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is a family $\tau = \langle \tau_A \colon \mathbf{F}(A) \to \mathbf{G}(A) \rangle_{A \in |\mathbf{K}|}$ of $\mathbf{K}'$-morphisms

$$\begin{array}{lll} \mathbf{K}\colon & \mathbf{K}'\colon & \\ A & \mathbf{F}(A) \xrightarrow{\tau_A} \mathbf{G}(A) & \\ & & \\ B & \mathbf{F}(B) \xrightarrow{\tau_B} \mathbf{G}(B) & \end{array}$$

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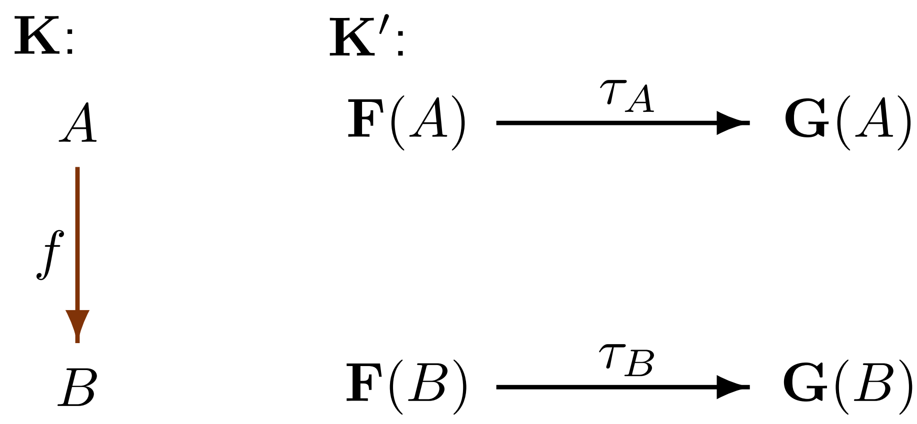

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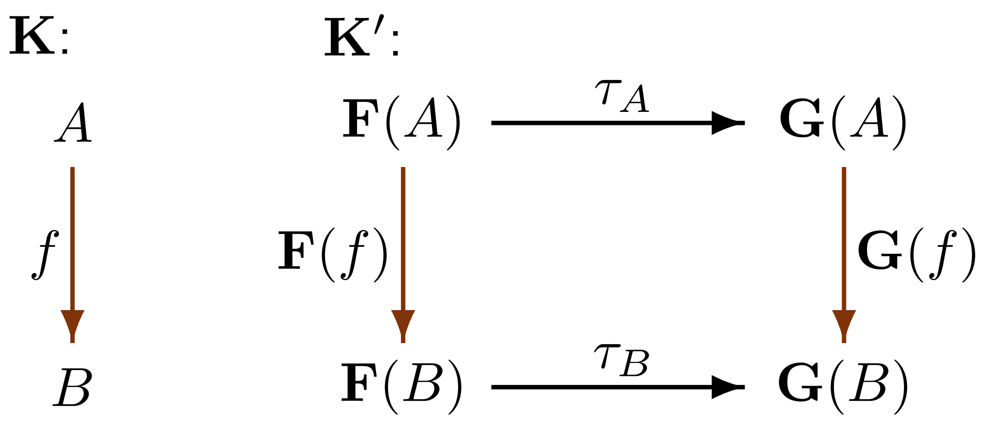

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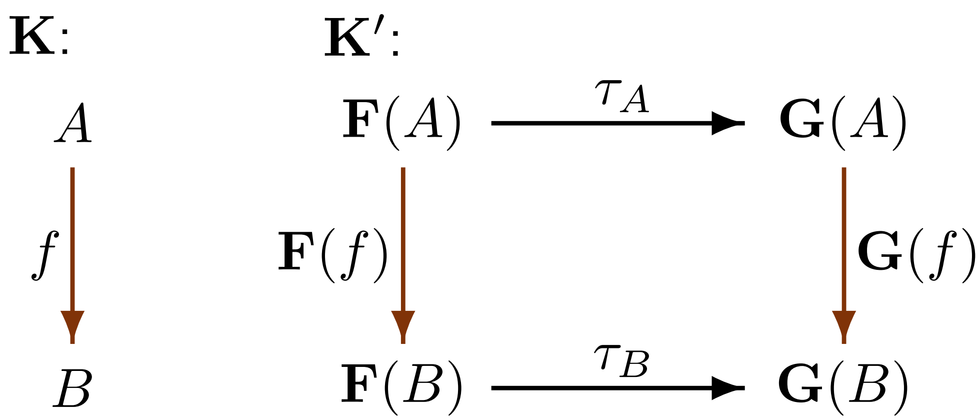

Then, τ is a *natural isomorphism* if for all $A \in |\mathbf{K}|$, τ_A is an isomorphism.

Examples

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- *identity transformations*: $id_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}$, where $\mathbf{F} \colon \mathbf{K} \to \mathbf{K}'$, for all objects $A \in |\mathbf{K}|$, $(id_{\mathbf{F}})_A = id_A \colon \mathbf{F}(A) \to \mathbf{F}(A)$

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$$X \qquad \mathbf{Id_{Set}}(X) \xrightarrow{sing_X} \mathbf{P}(X)$$

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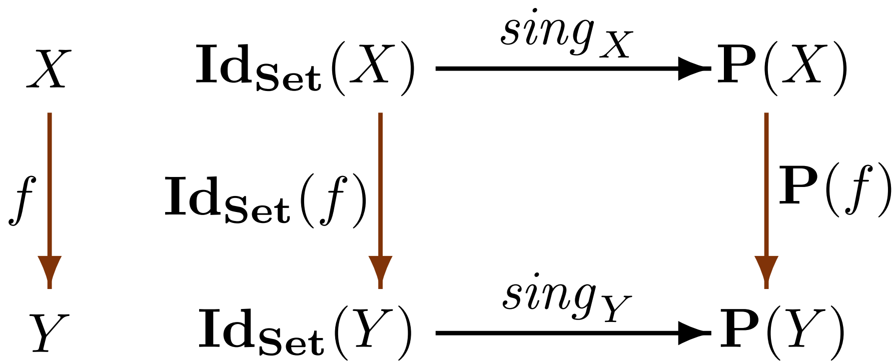

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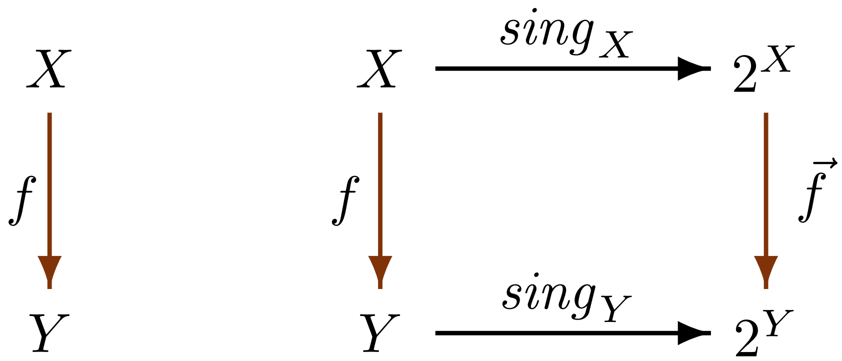

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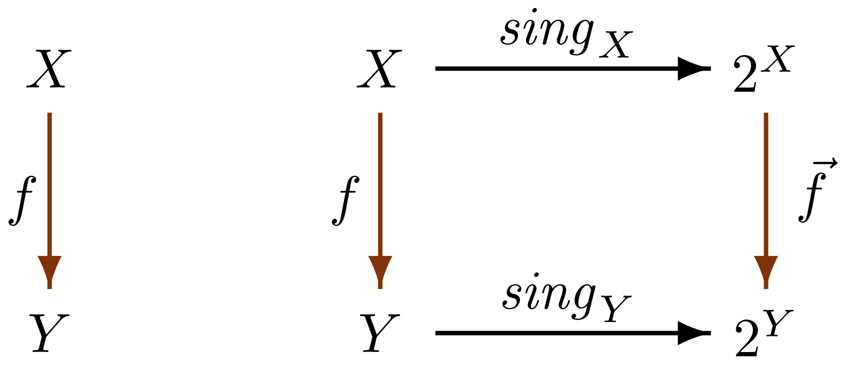

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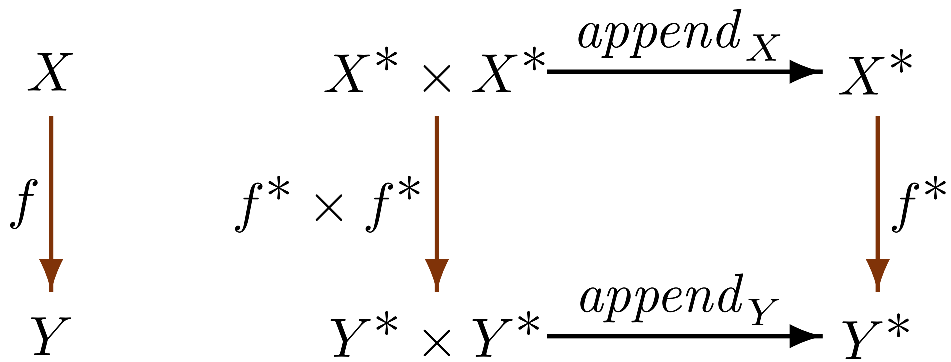

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  - $[\![\alpha_i]\!](X_1, \ldots, X_n) = X_i$
  - $[\![\mathtt{int}]\!](X_1, \ldots, X_n) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
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  - $[\![T_1 + T_2]\!](X_1, \ldots, X_n) = [\![T_1]\!](X_1, \ldots, X_n) + [\![T_2]\!](X_1, \ldots, X_n)$
  - $\ldots$
  - . . . recursive type definitions work as well. . .

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  - by induction on the structure of well-typed expressions

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For instance, for $rev\colon \alpha$ `list` $\to \alpha$ `list`, $even\colon$ `int` $\to$ `bool` and $l\colon$ `int list`:

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Theorems for free!
(see Wadler 89)

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EXERCISES:

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- Characterise all natural transformations from $\mathbf{Hom_K}(A, \_)$ to $\mathbf{Hom_K}(B, \_)$, for all objects $A, B \in |\mathbf{K}|$.

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- For $a \in \mathbf{F}(A)$, define $\tau^a \colon \mathbf{Hom}_{\mathbf{K}}(A, \_) \to \mathbf{F}$, as the family of functions $\tau^a_B \colon \mathbf{K}(A, B) \to \mathbf{F}(B)$, $B \in |\mathbf{K}|$,

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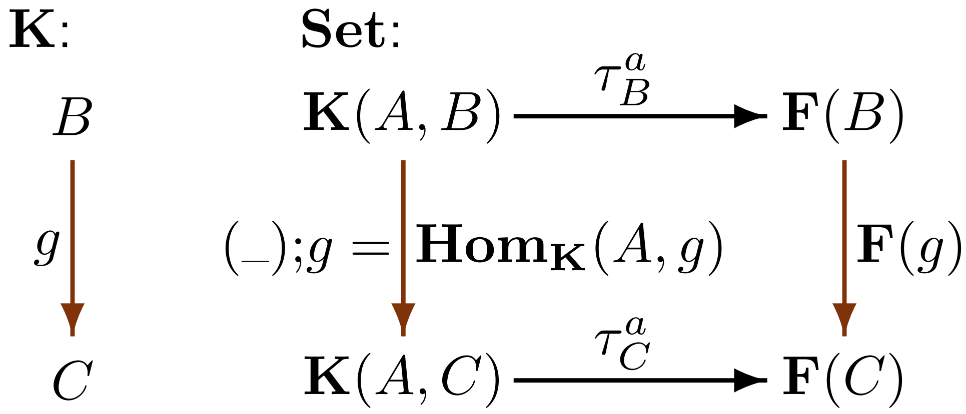

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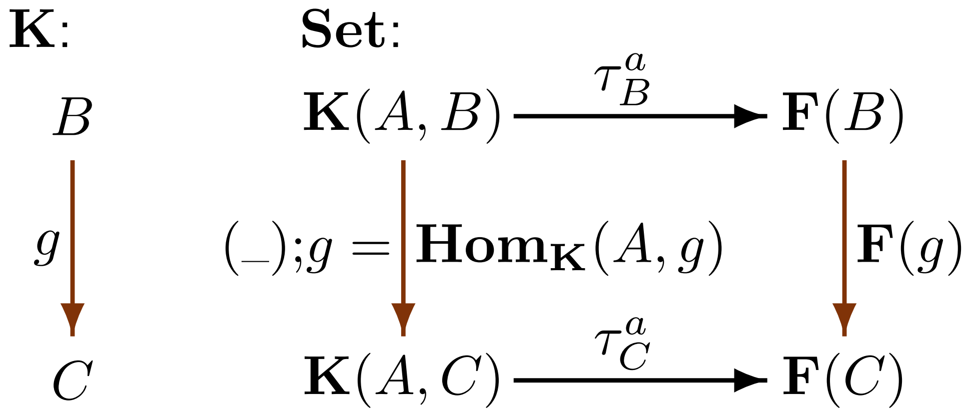

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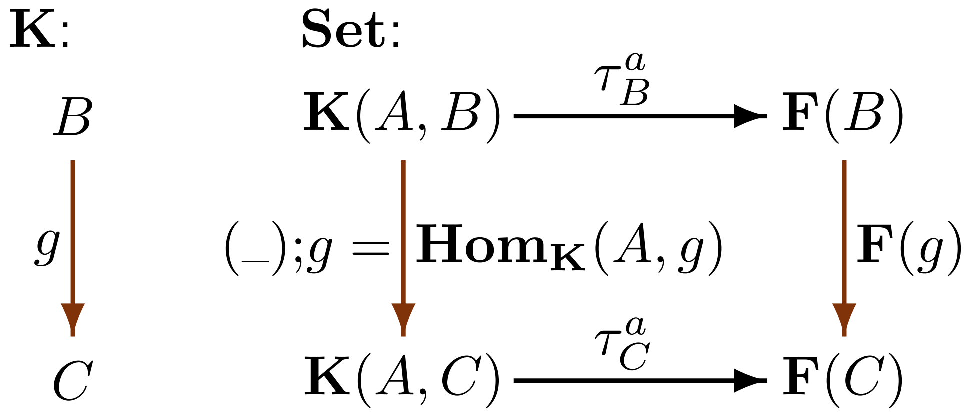

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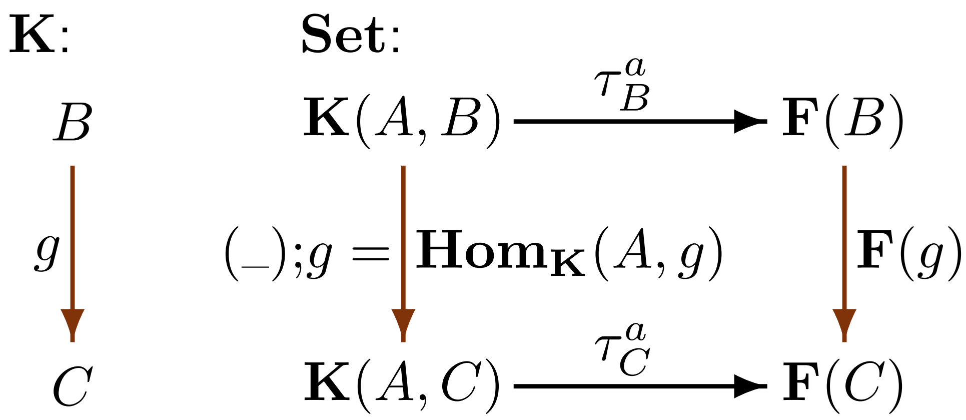

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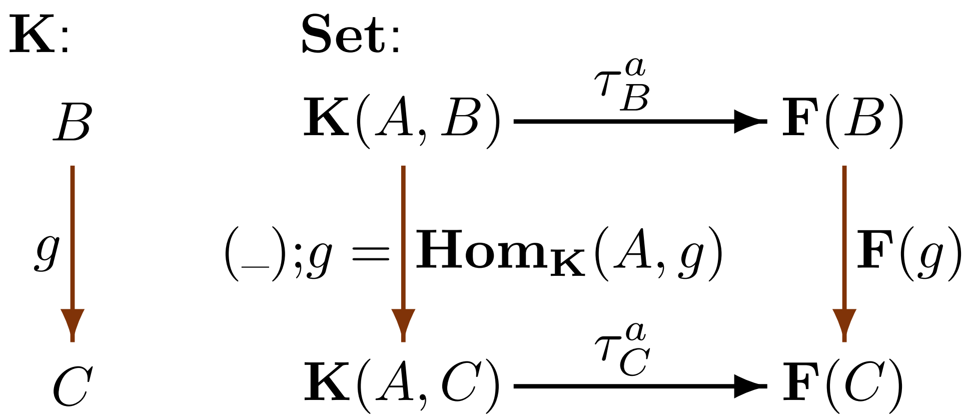

