Institutions

Tuning up the logical system

- various sets of formulae (Horn-clauses, first-order, higher-order, modal formulae, ...)

- various notions of algebra (partial algebras, relational structures, error algebras, Kripke structures, ...)

- various notions of signature (order-sorted, error, higher-order signatures, sets of propositional variables, ...)

- (various notions of signature morphisms)

> *No best logic for everything*

Solution:

> *Work with an arbitrary logical system*

Institutions

Abstract model theory
for specification and programming

Goguen & Burstall: $1980 \to 1992$

- a standard formalization of the concept of the underlying logical system for specification formalisms and most work on foundations of software specification and development from algebraic perspective;

- a formalization of the concept of a logical system for foundational studies:

  – truly abstract model theory

  – proof-theoretic considerations

  – building complex logical systems

Some institutional topics

- **Institutions: intuitions and motivations**

    Goguen & Burstall $\sim$1980 $\to$ 1992

- **Very abstract model theory**

    Tarlecki $\sim$1986, Diaconescu *et al* $\sim$2003 $\to$ ...

- **Structured specifications**

    Clear $\sim$1980, Sannella & Tarlecki $\sim$1984 $\to$ ..., Casl $\sim$2004

    for Casl see: LNCS 2900 & 2960

- **Moving between institutions**

    Goguen & Burstall $\sim$1983 $\to$ 1992, Tarlecki $\sim$1986, 1996, Goguen & Rosu $\sim$2002

- **Heterogeneous specifications**

    Sannella & Tarlecki $\sim$1988, Tarlecki $\sim$2000 $\to$ ..., Mossakowski $\sim$2002 $\to$ ...

    ...to be continued by **Till Mossakowski** (Hets)

    > ...apologies for missing some names and for inaccurate years...
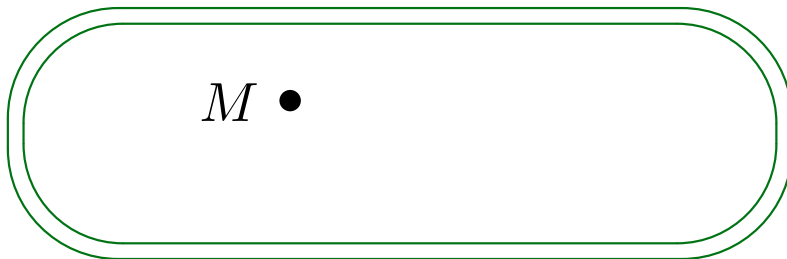
Institution: abstraction

Sen

$\varphi \bullet$

Mod

$M \bullet$
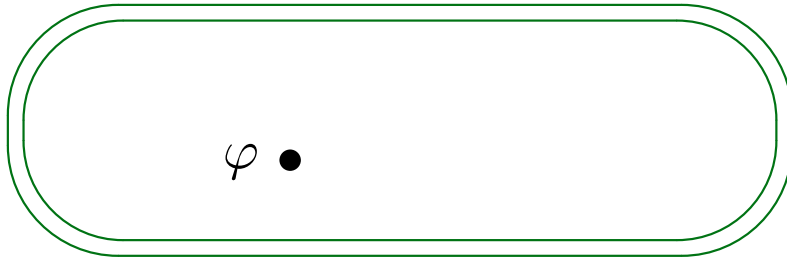
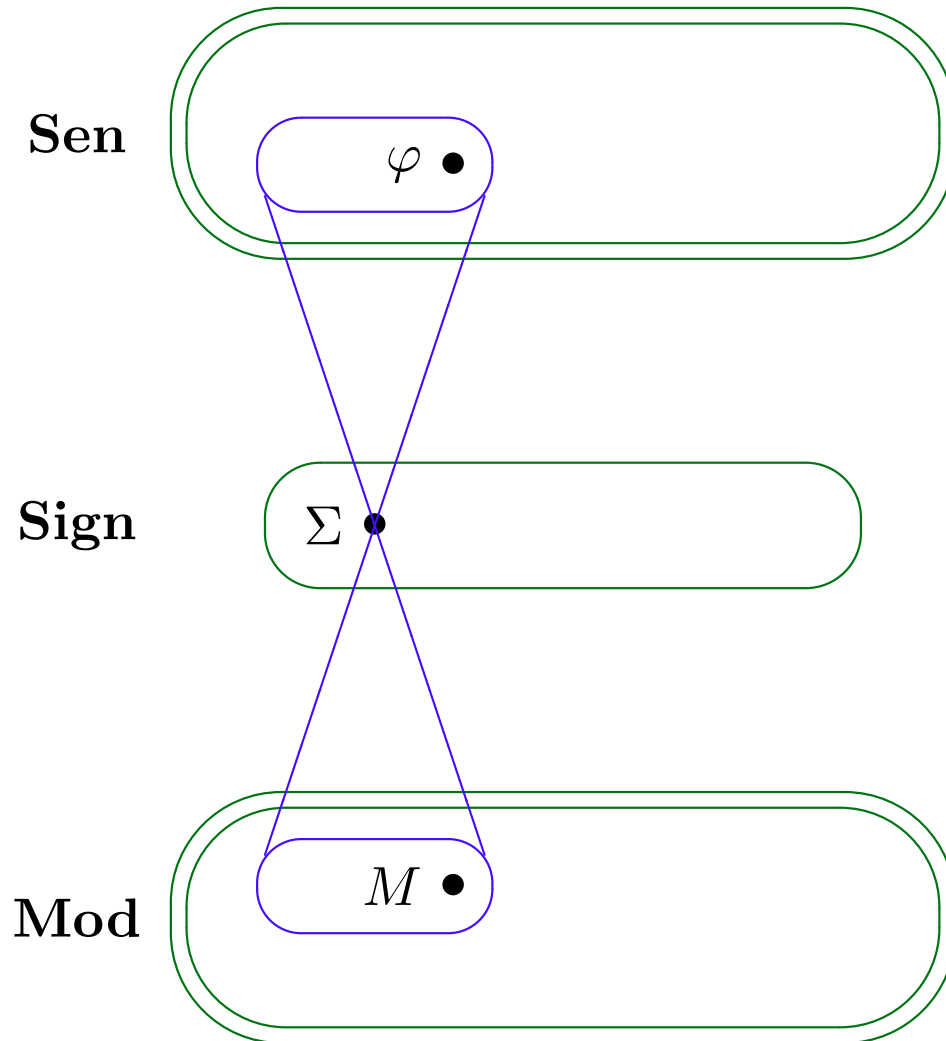plus *satisfaction relation:*

