Monads

Monads

monoids: a categorical generalisation

algebras: a categorical generalisation

Monads

category theory for
effects in typed functional programming

monoids: a categorical generalisation
algebras: a categorical generalisation

Monads

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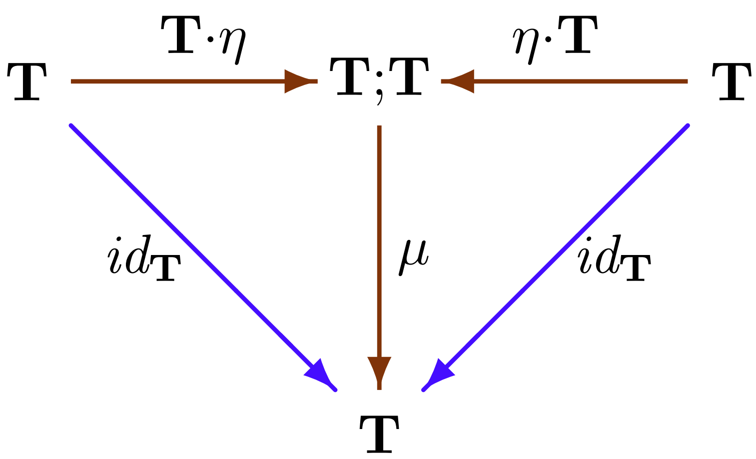

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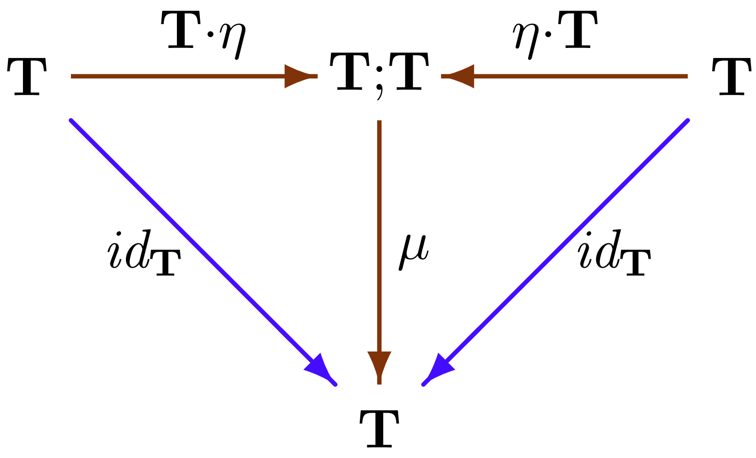

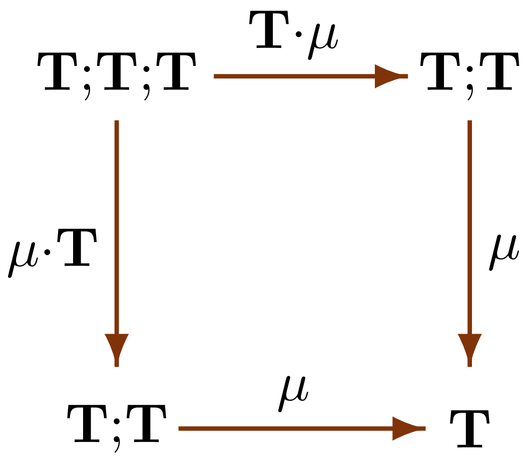

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- Monads in partial orders: *closure operators*

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  - $\mathbf{P}(X) = X + \{\bot\}$;

Examples of monads in **Set**

Simple Examples

- *Partiality* monad:
  - $\mathbf{P}(X) = X + \{\perp\}$; $\qquad \eta^{\mathbf{P}}_X \colon X \to X + \{\perp\}$

Simple Examples

Examples of monads in **Set**

- *Partiality* monad:
  - $\mathbf{P}(X) = X + \{\bot\}$; $\quad\quad\quad \eta^{\mathbf{P}}_X : X \to X + \{\bot\}$
  - $\eta^{\mathbf{P}}_X(x) = x$;

Simple Examples

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  - $\mathbf{P}(X) = X + \{\bot\}$; $\eta^{\mathbf{P}}_X \colon X \to X + \{\bot\}$
  - $\eta^{\mathbf{P}}_X(x) = x$; $\mu^{\mathbf{P}}_X \colon (X + \{\bot\}) + \{\bot\} \to X + \{\bot\}$

Simple Examples

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  - $\eta^{\mathbf{P}}_X(x) = x$; $\mu^{\mathbf{P}}_X \colon (X + \{\perp\}) + \{\perp\} \to X + \{\perp\}$
  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \perp$ for $x \notin X$.