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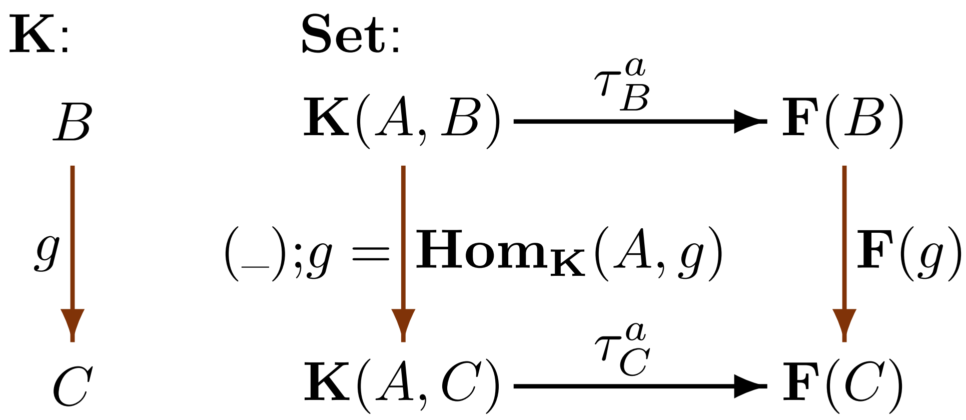

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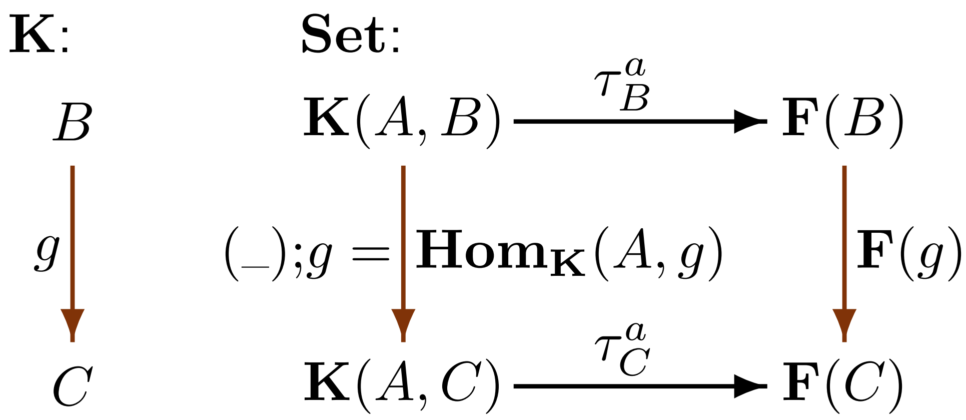

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  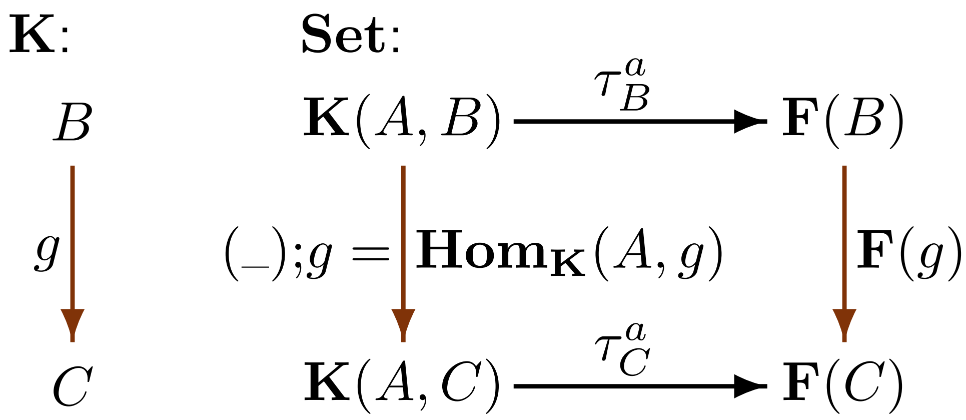

  Then $\tau^a_A(id_A) = a$, and so for distinct $a, a' \in \mathbf{F}(A)$, τ^a and $\tau^{a'}$ differ.

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  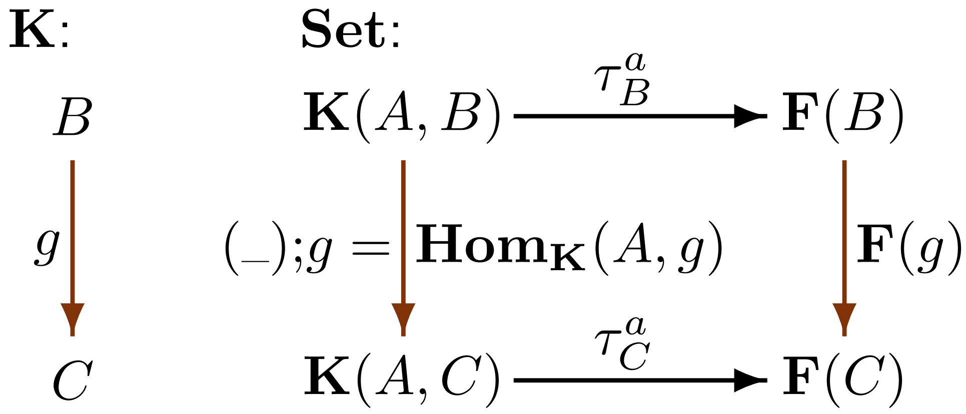

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  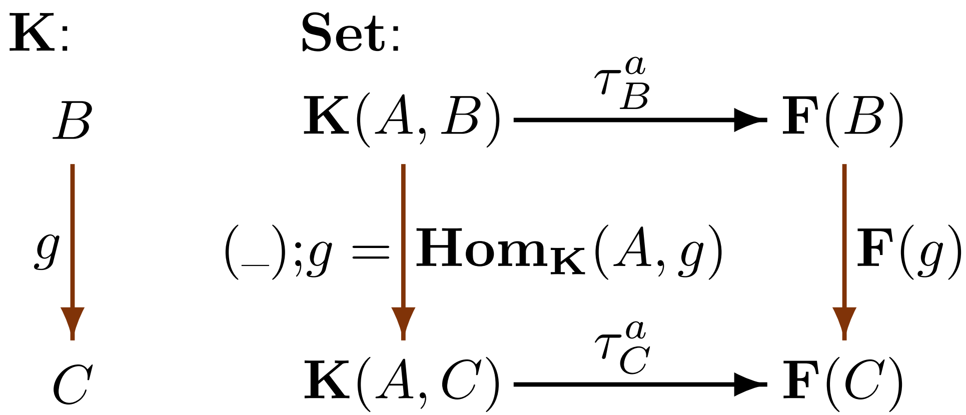

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  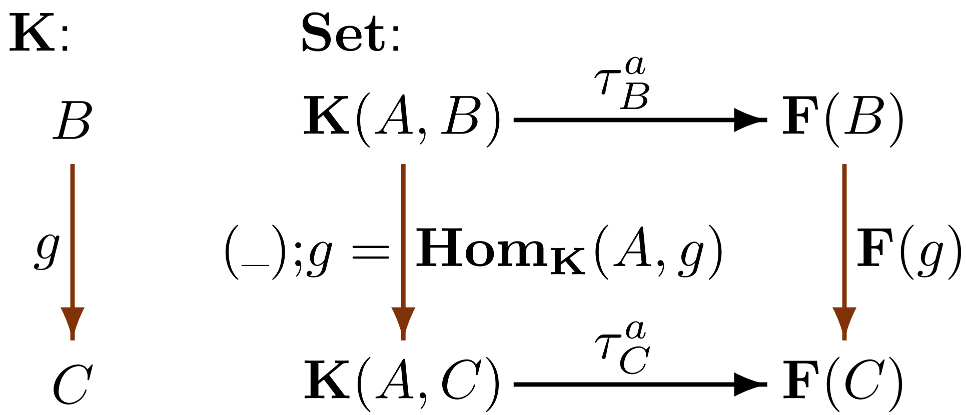

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- If $\tau \colon \mathbf{Hom_K}(A,\_) \to \mathbf{F}$ is a natural transformation then $\tau = \tau^a$, where we put $a = \tau_A(id_A)$,

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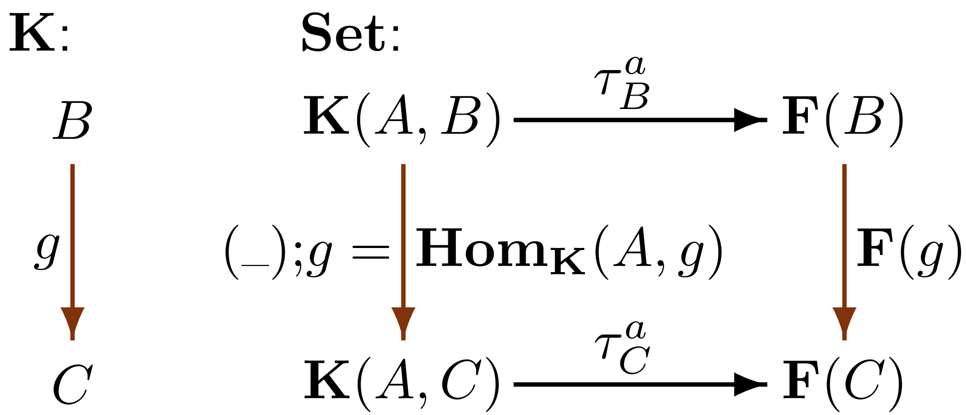

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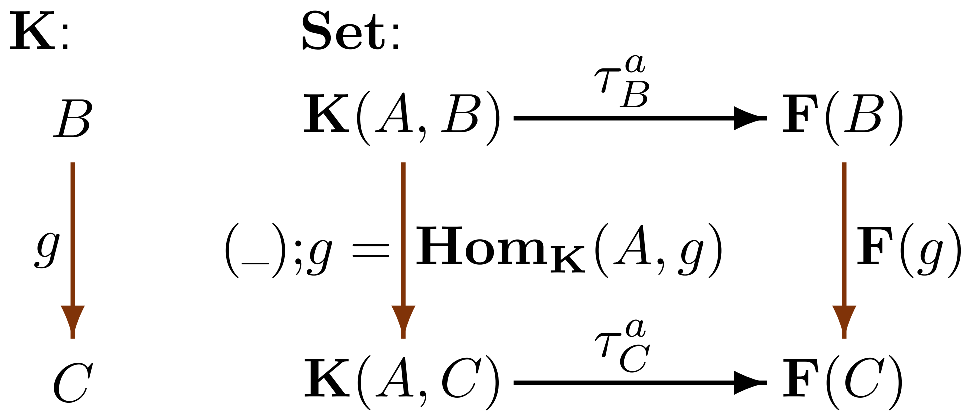

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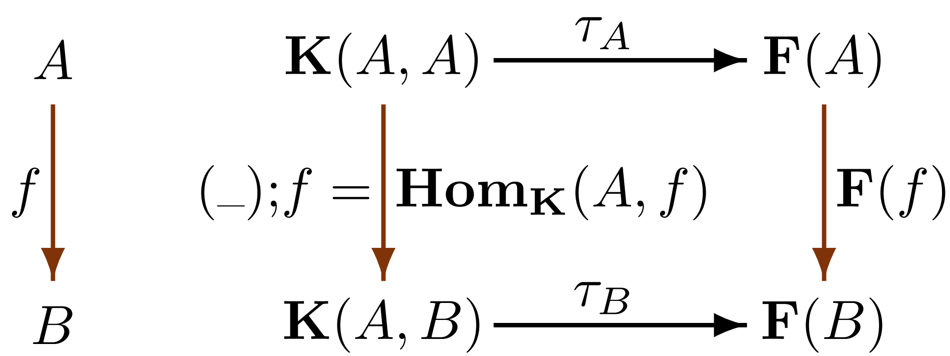


Compositions

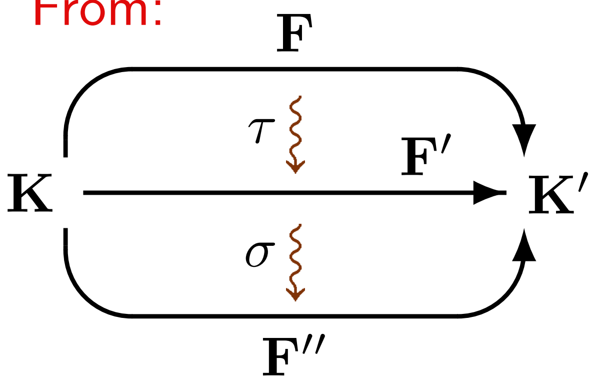
Compositions

vertical composition:

Compositions

vertical composition:

From:



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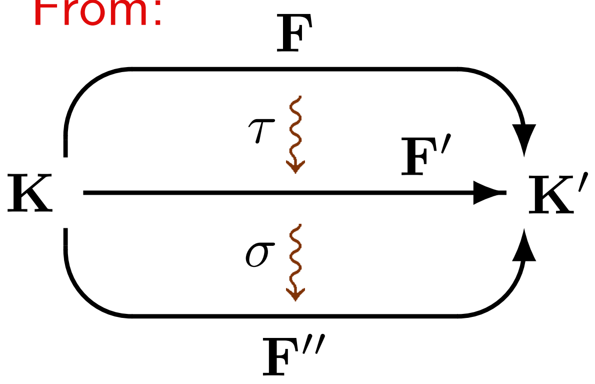
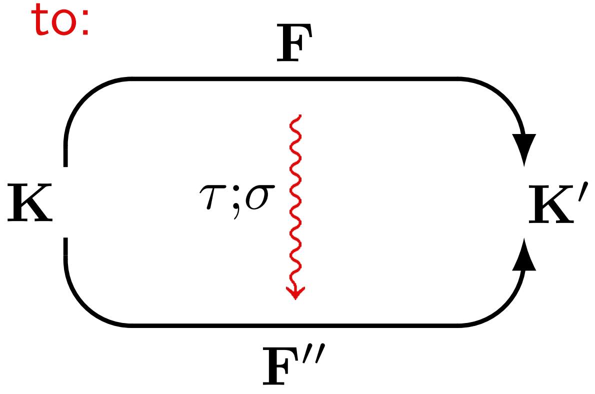


to:



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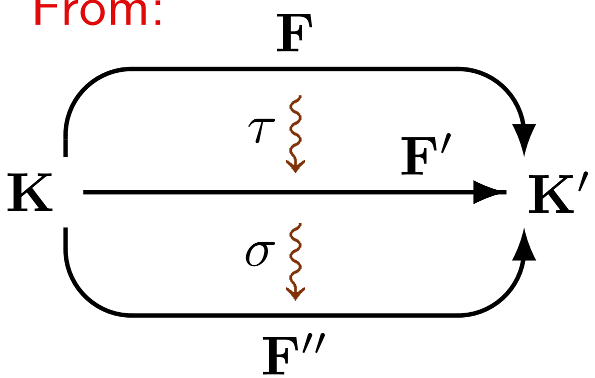

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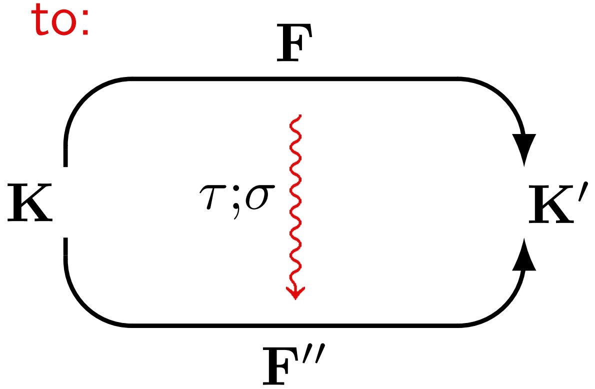

horizontal composition:

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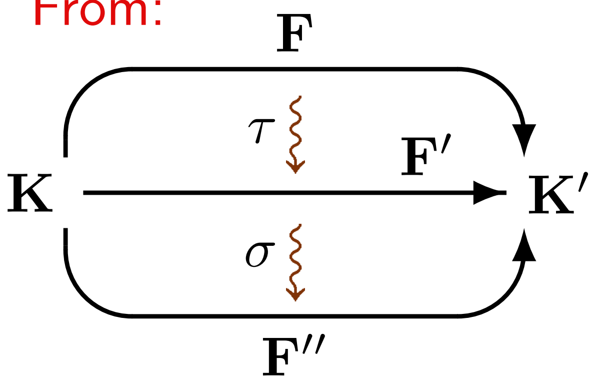

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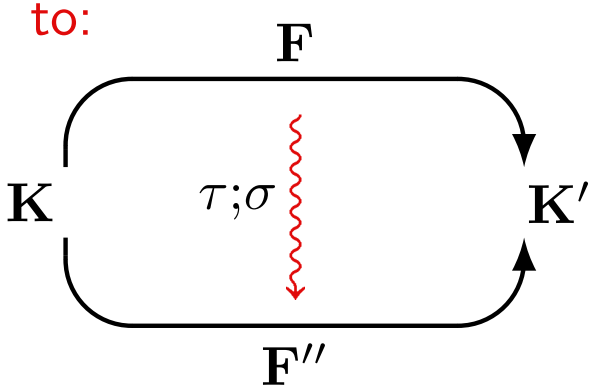

