ALTERNATING AUTOMATA, THE WEAK MONADIC THEORY OF THE TREE, AND ITS COMPLEXITY

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I. INTRODUCTION

This article continues the point of view that the "natural" theory of automata on trees is that of automata which are alternating in the sense of Muller and Schupp [2]. We recall the basic definitions below. In this article we use alternating automata to give a "direct" automata-theoretic characterization of the languages of k-ary trees which are weakly-definable, that is to say, definable by a formula in the weak monadic logic of the tree where one allows quantification only over finite sets. We define a "weak" acceptance condition and show that a language is weakly definable if and only if it is accepted by an alternating automaton using the weak acceptance condition. Secondly, alternating automata are closely related to complexity and we give a simple proof of a bound on the complexity of deciding formulas whose prenex normal form has n alternations of quantifiers.

The study of automata on infinite trees rests on the fundamental articles of Rabin [3,4]. In [4] Rabin gave an ingeneous characterization of weakly definable languages and our proof uses one direction of his result in an essential way. We thus begin with a discussion of acceptance conditions and explain Rabin's result. In his pioneering work on finite automata accepting infinite words Büchi worked with nondeterministic automata and supposed the acceptance condition to be defined by a subset F of the state set Q. An infinite calculation h of the automaton accepts if h contains some state from F infinitely often. The problem with nondeterministic automata is, of course, complementation, and given a Büchi automaton on infinite words, it is not generaly true that there is a deterministic Büchi automaton which accepts the same language. In order to be able to determinize, one must use the acceptance condition of Muller which is defined by a family $\mathcal{F}$ of subsets of the state set. An infinite calculation h of the automaton accepts if $\mathrm{Inf}(h) \in \mathcal{F}$ where $\mathrm{Inf}(h)$ is the set of states occurring infinitely often in h. McNaughton proved that any regular set of infinite words can be accepted by a deterministic Muller

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automaton. The relationship between Muller acceptance and complementation is not surprising when one notes that the denial of a Büchi acceptance condition is not a condition of the same type while the denial of the Muller condition defined by a family F is simply the condition defined by the complementary family F.

Rabin [3] showed that it was necessary to use Muller acceptance when considering automata on trees. In the tree case one cannot determinize and a simple solution for complementation requires alternating automata. Nonetheless, automata using the Büchi acceptance condition are used by Rabin [4] to characterize the weakly definable languages. Rabin calls such automata <u>special</u> but we shall call then Büchi and we say that a language is <u>Büchi</u> if it is accepted by a Büchi automaton. Rabin proves that a language L is weakly definable if and only if both L and $\bar{L}$ are Büchi. There are several characterizations of this general character in logic and set theory, ranging from the basic fact that a set S of natural numbers is recursive if and only if both S and $\bar{S}$ are recursively enumerable, to the considerably less evident fact that a set X of real numbers is Borel if and only if both X and $\bar{X}$ are analytic.

We shall consider a "weak acceptance condition" which would be extremely weak for nondeterministic automata. We shall consider alternating automata whose state set is written as a disjoint union $Q = \dot{\bigcup} Q_i$ and we suppose that there is a partial ordering on the collection of the Q_i. Furthermore, we suppose that the transition function is such that given a $q \in Q_i$, then if q' is any state occurring in $\delta_a(q)$ then $q' \in Q_j$ where $Q_j \leq Q_i$. Thus if h is an infinite individual history, from some point onwàrd all the states in h belong to the same set Q_i. We say that $Q_i = f(h)$ is the <u>finality</u> of h. We suppose that each Q_i is designated as accepting or rejecting. The history h is accepting if $f(h)$ is an accepting set. For alternating automata, weak acceptance gives exactly weak definability.

<u>Theorem I</u> Let L be a language of k-ary trees labelled from an alphabet Σ. Then L is weakly definable if and only if L is accepted by an alternating automaton using the weak acceptance condition.

