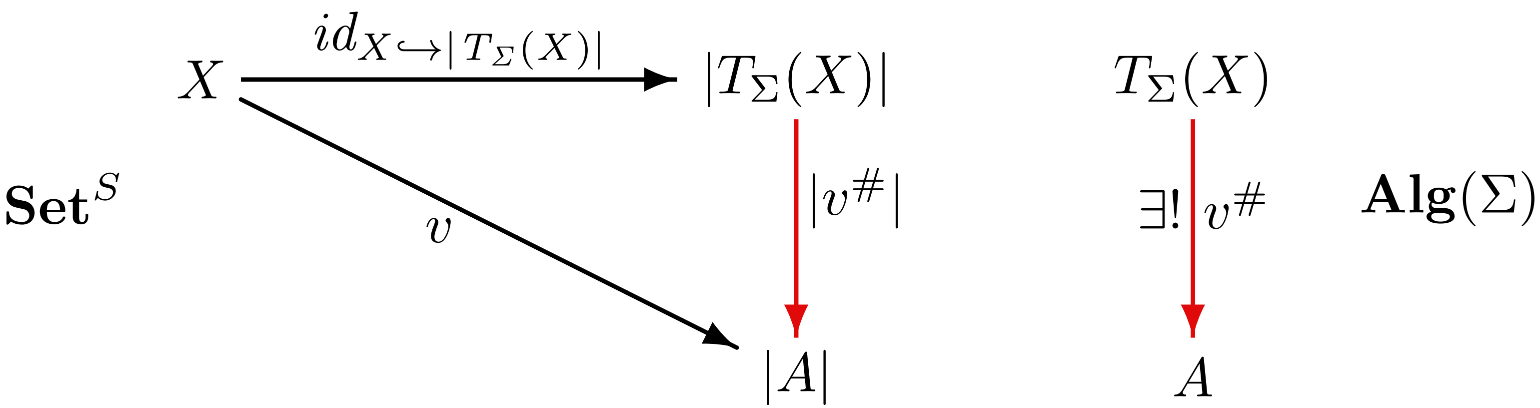
Adjunctions

Recall:

Term algebras

Fact: *For any S-sorted set X of variables, Σ-algebra A and valuation $v: X \to |A|$, there is a unique Σ-homomorphism $v^{\#}: T_\Sigma(X) \to A$ that extends v, so that*

$$id_{X \hookrightarrow |T_\Sigma(X)|};v^{\#} = v$$



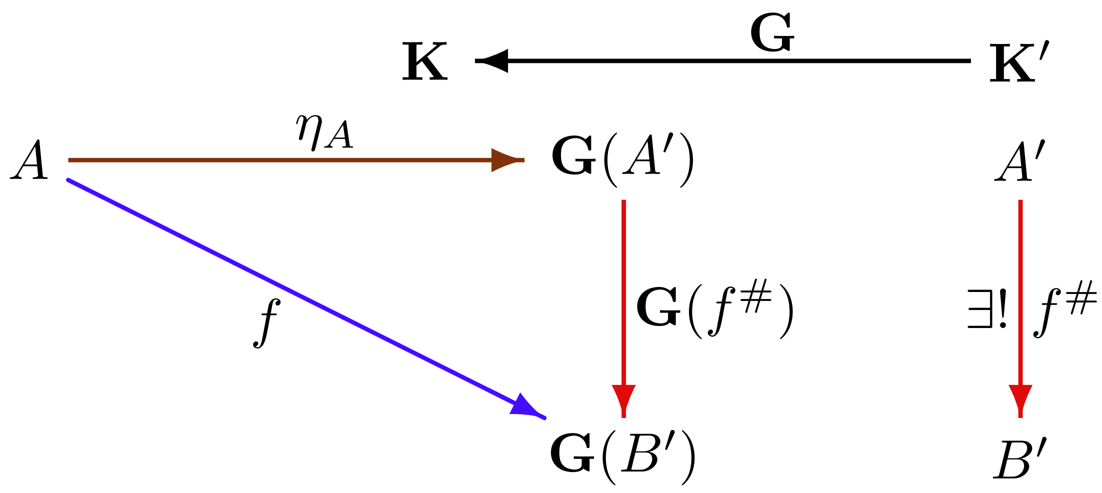
Free objects

Consider any functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$

Definition: *Given an object $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$ is a $\mathbf{K}'$-object $A' \in |\mathbf{K}'|$ together with a $\mathbf{K}$-morphism $\eta_A\colon A \to \mathbf{G}(A')$ (called unit morphism) such that given any $\mathbf{K}'$-object $B' \in |\mathbf{K}'|$ with $\mathbf{K}$-morphism $f\colon A \to \mathbf{G}(B')$, for a unique $\mathbf{K}'$-morphism $f^{\#}\colon A' \to B'$ we have*

$$\eta_A;\mathbf{G}(f^{\#}) = f$$

Paradigmatic example:
Term algebra $T_\Sigma(X)$ with unit $id_{X\hookrightarrow|T_\Sigma(X)|}\colon X \to |T_\Sigma(X)|$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$



Examples

- Consider inclusion $i\colon \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the ceiling of r, $\lceil r \rceil \in \mathbf{Int}$ is free over r w.r.t. i.

What about free objects w.r.t. the inclusion of rationals into reals?

- For any set $X \in |\mathbf{Set}|$, the "free monoid" $\mathbf{List}(X) = \langle X^*, {}^\frown, \epsilon \rangle$ is free over X w.r.t. $|\_|\colon \mathbf{Monoid} \to \mathbf{Set}$.
- For any graph $G \in |\mathbf{Graph}|$, the category of its paths, $\mathbf{Path}(G) \in |\mathbf{Cat}|$, is free over G w.r.t. the graph functor $G\colon \mathbf{Cat} \to \mathbf{Graph}$.
- Discrete topologies, completion of metric spaces, free groups, ideal completion of partial orders, ideal completion of free partial algebras, . . .

Makes precise these and other similar examples

Indicate unit morphisms!