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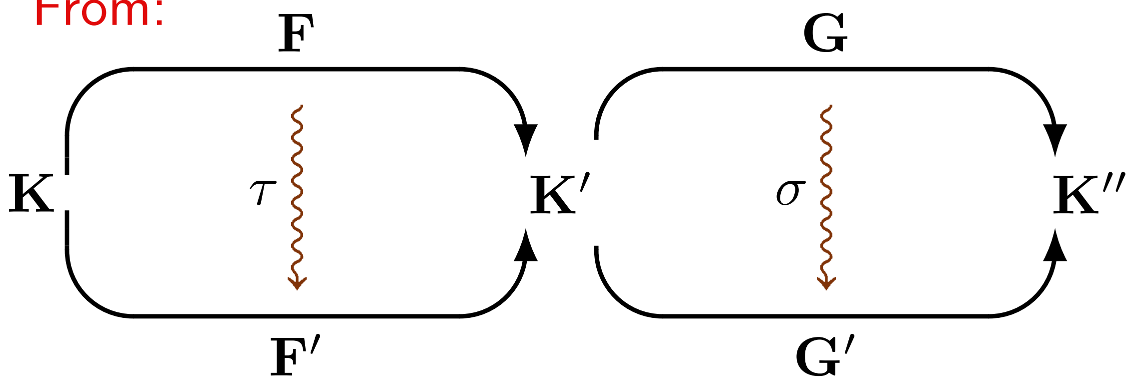

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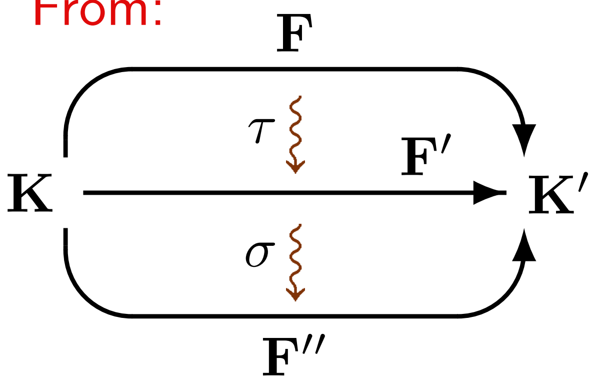

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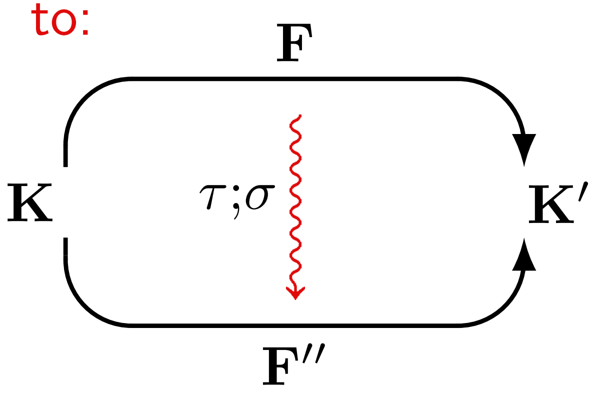

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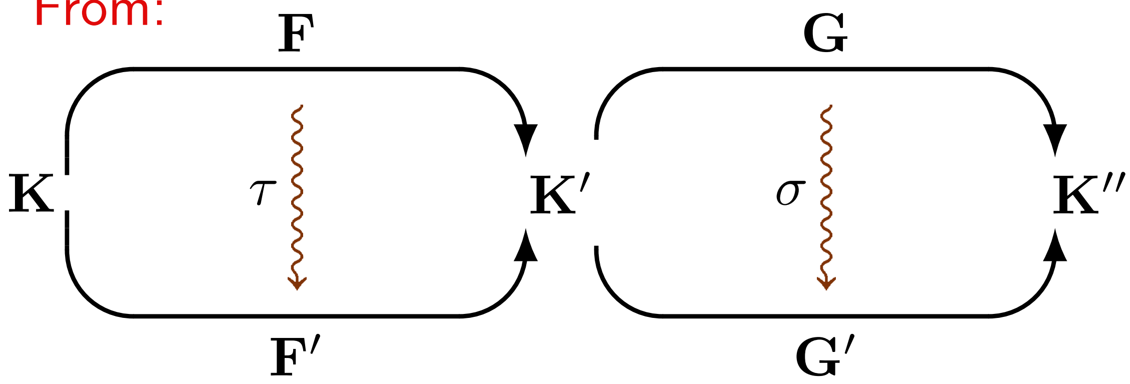

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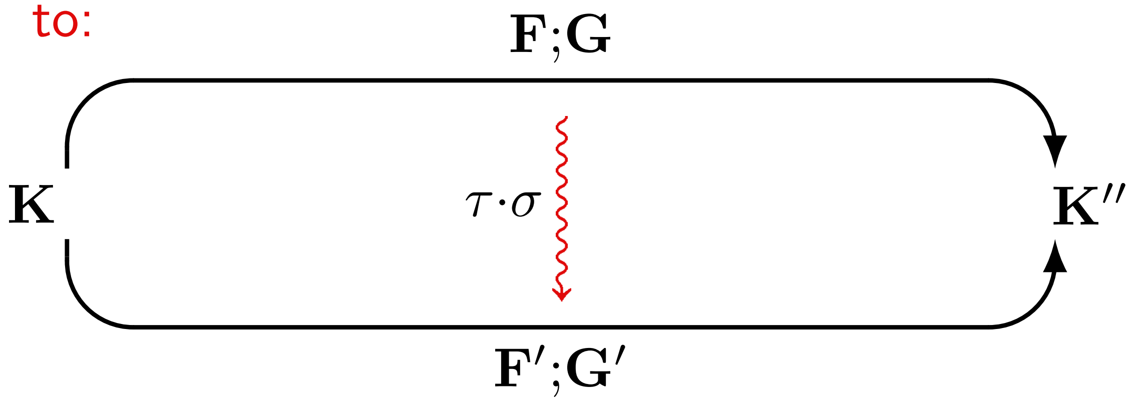
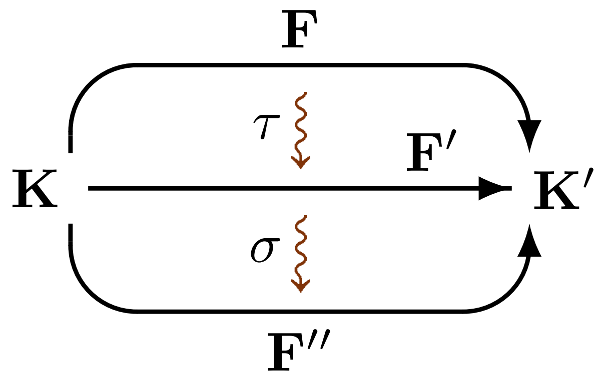


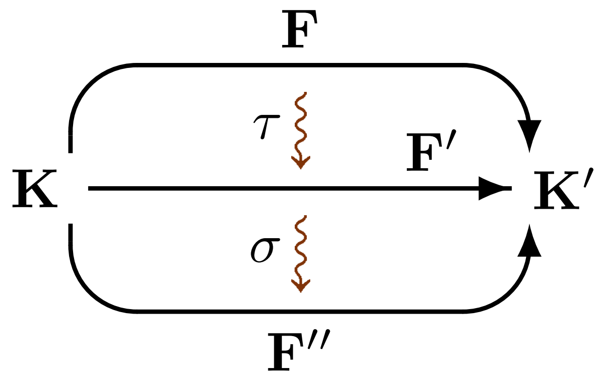
Vertical composition

Vertical composition



The *vertical composition* of natural transformations $\tau\colon \mathbf{F} \to \mathbf{F}'$ and $\sigma\colon \mathbf{F}' \to \mathbf{F}''$ between parallel functors $\mathbf{F}, \mathbf{F}', \mathbf{F}''\colon \mathbf{K} \to \mathbf{K}'$

Vertical composition



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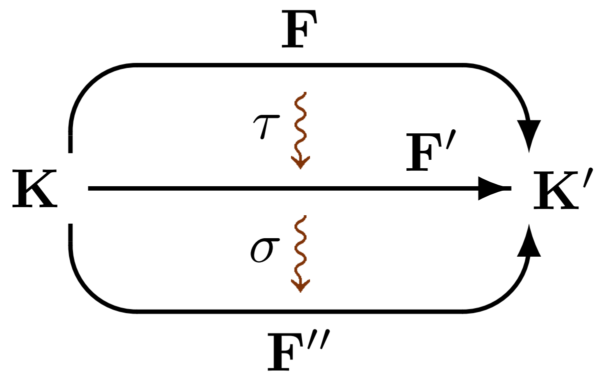
$$\tau;\sigma\colon \mathbf{F} \to \mathbf{F}''$$

Vertical composition



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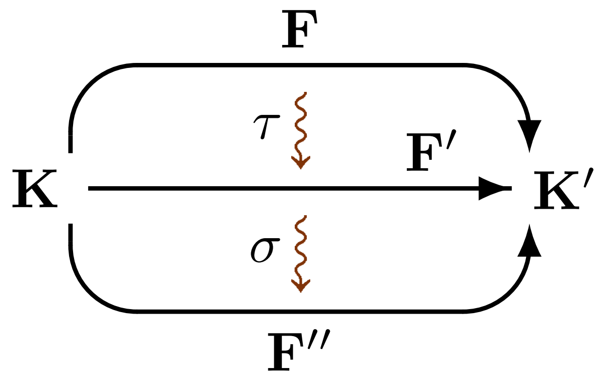
is a natural transformation given by $(\tau;\sigma)_A = \tau_A;\sigma_A$ for all $A \in |\mathbf{K}|$.

Vertical composition



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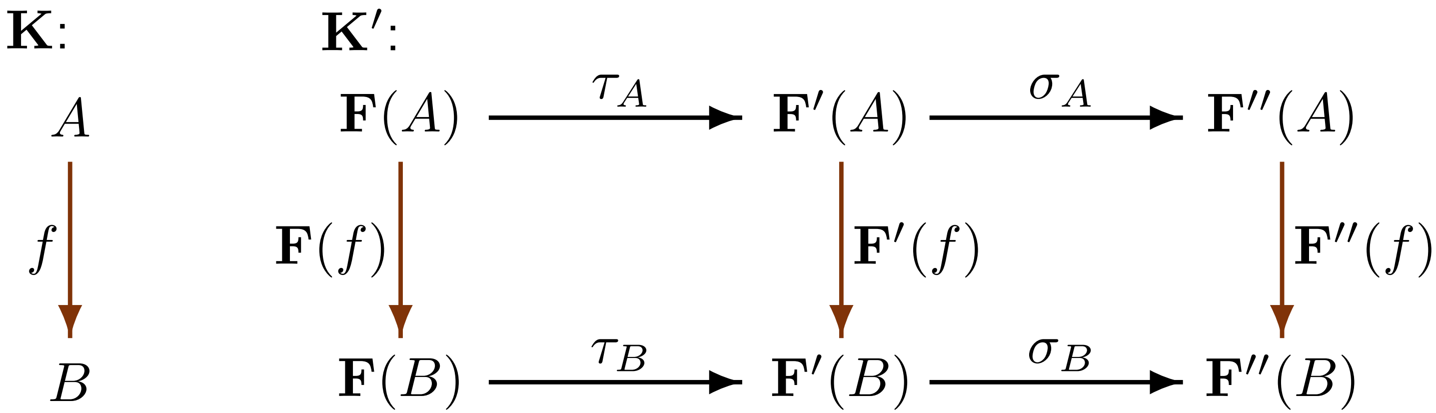
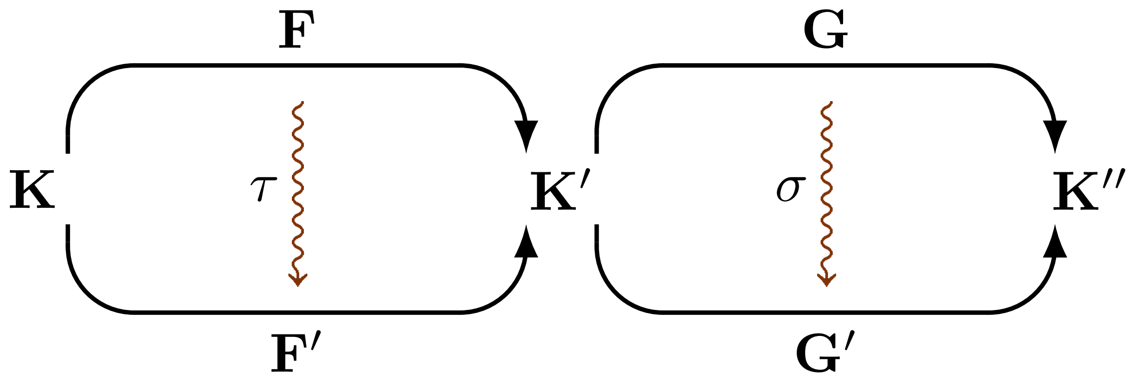


Horizontal composition

Horizontal composition



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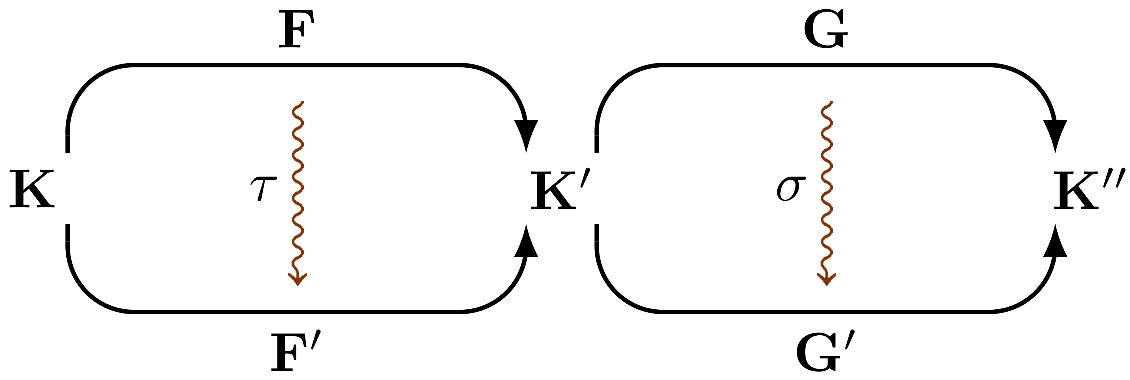
Horizontal composition



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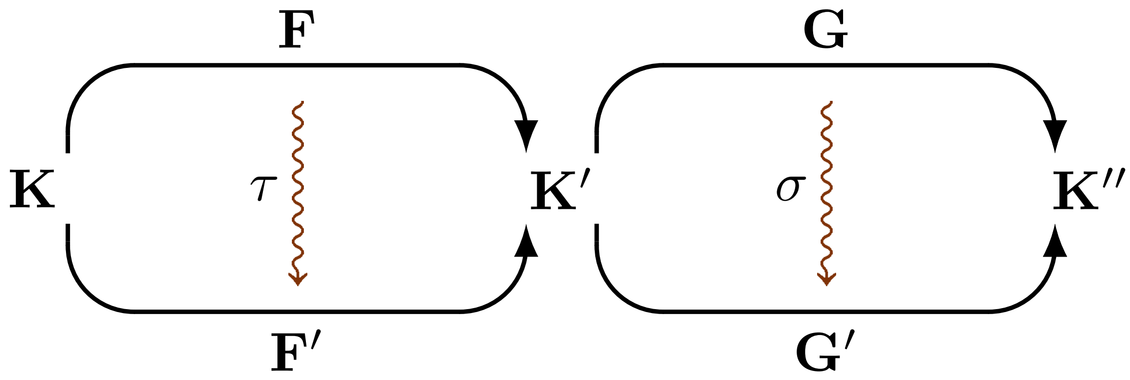
$$\tau{\cdot}\sigma\colon \mathbf{F};\mathbf{G} \to \mathbf{F}';\mathbf{G}'$$

Horizontal composition



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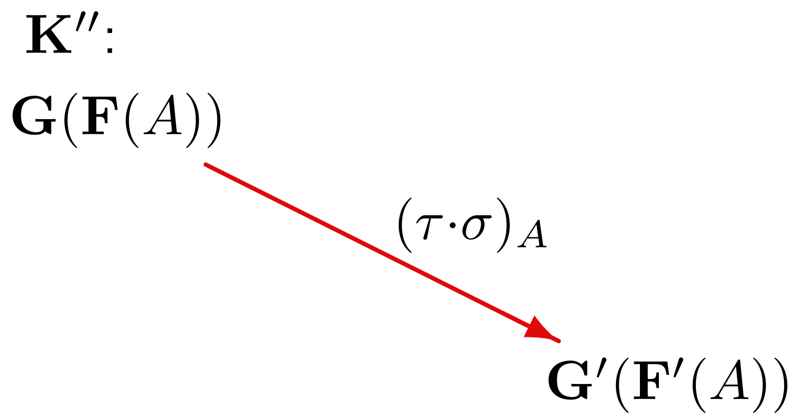