$$\boxed{M \models \varphi}$$

and so the usual Galois connection between classes of models and sets of sentences, with the standard notions induced ($Mod(\Phi)$, $Th(\mathcal{M})$, $Th(\Phi)$, $\Phi \models \varphi$, etc).

- Also, possibly adding (sound) consequence: $\Phi \vdash \varphi$ (implying $\Phi \models \varphi$) to deal with proof-theoretic aspects.
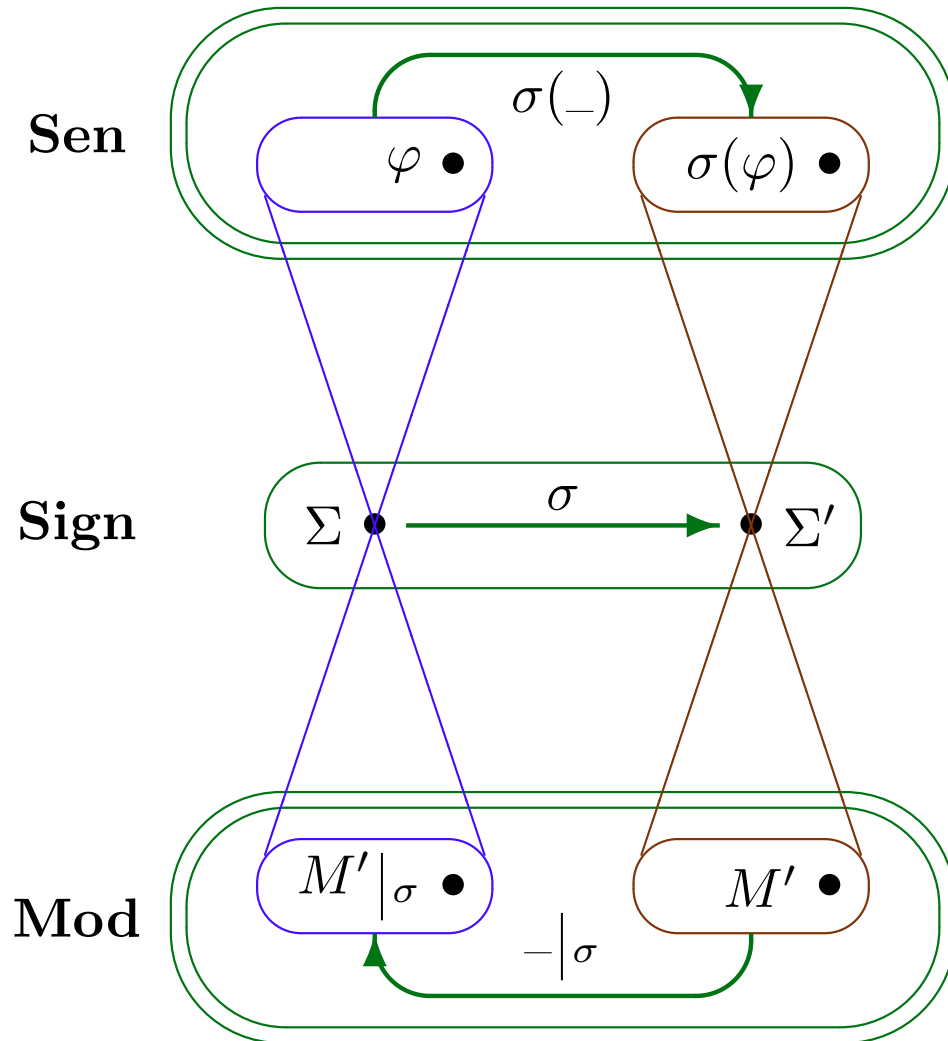
Institution: first insight

Sen

Sign

Mod

$\varphi \bullet$

$\Sigma \bullet$

$M \bullet$

plus *satisfaction relation:*

$$\boxed{M \models_\Sigma \varphi}$$

and so, for each signature, the usual Galois connection between classes of models and sets of sentences, with the standard notions induced ($Mod_\Sigma(\Phi)$, $Th_\Sigma(\mathcal{M})$, $Th_\Sigma(\Phi)$, $\Phi \models_\Sigma \varphi$, etc).

- Also, possibly adding (sound) consequence: $\Phi \vdash_\Sigma \varphi$ (implying $\Phi \models_\Sigma \varphi$) to deal with proof-theoretic aspects.

