Functors and natural transformations

functors	$\rightsquigarrow$	*category morphisms*
natural transformations	$\rightsquigarrow$	*functor morphisms*

Functors

A *functor* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ from a category $\mathbf{K}$ to a category $\mathbf{K}'$ consists of:

- a function $\mathbf{F}\colon |\mathbf{K}| \to |\mathbf{K}'|$, and
- for all $A, B \in |\mathbf{K}|$, a function $\mathbf{F}\colon \mathbf{K}(A,B) \to \mathbf{K}'(\mathbf{F}(A), \mathbf{F}(B))$

such that:

Make explicit categories in which we work at various places here

- $\mathbf{F}$ preserves identities, i.e.,

$$\mathbf{F}(id_A) = id_{\mathbf{F}(A)}$$

  for all $A \in |\mathbf{K}|$, and

- $\mathbf{F}$ preserves composition, i.e.,

$$\mathbf{F}(f;g) = \mathbf{F}(f);\mathbf{F}(g)$$

  for all $f\colon A \to B$ and $g\colon B \to C$ in $\mathbf{K}$.

We really should differentiate between various components of F

Examples

- *identity functors*: $\mathbf{Id_K}\colon \mathbf{K} \to \mathbf{K}$, for any category $\mathbf{K}$
- *inclusions*: $\mathbf{I_{K \hookrightarrow K'}}\colon \mathbf{K} \to \mathbf{K'}$, for any subcategory $\mathbf{K}$ of $\mathbf{K'}$
- *constant functors*: $\mathbf{C}_A\colon \mathbf{K} \to \mathbf{K'}$, for any categories $\mathbf{K}, \mathbf{K'}$ and $A \in |\mathbf{K'}|$, with $\mathbf{C}_A(f) = id_A$ for all morphisms f in $\mathbf{K}$
- *powerset functor*: $\mathbf{P}\colon \mathbf{Set} \to \mathbf{Set}$ given by
  - $\mathbf{P}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
  - $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(X')$ for all $f\colon X \to X'$ in $\mathbf{Set}$, $\mathbf{P}(f)(Y) = \{f(y) \mid y \in Y\}$ for all $Y \subseteq X$
- *contravariant powerset functor*: $\mathbf{P}_{-1}\colon \mathbf{Set}^{op} \to \mathbf{Set}$ given by
  - $\mathbf{P}_{-1}(X) = \{Y \mid Y \subseteq X\}$, for all $X \in |\mathbf{Set}|$
  - $\mathbf{P}_{-1}(f)\colon \mathbf{P}(X') \to \mathbf{P}(X)$ for all $f\colon X \to X'$ in $\mathbf{Set}$, $\mathbf{P}_{-1}(f)(Y') = \{x \in X \mid f(x) \in Y'\}$ for all $Y' \subseteq X'$

Examples, cont'd.

- *projection functors*: $\pi_1 : \mathbf{K} \times \mathbf{K}' \to \mathbf{K}$, $\pi_2 : \mathbf{K} \times \mathbf{K}' \to \mathbf{K}'$
- *list functor*: $\mathbf{List} : \mathbf{Set} \to \mathbf{Monoid}$, where $\mathbf{Monoid}$ is the category of monoids (as objects) with monoid homomorphisms as morphisms:
  - $\mathbf{List}(X) = \langle X^*, \frown, \epsilon \rangle$, for all $X \in |\mathbf{Set}|$, where X^* is the set of all finite lists of elements from X, $\frown$ is the list concatenation, and ϵ is the empty list.
  - $\mathbf{List}(f) : \mathbf{List}(X) \to \mathbf{List}(X')$ for $f : X \to X'$ in $\mathbf{Set}$,
    $\mathbf{List}(f)(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$ for all $x_1, \ldots, x_n \in X$
- *totalisation functor*: $\mathbf{Tot} : \mathbf{Pfn} \to \mathbf{Set}_*$, where $\mathbf{Set}_*$ is the subcategory of $\mathbf{Set}$ of sets with a distinguished element $*$ and $*$-preserving functions
  - $\mathbf{Tot}(X) = X \uplus \{*\}$
  - $\mathbf{Tot}(f)(x) = \begin{cases} f(x) & \text{if it is defined} \\ * & \text{otherwise} \end{cases}$

Define $\mathbf{Set}_*$ as the category of algebras

Examples, cont'd.

- *carrier set functors*: $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, yielding the algebra carriers and homomorphisms as functions between them
- *reduct functors*: $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$, for any signature morphism $\sigma\colon \Sigma \to \Sigma'$, as defined earlier
- *term algebra functors*: $\mathbf{T}_\Sigma\colon \mathbf{Set} \to \mathbf{Alg}(\Sigma)$ for all (single-sorted) algebraic signatures $\Sigma \in |\mathbf{AlgSig}|$

  Generalise to many-sorted signatures

  - $\mathbf{T}_\Sigma(X) = T_\Sigma(X)$ for all $X \in |\mathbf{Set}|$
  - $\mathbf{T}_\Sigma(f) = f^\#\colon T_\Sigma(X) \to T_\Sigma(X')$ for all functions $f\colon X \to X'$
- *diagonal functors*: $\Delta^G_{\mathbf{K}}\colon \mathbf{K} \to \mathbf{Diag}^G_{\mathbf{K}}$ for any graph G with nodes $N = |G|_{nodes}$ and edges $E = |G|_{edges}$, and category $\mathbf{K}$
  - $\Delta^G_{\mathbf{K}}(A) = D^A$, where D^A is the "constant" diagram, with $D^A_n = A$ for all $n \in N$ and $D^A_e = id_A$ for all $e \in E$
  - $\Delta^G_{\mathbf{K}}(f) = \mu^f\colon D^A \to D^B$, for all $f\colon A \to B$, where $\mu^f_n = f$ for all $n \in N$