Free equational models

- Recall: for any algebraic signature $\Sigma = \langle S, \Omega \rangle$, term algebra $T_\Sigma(X)$ is free over $X \in |\mathbf{Set}^S|$ w.r.t. the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$.
- For any set of Σ-equations Φ, for any set $X \in |\mathbf{Set}^S|$, there exist a model $\mathbf{F}_\Phi(X) \in Mod(\Phi)$ that is free over X w.r.t. the carrier functor $|\_|\colon \mathbf{Mod}(\langle \Sigma, \Phi \rangle) \to \mathbf{Set}^S$, where $\mathbf{Mod}(\langle \Sigma, \Phi \rangle)$ is the full subcategory of $\mathbf{Alg}(\Sigma)$ given by the models of Φ.
- For any algebraic signature morphism $\sigma\colon \Sigma \to \Sigma'$, for any Σ-algebra $A \in |\mathbf{Alg}(\Sigma)|$, there exist a Σ'-algebra $\mathbf{F}_\sigma(A) \in |\mathbf{Alg}(\Sigma')|$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$.
- For any equational specification morphism $\sigma\colon \langle \Sigma, \Phi \rangle \to \langle \Sigma', \Phi' \rangle$, for any model $A \in Mod(\Phi)$, there exist a model $\mathbf{F}_\sigma(A) \in Mod(\Phi')$ that is free over A w.r.t. the reduct functor $\_|_\sigma\colon \mathbf{Mod}(\langle \Sigma', \Phi' \rangle) \to \mathbf{Mod}(\langle \Sigma, \Phi \rangle)$.

Prove the above.

Facts

Consider a functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$, and object $A \in |\mathbf{K}|$, and an object $A' \in |\mathbf{K}'|$ free over A w.r.t. $\mathbf{G}$ with unit $\eta_A\colon A \to \mathbf{G}(A')$.

- A free objects over A w.r.t. $\mathbf{G}$ the initial objects in the comma category $(\mathbf{C}_A, \mathbf{G})$, where $\mathbf{C}_A\colon \mathbf{1} \to \mathbf{K}$ is the constant functor.
- A free object over A w.r.t. $\mathbf{G}$, if exists, is unique up to isomorphism.
- The function $(\_)^{\#}\colon \mathbf{K}(A, \mathbf{G}(B')) \to \mathbf{K}'(A', B')$ is bijective for each $B' \in |\mathbf{K}'|$.
- For any morphisms $g_1, g_2\colon A' \to B'$ in $\mathbf{K}'$, $g_1 = g_2$ iff $\eta_A;\mathbf{G}(g_1) = \eta_A;\mathbf{G}(g_2)$.

Colimits as free objects

Fact: *In a category $\mathbf{K}$, given a diagram D of shape $G(D)$, the colimit of D in $\mathbf{K}$ is a free object over D w.r.t. the diagonal functor $\Delta_{\mathbf{K}}^{G(D)}\colon \mathbf{K} \to \mathbf{Diag}_{\mathbf{K}}^{G(D)}$.*

Spell this out for initial objects, coproducts, coequalisers, and pushouts

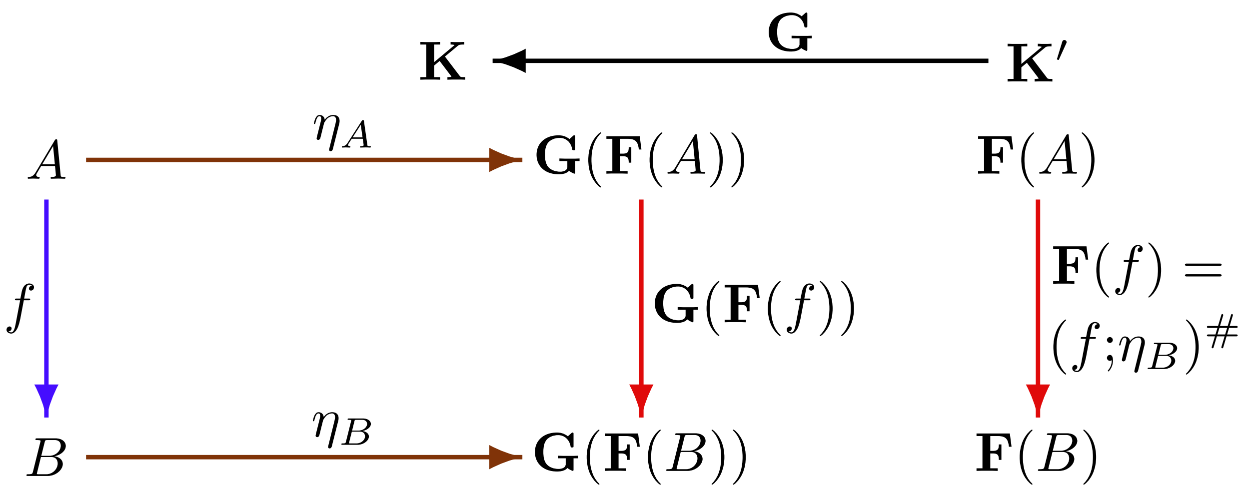
Left adjoints

Consider a functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$.