We shall simply say that an alternating automaton using the weak acceptance condition is a weak alternating automaton. This article has the following plan. We first recall the basic definitions about alternating automata from [2] and the complementation theorem, which remains valid for weak acceptance : if M accepts a language L then the dual automaton $\tilde{M}$ accepts $\bar{L}$. The class of languages accepted by weak alternating automata is thus closed under complementation. The proof of the theorem then proceeds in two steps. We first show that the class of languages accepted by weak alternating automata includes all weakly definable languages. Since we already have closure under complementation it suffices to establish closure under finite quantification and this is an easy lemma. We next show that a weak alternating automaton M can be simulated by a Büchi automaton N. Since we have closure under complementation, if L is accepted by a weak alternating automaton then both L and $\bar{L}$ are Büchi and L is thus weakly definable by Rabin's theorem. Note that we have used only one direction of Rabin's theorem and this direction is, in fact, the one with the shorter, more conceptual proof. Most of the space in Rabin's article [4] is devoted to showing that if L is weakly definable then both L and $\bar{L}$ are indeed Büchi, but this fact is a consequence of the two lemmas cited above. We view this as strengthering our contention that it is simply much easier to calculate with alternating automata. We then consider the complexity of deciding the truth of formulas with n alternations of quantifiers.

II. WEAK ALTERNATING AUTOMATA ON THE TREE

We review our conception of alternating automata as given in [2]. In Rabin's theory of nondeterministic automata on the binary tree, a single copy of the automaton begins in its initial state at the root of the tree. The automaton then splits into two copies, one moving to the left successor and the other moving to the right successor. The states of the two copies are given by a nondeterministic choice from the possibilities allowed by the transition function. In Rabin's notation, if the automaton is in state q_0 reading the letter a, the value of the transition function for (q_0, a) might be $((q_1, q_2)\ (q_0, q_3))$ where the left (right) member of a pair denotes the next state of the automaton moving to the left (right) successor vertex. We can represent this situation in our lattice formulation by using the free distributive lattice $L(\{0,1\} \times Q)$ generated by all the possible pairs (direction, state). Namely, we write :

(where, as usual, $\wedge$ has precedence over $\vee$).

$$\delta_a(q_0) = (0, q_1) \wedge (1, q_2) \quad \vee \quad (0, q_0) \wedge (1, q_3)$$

We interpret this expression as saying that the automaton has the choice of splitting into one copy in state q_1 going to the left successor and one copy in state q_2 going to the right successor or of splitting into one copy in state q_0 going to the left and one copy in state q_3 going to the right. We note that both "and" and "or" are present in the conception of an automaton on the binary tree.

In the general case of an alternating automaton we allow $\delta_a(q)$ to be an arbitrary element of the free distribution lattice $L(\{0, 1\} \times Q)$. For example, the dual of the expression above is :

$$\tilde{\delta}(q_0) = (0, q_1) \wedge (0, q_0) \vee (0, q_1) \wedge (1, q_3) \vee (0, q_0) \wedge (1, q_2) \vee (1, q_2) \wedge (1, q_3).$$

This expression illustrates that we do not require the automaton to send copies in all directions (although at least one copy must go in some direction) and that several copies may go in the same direction. One may think of an alternating automaton as a sort of completion of a nondeterministic automaton. It is only by going to $L(\{0, 1\} \times Q)$ that one can always calculate the dual of a given expression.

We review our conventions of the k-ary tree T_K viewed as a structure. The vertex set of T_K is the set K^* of all words on the direction alphabet $K = \{0, \ldots, k-1\}$, with the empty word being the origin of the tree. Given a vertex v and a letter $d \in K$ there is an edge e with label d from v to vd and v is the d-successor of v. The level $|v|$ of a vertex v is thus the length of v as a word. We thus think of the edges in T_K as being labelled by letters from K while the vertices are unlabelled.

Definition 1 A weak alternating automaton on K-ary Σ-trees is a tuple

$$M = \langle L(K \times Q), \Sigma, \delta, q_0, F \rangle$$

where K is a set of directions, the state set Q is written as a disjoint union $Q = \cup Q_i$ where there is a partial order $\geq$ on the collection of the Q_i, the set Σ is the input alphabet, the transition function

$$\delta : \Sigma \times Q \rightarrow L(K \times Q)$$

has the property that if $q \in Q_i$ and q' occurs in the expression $\delta(a,q)$ then $q' \in Q_j$ where $Q_j \leq Q_i$. The <u>final family</u> F is a list of these Q_i considered to be accepting. The <u>dual</u> of M is

$$\tilde{M} = \langle L(K \times Q),\ \Sigma,\ \tilde{\delta},\ q_0,\ \bar{F}\rangle$$

obtained by dualizing the transition function by interchanging $\wedge$ and $\vee$ as usual and taking the compliment $\bar{F}$ of F.