Institution: key insight

Sen

$\sigma(\_)$

$\varphi \bullet$ $\sigma(\varphi) \bullet$

Sign

$\Sigma \bullet$ $\xrightarrow{\sigma}$ $\bullet \Sigma'$

Mod

$M'|_\sigma \bullet$ $M' \bullet$

$\_|_\sigma$

imposing the *satisfaction condition:*

$$M' \models_{\Sigma'} \sigma(\varphi) \ \text{ iff } \ M'|_\sigma \models_\Sigma \varphi$$

*Truth is invariant
under change of notation*

*and independent of
any additional symbols around*

Institution

- a category **Sign** of *signatures*

- a functor $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$

  - $\mathbf{Sen}(\Sigma)$ is the set of Σ-*sentences*, for $\Sigma \in |\mathbf{Sign}|$

- a functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$

  - $\mathbf{Mod}(\Sigma)$ is the category of Σ-*models*, for $\Sigma \in |\mathbf{Sign}|$

- for each $\Sigma \in |\mathbf{Sign}|$, Σ-*satisfaction relation* $\models_\Sigma \,\subseteq\, |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$

subject to the *satisfaction condition*:

$$\boxed{\; M'|_\sigma \models_\Sigma \varphi \iff M' \models_{\Sigma'} \sigma(\varphi) \;}$$

where $\sigma\colon \Sigma \to \Sigma'$ in $\mathbf{Sign}$, $M' \in |\mathbf{Mod}(\Sigma')|$, $\varphi \in \mathbf{Sen}(\Sigma)$, $M'|_\sigma$ stands for $\mathbf{Mod}(\sigma)(M')$, and $\sigma(\varphi)$ for $\mathbf{Sen}(\sigma)(\varphi)$.

Typical institutions

- **EQ** — equational logic

- **FOEQ** — first-order logic (with predicates and equality)

- **PEQ**, **PFOEQ** — as above, but with partial operations

- **HOL** — higher-order logic

- logics of constraints (fitted via signature morphisms)

- **CASL** — the logic of CASL: partial first-order logic with equality, predicates, generation constraints, and subsorting

> CASL subsorting: the sets of sorts in signatures are *pre-ordered*;
> in every model M, $s \leq s'$ yields an injective *subsort embedding* (*coercion*) $em_M^{s \leq s'} : |M|_s \to |M|_{s'}$ such that $em_M^{s \leq s} = id_{|M|_s}$ for each sort s, and $em_M^{s \leq s'} ; em_M^{s' \leq s''} = em_M^{s \leq s''}$, for $s \leq s' \leq s''$; plus partial projections and subsort membership predicates derived from the embeddings.

Somewhat less typical institutions:

- modal logics

- three-valued logics

- programming language semantics:

  - **IMP**: imperative programming language with sets of computations as models and procedure declararions as sentences

  - **FPL**: functional programming language with partial algebras as models and the usual axioms with extended term syntax allowing for local recursive function definitions

Temporal logic

Institution **TL**:

- signatures $\mathcal{A}$: (finite) sets of *actions*;

- models $\mathcal{R}$: sets of *runs*, finite or infinite sequences of (sets of) actions;

- sentences φ: built from atomic statements a (action $a \in \mathcal{A}$ happens) using the usual propositional and temporal connectives, including $\mathbf{X}\varphi$ (an action happens and then φ holds) and $\varphi\mathbf{U}\psi$ (φ holds until ψ holds)

- satisfaction $\mathcal{R} \models \varphi$: φ holds at the beginning of every run in $\mathcal{R}$

WATCH OUT!   *Under some formalisations, satisfaction condition may fail!*

Care is needed in the exact choice of sentences considered, morphisms (between sets of actions) allowed, and reduct definitions.

Perhaps unexpected examples:

- no sentences

- no models

- no signatures

- trivial satisfaction relations

- sets of sentences as sentences

- sets of sentences as signatures

- classes of models as sentences

- sets of sentences as models

- ...

Let's fix an institution $\mathbf{I} = (\mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|})$ for a while.

Semantic entailment

$$\Phi \models_\Sigma \varphi$$

Σ-sentence φ is a semantic consequence of a set of Σ-sentences Φ

if φ holds in every Σ-models that satisfies Φ.

BTW:

- *Models* of a set of sentences: $Mod(\Phi) = \{M \in |\mathbf{Mod}(\Sigma)| \mid M \models \Phi\}$

- *Theory* of a class of models: $Th(\mathcal{C}) = \{\varphi \mid \mathcal{C} \models \varphi\}$

- $\Phi \models \varphi \iff \varphi \in Th(Mod(\Phi))$

- Mod and Th form a *Galois connection*

Semantic equivalences

Equivalence of sentences: for $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathbf{Sen}(\Sigma)$ and $\mathcal{M} \subseteq |\mathbf{Mod}(\Sigma)|$,

$$\boxed{\varphi \equiv_{\mathcal{M}} \psi}$$

if for all Σ-models $M \in \mathcal{M}$, $M \models \varphi$ iff $M \models \psi$. For $\varphi \equiv_{|\mathbf{Mod}(\Sigma)|} \psi$ we write:

$$\boxed{\varphi \equiv \psi}$$

Semantic equivalence

Equivalence of models: for $\Sigma \in |\mathbf{Sign}|$, $M, N \in |\mathbf{Mod}(\Sigma)|$, and $\Phi \subseteq \mathbf{Sen}(\Sigma)$,

$$\boxed{M \equiv_{\Phi} N}$$

if for all $\varphi \in \Phi$, $M \models \varphi$ iff $N \models \varphi$. For $M \equiv_{\mathbf{Sen}(\Sigma)} N$ we write:

$$\boxed{M \equiv N}$$

Elementary equivalence

Compactness, consistency, completeness...

- Institution $\mathbf{I}$ is *compact* if for each signature $\Sigma \in |\mathbf{Sign}|$, set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$, and Σ-sentences $\varphi \in \mathbf{Sen}(\Sigma)$,

$$\boxed{\text{if } \Phi \models \varphi \text{ then } \Phi_{fin} \models \varphi \text{ for some finite } \Phi_{fin} \subseteq \Phi}$$

- A set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$ is *consistent* if it has a model, i.e.,

$$\boxed{Mod(\Phi) \neq \emptyset}$$

- A set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$ is *complete* if it is a maximal consistent set of Σ-sentences, i.e., Φ is consistent and

$$\boxed{\text{for } \Phi \subseteq \Phi' \subseteq \mathbf{Sen}(\Sigma), \text{ if } \Phi' \text{ is consistent then } \Phi = \Phi'}$$

Fact: *Any complete set of Σ-sentences $\Phi \subseteq \mathbf{Sen}(\Sigma)$ is a theory:* $\Phi = Th(Mod(\Phi))$.