Fact: *Assume that for each object $A \in |\mathbf{K}|$ there is a free object over A w.r.t. $\mathbf{G}$, say $\mathbf{F}(A) \in |\mathbf{K}'|$ is free over A with unit $\eta_A\colon A \to \mathbf{G}(\mathbf{F}(A))$. Then the mapping:*

- $(A \in |\mathbf{K}|) \mapsto (\mathbf{F}(A) \in |\mathbf{K}'|)$
- $(f\colon A \to B) \mapsto ((f;\eta_B)^{\#}\colon \mathbf{F}(A) \to \mathbf{F}(B))$

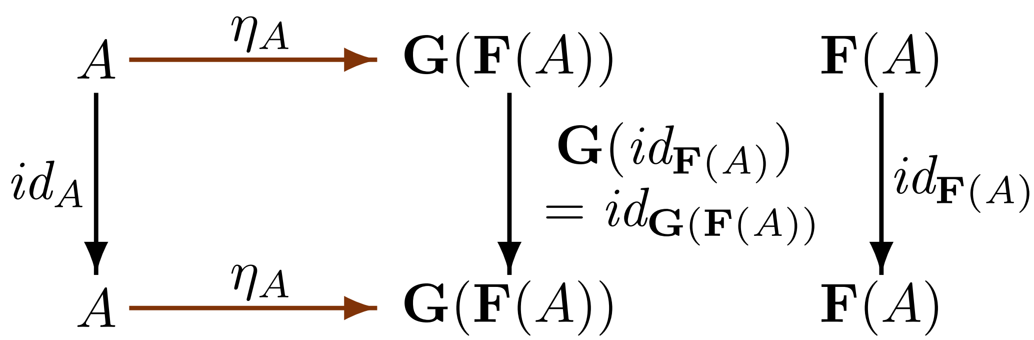
form a functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$. Moreover, $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ is a natural transformation.



Proof

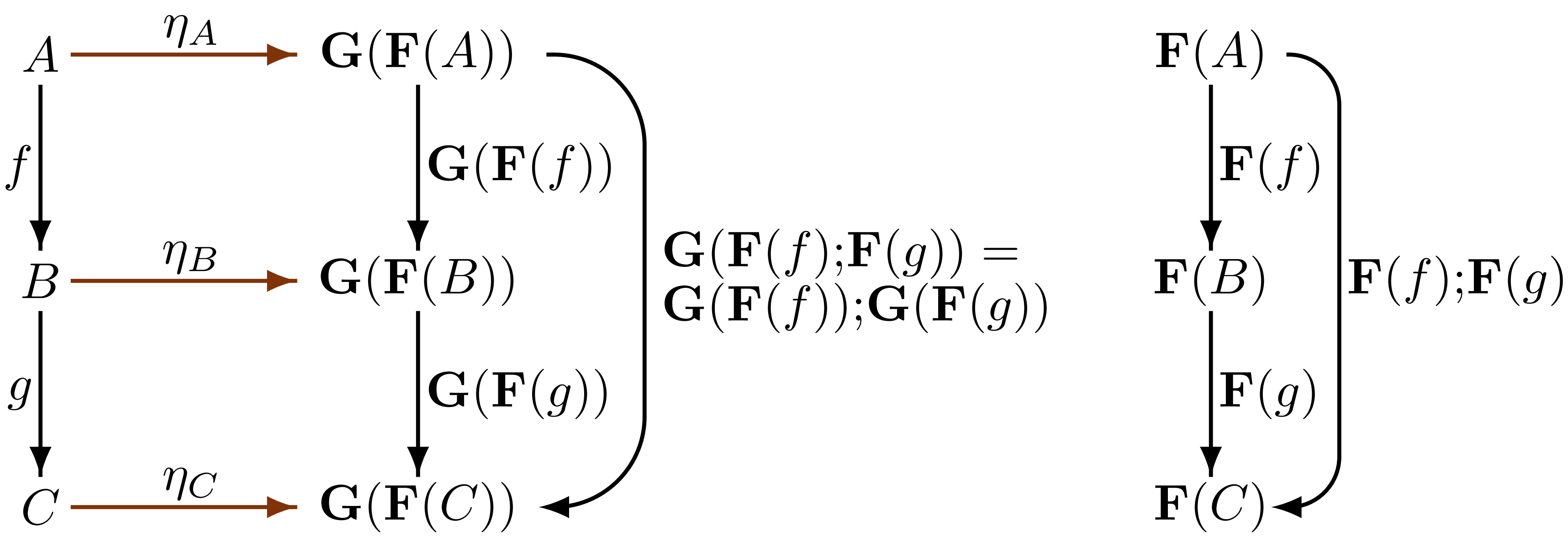
$\mathbf{F}$ preserves identities:

$$\mathbf{F}(id_A) = (id_A;\eta_A)^{\#} = id_{\mathbf{F}(A)}$$



$\mathbf{F}$ preserves composition:

$$\mathbf{F}(f;g) = (f;g;\eta_C)^{\#} = \mathbf{F}(f);\mathbf{F}(g)$$



Left adjoints

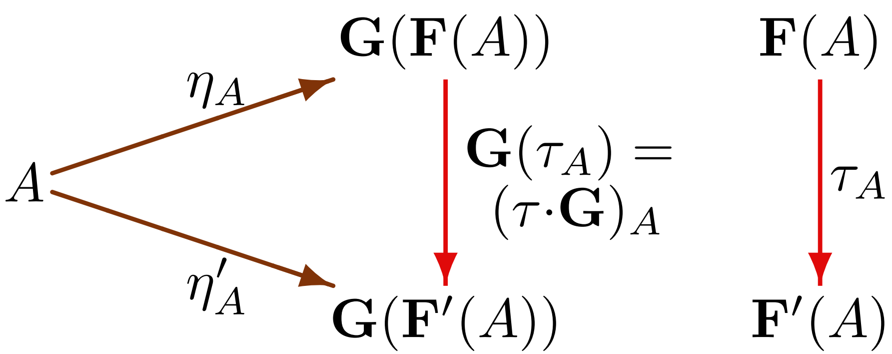
Definition: *A functor* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *is left adjoint to (a functor)* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *with unit (natural transformation)* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *if for all objects* $A \in |\mathbf{K}|$, $\mathbf{F}(A) \in |\mathbf{K}'|$ *is free over* A *with unit morphism* $\eta_A\colon A \to \mathbf{G}(\mathbf{F}(A))$.