The reader should consult [2] for complete details but the only result from [2] which we use in this article is the fact that acceptance of the complementary language by the dual automaton is an expression of a combinatorial relation between a machine and its dual and is independent of the acceptance condition, therefore remaining valid for weal alternating automata.

<u>Complementation Theorem</u>. Let M be a weak alternating automaton and let $L(M)$ be the language accepted by M. Then the dual automaton $\tilde{M}$ accepts the complement $L(M)$.

<u>Definition 2</u>. Let Δ and Σ be alphabets with $\Delta \supseteq \Sigma$ and let $\eta : \Delta \rightarrow \Sigma$ be a function which is the identity function on Σ. Let L be a language of k-ary trees labelled from Σ. We define the language $\eta_f(L)$ on the alphabet Σ which is obtained from L and η by <u>finite projection</u>. A tree t' belongs to $\eta_f(L)$ if there exists a tree $t \in L$ containing only <u>finitely</u> many vertices labelled from $\Delta - \Sigma$ and such that $t' = \eta(t)$ where, as usual, $\eta(t)$ denotes the result of replacing the label of each vetex of t by its image under η.

<u>Lemma 1</u> The class of languages accepted by weak alternating automata is closed under finite projection.

Proof Given a weak alternating automaton $M = \langle L(K \times Q), \Delta, \delta, q_0, F\rangle$ we can construct M' accepting $\eta_f(L(M))$ as follows. Since the weak acceptance condition is independent of past history, in order to simulate the behaviour of M up to a finite distance it is necessary to keep track of all the possible copies running in M only up to the information of current state. We think that two machines having the same information "merge" and since M is weak, one has not last any information.

The automaton M' begins in a nondeterministic mode where, at a given vertex, it keeps track of a possibility for an existing collection of machines in M up to the information of current state. Thus M' requires a state set of the form $P(Q)$ for its nondeterministic mode. For $S \in P(Q)$ and a letter a being read, M' chooses a preimage in $\eta^{-1}(a)$, a set of choices of the copies of M represented in S on this preimage, and sends in each direction d the collection S_d resulting from S and the choices made.

At any vertex M' also has the choice of guessing that it will not see any more letters form $\Delta-\Sigma$ in the subtree beginning at the vertex. If M' makes this choice it enters its alternating mode where it simply simulates M in an alternating fashion but will go into a special rejecting state q_r if it sees a letter of $\Delta-\Sigma$. In order to do this, M has a copy of Q which is disjoint from $P(Q)$. The ordering is $P(Q) > Q_i$ for each Q_i in the decomposition of Q in M. The transition form a "nondeterministic" state $S \neq \emptyset$ to the alternating mode is from S to $\bigwedge q_i$ where $q_i \in S$. M also has a special state $q_\emptyset$ indicating the absence of any copy of M and the transition form $\emptyset$ is to $q_\emptyset$. The transition function of M' on a state $q \in Q$ is exactly the same as that of M on letters from Σ but M' goes into a special rejecting state q_r on any letter form $\Delta-\Sigma$. A copy in $q_\emptyset$ or in q_r always stays in that state in every direction.

The ordering on the special states is $Q_i > \{q_r\}$ for each Q_i and $P(Q) > \{q_\emptyset\}$. The final family F' of M' consists of the final family F of M together with $\{q_\emptyset\}$. Note that $P(Q)$ is rejecting. Thus that an individual history h in M' has $f(h) \in F'$ requires that M' has made the transition to the alternating mode and that the simulation of M is accepting, and that no letters from $\Delta - \Sigma$ are encountered in the alternating mode. The König Infinity lemma assures that since M' has guessed on every branch the total subtree covered in the nondeterministic mode is indeed finite. We note that the construction of M' is really simply the subset construction. □

Lemma 2 A weak alternating automaton can be simulated by a Büchi automaton.