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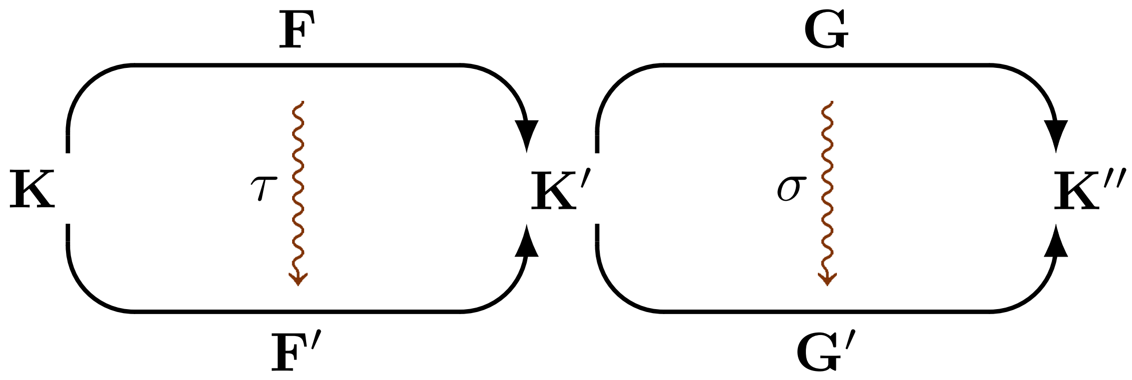

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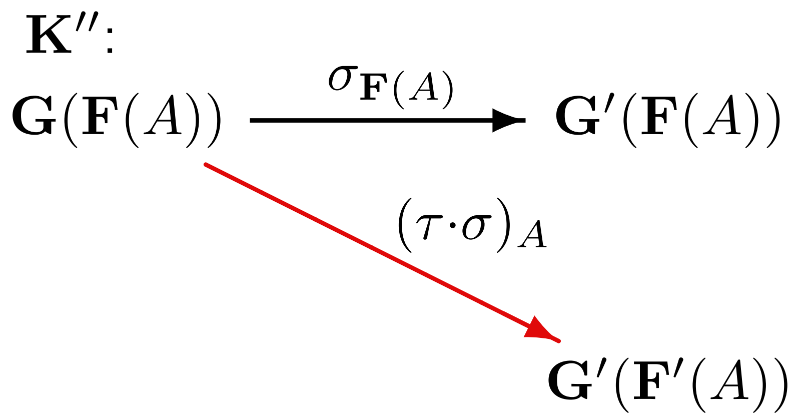

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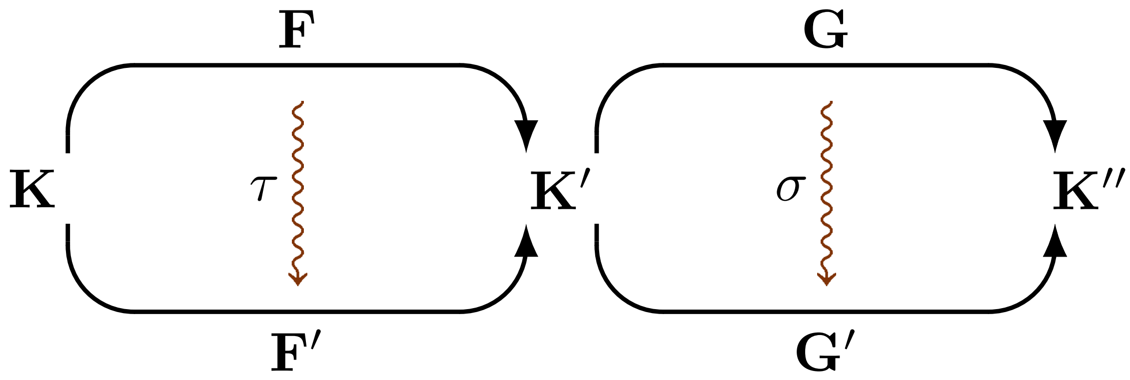

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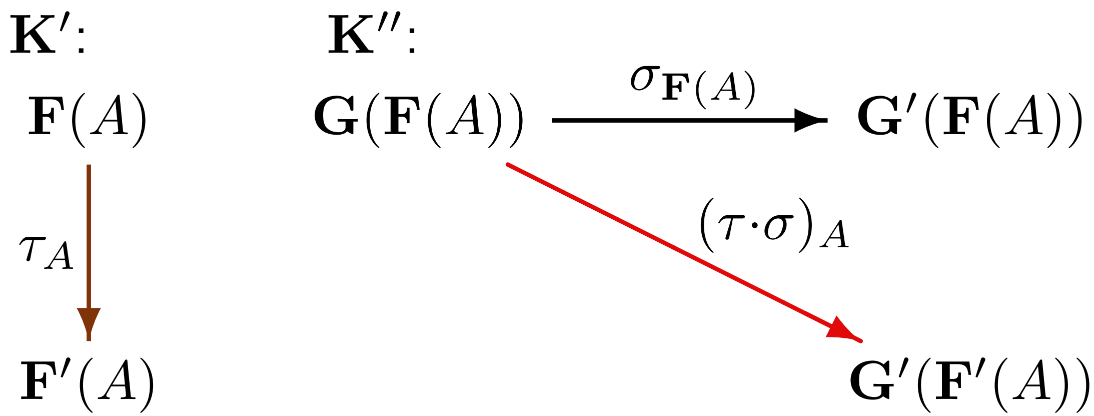

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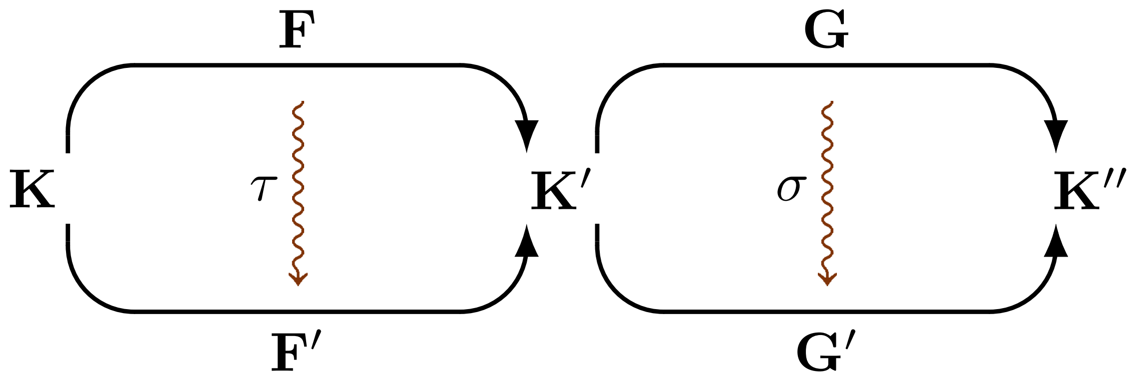

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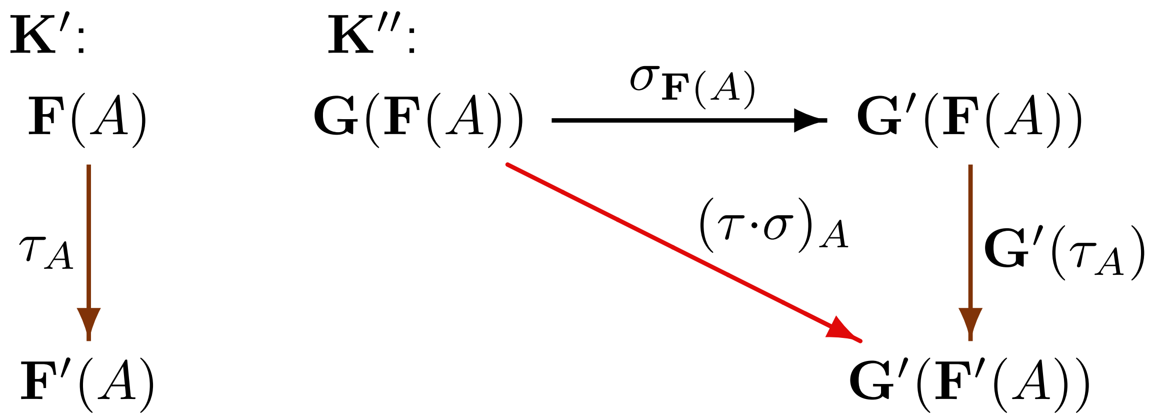

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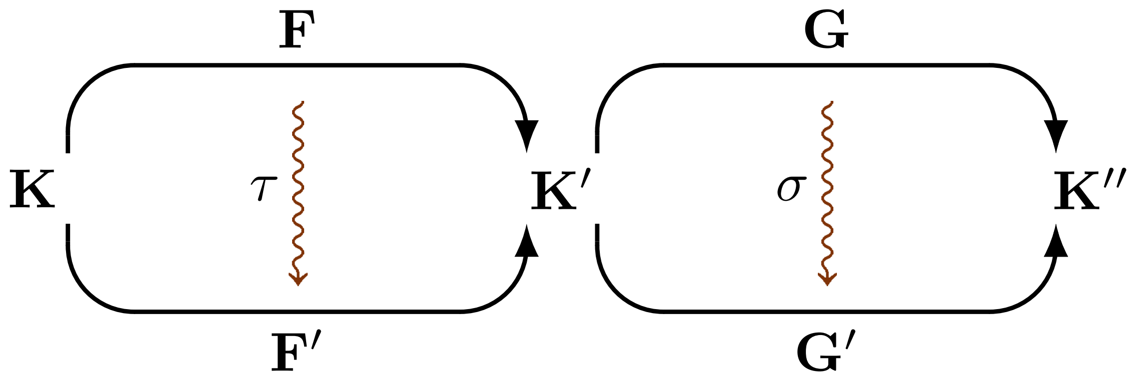

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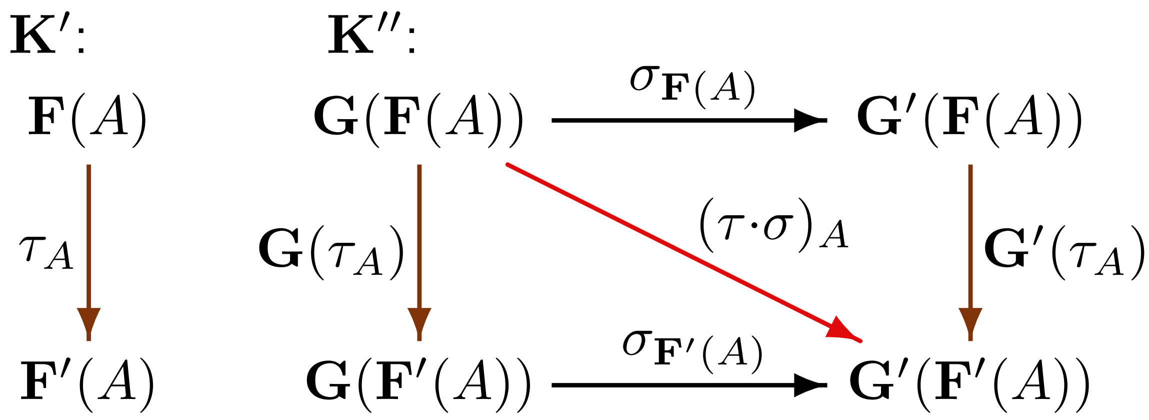

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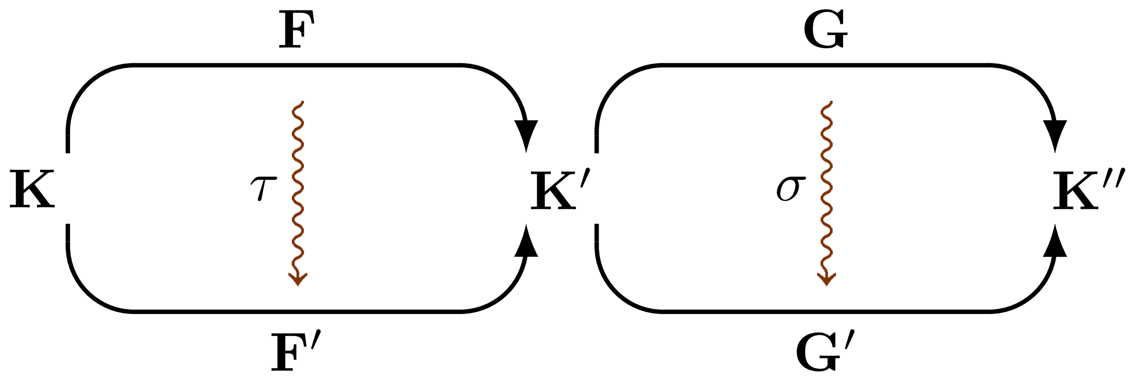

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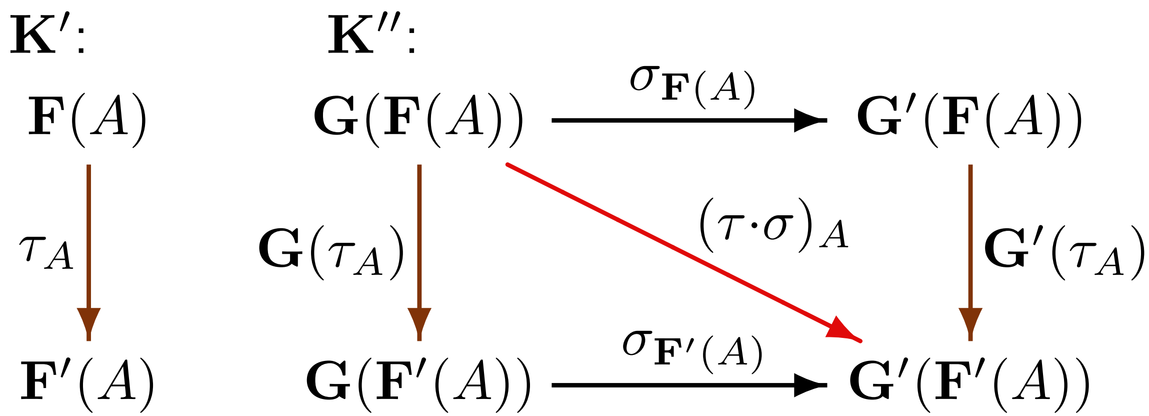

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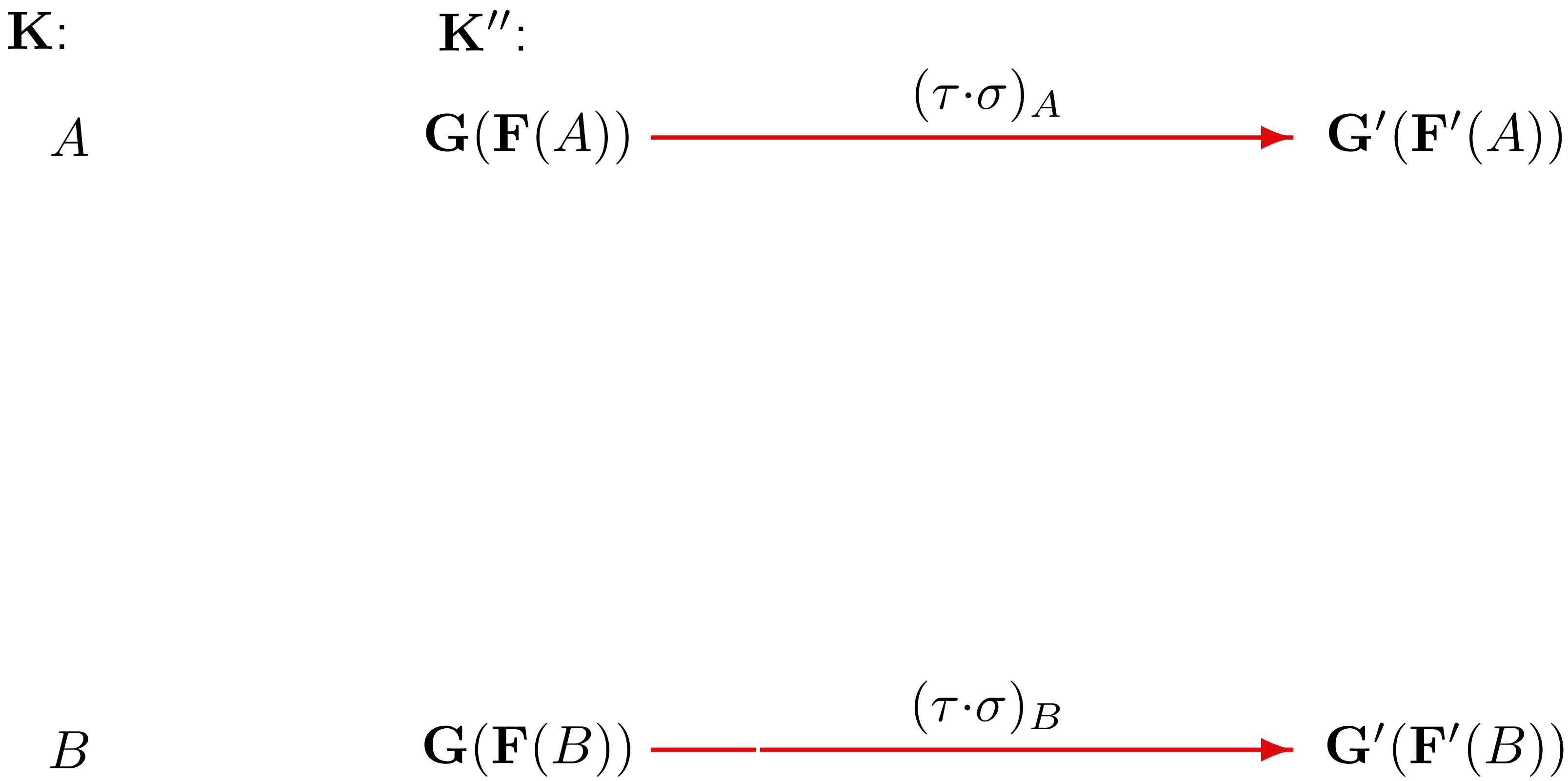