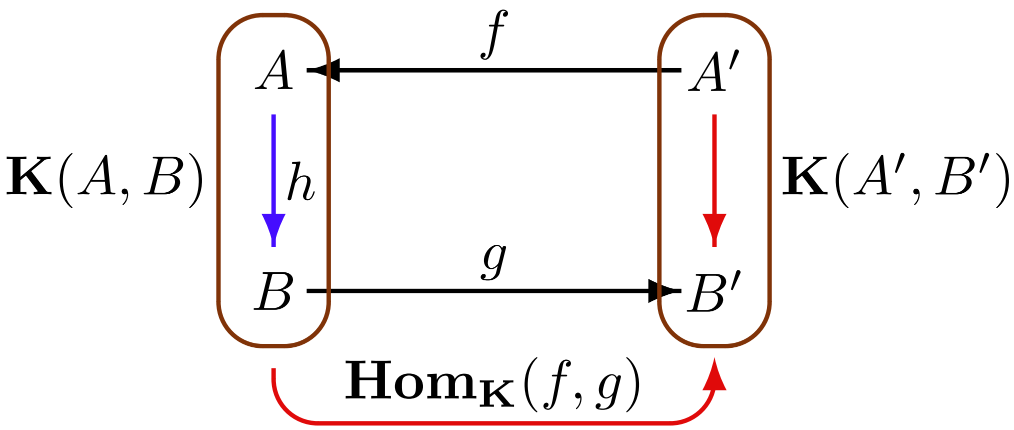
Hom-functors

Given a *locally small* category $\mathbf{K}$, define

$$\mathbf{Hom_K} \colon \mathbf{K}^{op} \times \mathbf{K} \to \mathbf{Set}$$

a binary *hom-functor*, contravariant on the first argument and covariant on the second argument, as follows:

- $\mathbf{Hom_K}(\langle A, B\rangle) = \mathbf{K}(A, B)$, for all $\langle A, B\rangle \in |\mathbf{K}^{op} \times \mathbf{K}|$, i.e., $A, B \in |\mathbf{K}|$
- $\mathbf{Hom_K}(\langle f, g\rangle) \colon \mathbf{K}(A, B) \to \mathbf{K}(A', B')$, for $\langle f, g\rangle \colon \langle A, B\rangle \to \langle A', B'\rangle$ in $\mathbf{K}^{op} \times \mathbf{K}$, i.e., $f \colon A' \to A$ and $g \colon B \to B'$ in $\mathbf{K}$, as a function given by $\mathbf{Hom_K}(\langle f, g\rangle)(h) = f;h;g$.



Also: $\mathbf{Hom_K}(A, \_) \colon \mathbf{K} \to \mathbf{Set}$
$\mathbf{Hom_K}(\_, B) \colon \mathbf{K}^{op} \to \mathbf{Set}$

Functors preserve. . .

- Check whether functors preserve:
  - monomorphisms
  - epimorphisms
  - (co)retractions
  - isomorphisms
  - (co)cones
  - (co)limits
  - . . .
- A functor is (finitely) continuous if it preserves all existing (finite) limits.

  Which of the above functors are (finitely) continuous?

Dualise!

Functors compose...

Given two functors $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}''$, their *composition* $\mathbf{F};\mathbf{G}\colon \mathbf{K} \to \mathbf{K}''$ is defined as expected:

- $(\mathbf{F};\mathbf{G})(A) = \mathbf{G}(\mathbf{F}(A))$ for all $A \in |\mathbf{K}|$
- $(\mathbf{F};\mathbf{G})(f) = \mathbf{G}(\mathbf{F}(f))$ for all $f\colon A \to B$ in $\mathbf{K}$

Cat, *the category of (sm)all categories*

– objects: (sm)all categories
– morphisms: functors between them
– composition: as above

Characterise isomorphisms in **Cat**

Define products, terminal objects, equalisers and pullback in **Cat**

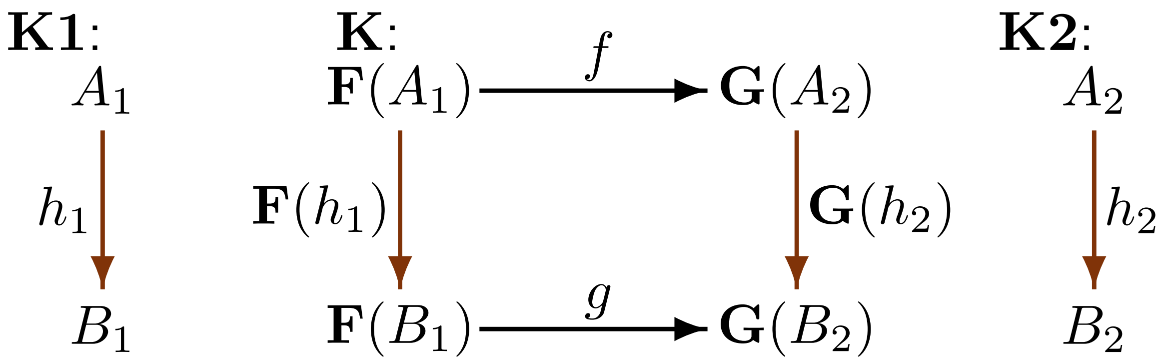
Try to define their duals

Comma categories

Given two functors with a common target, $\mathbf{F}\colon \mathbf{K1} \to \mathbf{K}$ and $\mathbf{G}\colon \mathbf{K2} \to \mathbf{K}$, define their *comma category*

$$(\mathbf{F}, \mathbf{G})$$

- objects: triples $\langle A_1, f\colon \mathbf{F}(A_1) \to \mathbf{G}(A_2), A_2\rangle$, where $A_1 \in |\mathbf{K1}|$, $A_2 \in |\mathbf{K2}|$, and $f\colon \mathbf{F}(A_1) \to \mathbf{G}(A_2)$ in $\mathbf{K}$
- morphisms: a morphism in $(\mathbf{F}, \mathbf{G})$ is any pair $\langle h_1, h_2\rangle\colon \langle A_1, f\colon \mathbf{F}(A_1) \to \mathbf{G}(A_2), A_2\rangle \to \langle B_1, g\colon \mathbf{F}(B_1) \to \mathbf{G}(B_2), B_2\rangle$, where $h_1\colon A_1 \to B_1$ in $\mathbf{K1}$, $h_2\colon A_2 \to B_2$ in $\mathbf{K2}$, and $\mathbf{F}(h_1);g = f;\mathbf{G}(h_2)$ in $\mathbf{K}$.
- composition: component-wise



Examples

- The category of graphs as a comma category:

$$\mathbf{Graph} = (\mathbf{Id}_{\mathbf{Set}}, \mathbf{CP})$$

where $\mathbf{CP}\colon \mathbf{Set} \to \mathbf{Set}$ is the (Cartesian) product functor ($\mathbf{CP}(X) = X \times X$ and $\mathbf{CP}(f)(\langle x, x' \rangle) = \langle f(x), f(x') \rangle$). Hint: write objects of this category as $\langle E, \langle source, target \rangle \colon E \to N \times N, N \rangle$

- The category of algebraic signatures as a comma category:

$$\mathbf{AlgSig} = (\mathbf{Id}_{\mathbf{Set}}, (\_)^+)$$

where $(\_)^+\colon \mathbf{Set} \to \mathbf{Set}$ is the non-empty list functor ($(X)^+$ is the set of all non-empty lists of elements from X, $(f)^+(\langle x_1, \ldots, x_n \rangle) = \langle f(x_1), \ldots, f(x_n) \rangle$). Hint: write objects of this category as $\langle \Omega, \langle arity, sort \rangle \colon \Omega \to S^+, S \rangle$

Define $\mathbf{K}^{\to}$, $\mathbf{K}{\downarrow}A$ as comma categories. The same for $\mathbf{Alg}(\Sigma)$.