Preservation of entailment

Fact:

$$\boxed{\Phi \models_\Sigma \varphi \implies \sigma(\Phi) \models_{\Sigma'} \sigma(\varphi)}$$

for $\sigma \colon \Sigma \to \Sigma'$, $\Phi \subseteq \mathbf{Sen}(\Sigma)$, $\varphi \in \mathbf{Sen}(\Sigma)$.

If the reduct $\_\big|_\sigma \colon |\mathbf{Mod}(\Sigma')| \to |\mathbf{Mod}(\Sigma)|$ is surjective, then

$$\boxed{\Phi \models_\Sigma \varphi \iff \sigma(\Phi) \models_{\Sigma'} \sigma(\varphi)}$$

Adding provability

Add to institution:

- *proof-theoretic entailment*:

$$\vdash_\Sigma \subseteq \mathcal{P}(\mathbf{Sen}(\Sigma)) \times \mathbf{Sen}(\Sigma)$$

for each signature $\Sigma \in |\mathbf{Sign}|$, closed under
- weakening, reflexivity, transitivity (cut)
- translation along signature morphisms

Require:

- *soundness*: $\Phi \vdash_\Sigma \varphi \implies \Phi \models_\Sigma \varphi$

(?) *completeness*: $\Phi \models_\Sigma \varphi \implies \Phi \vdash_\Sigma \varphi$

Presentations
(basic specifications)

$$\langle \Sigma, \Phi \rangle$$

- signature Σ, to determine the static module interface

- axioms (Σ-sentences) $\Phi \subseteq \mathbf{Sen}(\Sigma)$, to determine required module properties

Use strong enough logic to capture the "right" class of models, excluding undesirable "modules"

Presentation morphisms

Presentation morphism:

$$\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$$

is a signature morphism $\sigma : \Sigma \to \Sigma'$ such that for all $M' \in \mathbf{Mod}(\Sigma')$:

$$M' \in Mod(\Phi') \implies M'|_\sigma \in Mod(\Phi)$$

Then $\_|_\sigma : Mod(\Phi') \to Mod(\Phi)$

Fact: *A signature morphism $\sigma : \Sigma \to \Sigma'$ is a presentation morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ if and only if $\Phi' \models \sigma(\Phi)$.*

BTW: for all presentation morphisms $\Phi \models_\Sigma \varphi \implies \Phi' \models_{\Sigma'} \sigma(\varphi)$

Conservativity

A presentation morphism:

$$\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$$

is *conservative* if for all Σ-sentences φ: $\quad \Phi' \models_{\Sigma'} \sigma(\varphi) \implies \Phi \models_{\Sigma} \varphi$

A presentation morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ *admits model expansion* if for each $M \in Mod(\Phi)$ there exists $M' \in Mod(\Phi')$ such that $M'|_{\sigma} = M$

$$\text{(i.e., } \_|_{\sigma} : Mod(\Phi') \to Mod(\Phi) \text{ is surjective).}$$

Fact: *If $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ admits model expansion then it is conservative.*

In general, the equivalence does not hold!

Fact: *If $\langle \Sigma, \Phi \rangle$ is complete and $\langle \Sigma', \Phi' \rangle$ is consistent then any presentation morphism $\sigma : \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$ is conservative.*

Categories of presentations & of theories

- **Pres**: the *category of presentations* in **I** has presentations as objects and presentation morphisms as morphisms, with identities and composition inherited from **Sign**, the category of signatures.

- **TH**: the *category of theories* in **I** is the full subcateogry of **Pres** with theories (presentations with sets of sentences closed under consequence) as objects.

> **Pres** and **TH** are equivalent:
> $$id_\Sigma : \langle \Sigma, \Phi \rangle \rightarrow \langle \Sigma, Th(Mod(\Phi)) \rangle$$
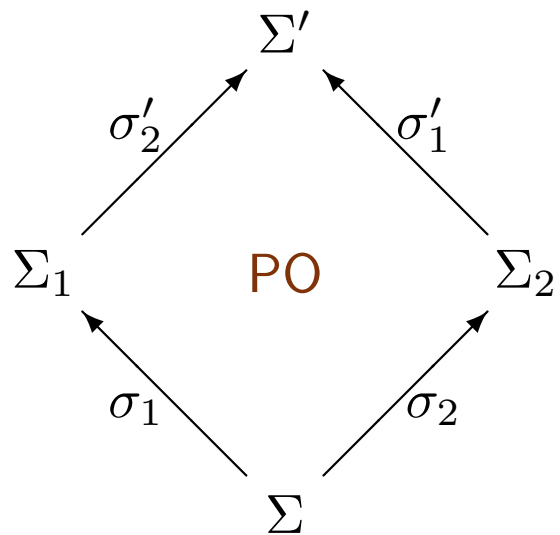> is an isomorphism in **Pres**

Fact: *The forgetful functors from* **Pres** *and* **TH**, *respectively, to* **Sign** *preserve and create colimits.*

Fact: *If the category* **Sign** *of signatures is cocomplete, so are the categories* **Pres** *of presentations and* **TH** *of theories.*