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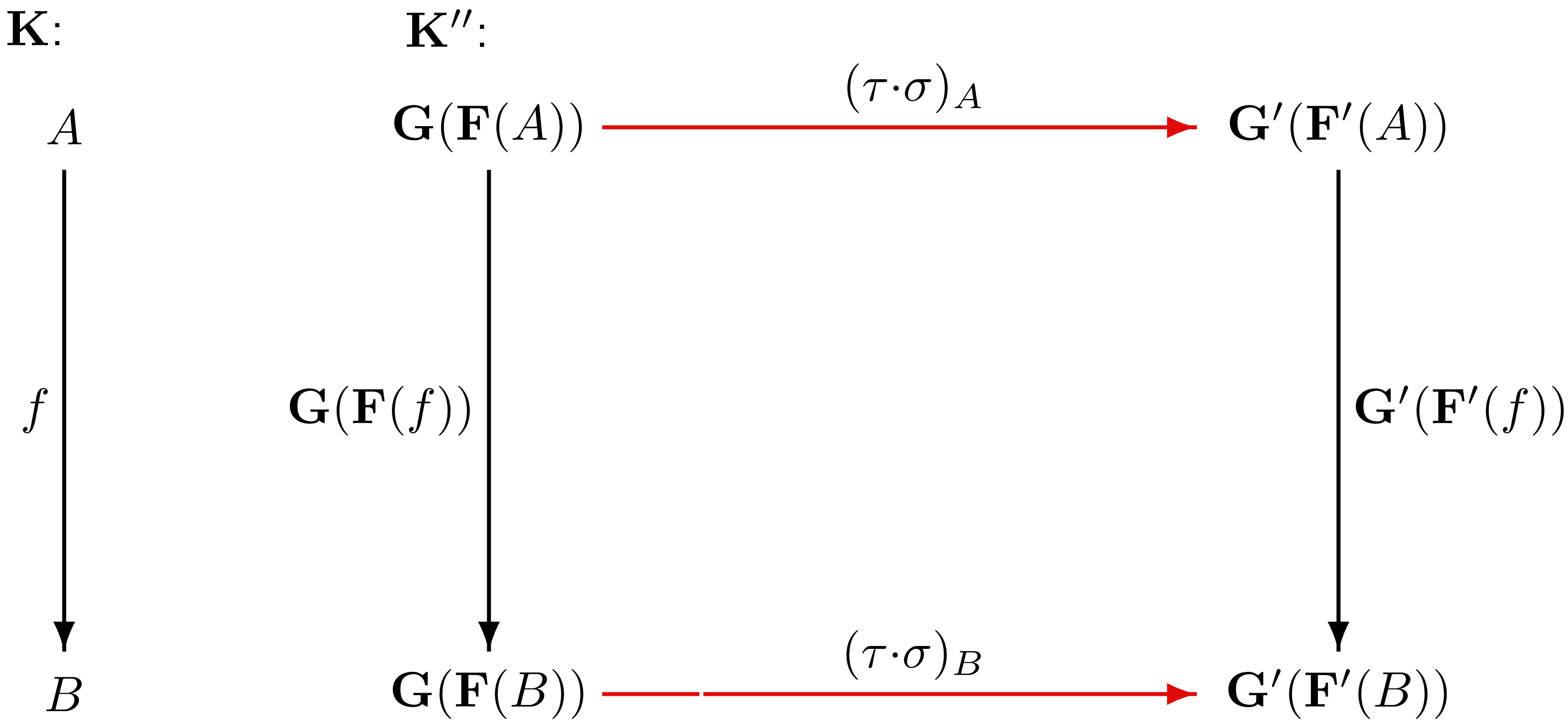

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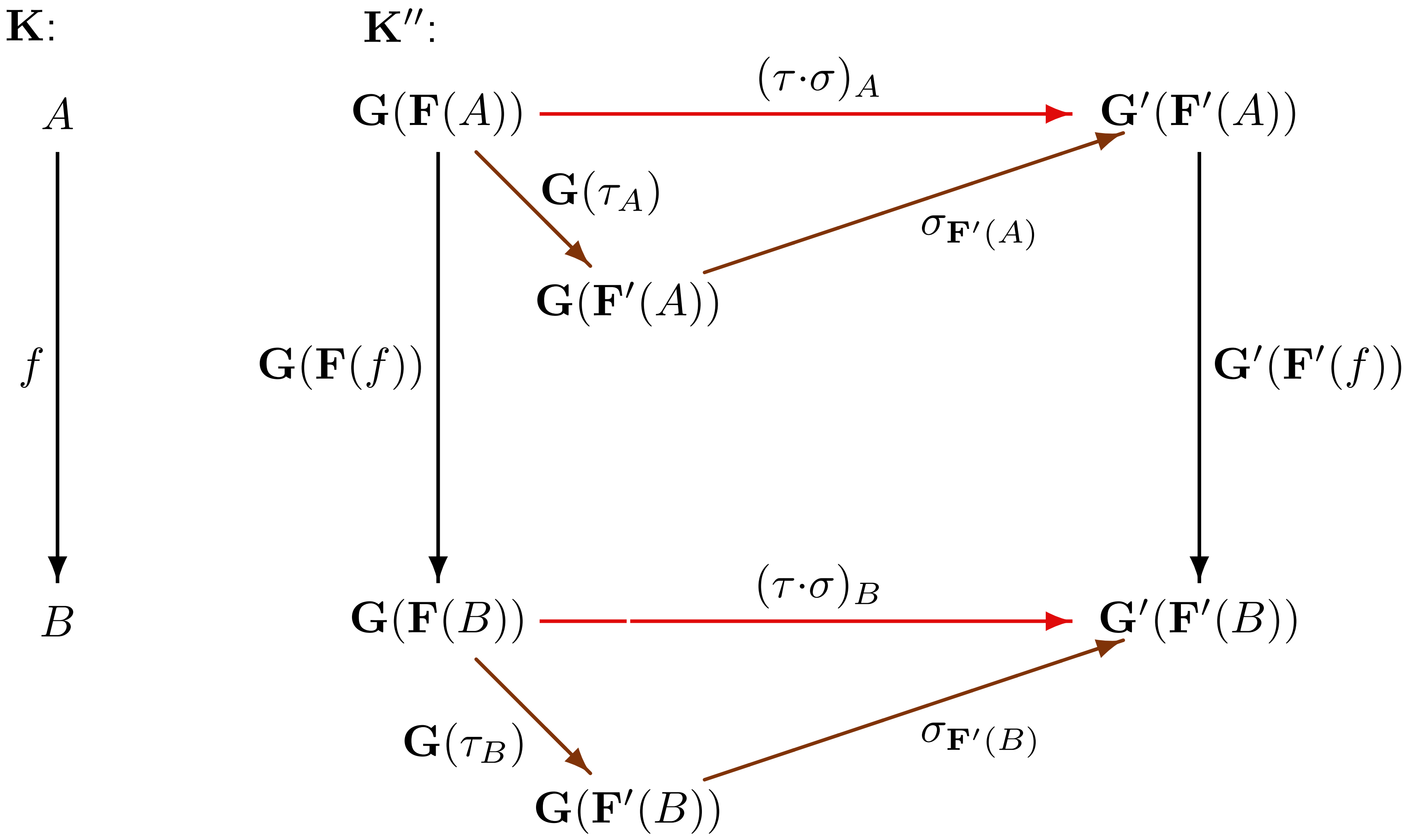

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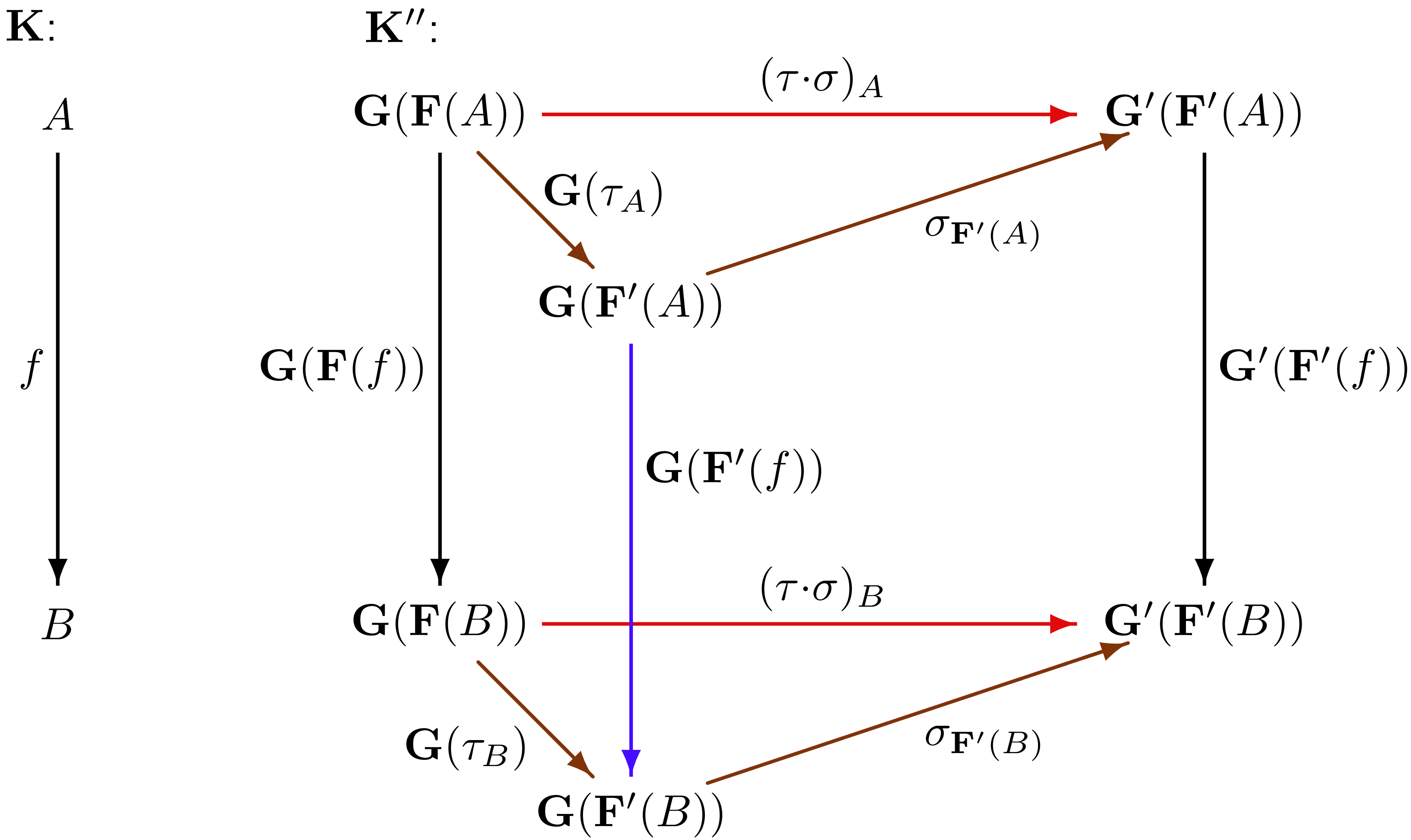

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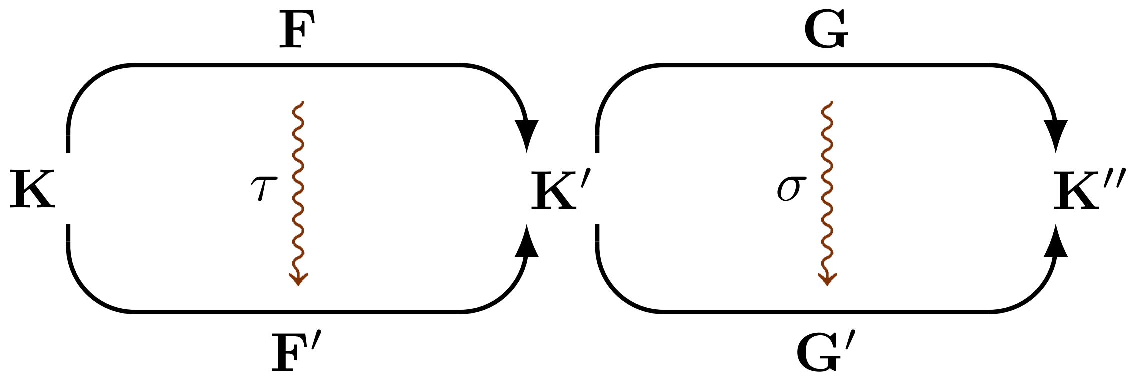

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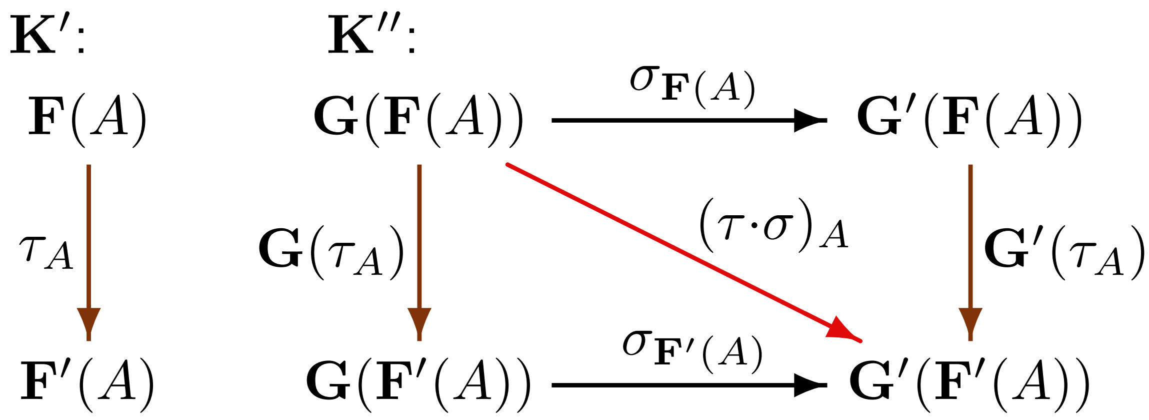

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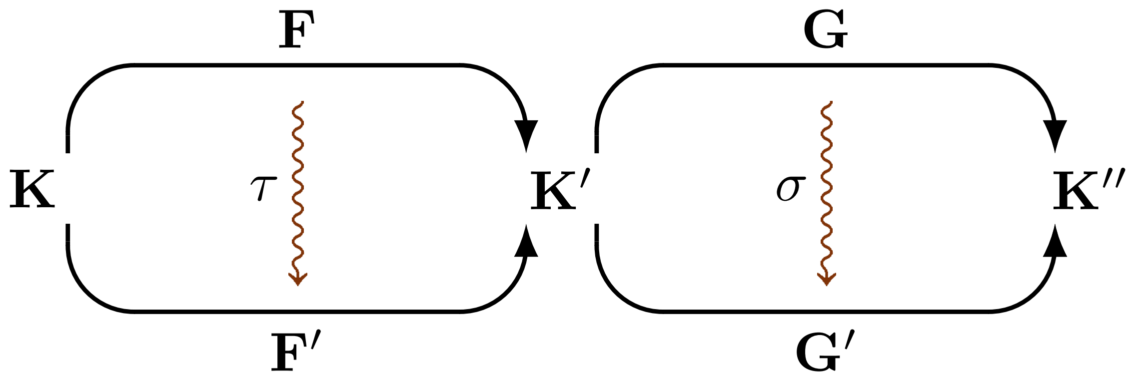

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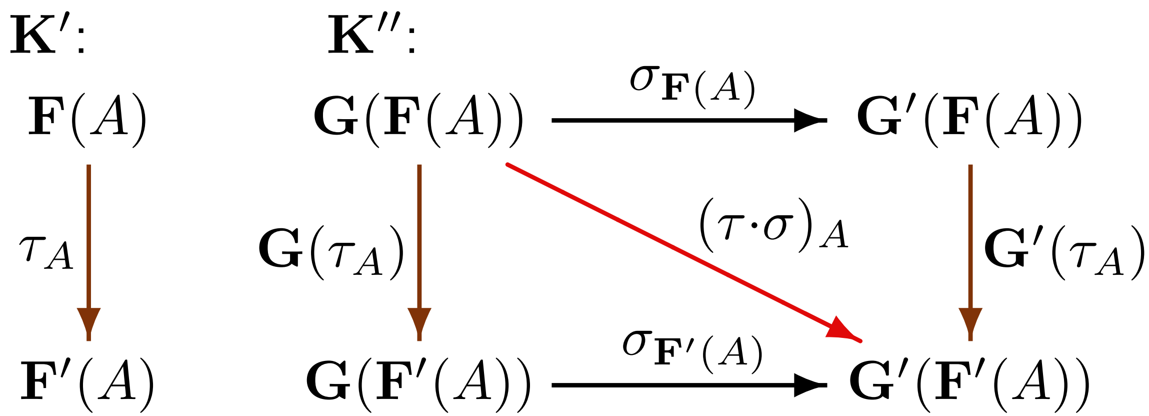