Cocompleteness of comma categories

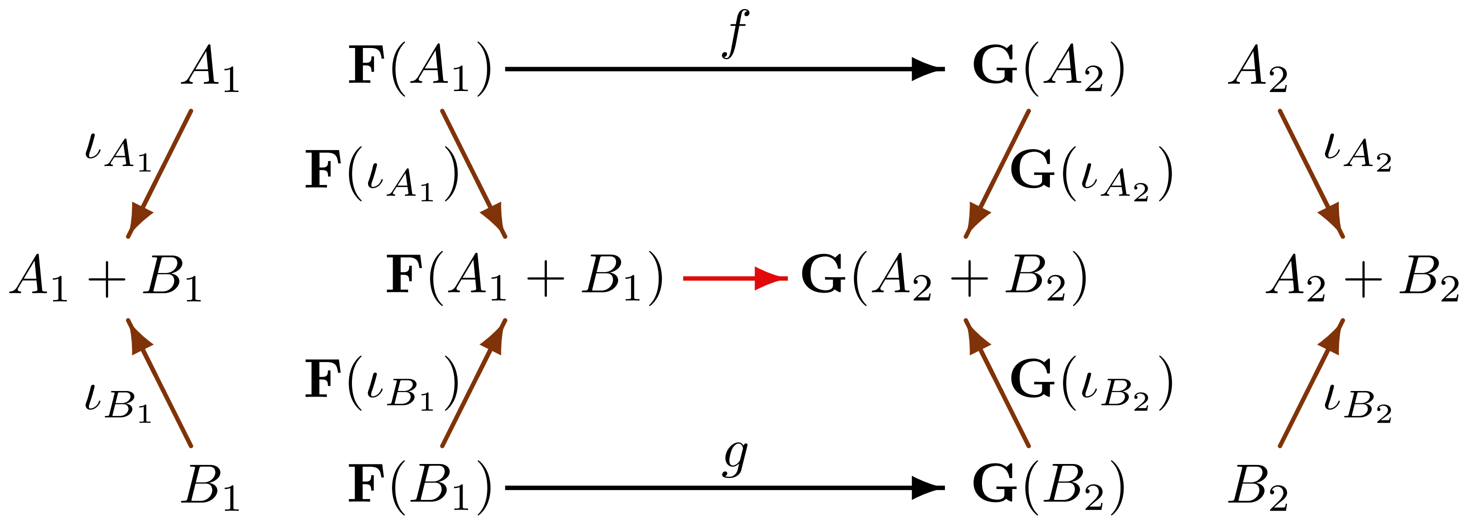
Fact: *If* $\mathbf{K1}$ *and* $\mathbf{K2}$ *are (finitely) cocomplete categories,* $\mathbf{F}\colon \mathbf{K1} \to \mathbf{K}$ *is a (finitely) cocontinuous functor, and* $\mathbf{G}\colon \mathbf{K2} \to \mathbf{K}$ *is a functor then the comma category* $(\mathbf{F}, \mathbf{G})$ *is (finitely) cocomplete.*

Proof (idea):

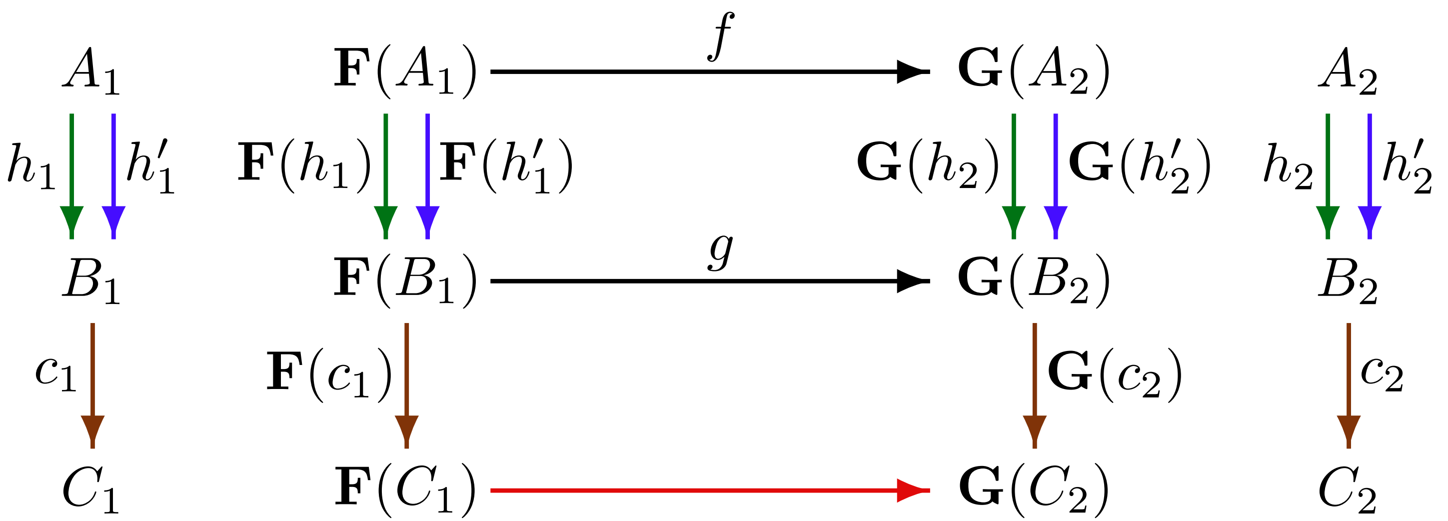
Construct coproducts and coequalisers in $(\mathbf{F}, \mathbf{G})$, using the corresponding constructions in $\mathbf{K1}$ and $\mathbf{K2}$, and cocontinuity of $\mathbf{F}$.