Proof hint

in **Sign**:

$$
\begin{array}{ccc}
 & \Sigma' & \\
\sigma'_2 \nearrow & & \nwarrow \sigma'_1 \\
\Sigma_1 & \text{PO} & \Sigma_2 \\
\sigma_1 \nwarrow & & \nearrow \sigma_2 \\
 & \Sigma &
\end{array}
$$

in **Pres**:

$$
\begin{array}{ccc}
 & \langle \Sigma', \sigma'_2(\Phi_1) \cup \sigma'_1(\Phi_2) \rangle & \\
\sigma'_2 \nearrow & & \nwarrow \sigma'_1 \\
\langle \Sigma_1, \Phi_1 \rangle & \text{PO} & \langle \Sigma_2, \Phi_2 \rangle \\
\sigma_1 \nwarrow & & \nearrow \sigma_2 \\
 & \langle \Sigma, \Phi \rangle &
\end{array}
$$
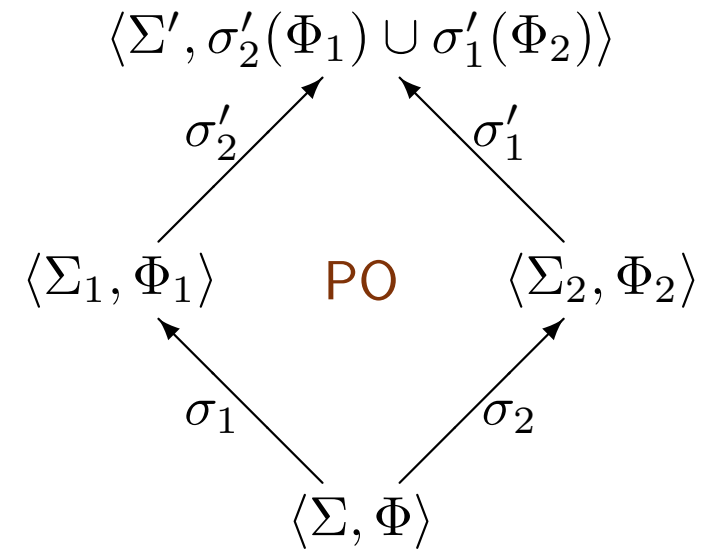
Logical connectives

- **I** *has negation* if for every signature $\Sigma \in |\mathbf{Sign}|$ and Σ-sentence $\varphi \in \mathbf{Sen}(\Sigma)$, there is a Σ-sentence "$\neg\varphi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\neg\varphi$" iff $M \not\models \varphi$.

- **I** *has conjunction* if for every signature $\Sigma \in |\mathbf{Sign}|$ and Σ-sentences $\varphi, \psi \in \mathbf{Sen}(\Sigma)$, there is a Σ-sentence "$\varphi \wedge \psi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\varphi \wedge \psi$" iff $M \models \varphi$ and $M \models \psi$.

- ... *implication*, *disjunction*, *falsity*, *truth* ...

Fact: *For any signature morphism $\sigma : \Sigma \to \Sigma'$ and Σ-sentence $\varphi \in \mathbf{Sen}(\Sigma)$ $\sigma($"$\neg\varphi$"$)$ and "$\neg\sigma(\varphi)$" are equivalent.*

Similarly, for Σ-sentences $\varphi, \psi \in \mathbf{Sen}(\Sigma)$), $\sigma($"$\varphi \wedge \psi$"$)$ and "$\sigma(\varphi) \wedge \sigma(\psi)$" are equivalent.

Similarly for other connectives...

For any institution **I**, define its closures: under negation $\mathbf{I}^{\neg}$, under conjunction $\mathbf{I}^{\wedge}$, etc.

Free variables and quantification

Standard algebra	Institution $\mathbf{I}$
algebraic signature $\Sigma = \langle S, \Omega \rangle$	signature $\Sigma \in |\mathbf{Sign}|$
S-sorted set of variables X	signature extension $\iota : \Sigma \to \Sigma(X)$
open Σ-formula φ with variables X	$\Sigma(X)$-sentence φ
Σ-algebra M	Σ-model $M \in |\mathbf{Mod}(\Sigma)|$
valuation of variables $v : X \to |M|$ in M	ι-expansion M^v of M, i.e., $M^v \in |\mathbf{Mod}(\Sigma(X)|)$, $M^v|_\iota = M$ *($M^v_x = v(x)$ for variable/constant $x \in X$)*
satisfaction of formula φ in M under v: $$M \models^v_\Sigma \varphi$$	satisfaction of "open formula" φ $$M^v \models_{\Sigma(X)} \varphi$$

A characterisation of such signature extensions:

$\sigma : \Sigma \to \Sigma'$ is *representable* iff $\mathbf{Mod}(\Sigma')$ has an initial model and

$\_|_\sigma : (\mathbf{Mod}(\Sigma') \!\uparrow\! M') \to (\mathbf{Mod}(\Sigma) \!\uparrow\! (M'|_\sigma))$ is iso for $M' \in |\mathbf{Mod}(\Sigma')|$

Let $\mathcal{I}$ be a class of signature morphisms. For decency, assume that it forms a subcategory of **Sign** and is closed under pushouts with arbitrary signature morphisms.

- **I** *has universal quantification along* $\mathcal{I}$ if for every signature morphism $\theta : \Sigma \to \Sigma'$ in $\mathcal{I}$ and Σ'-sentence $\psi \in \mathbf{Sen}(\Sigma')$, there is a Σ-sentence "$\forall \theta \cdot \psi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\forall \theta \cdot \psi$" iff for all Σ'-models with $M'|_\theta = M$, $M' \in |\mathbf{Mod}(\Sigma')|$, $M' \models \psi$.

- **I** *has existential quantification along* $\mathcal{I}$ if for $\theta : \Sigma \to \Sigma'$ in $\mathcal{I}$ and Σ'-sentence $\psi \in \mathbf{Sen}(\Sigma')$, there is a Σ-sentence "$\exists \theta \cdot \psi$" $\in \mathbf{Sen}(\Sigma)$ such that for all Σ-models $M \in |\mathbf{Mod}(\Sigma)|$, $M \models$ "$\exists \theta \cdot \psi$" iff for some Σ'-model $M' \in |\mathbf{Mod}(\Sigma')|$ with $M'|_\theta = M$, $M' \models \psi$.

