Rachunek lambda - ciąg dalszy

18 marca 2013

Siła wyrazu: logika zdaniowa

$$\textbf{true} = \lambda xy.x \qquad\qquad \textbf{false} = \lambda xy.y$$

$$\textbf{if } P \textbf{ then } Q \textbf{ else } R = PQR.$$

It works:

$$\textbf{if true then } Q \textbf{ else } R \twoheadrightarrow_\beta Q$$

$$\textbf{if false then } Q \textbf{ else } R \twoheadrightarrow_\beta R.$$

○

Ordered pair

Pair = Boolean selector:

$$
\begin{aligned}
\langle M, N \rangle &= \lambda x. x M N; \\
\pi_i &= \lambda x_1 x_2 . x_i \qquad (i = 1, 2); \\
\Pi_i &= \lambda p.\, p \pi_i \qquad (i = 1, 2).
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It works:

$$\Pi_1\langle M, N\rangle \to_\beta \langle M, N\rangle\pi_1 \twoheadrightarrow_\beta M.$$

○

Church's numerals

$$c_n = \mathbf{n} = \lambda fx.f^n(x),$$

$$\begin{aligned}
\mathbf{0} &= \lambda fx.x; \\
\mathbf{1} &= \lambda fx.fx; \\
\mathbf{2} &= \lambda fx.f(fx); \\
\mathbf{3} &= \lambda fx.f(f(fx)), \text{ etc.}
\end{aligned}$$

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- Test for zero: $\mathbf{zero} = \lambda m.m(\lambda y.\mathbf{false})\mathbf{true}$;

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and so on.

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- Given M and N, does $M =_\beta N$ hold?
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The standard theory

Adding equational axioms

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$$M = \mathbf{SI}(\mathbf{K}M)\mathbf{I} = \mathbf{KI}(\mathbf{K}M)\mathbf{I} = \mathbf{I}.$$

This extension is inconsistent.

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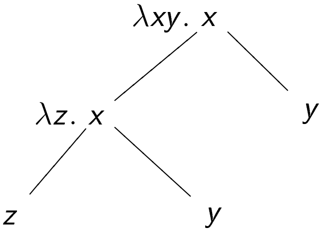
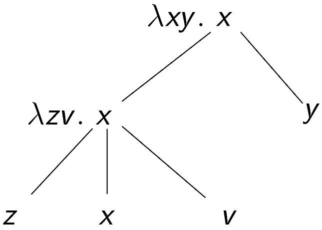
Böhm Theorem

*Let M, N be β-normal combinators with $M \neq_{\beta\eta} N$. Then $M\vec{P} =_\beta$ **true** and $N\vec{P} =_\beta$ **false**, for some $\vec{P}$.*

○

Böhm Trees (finite case)





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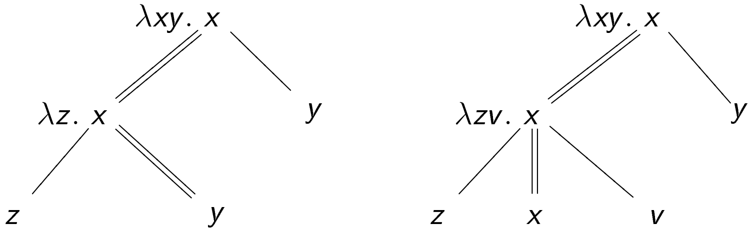
Böhm Trees: the difference



$M = \lambda xy.x(\lambda z.xzy)y$ $\qquad$ $N = \lambda xy.x(\lambda zv.xzxv)y$

Trick: Applying M to $\lambda uv.\langle u, v\rangle$ gives $\lambda y.\langle \lambda z.\langle z, y\rangle, y\rangle$.
And components can be extracted from a pair.

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Choose $Q = \lambda uvw.$ **true** and apply both sides to **false**, **I**, **true**:

true $\qquad$ **false**.

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Moral: A term without normal form can still behave in a well-defined way. In a sense it has a „value”.

Better idea: *Value = Head normal form.*
Undefined = without head normal form.

Solvability

A closed term is *solvable* iff $M\vec{P} =_\beta \mathbf{I}$, for some closed $\vec{P}$.

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Proof for closed terms:

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($\Leftarrow$) If $M =_\beta \lambda x_1 x_2 \ldots x_n.x_i R_1 \ldots R_m$ then $MP \ldots P = \mathbf{I}$, for $P = \lambda y_1 \ldots y_m.\mathbf{I}$.

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We can only *observe* their behaviour.