> *State and prove the dual fact,*
> *concerning completeness of comma categories*

Coproducts:



Coequalisers:



Indexed categories

An *indexed category* is a functor $\mathcal{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$.

Standard example: $\mathbf{Alg}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$

The Grothendieck construction: Given $\mathcal{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$, define a category $\mathbf{Flat}(\mathcal{C})$:

- objects: $\langle i, A\rangle$ for all $i \in |\mathbf{Ind}|$, $A \in |\mathcal{C}(i)|$
- morphisms: a morphism from $\langle i, A\rangle$ to $\langle j, B\rangle$, $\langle \sigma, f\rangle\colon \langle i, A\rangle \to \langle j, B\rangle$, consists of a morphism $\sigma\colon i \to j$ in $\mathbf{Ind}$ and a morphism $f\colon A \to \mathcal{C}(\sigma)(B)$ in $\mathcal{C}(i)$
- composition: given $\langle \sigma, f\rangle\colon \langle i, A\rangle \to \langle i', A'\rangle$ and $\langle \sigma', f'\rangle\colon \langle i', A'\rangle \to \langle i'', A''\rangle$, their composition in $\mathbf{Flat}(\mathcal{C})$, $\langle \sigma, f\rangle;\langle \sigma', f'\rangle\colon \langle i, A\rangle \to \langle i'', A''\rangle$, is given by

$$\langle \sigma, f\rangle;\langle \sigma', f'\rangle = \langle \sigma;\sigma', f;\mathcal{C}(\sigma)(f')\rangle$$

Fact: *If $\mathbf{Ind}$ is complete, $\mathcal{C}(i)$ are complete for all $i \in |\mathbf{Ind}|$, and $\mathcal{C}(\sigma)$ are continuous for all $\sigma\colon i \to j$ in $\mathbf{Ind}$, then $\mathbf{Flat}(\mathcal{C})$ is complete.*

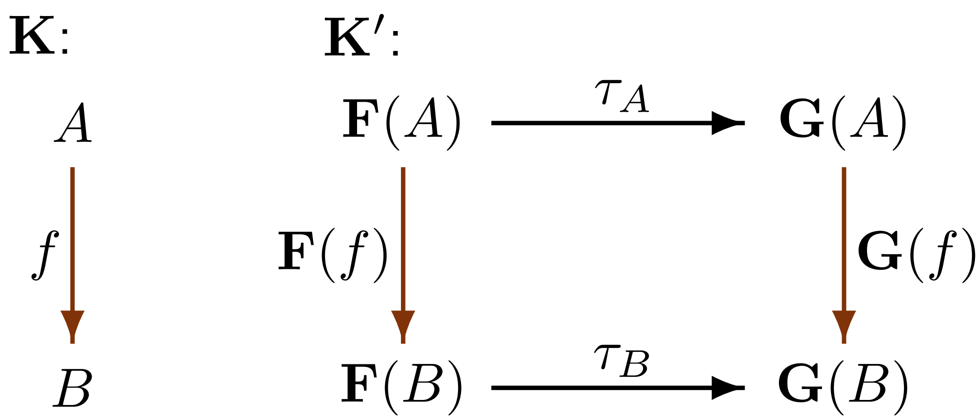
Try to formulate and prove a theorem concerning cocompleteness of $\mathbf{Flat}(\mathcal{C})$

Natural transformations

Given two parallel functors $\mathbf{F}, \mathbf{G} \colon \mathbf{K} \to \mathbf{K}'$, a *natural transformation* from $\mathbf{F}$ to $\mathbf{G}$

$$\boxed{\tau \colon \mathbf{F} \to \mathbf{G}}$$

is a family $\tau = \langle \tau_A \colon \mathbf{F}(A) \to \mathbf{G}(A) \rangle_{A \in |\mathbf{K}|}$ of $\mathbf{K}'$-morphisms such that for all $f \colon A \to B$ in $\mathbf{K}$ (with $A, B \in |\mathbf{K}|$), $\boxed{\tau_A;\mathbf{G}(f) = \mathbf{F}(f);\tau_B}$



Then, τ is a *natural isomorphism* if for all $A \in |\mathbf{K}|$, τ_A is an isomorphism.

Examples

- *identity transformations*: $id_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}$, where $\mathbf{F} \colon \mathbf{K} \to \mathbf{K}'$, for all objects $A \in |\mathbf{K}|$, $(id_{\mathbf{F}})_A = id_A \colon \mathbf{F}(A) \to \mathbf{F}(A)$
- *singleton functions*: $sing \colon \mathbf{Id_{Set}} \to \mathbf{P}$ $(\colon \mathbf{Set} \to \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $sing_X \colon X \to \mathbf{P}(X)$ is a function defined by $sing_X(x) = \{x\}$ for $x \in X$
- *singleton-list functions*: $sing^{\mathbf{List}} \colon \mathbf{Id_{Set}} \to |\mathbf{List}|$ $(\colon \mathbf{Set} \to \mathbf{Set})$, where $|\mathbf{List}| = \mathbf{List};|\_| \colon \mathbf{Set}(\to \mathbf{Monoid}) \to \mathbf{Set}$, and for all $X \in |\mathbf{Set}|$, $sing^{\mathbf{List}}_X \colon X \to X^*$ is a function defined by $sing^{\mathbf{List}}_X(x) = \langle x \rangle$ for $x \in X$
- *append functions*: $append \colon |\mathbf{List}|;\mathbf{CP} \to |\mathbf{List}|$ $(\colon \mathbf{Set} \to \mathbf{Set})$, where for all $X \in |\mathbf{Set}|$, $append_X \colon (X^* \times X^*) \to X^*$ is the usual append function (list concatenation) polymorphic functions between algebraic types

Polymorphic functions

Work out the following generalisation of the last two examples:

– for each algebraic type scheme $\forall \alpha_1 \ldots \alpha_n \cdot T$, built in `Standard ML` using at least products and algebraic data types (no function types though), define the corresponding functor $[\![T]\!] \colon \mathbf{Set}^n \to \mathbf{Set}$