Proof Let $M = \langle L(K \times Q), \Sigma, \delta, q_0, F\rangle$ be a weak alternating automaton. We can construct a Büchi automaton B simultating M as follows. As in the proof of Lemma 1, B uses a copy of $P(Q)$ to keep track of the possibilities of machines running in M up to the information of current state. But we must now test that all the individual histories of machines in M are accepting. Let $\bar{F} = \{Q_0, \ldots, Q_{r-1}\}$ be a consecutive list of those sets in the decomposition of Q which are not accepting. (This indexing has nothing to do with the partial order on the Q_i). Let Z_r be a copy of the integers modulo r. An individual history h is rejecting if and only if it eventually arrives in some set $Q_i \in \bar{F}$ and there after remains in Q_i. Let $V = \{S : S \subseteq Q_i, 0 \leq i \leq r-1\}$ be the collection of subsets of the Q_i occurring in $\bar{F}$. The state set of B is $P(Q) \times Z_r \times V$. The $P(Q)$-component is the simulation track, the Z_r-component is the testing-index and the V-component is the testing-track. The initial state of B is $(\{q_0\}, 0, \emptyset)$. As in the previous lemma, if a copy of B reads a letter a at a vertex and S is the first component of the current state, then B selects a set of possible choices for the copies of M represented in S and, in each direction d, puts the collection S_d, resulting from S and the choices made, in the simulation track.

Suppose that the testing track contains $\emptyset$. In this case, B advances the testing index (modulo r) by one, say to i, and, for each direction d, puts $S_d \cap Q_i$ in the testing track.

Suppose now that the testing index is i and that the testing track contains a non-empty set $C \subseteq S$. In each direction d, B puts in the testing track the set C_d which consists of those states in $S_d \cap Q_i$ which arise from a state in C. Thus C_d records these copies in C which remain in a state from Q_i according to the selection of choices made for the simulation track. (Note that states not in C may give rise to states in Q_i but these are not recorded in C_d). If $C_d = \emptyset$ the testing track is discharged. The acceptance condition for B is that one encounters $\emptyset$ infinitely often in the testing track. This condition exactly prevents an infinite history of a copy of M from forever remaining in a rejecting set Q_i. Thus B accepts an input if and only if M does. □

III. - THE COMPLEXITY OF WEAK MONADIC THEORIES

Alternating automata give a very simple proof of an (n+1)-exponential complexity bound on deciding the class F_n of formulas of the weak monadic theory which are in prenex normal form and which have n alternations of quantifiers. The natural object of study here is, in fact, the monadic theory of the k-ary tree T_K together with an arbitrary partition $\Pi = \{S_1, \ldots, S_p\}$ of T_K into finitely many subsets. We suppose that the monadic language contains only set variables, the set constants $S_1, \ldots, S_p$, the constant v_o denoting the set whose only element is the origin, for each $d \in K$, the set-valued successor function σ_d, yielding for a subset $X \subseteq T_K$ the set $X\sigma_d = \{xd \; ; \; x \in X\}$ of d-successors of elements of X, and the relation $\subseteq$ of set inclusion and the unary relation $|X| = 1$. (Compare Muller and Schupp [1]).

The idea of an automaton M (of any type whatever) with an oracle for Π is very simple. The automaton M has a collection $\{\delta_i\}$ of transition functions and M applies δ_i at exactly those points belonging to the set s_i of the partition. For alternating automata, the dual automaton has the collection $\{\widetilde{\delta}_i\}$. The lemmas which we have proved and the complementation theorem relativize immediately to the case of automata with oracles.

Since for weak alternating automata, complementation costs no states while a block of existential quantifiers costs one exponential, the automaton M_Φ associated with a formula in $F(n,\Pi)$, the sentences of $WMT(T_K,\Pi)$ in prenex normal form with n alternations of quantifiers, have a state set whose size is n exponentials. One application of Lemma 2 converts to a Büchi automaton. The emptiness problem for Büchi automata without oracle is polynomial by Rabin [4]. Thus we have

Theorem II. Let $\Pi = \{S_1, \ldots, S_p\}$ be an arbitrary partition of T_K. Let $F(n,\Pi)$ be the set of formulas of $WMT(T_K,\Pi)$ having n alternations of quantifiers in prenex normal form. There is an (n+1)-exponential reduction of deciding $F(n,\Pi)$ to the emptiness problem for Büchi automata with an oracle for Π.

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