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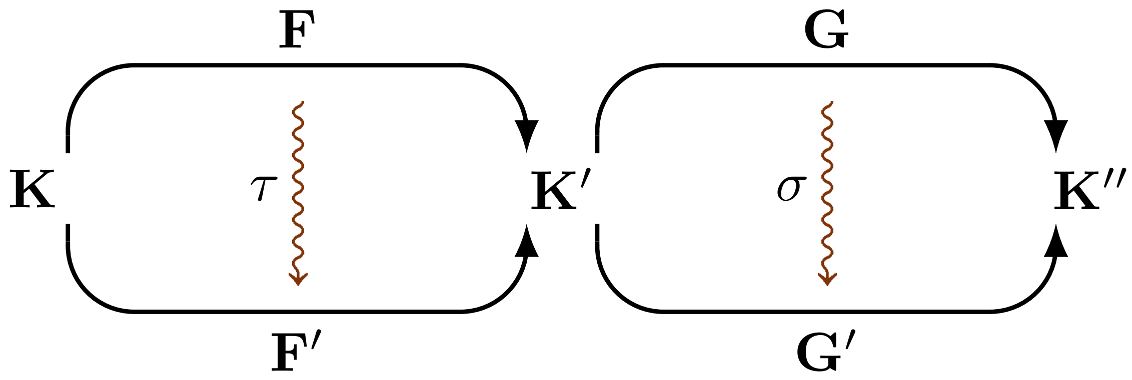

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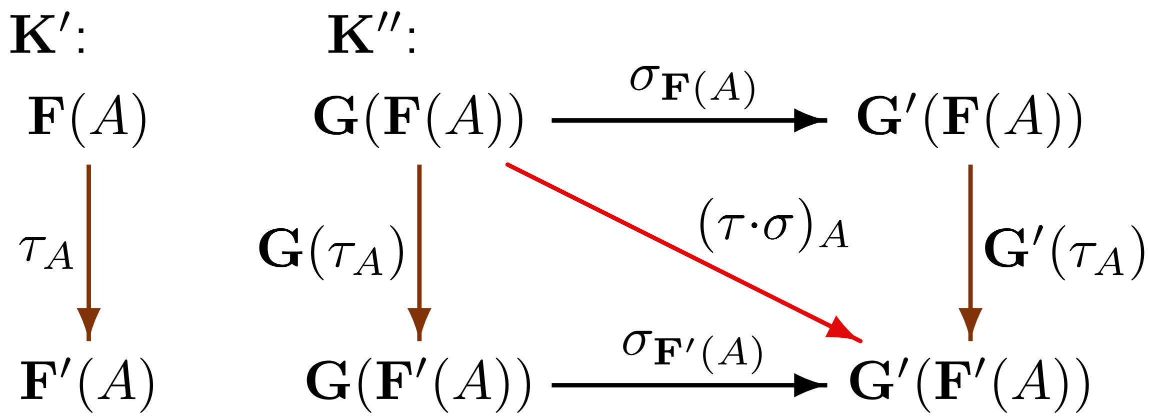

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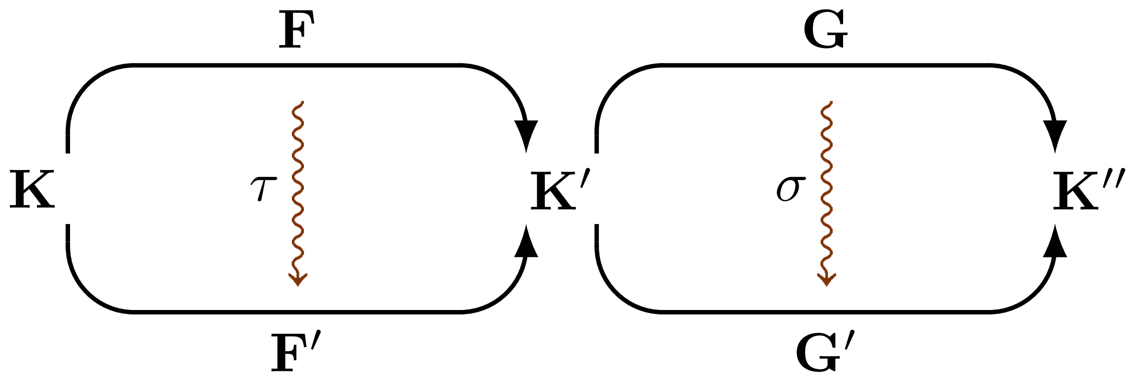

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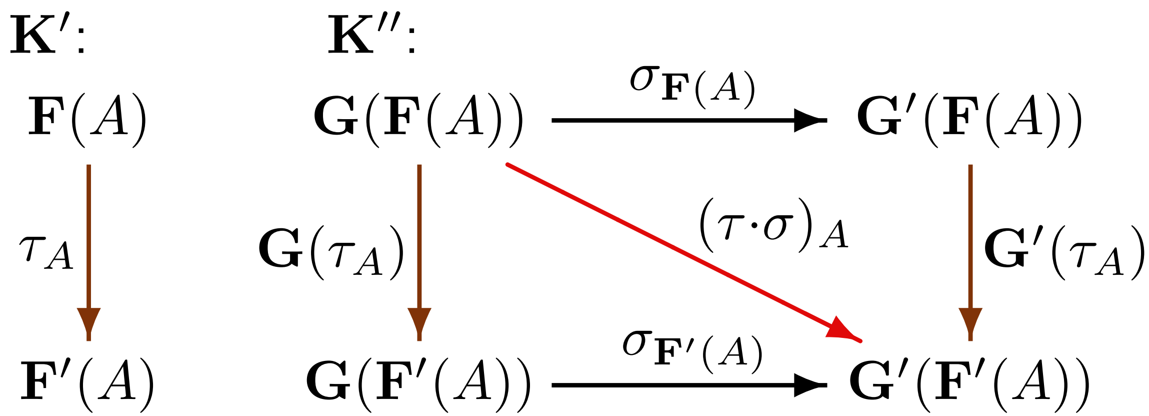

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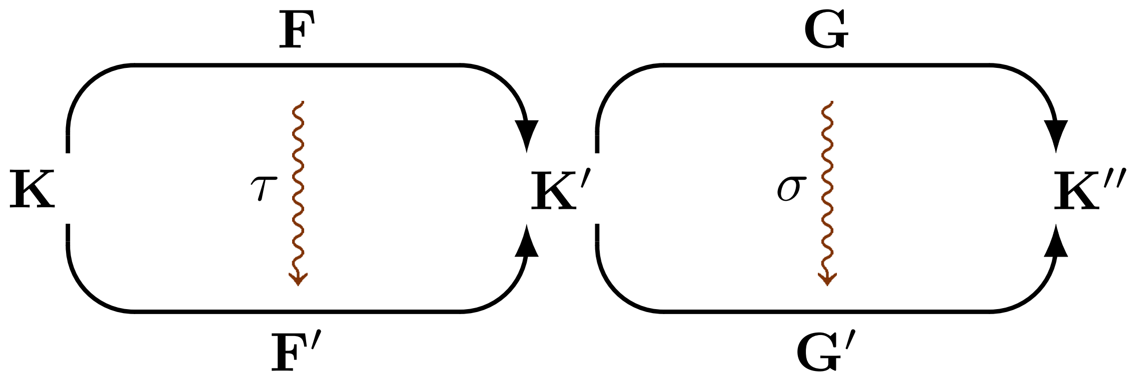

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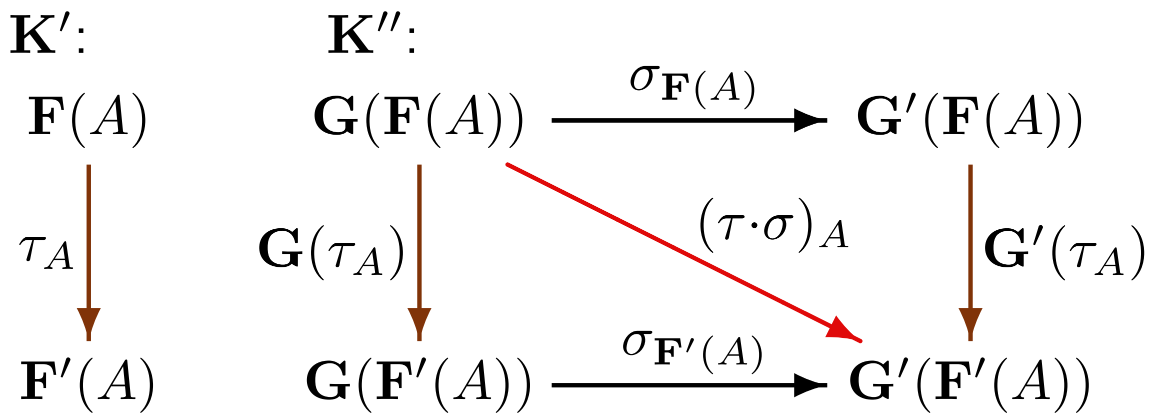

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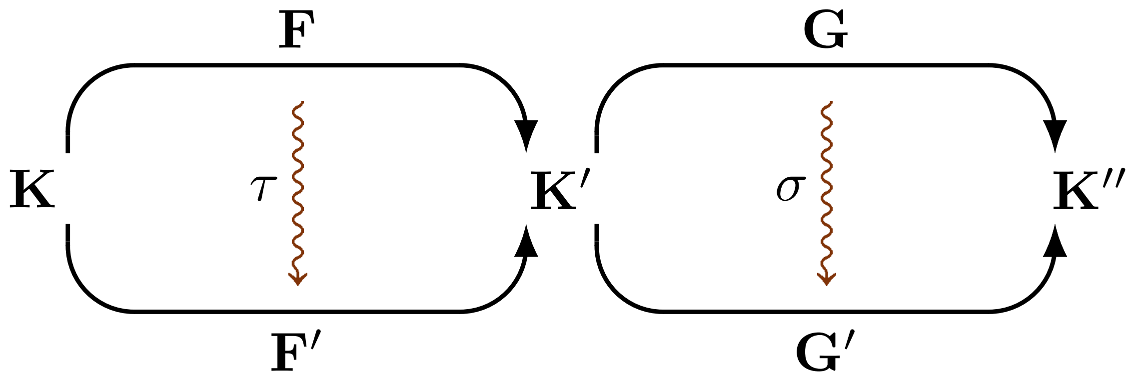

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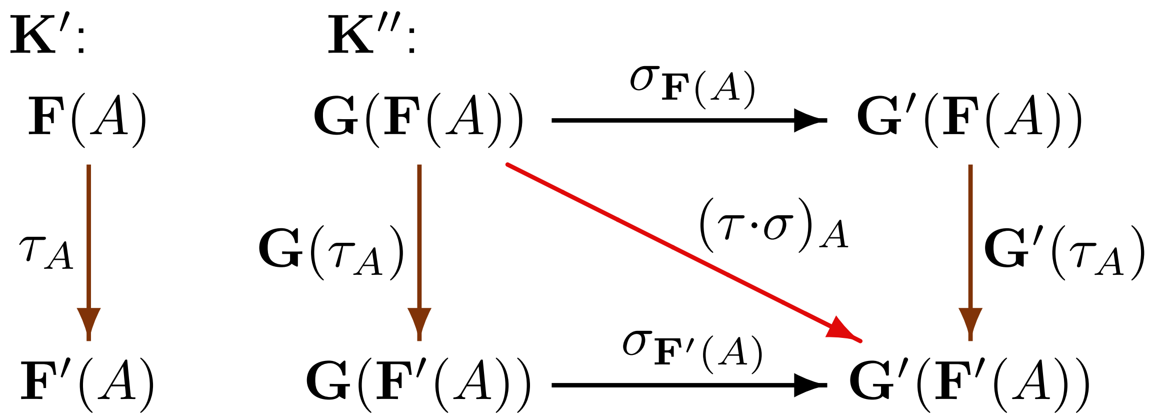

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Exercises:

- View the category of S-sorted sets, $\mathbf{Set}^S$, as a functor category.
- Check whether $\mathbf{K}^{\mathbf{K}'}$ is (finitely) (co)complete whenever $\mathbf{K}$ is so.

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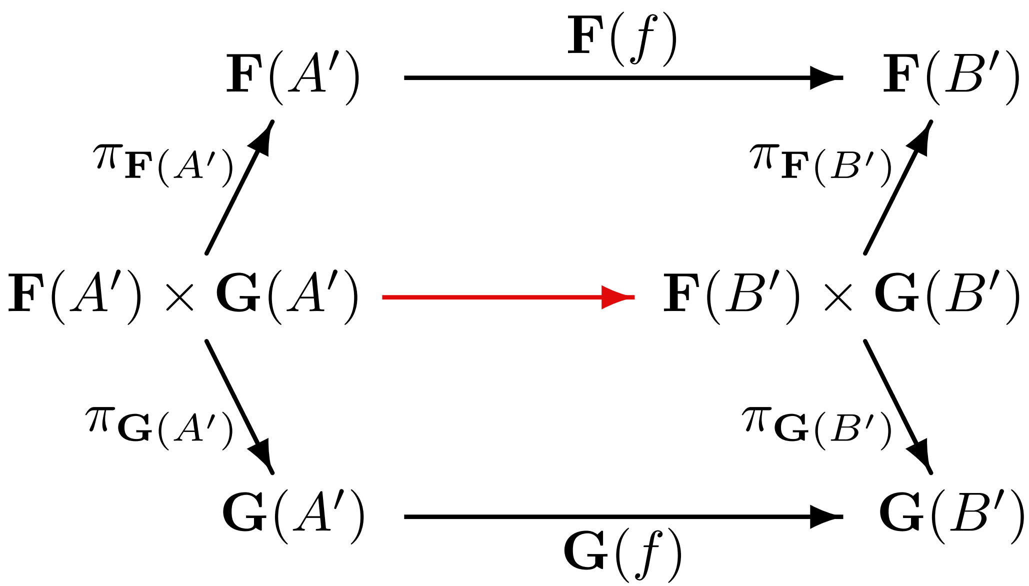

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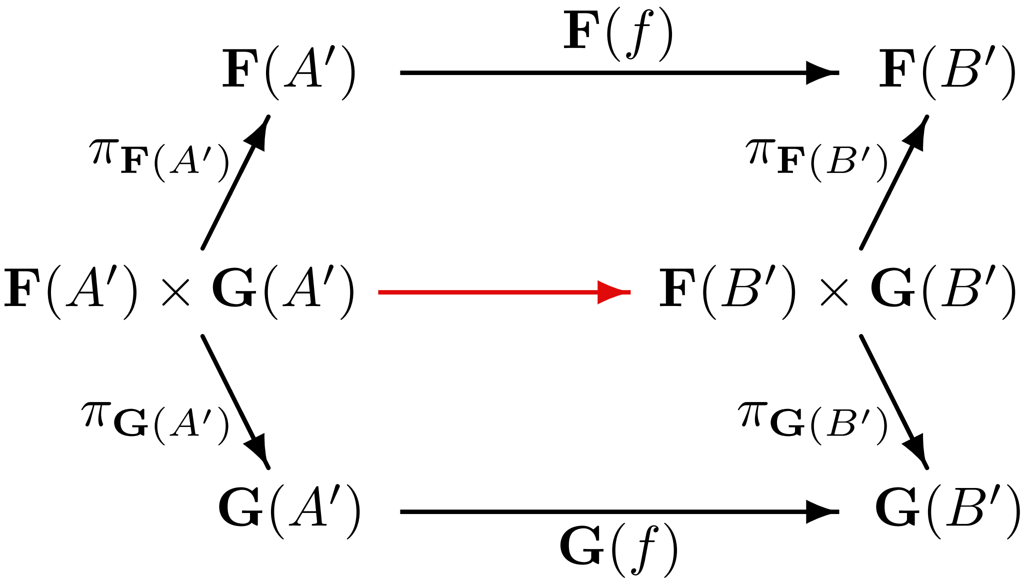

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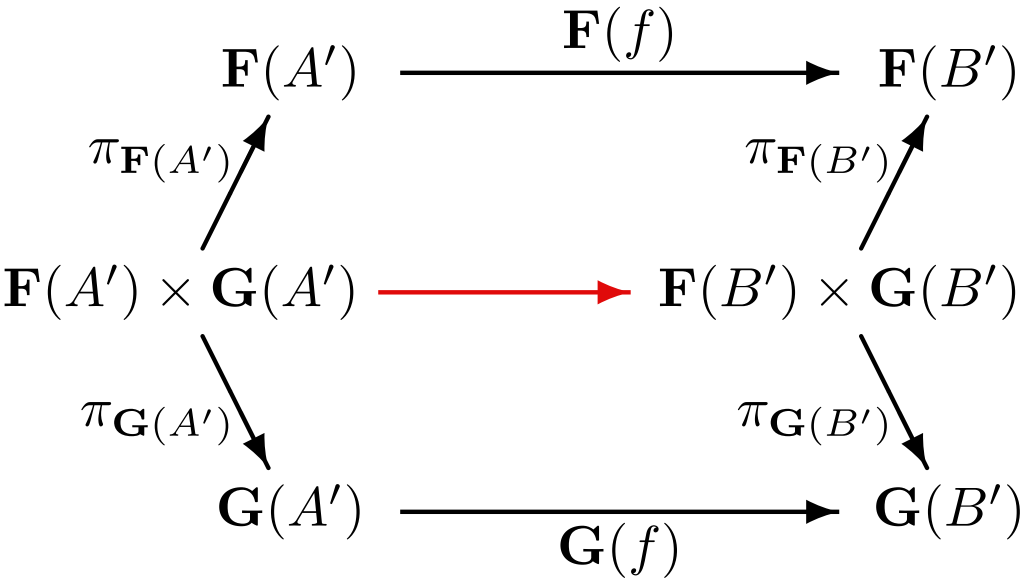