– argue that in a representative subset of `Standard ML`, for each polymorphic expression $E \colon \forall \alpha_1 \ldots \alpha_n \cdot T \to T'$ its semantics is a natural transformation $[\![E]\!] \colon [\![T]\!] \to [\![T']\!]$

Theorems for free!
(see Wadler 89)

Yoneda lemma

Given a locally small category $\mathbf{K}$, functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{Set}$ and object $A \in |\mathbf{K}|$:

$$Nat(\mathbf{Hom}_{\mathbf{K}}(A,\_),\mathbf{F}) \cong \mathbf{F}(A)$$

natural transformations from $\mathbf{Hom}_{\mathbf{K}}(A,\_)$ *to* $\mathbf{F}$*, between functors from* $\mathbf{K}$ *to* $\mathbf{Set}$*, are given exactly by the elements of the set* $\mathbf{F}(A)$

EXERCISES:

- Dualise: for $\mathbf{G}\colon \mathbf{K}^{op} \to \mathbf{Set}$,

$$Nat(\mathbf{Hom}_{\mathbf{K}}(\_,A),\mathbf{G}) \cong \mathbf{G}(A)$$

.

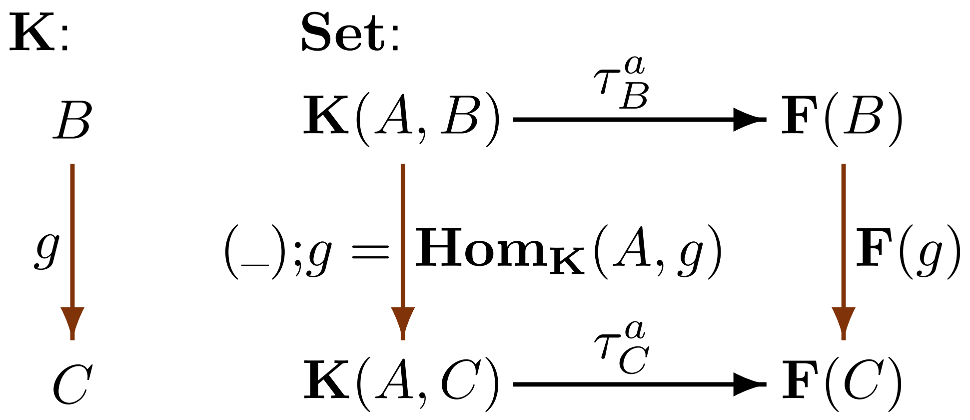
- Characterise all natural transformations from $\mathbf{Hom}_{\mathbf{K}}(A,\_)$ to $\mathbf{Hom}_{\mathbf{K}}(B,\_)$, for all objects $A, B \in |\mathbf{K}|$.

Proof

- For $a \in \mathbf{F}(A)$, define $\tau^a \colon \mathbf{Hom_K}(A, \_) \to \mathbf{F}$, as the family of functions $\tau^a_B \colon \mathbf{K}(A,B) \to \mathbf{F}(B)$ given by $\tau^a_B(f) = \mathbf{F}(f)(a)$ for $f \colon A \to B$ in $\mathbf{K}$.

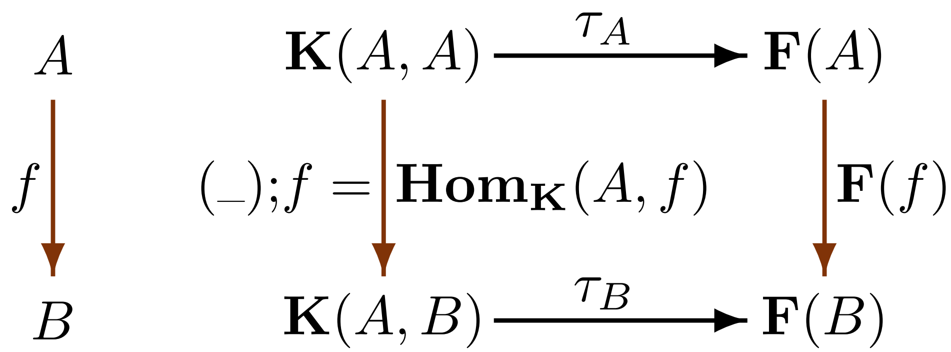
  This is a natural transformation, since for $g \colon B \to C$ and then $f \colon A \to B$,

  $$\mathbf{F}(g)(\tau^a_B(f)) = \mathbf{F}(g)(\mathbf{F}(f)(a))$$
  $$= \mathbf{F}(f;g)(a) = \tau^a_C(f;g)$$
  $$= \tau^a_C(\mathbf{Hom_K}(A,g)(f))$$

  Then $\tau^a_A(id_A) = a$, and so for distinct $a, a' \in \mathbf{F}(A)$, τ^a and $\tau^{a'}$ differ.

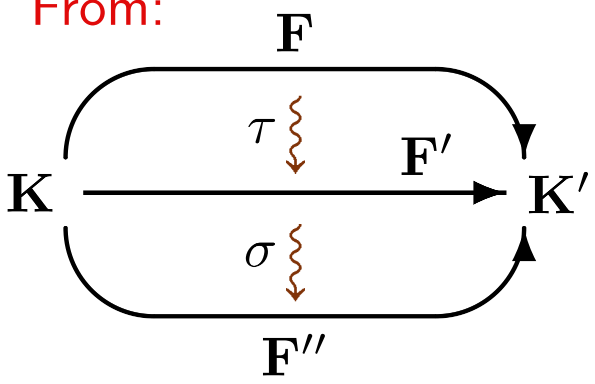
- If $\tau \colon \mathbf{Hom_K}(A, \_) \to \mathbf{F}$ is a natural transformation then $\tau = \tau^a$, where we put $a = \tau_A(id_A)$, since for $B \in |\mathbf{K}|$ and $f \colon A \to B$, $\tau_B(f) = \mathbf{F}(f)(\tau_A(id_A))$ by naturality of τ:

Compositions

vertical composition:

From:



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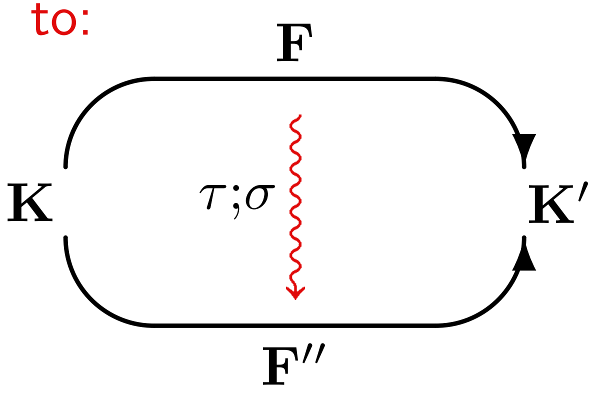
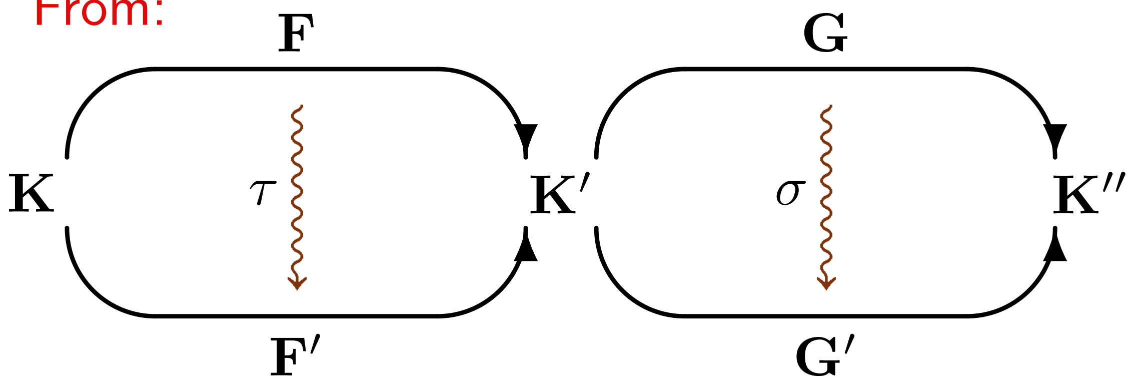


horizontal composition:

From:



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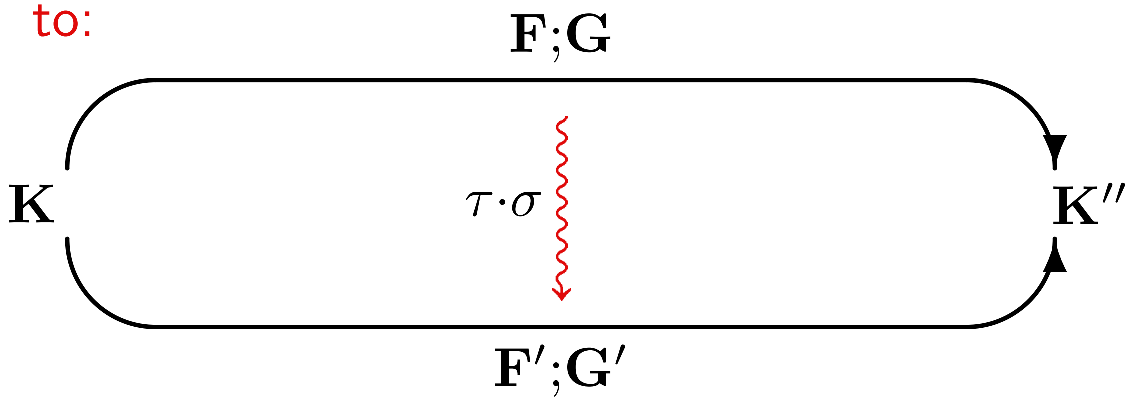
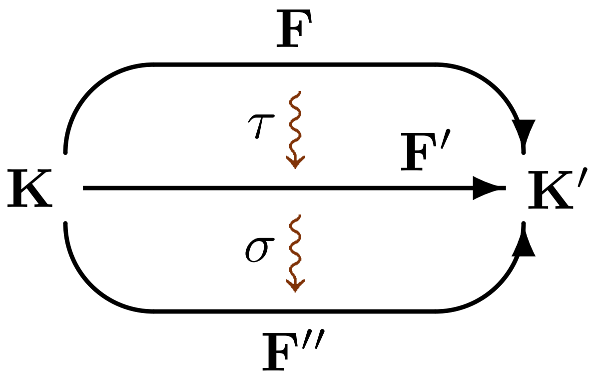