Simple Examples

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  - $\mathbf{P}(X) = X + \{\bot\}$; $\eta^{\mathbf{P}}_X : X \to X + \{\bot\}$
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  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \bot$ for $x \notin X$.
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  - $\mathcal{E}(X) = X + E$;

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- *Exceptions* monad;
  - $\mathcal{E}(X) = X + E$; $\qquad \eta^{\mathcal{E}}_X \colon X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$;

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  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \bot$ for $x \notin X$.
- *Exceptions* monad;
  - $\mathcal{E}(X) = X + E$; $\qquad \eta^{\mathcal{E}}_X : X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$; $\qquad \mu^{\mathcal{E}}_X : (X + E) + E \to X + E$

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  - $\eta^{\mathbf{P}}_X(x) = x$; $\quad \mu^{\mathbf{P}}_X \colon (X + \{\bot\}) + \{\bot\} \to X + \{\bot\}$
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  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.

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  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \bot$ for $x \notin X$.
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  - $\eta^{\mathcal{E}}_X(x) = x$; $\qquad \mu^{\mathcal{E}}_X \colon (X + E) + E \to X + E$
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- *Nondeterminism* monad:

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  - $\mathcal{E}(X) = X + E$; $\qquad \eta^{\mathcal{E}}_X \colon X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$; $\qquad \mu^{\mathcal{E}}_X \colon (X + E) + E \to X + E$
  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.
- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$;

Simple Examples

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  - $\mathcal{E}(X) = X + E$; $\qquad \eta^{\mathcal{E}}_X \colon X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$; $\qquad \mu^{\mathcal{E}}_X \colon (X + E) + E \to X + E$
  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.
- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$; $\qquad \eta^{\mathcal{P}}_X \colon X \to 2^X$

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  - $\mathbf{P}(X) = X + \{\bot\}$; $\quad \eta^{\mathbf{P}}_X \colon X \to X + \{\bot\}$
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  - $\eta^{\mathcal{E}}_X(x) = x$; $\quad \mu^{\mathcal{E}}_X \colon (X + E) + E \to X + E$
  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.
- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$; $\quad \eta^{\mathcal{P}}_X \colon X \to 2^X$
  - $\eta^{\mathcal{P}}_X(x) = \{x\}$;

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- *Partiality* monad:
  - $\mathbf{P}(X) = X + \{\bot\}$; $\eta^{\mathbf{P}}_X \colon X \to X + \{\bot\}$
  - $\eta^{\mathbf{P}}_X(x) = x$; $\mu^{\mathbf{P}}_X \colon (X + \{\bot\}) + \{\bot\} \to X + \{\bot\}$
  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \bot$ for $x \notin X$.
- *Exceptions* monad;
  - $\mathcal{E}(X) = X + E$; $\eta^{\mathcal{E}}_X \colon X \to X + E$
  - $\eta^{\mathcal{E}}_X(x) = x$; $\mu^{\mathcal{E}}_X \colon (X + E) + E \to X + E$
  - $\mu^{\mathcal{E}}_X(x) = x$ for $x \in X$, $\mu^{\mathcal{E}}_X(e) = e$ for $e \in E$.
- *Nondeterminism* monad:
  - $\mathcal{P}(X) = 2^X$; $\eta^{\mathcal{P}}_X \colon X \to 2^X$
  - $\eta^{\mathcal{P}}_X(x) = \{x\}$; $\mu^{\mathcal{P}}_X \colon 2^{2^X} \to 2^X$

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  - $\mu^{\mathbf{P}}_X(x) = x$ for $x \in X$, $\mu^{\mathbf{P}}_X(x) = \bot$ for $x \notin X$.
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  - $\mathcal{P}(X) = 2^X$; $\qquad \eta^{\mathcal{P}}_X : X \to 2^X$
  - $\eta^{\mathcal{P}}_X(x) = \{x\}$; $\qquad \mu^{\mathcal{P}}_X : 2^{2^X} \to 2^X$
  - $\mu^{\mathcal{P}}_X(U) = \bigcup U$ for $U \in 2^{2^X}$.

