = Q$. Lemma 2 yields that $Qx^{**} = F_0(x^{**})f_0$. Hence $\text{Ker}Q = \text{Ker}F_0 = X$ (we use here that X is quasi-reflexive of order 1). Therefore $\text{Ker}P^* \subset X$. By Lemma 1 Fix $(T^{**}) = P^*(X^{**})$, and

$$X^{**} = P^*X^{**} \oplus \text{Ker}P^* = \text{Fix}\ (T^{**}) \oplus \text{Ker}P^* = [f_0] \oplus \text{Fix}\ (T) \oplus \ker P^*,$$

and hence

$$X = \text{Fix}\ (T) \oplus \text{Ker}P^* \tag{2}$$

Mean ergodicity of T^* implies

$$P^*x^{**} = w^* - \lim \frac{1}{n}\sum_{k=1}^{n} T^{**k}x^{**}.$$

If $P^*x^{**} = 0$ then $x^{**} = x \in X$, and we have $\frac{1}{n}\sum_{k=1}^{n} T^k x \to 0$ weakly, hence in norm (e.g. [9, p. 72]). The decomposition (2) yields that T is ME. $\square$

Theorem 2. *Let X be quasi-reflexive of order 1, with $|||\cdot|||$ the norm defined by Corollary 2, and let T be a contraction on $(X, |||\cdot|||)$; then T^* is mean ergodic.*

Proof. By Proposition 2 it remains to prove the theorem only for $T = I + W$ with W weakly compact, which we now assume. From Theorem 1 we know that T is mean ergodic, and denote $Ex := \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k x$. Then E is a projection onto Fix (T), and $E^2 = E = ET = TE$. Hence by Lemma 1 E^* is a projection of X^* onto Fix (T^*), and E^{**} projects X^{**} into Fix (T^{**}), so we have $E^{**}f_0 \in$ Fix (T^{**}). By Lemma 3 $f_0 \in$ Fix (T^{**}), and the decomposition $X^{**} = X \oplus [f_0]$ yields that T^{**} is mean ergodic. The ergodic decomposition of X by mean ergodicity of T yields

$$X^{**} = X \oplus [f_0] = \text{Fix } (T) \oplus \overline{(I-T)X} \oplus [f_0] \tag{3}$$

so Fix $(T^{**}) =$ Fix $(T) \oplus [f_0]$. Hence $E^{**}f_0 = \lambda f_0 + y_0$ with $y_0 \in$ Fix (T). Since $E^{**} \mid X = E$, we have $E^{**}y_0 = y_0 \in X$. The functional F_0 is in $X^{\perp}$, so we have

$$\lambda = F_0(\lambda f_0 + y_0) = F_0(E^{**}f_0) = F_0(E^{**}E^{**}f_0) = F_0(\lambda^2 f_0 + \lambda y_0 + E^{**}y_0) = \lambda^2.$$

Case (i): $\lambda = 1$. In this case $F_0(E^{**}f_0) = 1$, and since $|||f_0||| = 1$ and $|||E||| \le 1$, Corollary 2(i) yields $E^{**}f_0 = f_0$. We have observed that T^{**} is mean ergodic with the decomposition (3). Since $E^{**} \mid X = E$ and $E^{**}f_0 = f_0$, we obtain that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^{**k}x^{**} = E^{**}x^{**}$. Hence for $x^* \in X^*$ we have that for every $x^{**} \in X^{**}$

$$x^{**}\left(\frac{1}{n}\sum_{k=1}^{n} T^{*k}x^*\right) = \left(\frac{1}{n}\sum_{k=1}^{n} T^{**k}x^{**}\right)(x^*) \to E^{**}x^{**}(x^*) = x^{**}(E^*x^*).$$

Thus $\frac{1}{n}\sum_{k=1}^{n} T^{*k}x^*$ converges weakly to E^*x^*, and therefore in norm [9, p. 72]. Hence T^* is mean ergodic.

Case (ii): $\lambda = 0$. In this case we have $E^{**}f_0 \in$ Fix (T), so $E^{**}(\text{Fix } (T^{**})) \subset$ Fix (T) (and equality holds). Let $y^{**} \in$ Fix (T^{**}), and put $y = E^{**}y^{**} \in$ Fix (T). Since (for any contraction on a Banach space) Fix (T^*) separates Fix (T), there exists $y^* \in$ Fix (T^*) such that $y^*(y) \ne 0$. By Lemma 1 $E^*y^* = y^*$, which yields

$$y^{**}(y^*) = y^{**}(E^*y^*) = E^{**}y^{**}(y^*) = y^*(y) \ne 0.$$

Hence Fix (T^*) separates Fix (T^{**}), so by Sine's criterion T^* is mean ergodic. $\square$

Lemma 4. *Let X be a Banach space such that all contractions are mean ergodic. Then every strongly continuous semi-group of contractions $\{T_t\}_{t\ge 0}$ on X is mean ergodic, i.e.*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t T_s x\, ds \quad \textit{exists} \quad \forall x \in X.$$

Proof. For the sake of completeness we indicate the standard proof. For $x \in X$ put $y = \int_0^1 T_s x\, ds$. By the semi-group property $T_k = T_1^k$ and we obtain

$$\frac{1}{t}\int_0^t T_s x\, ds = \frac{[t]}{t} \cdot \frac{1}{[t]} \sum_{k=0}^{[t]-1} T_1^k y + \frac{1}{t}\int_{[t]}^t T_s x\, ds \longrightarrow E(T_1)y$$

by mean ergodicity of the contraction T_1. □

Remark. Mugnolo [14] gave a semi-group analogue of the result of [6].