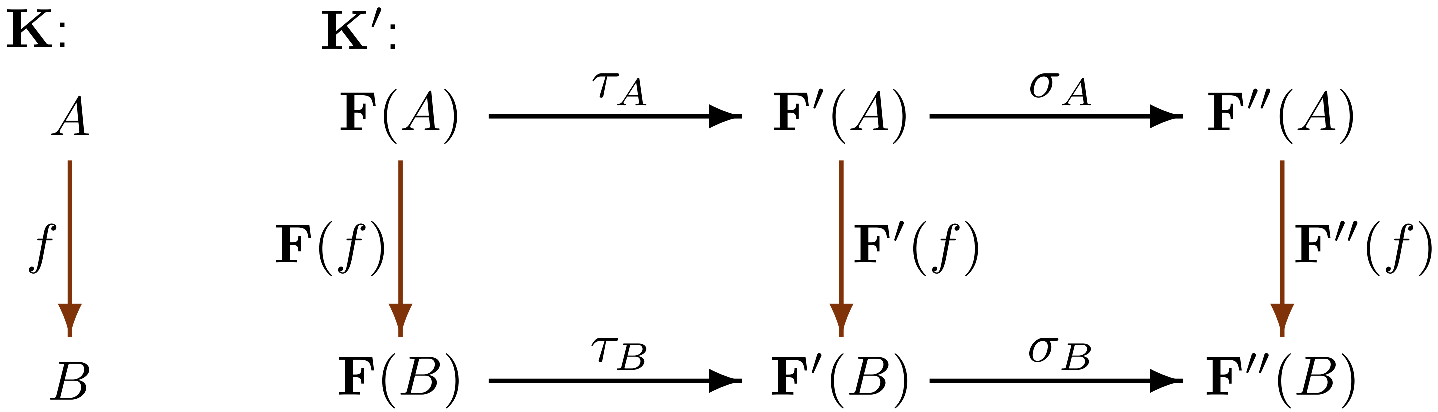
Vertical composition



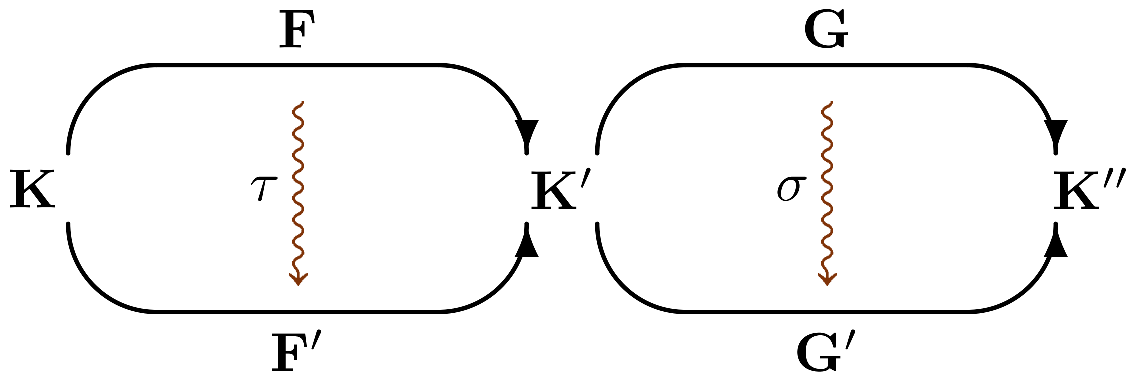
The *vertical composition* of natural transformations $\tau\colon \mathbf{F} \to \mathbf{F}'$ and $\sigma\colon \mathbf{F}' \to \mathbf{F}''$ between parallel functors $\mathbf{F}, \mathbf{F}', \mathbf{F}''\colon \mathbf{K} \to \mathbf{K}'$

$$\tau;\sigma\colon \mathbf{F} \to \mathbf{F}''$$

is a natural transformation given by $(\tau;\sigma)_A = \tau_A;\sigma_A$ for all $A \in |\mathbf{K}|$.



Horizontal composition



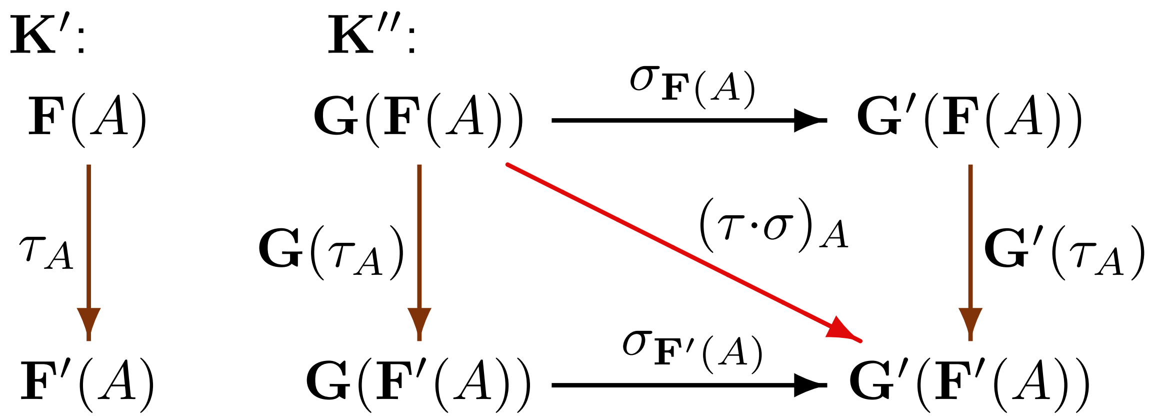
The *horizontal composition* of natural transformations $\tau\colon \mathbf{F} \to \mathbf{F}'$ and $\sigma\colon \mathbf{G} \to \mathbf{G}'$ between composable pairs of parallel functors $\mathbf{F}, \mathbf{F}'\colon \mathbf{K} \to \mathbf{K}'$, $\mathbf{G}, \mathbf{G}'\colon \mathbf{K}' \to \mathbf{K}''$

$$\boxed{\tau{\cdot}\sigma\colon \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}'}$$

is a natural transformation given by $\boxed{(\tau{\cdot}\sigma)_A = \mathbf{G}(\tau_A);\sigma_{\mathbf{F}'(A)} = \sigma_{\mathbf{F}(A)};\mathbf{G}'(\tau_A)}$ for all $A \in |\mathbf{K}|$.

Multiplication by functor:

- $\tau{\cdot}\mathbf{G} = \tau{\cdot}id_{\mathbf{G}}\colon \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}$, i.e., $(\tau{\cdot}\mathbf{G})_A = \mathbf{G}(\tau_A)$
- $\mathbf{F}{\cdot}\sigma = id_{\mathbf{F}}{\cdot}\sigma\colon \mathbf{F};\mathbf{G} \to \mathbf{F};\mathbf{G}'$, i.e., $(\mathbf{F}{\cdot}\sigma)_A = \sigma_{\mathbf{F}(A)}$



Show that indeed, $\tau{\cdot}\sigma$ is a natural transformation

Functor categories

Given two categories $\mathbf{K}, \mathbf{K}'$, define the *category of functors from $\mathbf{K}'$ to $\mathbf{K}$*, $\mathbf{K}^{\mathbf{K}'}$, as follows:

- objects: functors from $\mathbf{K}'$ to $\mathbf{K}$
- morphisms: natural transformations between them
- composition: vertical composition of the natural transformations

Exercises:

- View the category of S-sorted sets, $\mathbf{Set}^S$, as a functor category
- Show how any functor $\mathbf{F} : \mathbf{K}'' \to \mathbf{K}'$ induces a functor $(\mathbf{F};\_) : \mathbf{K}^{\mathbf{K}'} \to \mathbf{K}^{\mathbf{K}''}$
- Check whether $\mathbf{K}^{\mathbf{K}'}$ is (finitely) (co)complete whenever $\mathbf{K}$ is so.
- Check when $(\mathbf{F};\_) : \mathbf{K}^{\mathbf{K}'} \to \mathbf{K}^{\mathbf{K}''}$ is (finitely) (co)continuous, for a given functor $\mathbf{F} : \mathbf{K}'' \to \mathbf{K}'$

Yoneda embedding

Given a category $\mathbf{K}$, define

$$\mathcal{Y}\colon \mathbf{K} \to \mathbf{Set}^{\mathbf{K}^{op}}$$

- $\mathcal{Y}(A) = \mathbf{Hom_K}(\_, A)\colon \mathbf{K}^{op} \to \mathbf{Set}$, for $A \in |\mathbf{K}|$
- $\mathcal{Y}(f)_X = (\_;f)\colon \mathbf{Hom_K}(X, A) \to \mathbf{Hom_K}(X, B)$, for $f\colon A \to B$ in $\mathbf{K}$, for $X \in |\mathbf{K}^{op}|$.