Typical examples

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- *List* monad:

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- *List* monad:
  - $\mathcal{L}(X) = X^*$;

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- *List* monad:
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- *List* monad:
  - $\mathcal{L}(X) = X^*$;
  - $\eta^{\mathcal{L}}_X(x) = \langle x \rangle$;

$\eta^{\mathcal{L}}_X \colon X \to X^*$

Typical examples

- *List* monad:
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  - $\eta^{\mathcal{L}}_X(x) = \langle x \rangle$; $\qquad \mu^{\mathcal{L}}_X : (X^*)^* \to X^*$

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  - $\mathcal{L}(X) = X^*$; $\qquad \eta^{\mathcal{L}}_X \colon X \to X^*$
  - $\eta^{\mathcal{L}}_X(x) = \langle x \rangle$; $\qquad \mu^{\mathcal{L}}_X \colon (X^*)^* \to X^*$
  - $\mu^{\mathcal{L}}_X(\langle l_1, \ldots, l_n \rangle) = append(l_1, \ldots append(l_{n-1}, l_n) \ldots)$.

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  - $\mu^{\mathcal{L}}_X(\langle l_1, \ldots, l_n \rangle) = append(l_1, \ldots append(l_{n-1}, l_n) \ldots)$.
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  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$;

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  - $\mu^{\mathcal{L}}_X(\langle l_1, \ldots, l_n \rangle) = append(l_1, \ldots append(l_{n-1}, l_n) \ldots)$.
- *Term* monad:
  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$; $\qquad \eta^{\mathcal{T}_\Sigma}_X : X \to T_\Sigma(X)$

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- *Term* monad:
  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$; $\qquad \eta^{\mathcal{T}_\Sigma}_X : X \to T_\Sigma(X)$
  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$;

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- *Term* monad:
  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$; $\quad \eta^{\mathcal{T}_\Sigma}_X : X \to T_\Sigma(X)$
  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$; $\quad \mu^{\mathcal{T}_\Sigma}_X : T_\Sigma(T_\Sigma(X)) \to T_\Sigma(X)$

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  - $\eta^{\mathcal{L}}_X(x) = \langle x \rangle$; $\qquad \mu^{\mathcal{L}}_X : (X^*)^* \to X^*$
  - $\mu^{\mathcal{L}}_X(\langle l_1, \ldots, l_n \rangle) = append(l_1, \ldots append(l_{n-1}, l_n) \ldots)$.
- *Term* monad:
  - $\mathcal{T}_\Sigma(X) = T_\Sigma(X)$; $\qquad \eta^{\mathcal{T}_\Sigma}_X : X \to T_\Sigma(X)$
  - $\eta^{\mathcal{T}_\Sigma}_X(x) = x$; $\qquad \mu^{\mathcal{T}_\Sigma}_X : T_\Sigma(T_\Sigma(X)) \to T_\Sigma(X)$
  - $\mu^{\mathcal{T}_\Sigma}_X(t) = t[id_{T_\Sigma(X)}]$ for $t \in T_\Sigma(T_\Sigma(X))$.

Difficult(?) examples

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- *Side-effects* monad:

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  - $\mathcal{S}(X) = (X \times S)^S$;

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Difficult(?) examples

Examples of monads in **Set**

- *Side-effects* monad:
  - $\mathcal{S}(X) = (X \times S)^S$;
  - $\eta^{\mathcal{S}}_X(x)(s) = \langle x, s \rangle$;

$\eta^{\mathcal{S}}_X : X \to (X \times S)^S$

Difficult(?) examples

- *Side-effects* monad:
  - $\mathcal{S}(X) = (X \times S)^S$; $\qquad \eta^{\mathcal{S}}_X : X \to (X \times S)^S$
  - $\eta^{\mathcal{S}}_X(x)(s) = \langle x, s \rangle$; $\qquad \mu^{\mathcal{S}}_X : ((X \times S)^S \times S)^S \to (X \times S)^S$

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- *Side-effects* monad:
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  - $\mu^{\mathcal{S}}_X(f)(s) = g(s')$ where $f(s) = \langle g, s' \rangle$, for $f \in ((X \times S)^S \times S)^S$.