○

Observational equivalence

Terms M, N with $\mathsf{FV}(M) \cup \mathsf{FV}(N) = \vec{x}$, are *observationally equivalent* ($M \equiv N$) when, for all closed P:

$$P(\lambda\vec{x}.M) \text{ is solvable} \quad \Longleftrightarrow \quad P(\lambda\vec{x}.N) \text{ is solvable}$$

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Put it differently:

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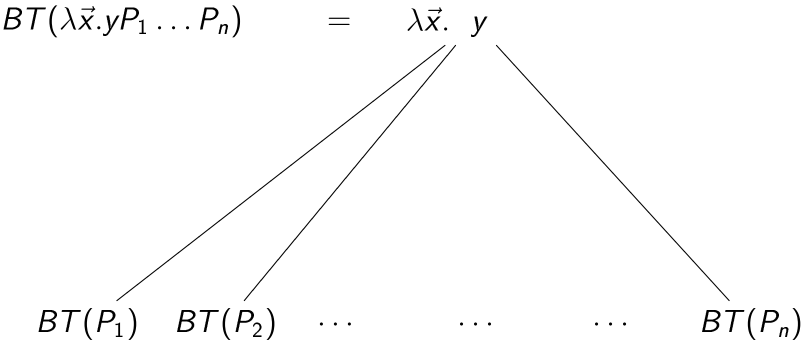
Note: If $M =_\eta N$ then $M \equiv N$.

Böhm Trees



If M has a hnf N then $BT(M) = BT(N)$.

If M is unsolvable then $BT(M) = \bot$.

○

Example: $\mathbf{J} = \mathbf{Y}(\lambda fxy.\, x(fy))$

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The tree $BT(\mathbf{J})$ consists of one infinite path:

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○

When are terms observationally equivalent?

Böhm trees B i B' are *η-equivalent* ($B \approx_\eta B'$), if there are two (possibly infinite) sequences of η-expansions:

$$B = B_0 \;{}_\eta\!\leftarrow B_1 \;{}_\eta\!\leftarrow B_2 \;{}_\eta\!\leftarrow B_3 \;{}_\eta\!\leftarrow \cdots$$

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Theorem

Terms M and N are observationally equivalent if and only if $BT(M) \approx_\eta BT(N)$.

Semantics

Goal: Interpret any term M as an element $[\![M]\!]$ of some structure A, so that $M =_\beta N$ implies $[\![M]\!] = [\![N]\!]$.

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More precisely, $[\![M]\!]$ may depend on a *valuation*:

$$v : Var \rightarrow A.$$

Write $[\![M]\!]_v$, for the value of M under v.

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Write $[\![M]\!]_v$ instead of $[\![\ ]\!](M, v)$.

Postulates:

(a) $[\![x]\!]_v = v(x)$;

(b) $[\![PQ]\!]_v = [\![P]\!]_v \cdot [\![Q]\!]_v$;

(c) $[\![\lambda x.P]\!]_v \cdot a = [\![P]\!]_{v[x \mapsto a]}$, for any $a \in A$;

(d) If $v|_{\mathrm{FV}(P)} = u|_{\mathrm{FV}(P)}$, then $[\![P]\!]_v = [\![P]\!]_u$.

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Weakly extensional interpretation:

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Meaning: Abstraction makes sense algebraically.

(N.B. $[\![\lambda x.M]\!]_v \approx [\![\lambda x.N]\!]_v$ iff $[\![M]\!]_{v[x \mapsto a]} = [\![N]\!]_{v[x \mapsto a]}$, all a.)

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Lambda-model

Lambda-model: Weakly extensional lambda-interpretation:

$$[\![\lambda x.M]\!]_v \approx [\![\lambda x.N]\!]_v \quad \text{implies} \quad [\![\lambda x.M]\!]_v = [\![\lambda x.N]\!]_v$$

$\circ$

Very Important Lemma

Lemma

In every lambda-model,

$$[\![M[x := N]]\!]_v = [\![M]\!]_{v[x \mapsto [\![N]\!]_v]}.$$

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Proof: Induction wrt M. Case of λ with $x \notin \mathrm{FV}(N)$.

$[\![(\lambda y\, P)[x := N]]\!]_{v[x \mapsto [\![N]\!]_v]} \cdot a = [\![\lambda y . P[x := N]]\!]_v \cdot a$

$= [\![P[x := N]]\!]_{v[y \mapsto a]} = [\![P]\!]_{v[y \mapsto a][x \mapsto [\![N]\!]_{v[y \mapsto a]}]}$

$= [\![P]\!]_{v[y \mapsto a][x \mapsto [\![N]\!]_v]} = [\![\lambda y . P]\!]_{v[x \mapsto [\![N]\!]_v]} \cdot a$, for all a.

Therefore $[\![(\lambda y\, P)[x := N]]\!]_{v[x \mapsto [\![N]\!]_v]} = [\![(\lambda y . P)]\!]_{v[x \mapsto [\![N]\!]_v]}$.