Fact: *The category of presheaves* $\mathbf{Set}^{\mathbf{K}^{op}}$ *is complete and cocomplete.*

Fact: $\mathcal{Y}\colon \mathbf{K} \to \mathbf{Set}^{\mathbf{K}^{op}}$ *is full and faithful.*

Diagrams as functors

Each diagram D over graph G in category $\mathbf{K}$ yields a functor $\mathbf{F}_D\colon \mathbf{Path}(G) \to \mathbf{K}$ given by:

- $\mathbf{F}_D(n) = D_n$, for all nodes $n \in |G|_{nodes}$
- $\mathbf{F}_D(n_0 e_1 n_1 \ldots n_{k-1} e_k n_k) = D_{e_1};\ldots;D_{e_k}$, for paths $n_0 e_1 n_1 \ldots n_{k-1} e_k n_k$ in G

Moreover:

- for distinct diagrams D and D' of shape G, $\mathbf{F}_D$ and $\mathbf{F}_{D'}$ are different
- all functors from $\mathbf{Path}(G)$ to $\mathbf{K}$ are given by diagrams over G

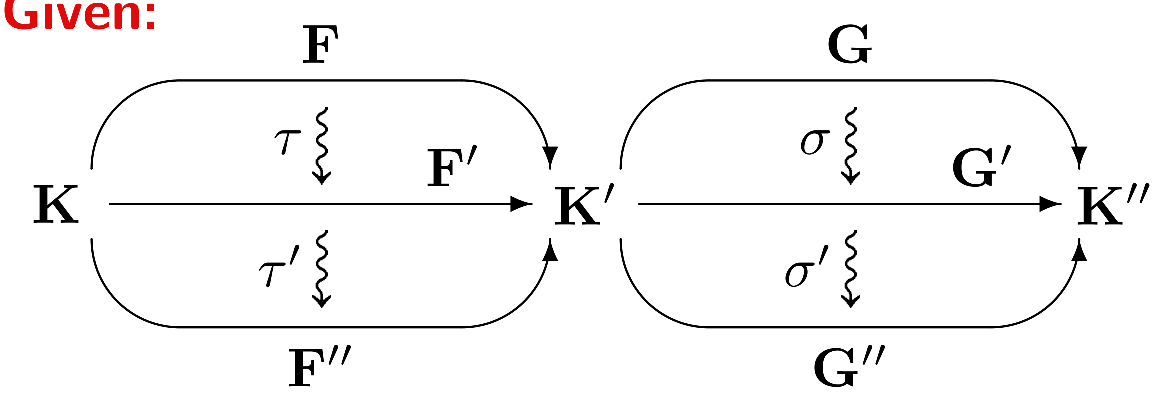
Diagram morphisms $\mu\colon D \to D'$ between diagrams of the same shape G are exactly natural transformations $\mu\colon \mathbf{F}_D \to \mathbf{F}_{D'}$.

$$\mathbf{Diag}^G_{\mathbf{K}} \cong \mathbf{K}^{\mathbf{Path}(G)}$$

Diagrams are functors from small (shape) categories

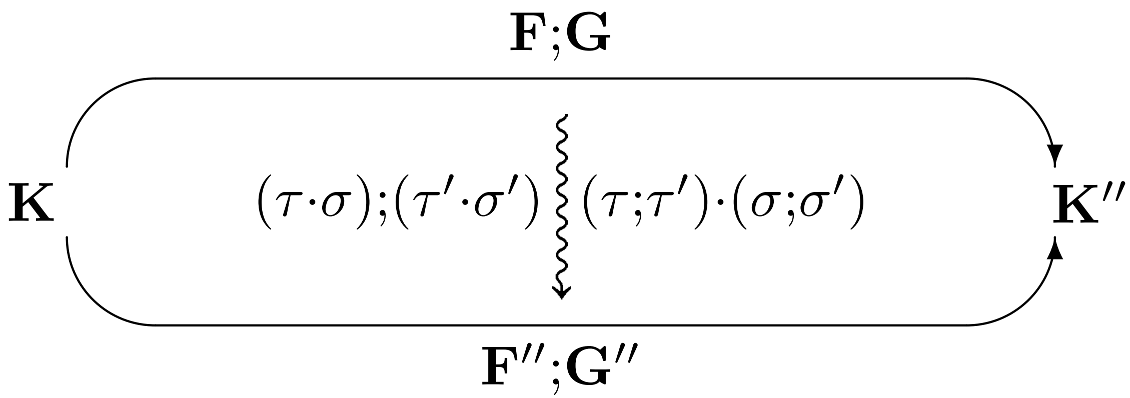
Double law

Given:



then:

$$(\tau\cdot\sigma);(\tau'\cdot\sigma') = (\tau;\tau')\cdot(\sigma;\sigma')$$



This holds in **Cat**, which is a paradigmatic example of a two-category.

A category $\mathbf{K}$ is a *two-category* when for all objects $A, B \in |\mathbf{K}|$, $\mathbf{K}(A,B)$ is again a category, with *1-morphisms* (the usual $\mathbf{K}$-morphisms) as objects and *2-morphisms* between them. Those 2-morphisms compose vertically (in the categories $\mathbf{K}(A,B)$) and horizontally, subject to the double law as stated here.

In two-category **Cat**, we have $\mathbf{Cat}(\mathbf{K}',\mathbf{K}) = \mathbf{K}^{\mathbf{K}'}$.

Equivalence of categories

- Two categories $\mathbf{K}$ and $\mathbf{K}'$ are *isomorphic* if there are functors $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ such that $\mathbf{F};\mathbf{G} = \mathbf{Id}_{\mathbf{K}}$ and $\mathbf{G};\mathbf{F} = \mathbf{Id}_{\mathbf{K}'}$.
- Two categories $\mathbf{K}$ and $\mathbf{K}'$ are *equivalent* if there are functors $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and natural isomorphisms $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ and $\epsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$.
- A category is *skeletal* if any two isomorphic objects are identical.
- A *skeleton* of a category is any of its maximal skeletal subcategory.

Fact: *Two categories are equivalent iff they have isomorphic skeletons.*

All "categorical" properties are preserved under equivalence of categories