Fact: *For any* $\sigma : \Sigma \to \Sigma_1$, $\sigma($"$\forall \theta \cdot \psi$"$)$ *and* "$\forall \theta' \cdot \sigma'(\psi)$" *are equivalent,*
where the following is a pushout in **Sign** *with* $\theta' \in \mathcal{I}$:
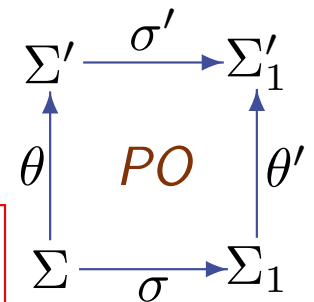
Similarly for existential quantification.

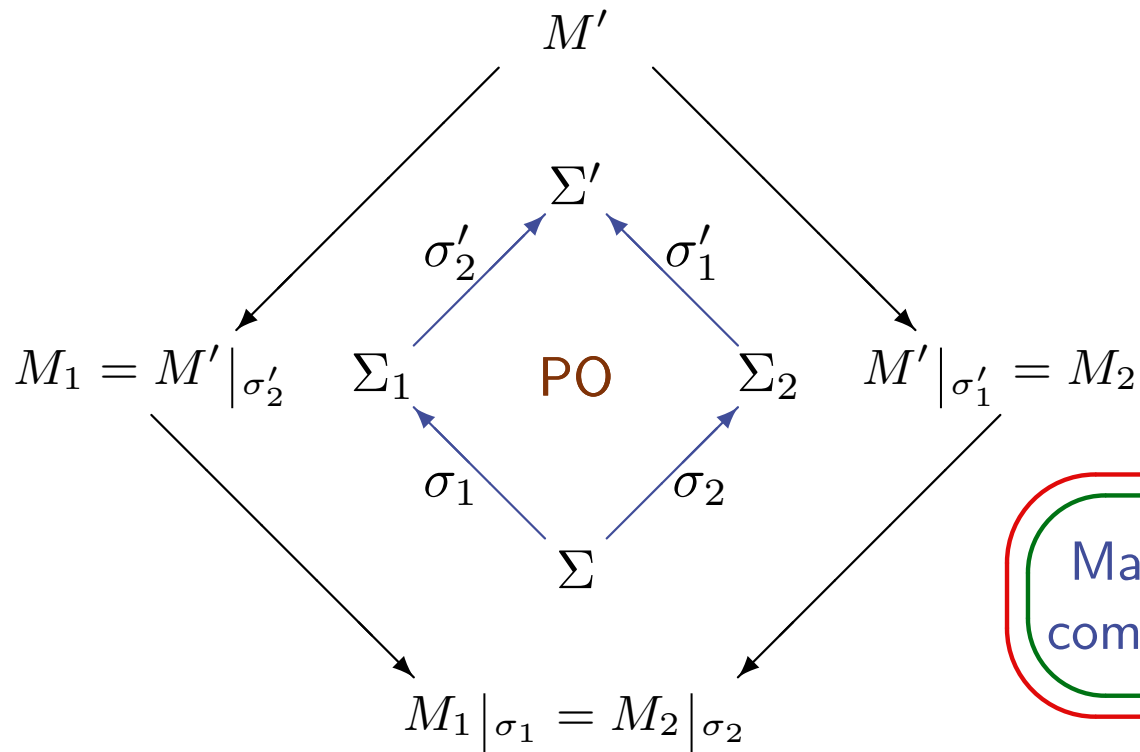AMALGAMATION NEEDED!

Define $\mathbf{I}^{FO}$, "first-order closure" of $\mathbf{I}$

$$
\begin{array}{ccc}
\Sigma' & \xrightarrow{\sigma'} & \Sigma'_1 \\
\theta \uparrow & PO & \uparrow \theta' \\
\Sigma & \xrightarrow[\sigma]{} & \Sigma_1
\end{array}
$$

Amalgamation for algebras

$$A'$$

$$\Sigma_1 \cup \Sigma_2$$

$$A_1 = A'\big|_{\Sigma_1} \qquad \Sigma_1 \qquad \text{PO} \qquad \Sigma_2 \qquad A'\big|_{\Sigma_2} = A_2$$