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  - $\mathcal{K}(X) = A^{(A^X)}$;

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- *Continuation* monad:
  - $\mathcal{K}(X) = A^{(A^X)}$; $\quad \eta^{\mathcal{K}}_X \colon X \to A^{(A^X)}$
  - $\eta^{\mathcal{K}}_X(x)(k) = k(x)$;

Difficult(?) examples

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  - $\mathcal{K}(X) = A^{(A^X)}$; $\eta^{\mathcal{K}}_X \colon X \to A^{(A^X)}$
  - $\eta^{\mathcal{K}}_X(x)(k) = k(x)$; $\mu^{\mathcal{K}}_X \colon A^{(A^{(A^{(A^X)})})} \to A^{(A^X)}$

Difficult(?) examples

- *Side-effects* monad:
  - $\mathcal{S}(X) = (X \times S)^S$; $\qquad \eta^{\mathcal{S}}_X \colon X \to (X \times S)^S$
  - $\eta^{\mathcal{S}}_X(x)(s) = \langle x, s\rangle$; $\qquad \mu^{\mathcal{S}}_X \colon ((X \times S)^S \times S)^S \to (X \times S)^S$
  - $\mu^{\mathcal{S}}_X(f)(s) = g(s')$ where $f(s) = \langle g, s'\rangle$, for $f \in ((X \times S)^S \times S)^S$.
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  - $\mu^{\mathcal{K}}_X(f)(k) = f(\lambda g \in A^{(A^X)}.g(k))$, for $f \in A^{(A^{(A^{(A^X)})})}$.

Instead of more examples

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Adjunctions give rise to monads

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(i.e. $\mu^{\mathbf{T}}_X = \mathbf{G}(\varepsilon_{\mathbf{F}(X)})\colon \mathbf{G}(\mathbf{F}(\mathbf{G}(\mathbf{F}(X)))) \to \mathbf{G}(\mathbf{F}(X))$*)*

Proof

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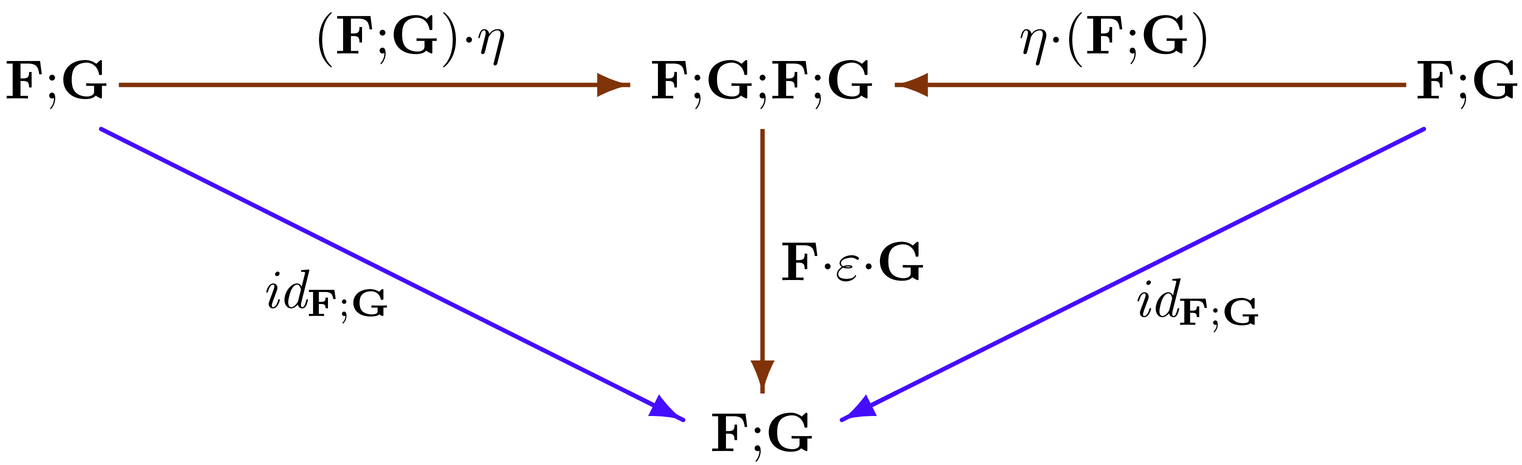