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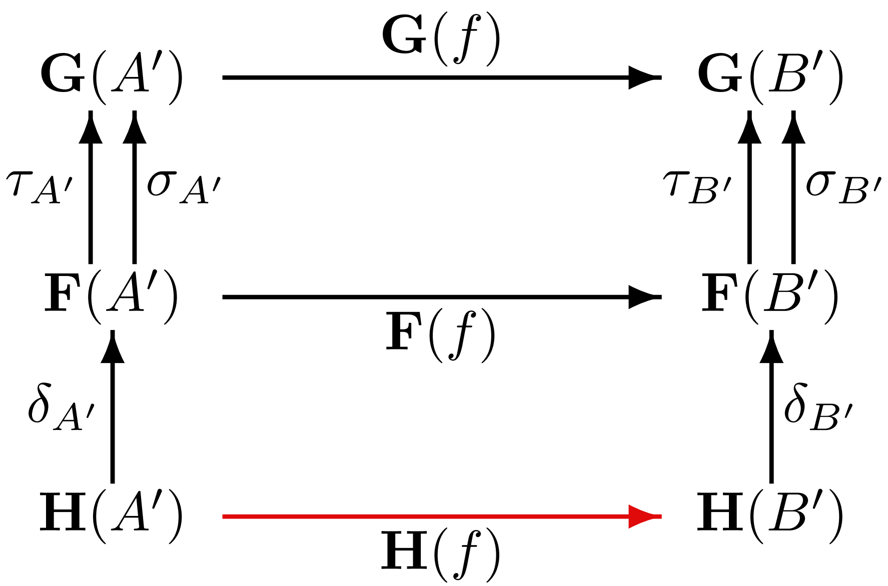

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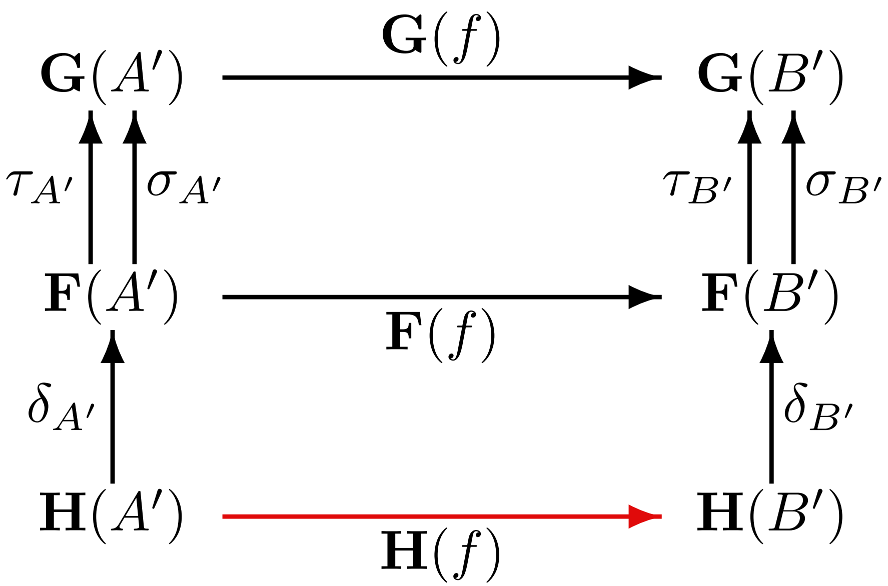

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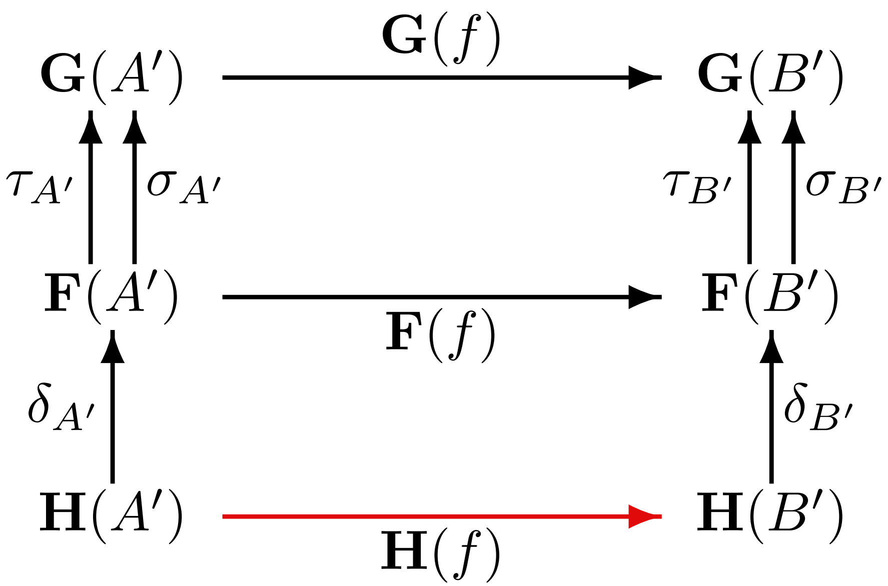

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This yields a natural transformation:
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Given two categories $\mathbf{K}, \mathbf{K}'$, define the *category of functors from* $\mathbf{K}'$ *to* $\mathbf{K}$, $\mathbf{K}^{\mathbf{K}'}$, as follows:

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- View the category of S-sorted sets, $\mathbf{Set}^S$, as a functor category.
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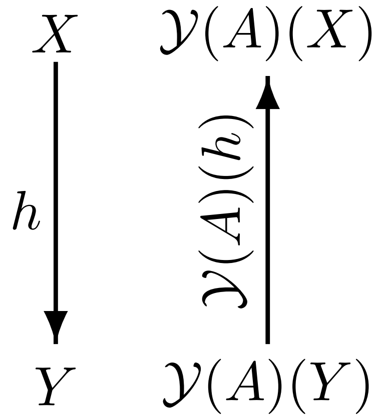

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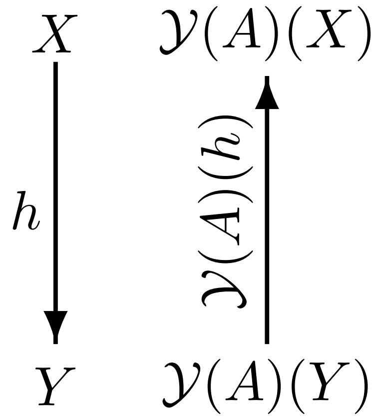

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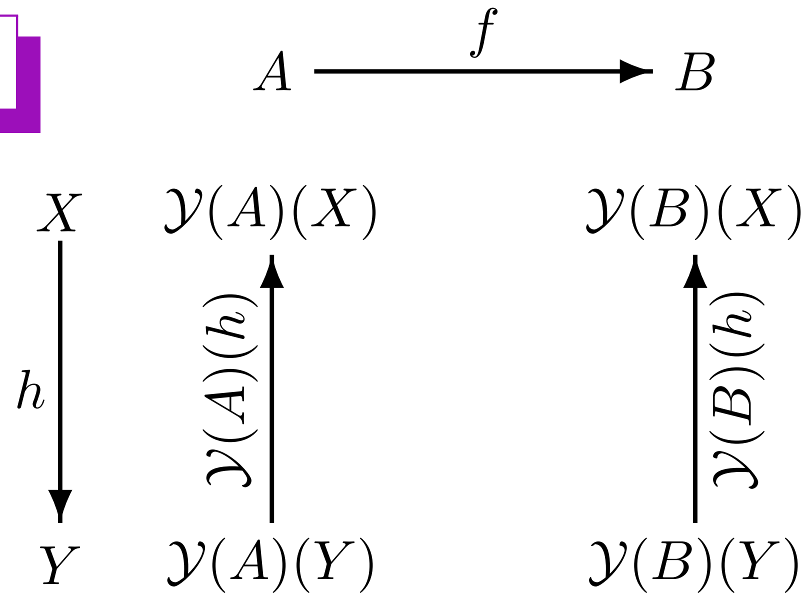

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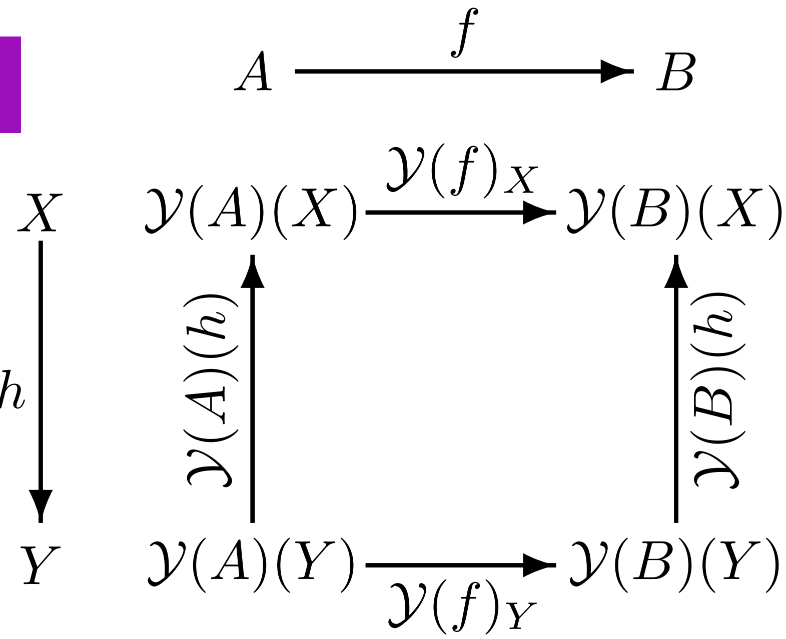

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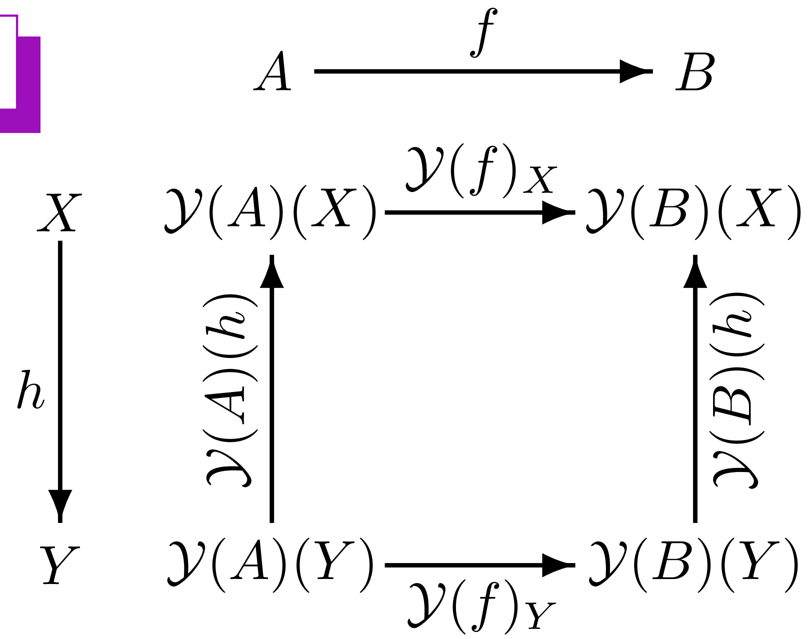

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Diagrams are functors from small (shape) categories

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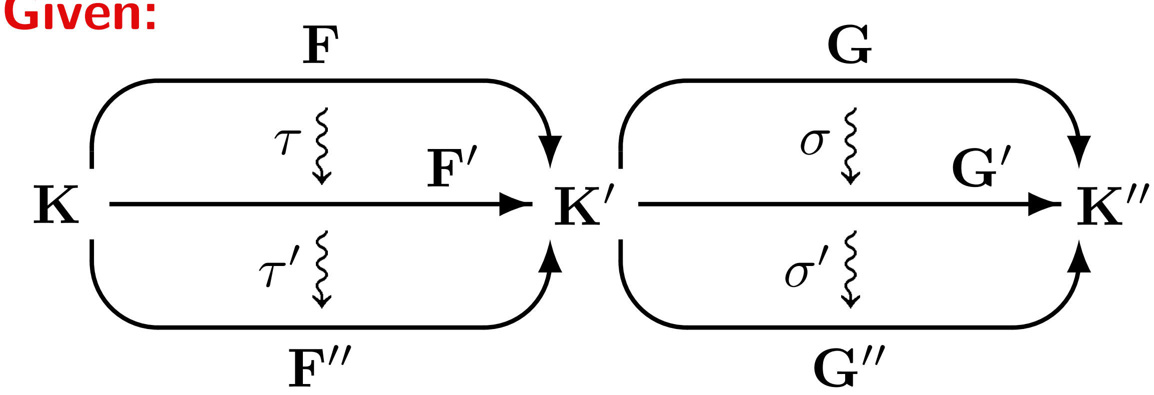

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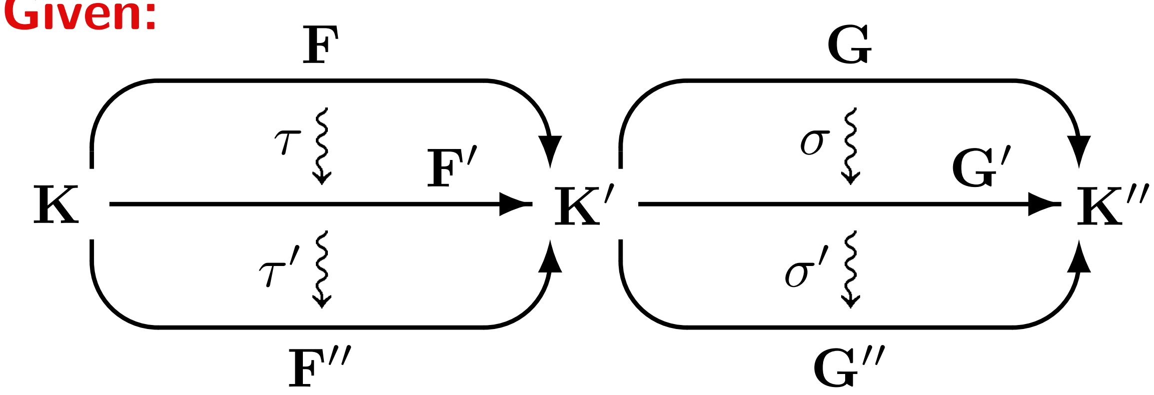

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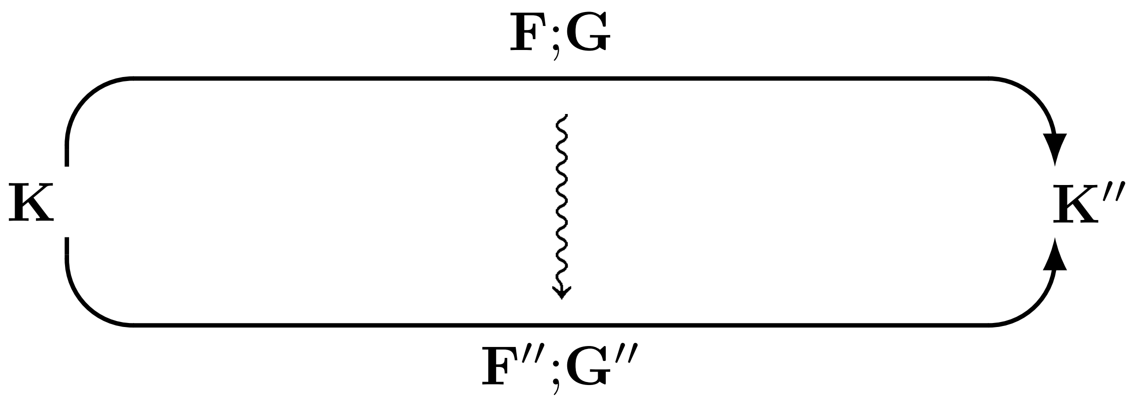

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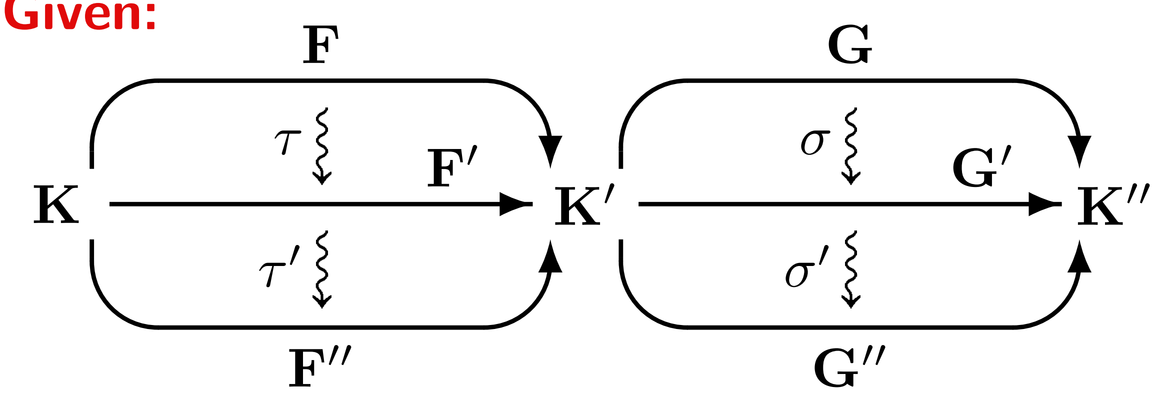