$$\Sigma_1 \cap \Sigma_2$$
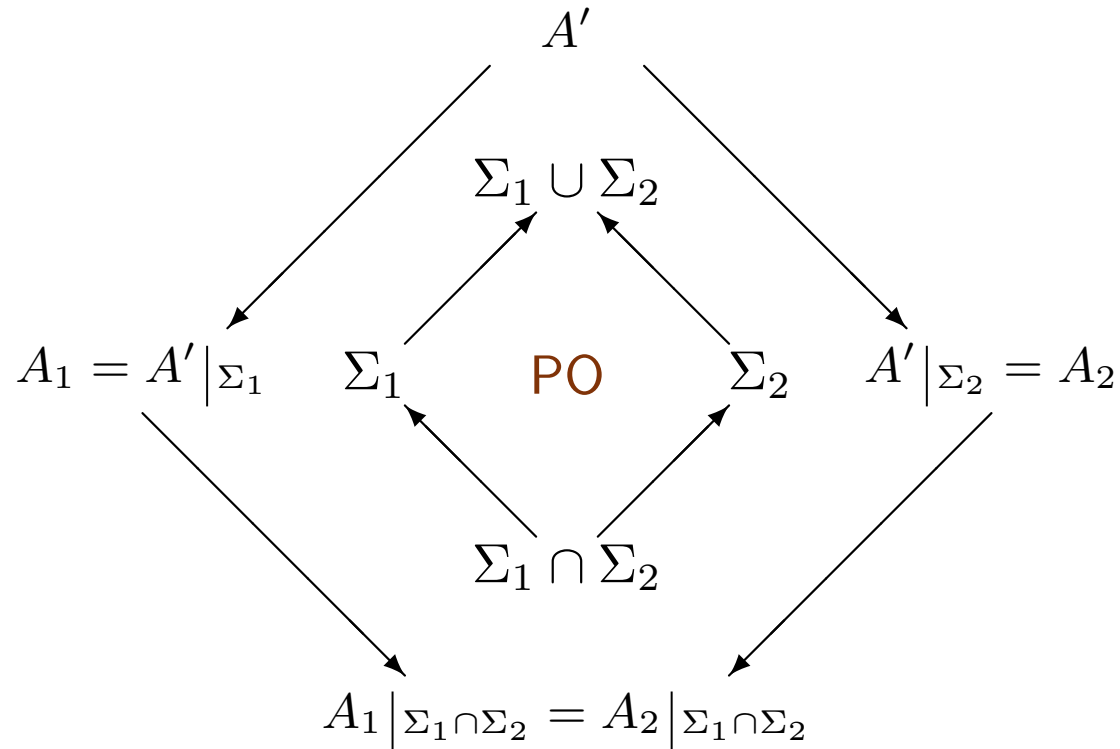
$$A_1\big|_{\Sigma_1 \cap \Sigma_2} = A_2\big|_{\Sigma_1 \cap \Sigma_2}$$

Fact: *For any algebras $A_1 \in |\mathbf{Alg}(\Sigma_1)|$ and $A_2 \in |\mathbf{Alg}(\Sigma_2)|$ with common interpretation of common symbols $A_1\big|_{\Sigma_1 \cap \Sigma_2} = A_2\big|_{\Sigma_1 \cap \Sigma_2}$, there is a unique "union" of A_1 and A_2, $A' \in |\mathbf{Alg}(\Sigma_1 \cup \Sigma_2)|$ with $A'\big|_{\Sigma_1} = A_1$ and $A'\big|_{\Sigma_2} = A_2$.*

Amalgamation

$$M'$$

$$\Sigma'$$

$$\sigma_2' \nearrow \qquad \nwarrow \sigma_1'$$

$$M_1 = M'\big|_{\sigma_2'} \qquad \Sigma_1 \qquad \text{PO} \qquad \Sigma_2 \qquad M'\big|_{\sigma_1'} = M_2$$

$$\sigma_1 \nwarrow \qquad \nearrow \sigma_2$$

$$\Sigma$$

$$M_1\big|_{\sigma_1} = M_2\big|_{\sigma_2}$$

> May be sensibly stated for any
> commuting square of morphisms

In **I**, *amalgamation property* holds for the pushout above if for all $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ and $M_2 \in |\mathbf{Mod}(\Sigma_2)|$ with $M_1\big|_{\sigma_1} = M_2\big|_{\sigma_2}$, there is a unique $M' \in |\mathbf{Mod}(\Sigma')|$ with $M'\big|_{\sigma_1'} = M_2$ and $M'\big|_{\sigma_2'} = M_1$.

Adding amalgamation

Assume:

- the model functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$ is *continuous* (maps colimits of signatures to limits of model categories)
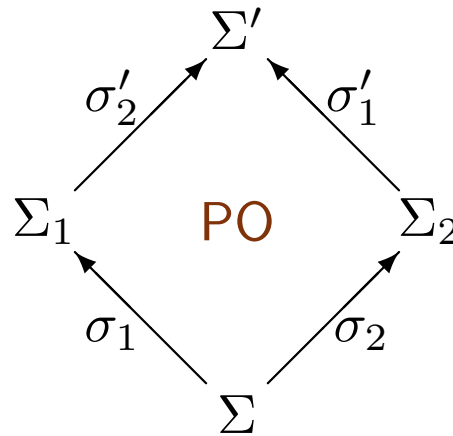
Fact: $\mathbf{Alg}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$ *is continuous.*

Amalgamation property: *Amalgamation property follows for a pushout in* $\mathbf{Sign}$ *if* $\mathbf{Mod}$ *maps it to a pullback in* $\mathbf{Cat}$:

$$
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\;\sigma_2'\;} & \Sigma' \\
\sigma_1 \big\uparrow & \text{PO} & \big\uparrow \sigma_1' \\
\Sigma & \xrightarrow{\;\sigma_2\;} & \Sigma_2
\end{array}
\qquad \xmapsto{\ \mathbf{Mod}\ } \qquad
\begin{array}{ccc}
\mathbf{Mod}(\Sigma_1) & \xleftarrow{\;-|_{\sigma_2'}\;} & \mathbf{Mod}(\Sigma') \\
{-|_{\sigma_1}} \big\downarrow & \text{PB} & \big\downarrow {-|_{\sigma_1'}} \\
\mathbf{Mod}(\Sigma) & \xleftarrow{\;-|_{\sigma_2}\;} & \mathbf{Mod}(\Sigma_2)
\end{array}
$$