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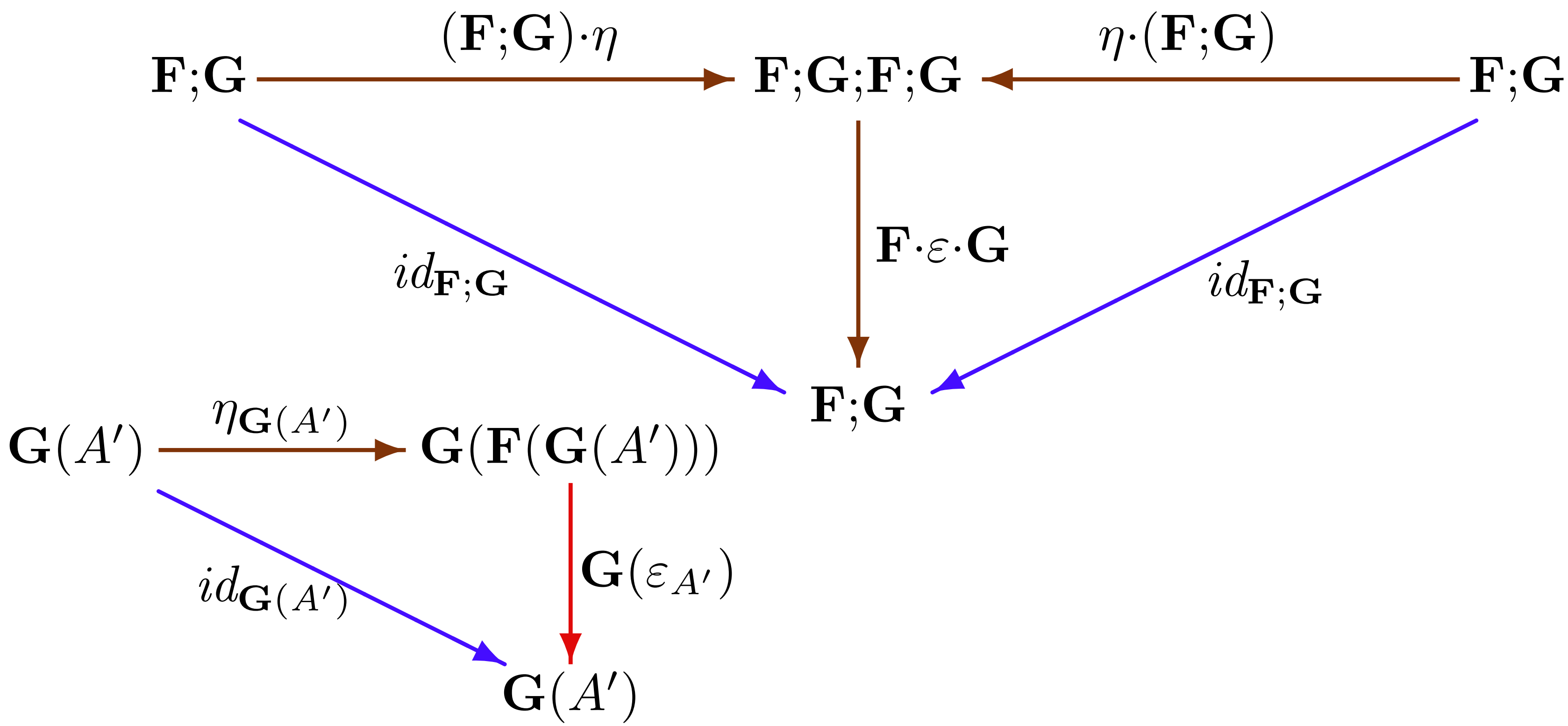