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Soundness

Proposition

Every lambda-model is a "lambda-algebra":

$$M =_\beta N \quad \textit{implies} \quad [\![M]\!]_v = [\![N]\!]_v$$

Proof: Induction wrt $M =_\beta N$. Non-immediate cases are two:

(Beta)

$$[\![(\lambda x.P)Q]\!]_v = [\![\lambda x.P]\!]_v \cdot [\![Q]\!]_v = [\![P]\!]_{v[x \mapsto [\![Q]\!]_v]} = [\![P[x := Q]]\!]_v.$$

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(Xi)

Let $P =_\beta Q$ and let $M = \lambda x.P$, $N = \lambda x.Q$. Then

$[\![M]\!]_v \cdot a = [\![P]\!]_{v[x \mapsto a]} = [\![Q]\!]_{v[x \mapsto a]} = [\![N]\!]_v \cdot a$, for all a.

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Completeness

Theorem

The following are equivalent:

1) $M =_\beta N$;
2) $\mathcal{A} \models M = N$, *for every lambda-model* $\mathcal{A}$.

Proof.

(1)$\Rightarrow$(2) By soundness.

(2)$\Rightarrow$(1) Because term model is a lambda-model. □

Complete partial orders

Let $\langle A, \leq \rangle$ be a partial order.

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It follows that every cpo has a least element $\bot = \sup \varnothing$.

Complete partial orders

Let $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ be cpos, and $f : A \rightarrow B$.

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$[A \to B]$ is the set of all continuous functions from A to B

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Complete partial orders

If $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ are cpos then:

- The product $A \times B$ is a cpo with
  $$\langle a, b \rangle \leq \langle a', b' \rangle \text{ iff } a \leq a' \text{ and } b \leq b'.$$
- The function space $[A \rightarrow B]$ is a cpo with
  $$f \leq g \text{ iff } \forall a.\, f(a) \leq g(a).$$

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Continuous functions

Lemma

A function $f : A \times B \to C$ *is continuous iff it is continuous wrt both arguments, i.e. all functions of the form* $\lambda a.\, f(a, b)$ *and* $\lambda b.\, f(a, b)$ *are continuous.*

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Step 2: Therefore $\sup X = \langle \sup X_1, \sup X_2 \rangle = \langle a_0, b_0 \rangle$.
We show that $\langle f(a_0), f(b_0) \rangle$ is the supremum of $f(X)$.

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We show that $\langle f(a_0), f(b_0)\rangle$ is the supremum of $f(X)$.
Let $c \geq f(X)$, then $c \geq f\langle a, b\rangle$ for all $\langle a, b\rangle \in X_1 \times X_2$.
Fix a, to get $c \geq \sup_b f(a, b) = f(a, b_0)$.
Fix b_0, to get $c \geq \sup_a f(a, b_0) = f(a_0, b_0)$. $\square$

Continuous functions

Lemma

The application $App : [A \to B] \times A \to B$ *is continuous.*

Proof: Uses the previous lemma.

Lemma

The abstraction $Abs : [(A \times B) \to C] \to [A \to [B \to C]]$, *given by* $Abs(F)(a)(b) = F(a, b)$, *is continuous.*

Reflexive cpo

The cpo D is *reflexive* iff there are continuous functions $F : D \to [D \to D]$ and $G : [D \to D] \to D$,

with $F \circ G = \mathrm{id}_{[D \to D]}$.

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Then F must be onto and G is injective.

The following are equivalent conditions:

"$G \circ F = \mathrm{id}_D$", "$G$ onto", "F injective".

∘

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Define interpretation as

- $[\![x]\!]_v = v(x)$;
- $[\![PQ]\!]_v = [\![P]\!]_v \cdot [\![Q]\!]_v$;
- $[\![\lambda x.P]\!]_v = G(\lambda\!\!\lambda a.[\![P]\!]_{v[x \mapsto a]})$.

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Fact: This is a (well-defined) lambda interpretation.
(Use continuity of *App* and *Abs*.)

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Reflexive cpo

Theorem

A reflexive cpo is a lambda-model.

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Proof.

Prove weak extensionality: let $[\![\lambda x.M]\!]_v \cdot a = [\![\lambda x.N]\!]_v \cdot a$, all a. Note that $[\![\lambda x.M]\!]_v \cdot a = G(\lambda\!\!\lambda a.[\![M]\!]_{v[x\mapsto a]}) \cdot a = [\![M]\!]_{v[x\mapsto a]}$, and thus $\lambda\!\!\lambda a.[\![M]\!]_{v[x\mapsto a]} = \lambda\!\!\lambda a.[\![N]\!]_{v[x\mapsto a]}$. By the injectivity of G, it follows that $[\![\lambda x.M]\!]_v = [\![\lambda x.N]\!]_v$. □