Examples

- The term-algebra functor $T_\Sigma\colon \mathbf{Set}^S \to \mathbf{Alg}(\Sigma)$ is left adjoint to the carrier functor $|\_|\colon \mathbf{Alg}(\Sigma) \to \mathbf{Set}^S$, for any algebraic signature $\Sigma = \langle S, \Omega \rangle$.
- The ceiling $\lceil \_ \rceil\colon \mathbf{Real} \to \mathbf{Int}$ is left adjoint to the inclusion $i\colon \mathbf{Int} \hookrightarrow \mathbf{Real}$ of integers into reals.
- The path-category functor $\mathbf{Path}\colon \mathbf{Graph} \to \mathbf{Cat}$ is left adjoint to the graph functor $G\colon \mathbf{Cat} \to \mathbf{Graph}$.
- ... other examples given by the examples of free objects above ...

Uniqueness of left adjoints

Fact: *A left adjoint to any functor* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$*, if exists, is determined uniquely up to a natural isomorphism: if* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{F}'\colon \mathbf{K} \to \mathbf{K}'$ *are left adjoint to* $\mathbf{G}$ *with units* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\eta'\colon \mathbf{Id_K} \to \mathbf{F}';\mathbf{G}$*, respectively, then there exists a natural isomorphism* $\tau\colon \mathbf{F} \to \mathbf{F}'$ *such that* $\eta;(\tau{\cdot}\mathbf{G}) = \eta'$*.*



Proof: For each $A \in |\mathbf{K}|$, $\tau_A = (\eta'_A)^{\#}$.

Put also $\tau_A^{-1} = (\eta_A)^{\#'}$.

Then show:

- $\tau_A;\tau_A^{-1} = id_{\mathbf{F}(A)}$ and $\tau_A^{-1};\tau_A = id_{\mathbf{F}'(A)}$
- $\tau\colon \mathbf{F} \to \mathbf{F}'$ is indeed a natural transformation

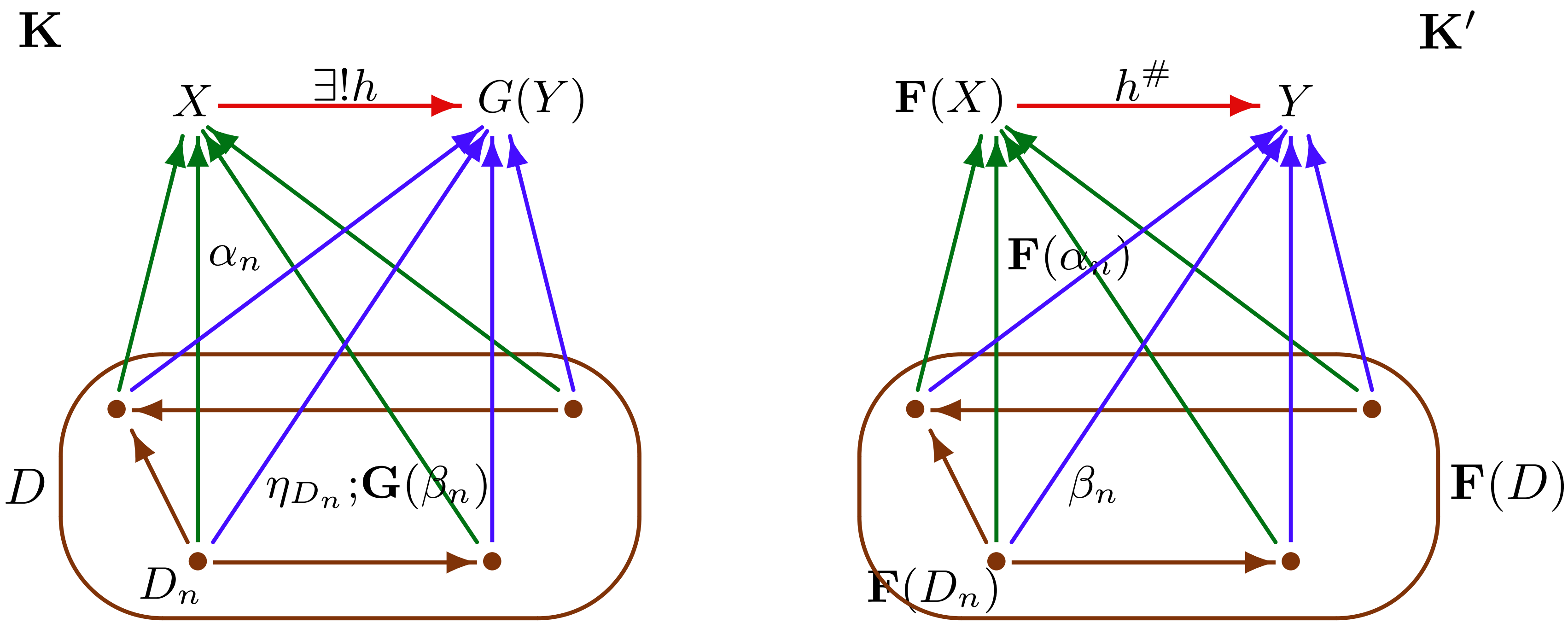
- For $f\colon A \to B$, $\mathbf{F}(f) = (f;\eta_B)^{\#}$.
- For $g_1, g_2\colon \mathbf{F}(A) \to \bullet$, if $\eta_A;\mathbf{G}(g_1) = \eta_A;\mathbf{G}(g_2)$ then $g_1 = g_2$.

Left adjoints and colimits

Let $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ with unit $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$.