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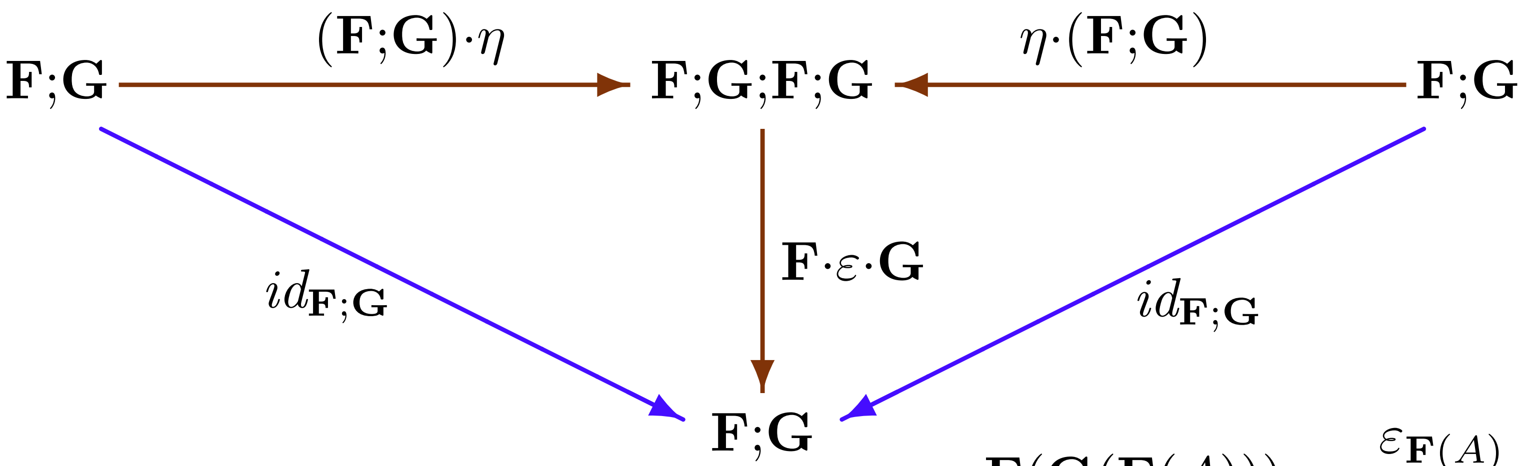

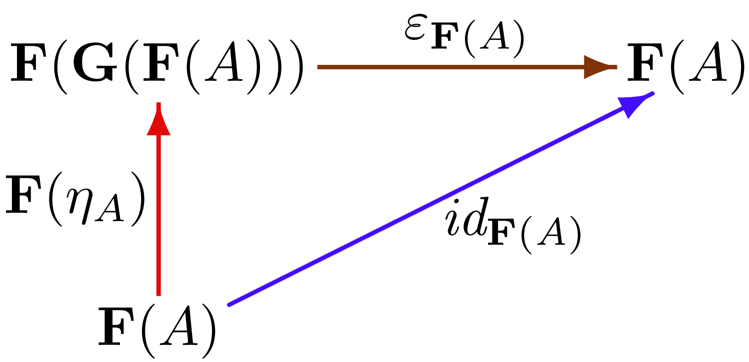

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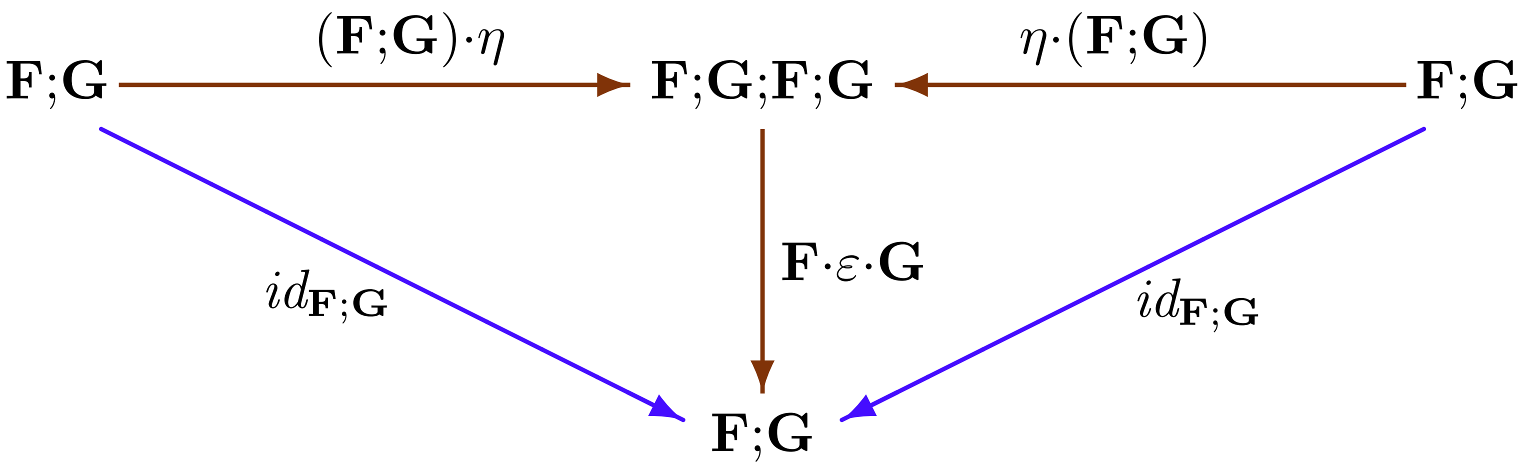