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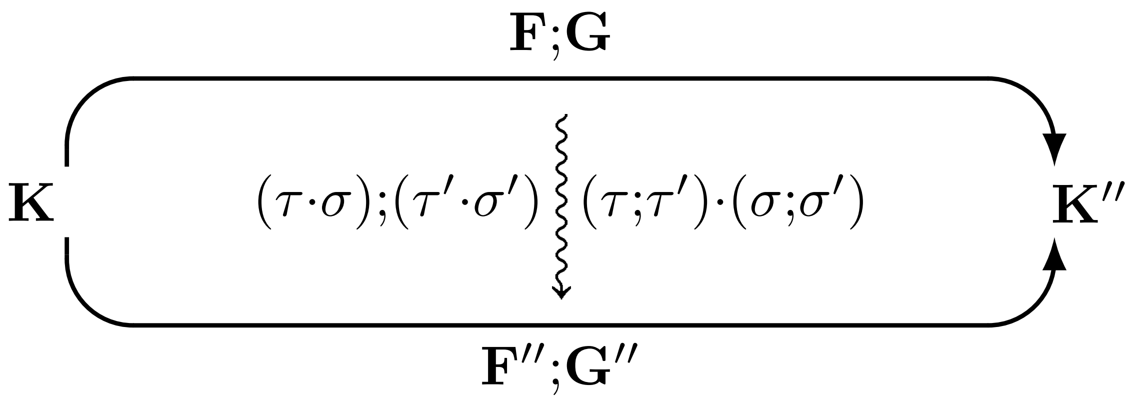

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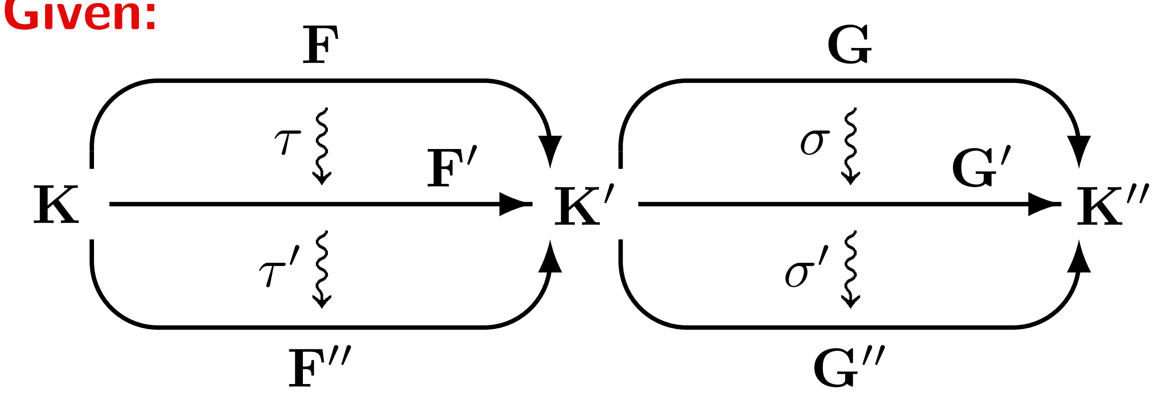

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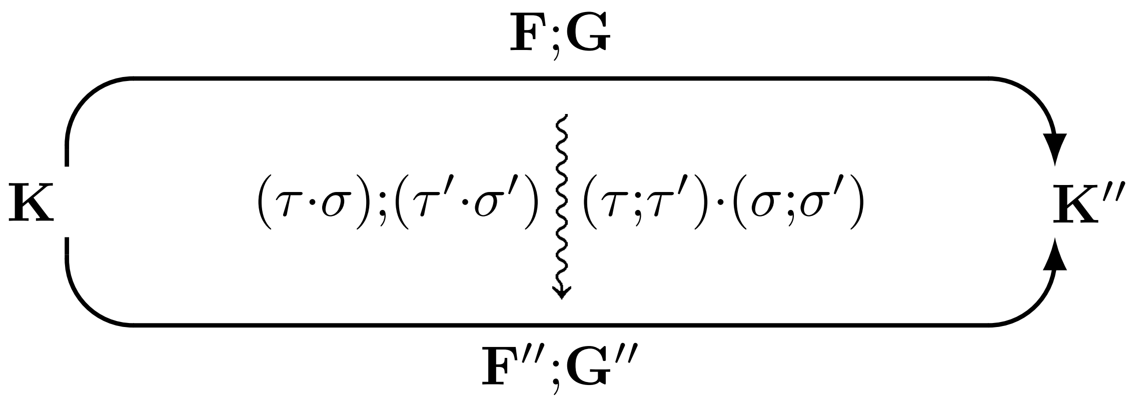

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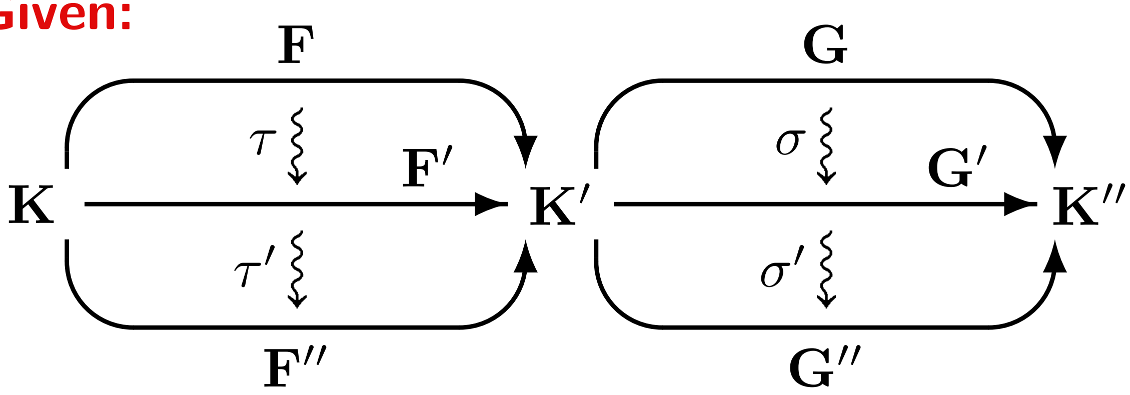

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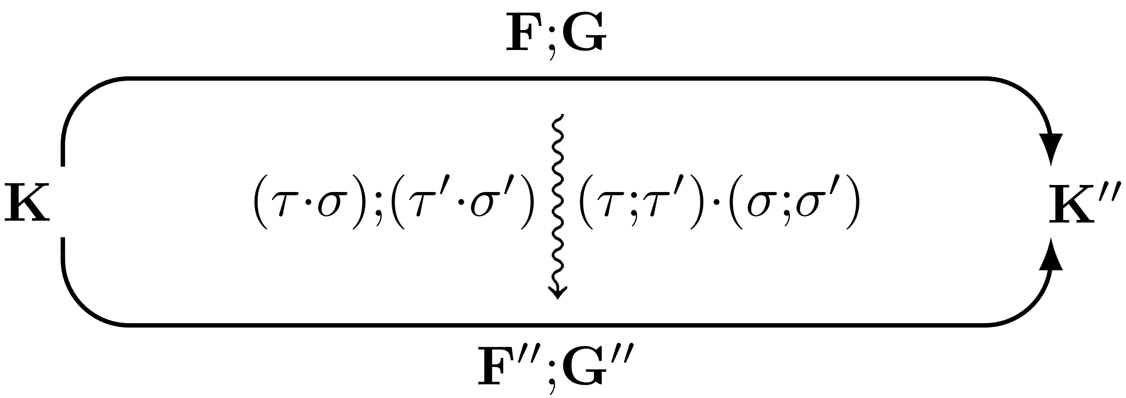

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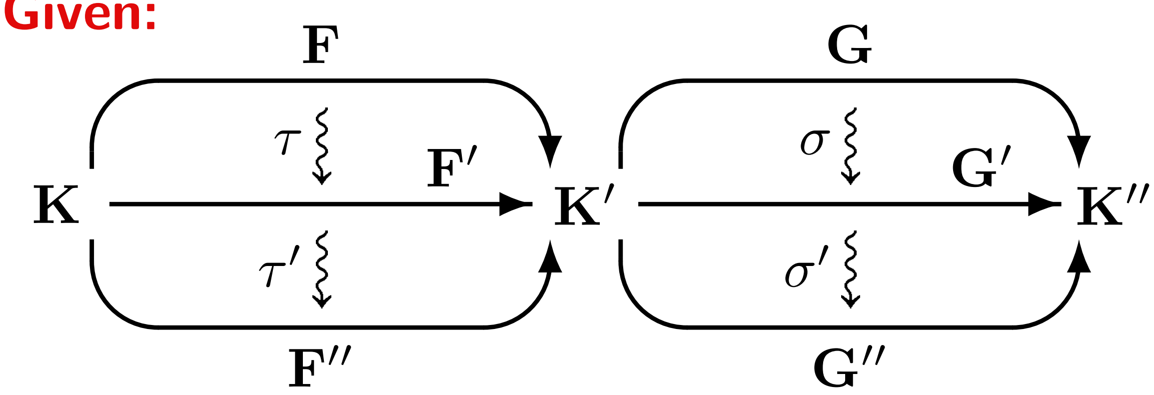

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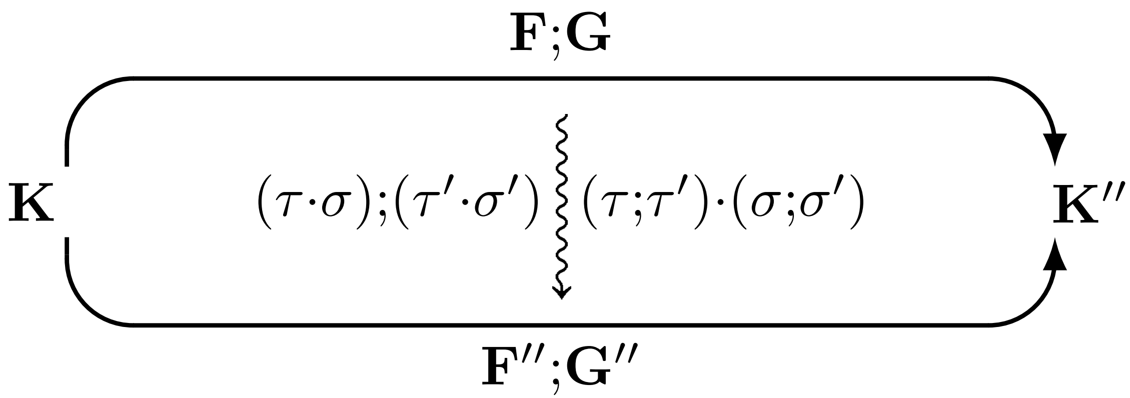

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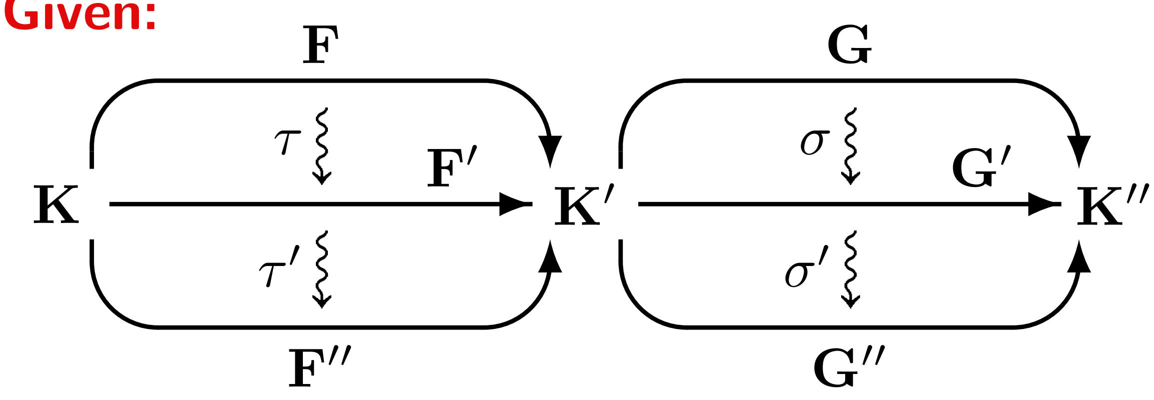

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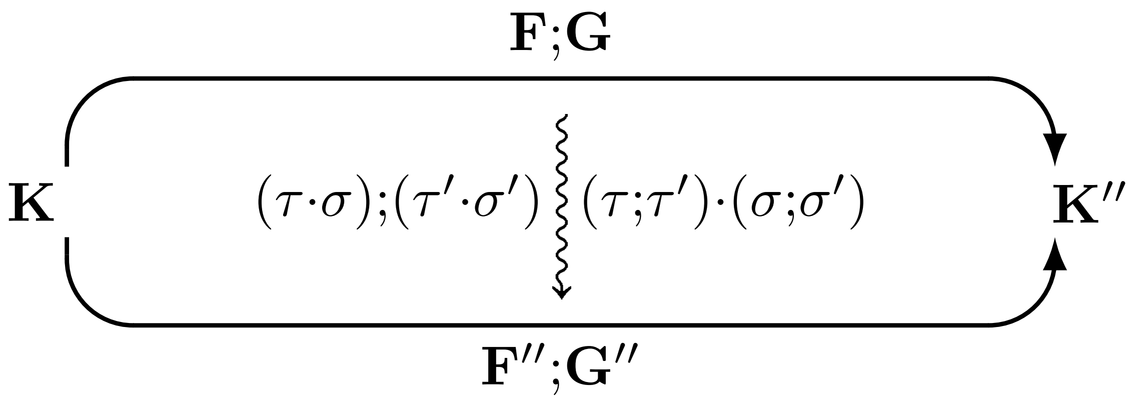

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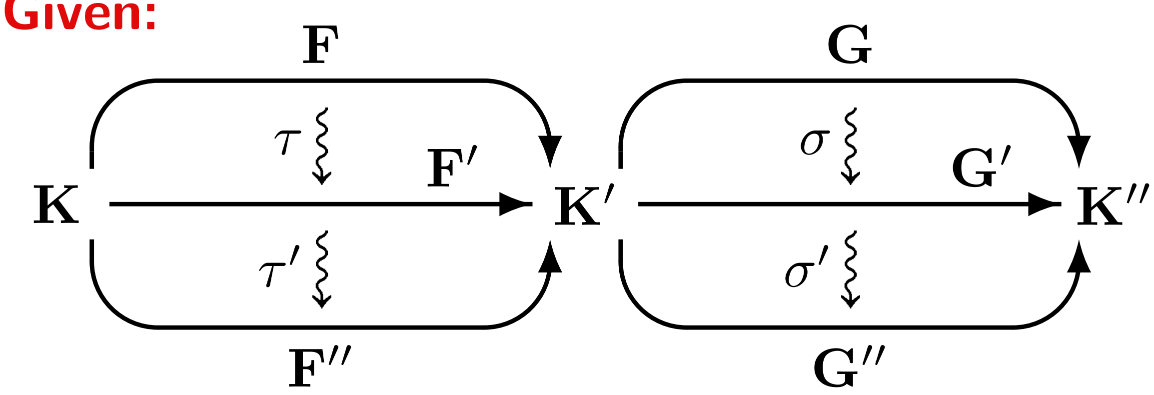

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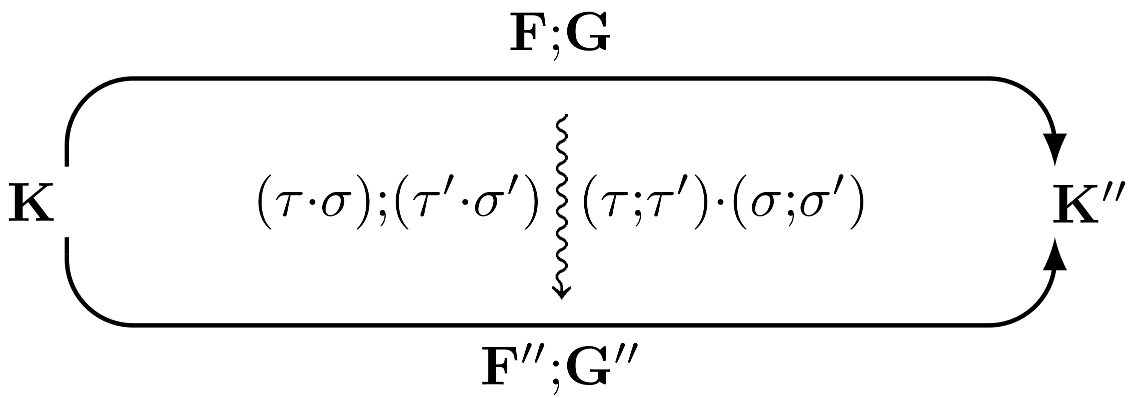

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All "categorical" properties are preserved under equivalence of categories