Fact: $\mathbf{F}$ *is cocontinuous (preserves colimits).*

Proof:

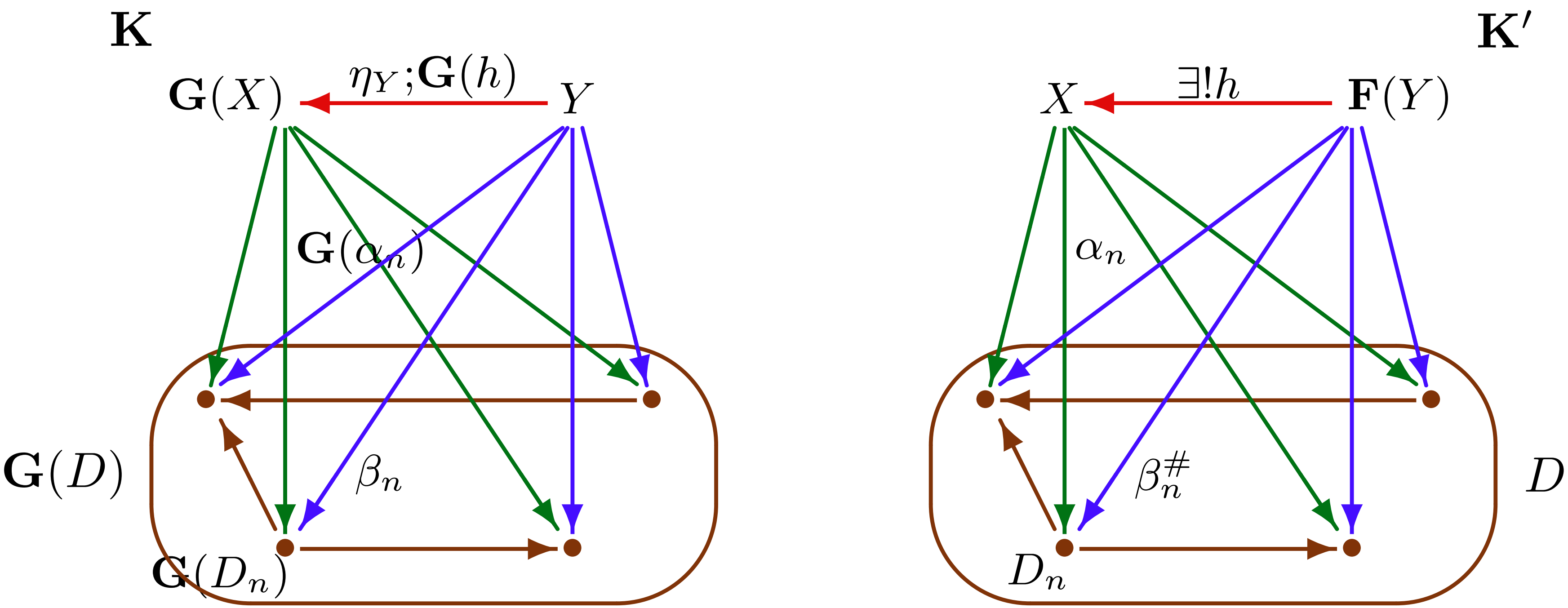


Left adjoints and limits

Let $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ with unit $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$.

Fact: $\mathbf{G}$ *is continuous (preserves limits).*

Proof:



Existence of left adjoints

Fact: *Let $\mathbf{K}'$ be a locally small complete category. Then a functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ has a left adjoint iff*

1. *$\mathbf{G}$ is continuous, and*
2. *for each $A \in |\mathbf{K}|$ there exists a set $\{f_i \colon A \to \mathbf{G}(X_i) \mid i \in \mathcal{I}\}$ (of objects $X_i \in |\mathbf{K}'|$ with morphisms $f_i \colon A \to \mathbf{G}(X_i)$, $i \in \mathcal{I}$) such that for each $B \in |\mathbf{K}'|$ and $h \colon A \to \mathbf{G}(B)$, for some $f \colon X_i \to B$, $i \in \mathcal{I}$, we have $h = f_i;f$.*

Proof:

"$\Rightarrow$": Let $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ be left adjoint to $\mathbf{G}$ with unit $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$. Then 1 follows by the previous fact, and for 2 just put $\mathcal{I} = \{*\}$, $X_* = \mathbf{F}(A)$, and $f_* = \eta_A \colon A \to \mathbf{G}(\mathbf{F}(A))$

"$\Leftarrow$": It is enough to show that for each $A \in |\mathbf{K}|$ the comma category $(\mathbf{C}_A, \mathbf{G})$ has an initial object. Under our assumptions, $(\mathbf{C}_A, \mathbf{G})$ is complete. The rest follows by the next fact.

On the existence of initial objects

Fact: *A locally small complete category* $\mathbf{K}$ *has an initial object if there exists a set of objects* $\mathcal{I} \subseteq |\mathbf{K}|$ *such that for all* $B \in |\mathbf{K}|$*, for some* $X \in \mathcal{I}$ *there is* $f \colon X \to B$*.*

Proof: Let $P \in |\mathbf{K}|$ be a product of $\mathcal{I}$, with projections $p_X \colon P \to X$ for $X \in \mathcal{I}$. Let $e \colon E \to P$ be an "equaliser" (limit) of all morphisms in $\mathbf{K}(P, P)$. Then E is initial in $\mathbf{K}$, since for any $B \in |\mathbf{K}|$:

- $e;p_X;f \colon E \to B$, where $f \colon X \to B$ for some $X \in \mathcal{I}$.
- Given $g_1, g_2 \colon E \to B$, take their equaliser $e' \colon E' \to E$. As in the previous item, we have $h \colon P \to E'$. Then $h;e;e' \colon P \to P$, and by the construction of $e \colon E \to P$, $e;h;e';e = e;id_P = id_E;e$. Now, since e is mono, $e;h;e' = id_E$, and so e' is a mono retraction, hence an isomorphism, which proves $g_1 = g_2$.