We can now reinforce our negative answer to Sucheston's question.

Theorem 3. *Every separable Banach space Z which is quasi-reflexive of order 1 has an equivalent norm such that all contractions on Z and all contractions on Z^* (in the induced dual norm) are mean ergodic. Moreover, every strongly continuous semi-group of contractions on Z or on Z^* (in the above norms) is mean ergodic.*

Proof. Let Z be a separable Banach space quasi-reflexive of order 1 and let $X = Z^*$. Then also X is quasi-reflexive of order 1 (e.g. [4, p. 10]), and by the construction of the norm $|||\cdot|||$ on X obtained in Corollary 2, there is an equivalent norm on Z, denoted by $|\cdot|$, for which $(X, |||\cdot|||) = (Z, |\cdot|)^*$. Let R be a contraction on $(Z, |\cdot|)$. Then $T = R^*$ is a contraction on $(X, |||\cdot|||)$, and by Theorem 2 $T^* = R^{**}$ is mean ergodic on $X^* = Z^{**}$. But $R^{**} \mid Z = R$, so the mean ergodicity of R^{**} yields mean ergodicity of R.

If T is a contraction on $(Z, |\cdot|)^* = (X, |||\cdot|||)$, then by Theorem 1 T is mean ergodic on X.

The mean ergodicity of strongly continuous contraction semi-groups follows from the above and the previous lemma. □

Browder [1, Lemma 5] proved that for T power-bounded on a reflexive Banach space Y we have

$$x \in (I-T)Y \qquad \text{if and only if} \qquad \sup_n \Big\| \sum_{k=0}^n T^k x \Big\| < \infty. \tag{4}$$

Lin and Sine [11] showed that (4) holds also for $Y = L_1$ and T any *contraction* (even not mean ergodic), and gave an example of a mean ergodic contraction T on a subspace of L_1 for which (4) fails. Thus (4) and mean ergodicity are incomparable. When (4) holds we say that T *satisfies Browder's condition.*

Lemma 5. *Let T be a power-bounded operator on a Banach space Y. If T^{**} is mean ergodic (on Y^{**}), then (4) holds.*

Proof. Let $\sup_n \| \sum_{k=0}^n T^k x\| < \infty$. Browder's result was extended in [10] (see also [11]) to show that (4) holds when Y is a dual space and T is a dual operator. We apply this to T^{**} and obtain that $x \in (I - T^{**})Y^{**}$. The assumption that T^{**} is mean ergodic yields, by [11, Theorem 1], that a solution $y^{**} \in Y^{**}$ of the equation $(I - T^{**})y^{**} = x$ is given by

$$y^{**} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \sum_{k=0}^{n-1} T^{**k} x = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \sum_{k=0}^{n-1} T^k x\ ,$$

which shows that $y^{**} \in Y$ and $x \in (I-T)Y$. □

Theorem 4. *Let Z be a separable Banach space which is quasi-reflexive of order 1, endowed with the equivalent norm defined in Theorem 3. Then every contraction on Z and every contraction on $X = Z^*$ satisfies Browder's condition.*

Proof. Let T be a contraction on Z. Then T^* is a contraction on X, and by Theorem 2 T^{**} is mean ergodic on $X^* = Z^{**}$. Hence T satisfies Browder's condition by the previous lemma.

Now let T be a contraction on $Z^* = (X, |||\cdot|||)$ and let $x \in X$ satisfy $\sup_n |||\sum_{k=0}^{n} T^k x||| < \infty$; then there exists $y^{**} \in X^{**}$ with $(I - T^{**})y^{**} = x$. Recall (Corollary 1) that $T = \lambda I + W$ with W weakly compact (and $|\lambda| \leq 1$).

Case (i): $\lambda \neq 1$. Since $W^{**}y^{**} = z \in X$, we obtain

$$x = (I - T^{**})y^{**} = (1 - \lambda)y^{**} - W^{**}y^{**} = (1 - \lambda)y^{**} - z\,,$$

which yields $y^{**} = (1 - \lambda)^{-1}(x + z) \in X$, so $x \in (I - T)X$.

Case (ii): $\lambda = 1$. Let $y^{**} = y + \alpha f_0$, with $y \in X$. By Lemma 3 $T^{**}f_0 = f_0$, so $x = (I - T^{**})(y + \alpha f_0) = (I - T)y$. $\square$

5 Problems

It was proved in [6] that if Y is a Banach space such that every power-bounded operator defined on any closed subspace of Y is mean ergodic (on that subspace), then Y is reflexive (no basis assumed). The question (related to another question in [18]) is this: *If Y is a Banach space such that every* contraction *defined on any closed subspace is mean ergodic, is Y reflexive?*

If the answer to the above is negative, what if both Y and Y^* have the above property?

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