$\mathbf{I}$ has the *interpolation property* for a pushout in $\mathbf{Sign}$

$$
\begin{array}{ccc}
 & \Sigma' & \\
\sigma_2' \nearrow & & \nwarrow \sigma_1' \\
\Sigma_1 & \text{PO} & \Sigma_2 \\
\nwarrow \sigma_1 & & \nearrow \sigma_2 \\
 & \Sigma &
\end{array}
$$

if for all $\varphi_1 \in \mathbf{Sen}(\Sigma_1)$ and $\varphi_2 \in \mathbf{Sen}(\Sigma_2)$ such that $\sigma_2'(\varphi_1) \models_{\Sigma'} \sigma_1'(\varphi_2)$ there is $\theta \in \mathbf{Sen}(\Sigma)$ such that $\varphi_1 \models_{\Sigma_1} \sigma_1(\theta)$ and $\sigma_2(\theta) \models_{\Sigma_2} \varphi_2$.
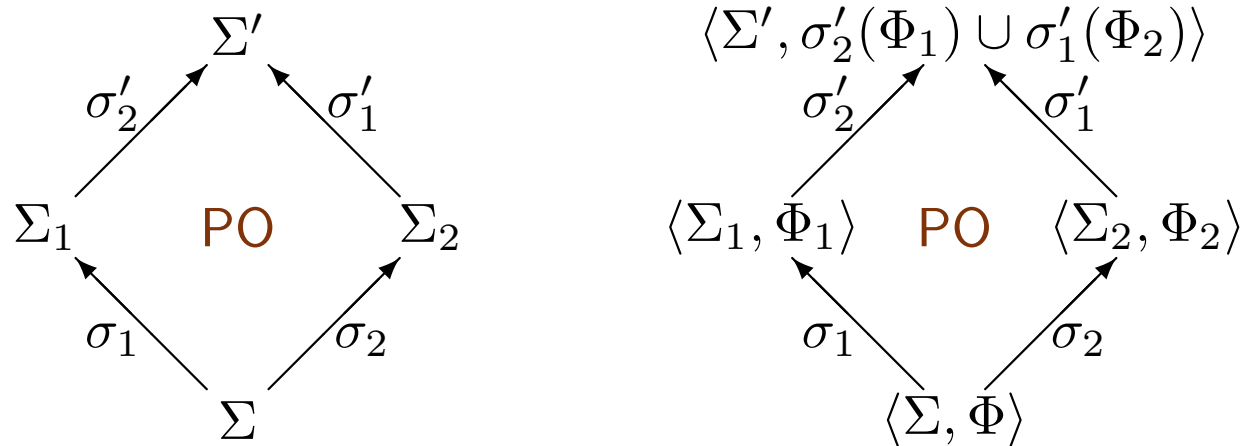
Fact: $\mathbf{FOEQ}$ *has the interpolation property for all pushouts of pairs of morphisms, where at least one of the morphisms is injective on sorts.*

Spell out a version with a set of interpolants

Craig interpolation theorem

$\mathbf{I}$ has the *consistency property* for a pushout in $\mathbf{Sign}$

$$
\begin{array}{ccc}
 & \Sigma' & \\
\sigma_2' \nearrow & & \nwarrow \sigma_1' \\
\Sigma_1 & \text{PO} & \Sigma_2 \\
\nwarrow \sigma_1 & & \nearrow \sigma_2 \\
 & \Sigma &
\end{array}
\qquad
\begin{array}{ccc}
 & \langle \Sigma', \sigma_2'(\Phi_1) \cup \sigma_1'(\Phi_2) \rangle & \\
\sigma_2' \nearrow & & \nwarrow \sigma_1' \\
\langle \Sigma_1, \Phi_1 \rangle & \text{PO} & \langle \Sigma_2, \Phi_2 \rangle \\
\nwarrow \sigma_1 & & \nearrow \sigma_2 \\
 & \langle \Sigma, \Phi \rangle &
\end{array}
$$

if for all $\Phi \subseteq \mathbf{Sen}(\Sigma)$ and consistent $\Phi_1 \subseteq \mathbf{Sen}(\Sigma_1)$ and $\Phi_2 \subseteq \mathbf{Sen}(\Sigma_2)$ such that $\sigma_1 : \langle \Sigma, \Phi \rangle \to \langle \Sigma_1, \Phi_1 \rangle$ is a conservative presentation morphism and $\sigma_2 : \langle \Sigma, \Phi \rangle \to \langle \Sigma_2, \Phi_2 \rangle$ is a presentation morphism, $\langle \Sigma', \sigma_2'(\Phi_1) \cup \sigma_1'(\Phi_2) \rangle$ is consistent. Robinson consistency theorem (for first-order logic)

Fact: *In any compact institution with falsity, negation and conjunction, Craig interpolation and Robinson consistency properties are equivalent.*

The method of diagrams

Institution $\mathbf{I}$	Standard algebra
Given a signature Σ and Σ-model M, build signature extension $\iota : \Sigma \to \Sigma(M)$ and a $\Sigma(M)$-presentation E_M so that the reduct by ι yields isomorphism $\mathbf{Mod}(\Sigma(M), E_M) \to (\mathbf{Mod}(\Sigma){\uparrow}M)$ ...and everything is natural ...	*(adding elements of $\lvert M \rvert$ as constants)* *(all ground atoms true in M^M, the natural ι-expansion of M)* *(then the reduct by ι yields isomorphism $\mathbf{Alg}(\Sigma(M), E_M) \to (\mathbf{Alg}(\Sigma){\uparrow}M)$)* *(everything is natural)*
Now: M has a "canonical" ι-expansion which is initial in $\mathbf{Mod}(\Sigma(M), E_M)$	*(M^M, reachable ι-expansion of M, is initial in $\mathbf{Alg}(\Sigma(M), E_M)$)*

Equipped with the method of diagrams, one can do a lot!

Abstract abstract model theory

Providing new insights and abstract formulations
for classical model-theoretic concepts and results

- amalgamation over pushouts

- the method of elementary diagrams

- existence of free extensions

- interpolation results

- Birkhoff variety theorem(s)

- Beth definability theorem

- logical connectives, free variables, quantification

- completeness for *any* first-order logic

- . . .

in any institution
with various bits of extra structure,
under some technical assumptions. . .

WORK IN AN ARBITRARY INSTITUTION

. . . adding extra structure and assumptions only if really needed . . .

Revised rough analogy

module interface	$\rightsquigarrow$	signature
module	$\rightsquigarrow$	model
module specification	$\rightsquigarrow$	class of models