Cofree objects

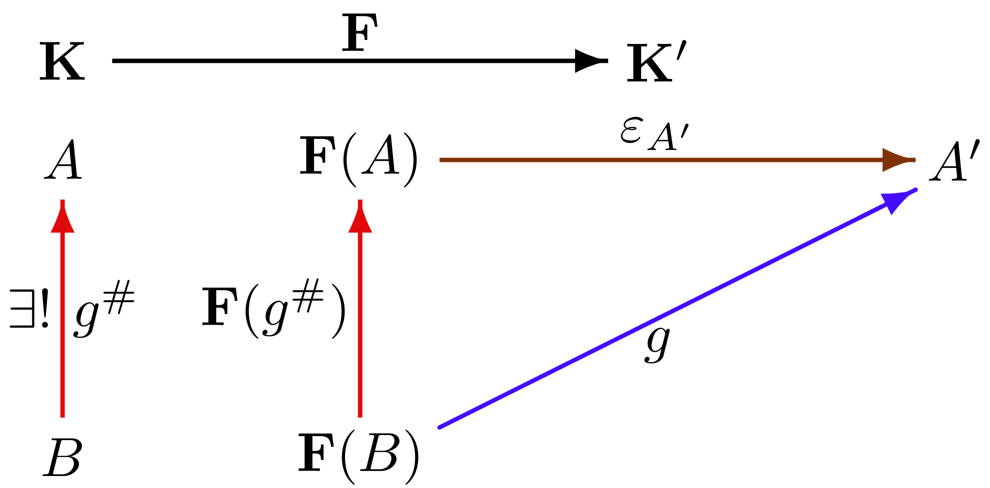
Consider any functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$

Definition: *Given an object $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$ is a $\mathbf{K}$-object $A \in |\mathbf{K}|$ together with a $\mathbf{K}$-morphism $\varepsilon_{A'}\colon \mathbf{F}(A) \to A'$ (called counit morphism) such that given any $\mathbf{K}$-object $B \in |\mathbf{K}|$ with $\mathbf{K}'$-morphism $g\colon \mathbf{F}(B) \to A'$, for a unique $\mathbf{K}$-morphism $g^{\#}\colon B \to A$ we have*

$$\mathbf{F}(g^{\#});\varepsilon_{A'} = g$$

Paradigmatic example:

Function spaces, coming soon



Examples

- Consider inclusion $i\colon \mathbf{Int} \hookrightarrow \mathbf{Real}$, viewing $\mathbf{Int}$ and $\mathbf{Real}$ as (thin) categories, and i as a functor between them. For any real $r \in \mathbf{Real}$, the floor of r, $\lfloor r \rfloor \in \mathbf{Int}$ is cofree under r w.r.t. i.

What about cofree objects w.r.t. the inclusion of rationals into reals?

- Fix a set $X \in |\mathbf{Set}|$. Consider functor $\mathbf{F}_X\colon \mathbf{Set} \to \mathbf{Set}$ defined by:
  - for any set $A \in |\mathbf{Set}|$, $\mathbf{F}_X(A) = A \times X$
  - for any function $f\colon A \to B$, $\mathbf{F}_X(f)\colon A \times X \to B \times X$ is a function given by $\mathbf{F}_X(f)(\langle a, x \rangle) = \langle f(a), x \rangle$.

  Then for any set $A \in |\mathbf{Set}|$, the powerset $A^X \in |\mathbf{Set}|$ (i.e., the set of all functions from X to A) is a cofree objects under A w.r.t. $\mathbf{F}_X$. The counit morphism $\varepsilon_A\colon \mathbf{F}_X(A^X) = A^X \times X \to A$ is the evaluation function: $\varepsilon_A(\langle f, x \rangle) = f(x)$.

A generalisation to deal with exponential objects will (not) be discussed later

Facts

Dual to those for free objects: Consider a functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$, object $A' \in |\mathbf{K}'|$, and an object $A \in |\mathbf{K}|$ cofree under A' w.r.t. $\mathbf{F}$ with counit $\varepsilon_{A'}\colon \mathbf{F}(A) \to A'$.

- Cofree objects under A' w.r.t. $\mathbf{F}$ are the terminal objects in the comma category $(\mathbf{F}, \mathbf{C}_{A'})$, where $\mathbf{C}_{A'}\colon \mathbf{1} \to \mathbf{K}'$ is the constant functor.
- A cofree object under A' w.r.t. $\mathbf{F}$, if exists, is unique up to isomorphism.
- The function $(\_)^{\#}\colon \mathbf{K}'(\mathbf{F}(B), A') \to \mathbf{K}(B, A)$ is bijective for each $B \in |\mathbf{K}|$.
- For any morphisms $g_1, g_2\colon B \to A$ in $\mathbf{K}$, $g_1 = g_2$ iff $\mathbf{F}(g_1);\varepsilon_{A'} = \mathbf{F}(g_2);\varepsilon_{A'}$.

Limits as cofree objects

Fact: *In a category* $\mathbf{K}$, *given a diagram* D *of shape* $G(D)$, *the limit of* D *in* $\mathbf{K}$ *is a cofree object under* D *w.r.t. the diagonal functor* $\Delta_{\mathbf{K}}^{G(D)}\colon \mathbf{K} \to \mathbf{Diag}_{\mathbf{K}}^{G(D)}$.

Spell this out for terminal objects, products, equalisers, and pullbacks

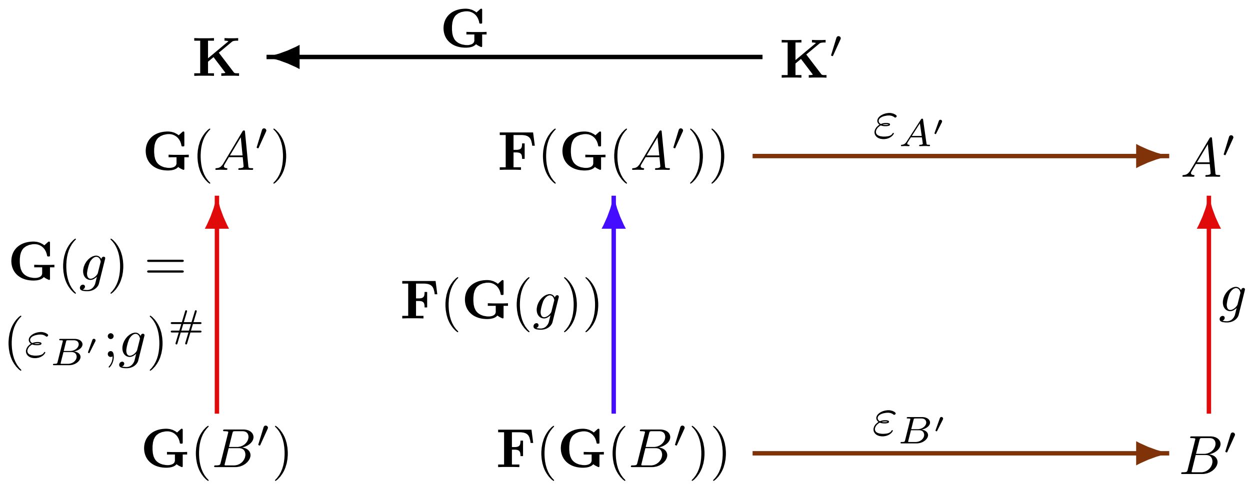
Right adjoints

Consider a functor $\mathbf{F} \colon \mathbf{K} \to \mathbf{K}'$.

Fact: *Assume that for each object $A' \in |\mathbf{K}'|$ there is a cofree object under A' w.r.t. $\mathbf{F}$, say $\mathbf{G}(A') \in |\mathbf{K}|$ is cofree under A' with counit $\varepsilon_{A'} \colon \mathbf{F}(\mathbf{G}(A')) \to A'$. Then the mapping:*

- $(A' \in |\mathbf{K}'|) \mapsto (\mathbf{G}(A') \in |\mathbf{K}|)$
- $(g \colon B' \to A') \mapsto ((\varepsilon_{B'};g)^{\#} \colon \mathbf{G}(B') \to \mathbf{G}(A'))$

form a functor $\mathbf{G} \colon \mathbf{K}' \to \mathbf{K}$. Moreover, $\varepsilon \colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ is a natural transformation.



Right adjoints

Definition: *A functor* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *is right adjoint to (a functor)* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *with counit (natural transformation)* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *if for all objects* $A' \in |\mathbf{K}'|$*,* $\mathbf{G}(A') \in |\mathbf{K}|$ *is cofree under* A' *with counit morphism* $\varepsilon_{A'}\colon \mathbf{F}(\mathbf{G}(A')) \to A'$*.*

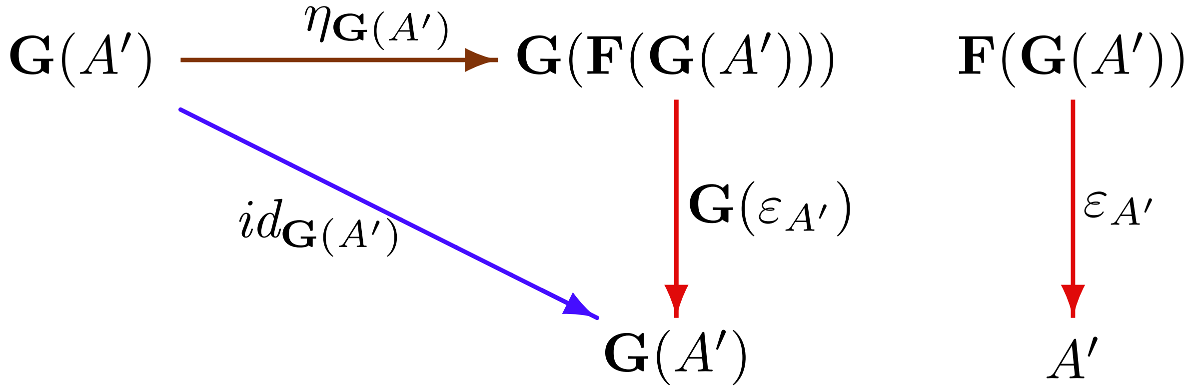
Fact: *A right adjoint to any functor* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$*, if exists, is determined uniquely up to a natural isomorphism: if* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *and* $\mathbf{G}'\colon \mathbf{K}' \to \mathbf{K}$ *are right adjoint to* $\mathbf{F}$ *with counits* $\varepsilon\colon \mathbf{G};\mathbf{F}$ *and* $\varepsilon'\colon \mathbf{G}';\mathbf{F}$*, respectively, then there exists a natural isomorphism* $\tau\colon \mathbf{G} \to \mathbf{G}'$ *such that* $(\tau \cdot \mathbf{F});\varepsilon' = \varepsilon$*.*

Fact: *Let* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *be right adjoint to* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *with counit* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$*. Then* $\mathbf{G}$ *is continuous (preserves limits) and* $\mathbf{F}$ *is cocontinuous (preserves colimits).*

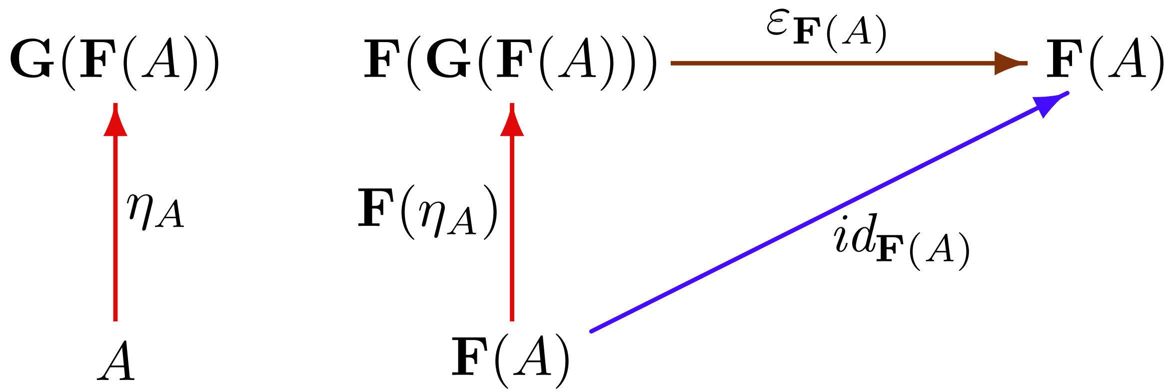
From left adjoints to adjunctions

Fact: *Let* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *be left adjoint to* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *with unit* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$.
Then there is a natural transformation $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$



- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

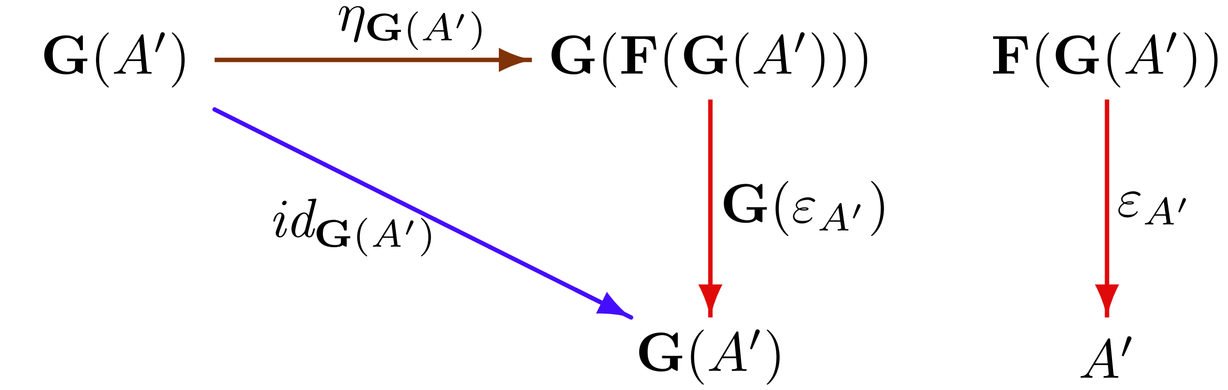


Proof (idea):
Put $\varepsilon_{A'} = (id_{\mathbf{G}(A')})^{\#}$.

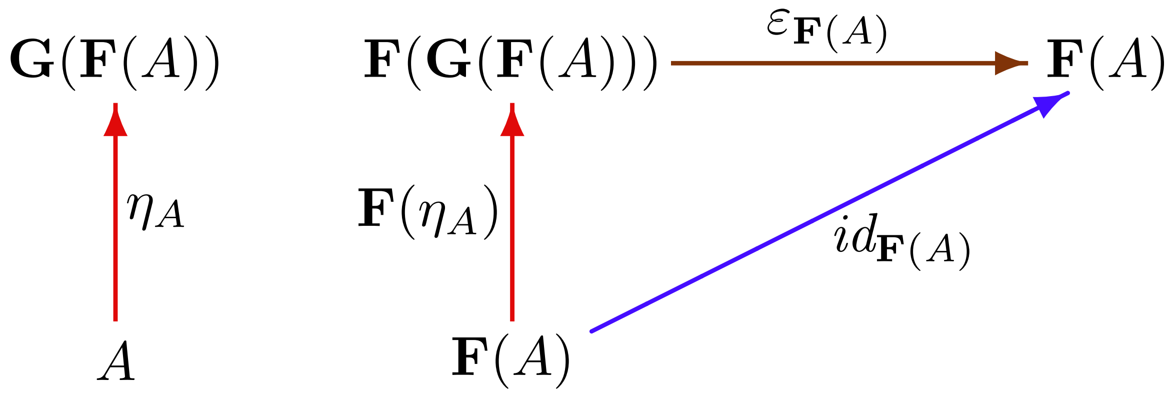
From right adjoints to adjunctions

Fact: *Let* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *be right adjoint to* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *with counit* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$*.*
Then there is a natural transformation $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *such that:*

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$



- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$



Proof (idea):
Put $\eta_A = (id_{\mathbf{F}(A)})^{\#}$.

From adjunctions to left and right adjoints

Fact: *Consider two functors* $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *with natural transformations* $\eta\colon \mathbf{Id_K} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id_{K'}}$ *such that:*

- $(\mathbf{G}\cdot\eta);(\varepsilon\cdot\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta\cdot\mathbf{F});(\mathbf{F}\cdot\varepsilon) = id_{\mathbf{F}}$

Then:

- $\mathbf{F}$ *is left adjoint to* $\mathbf{G}$ *with unit* η*.*
- $\mathbf{G}$ *is right adjoint to* $\mathbf{F}$ *with counit* ε*.*

Proof: For $A \in |\mathbf{K}|$, $B' \in |\mathbf{K}'|$ and $f\colon A \to \mathbf{G}(B')$, define $f^\# = \mathbf{F}(f);\varepsilon_{B'}$. Then $f^\#\colon \mathbf{F}(A) \to B'$ satisfies $\eta_A;\mathbf{G}(f^\#) = f$ and is the only such morphism in $\mathbf{K}'(\mathbf{F}(A), B')$. This proves that $\mathbf{F}(A)$ is free over A with unit η_A, and so indeed, $\mathbf{F}$ is left adjoint to $\mathbf{G}$ with unit η.

The proof that $\mathbf{G}$ is right adjoint to $\mathbf{F}$ with counit ε is similar.

Adjunctions

Definition: *An adjunction between categories* $\mathbf{K}$ *and* $\mathbf{K}'$ *is*

$$\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle$$

where $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ *and* $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ *are functors, and* $\eta\colon \mathbf{Id}_{\mathbf{K}} \to \mathbf{F};\mathbf{G}$ *and* $\varepsilon\colon \mathbf{G};\mathbf{F} \to \mathbf{Id}_{\mathbf{K}'}$ *natural transformations such that:*

- $(\mathbf{G}{\cdot}\eta);(\varepsilon{\cdot}\mathbf{G}) = id_{\mathbf{G}}$
- $(\eta{\cdot}\mathbf{F});(\mathbf{F}{\cdot}\varepsilon) = id_{\mathbf{F}}$

Equivalently, such an adjunction may be given by:

- Functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and all $A \in |\mathbf{K}|$, a free object over A w.r.t. $\mathbf{G}$.
- Functor $\mathbf{G}\colon \mathbf{K}' \to \mathbf{K}$ and its left adjoint.
- Functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and all $A' \in |\mathbf{K}'|$, a cofree object under A' w.r.t. $\mathbf{F}$.
- Functor $\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$ and its right adjoint.