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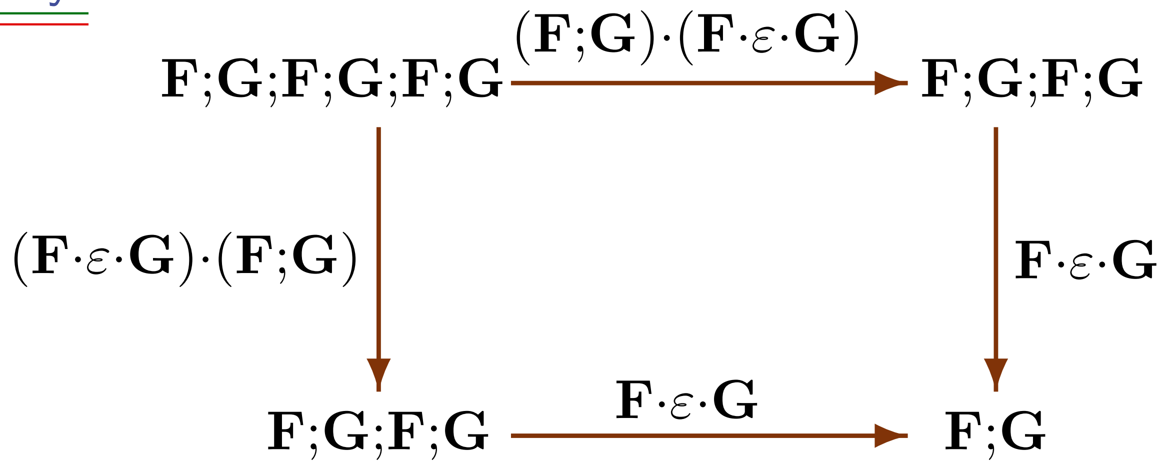

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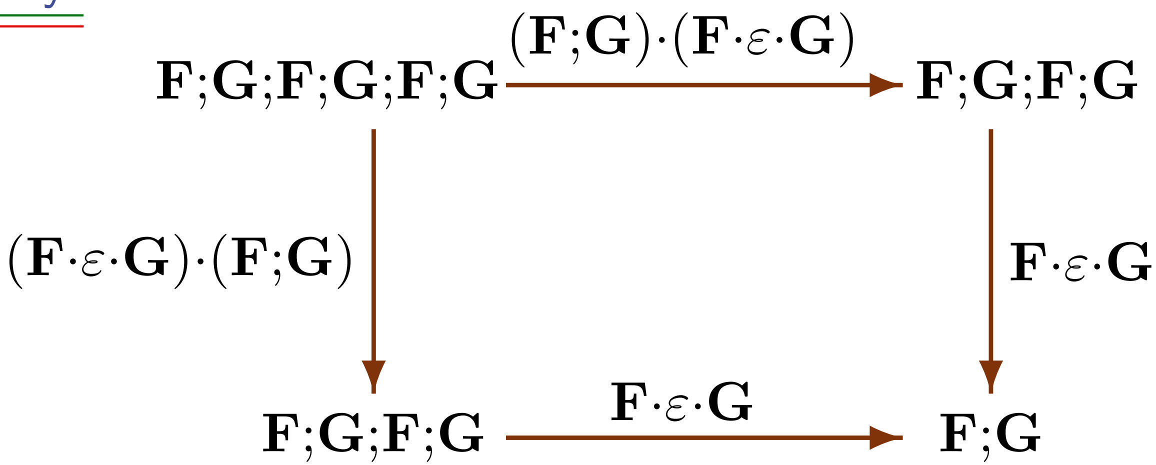

Follows by the commutativity of the diagrams below:

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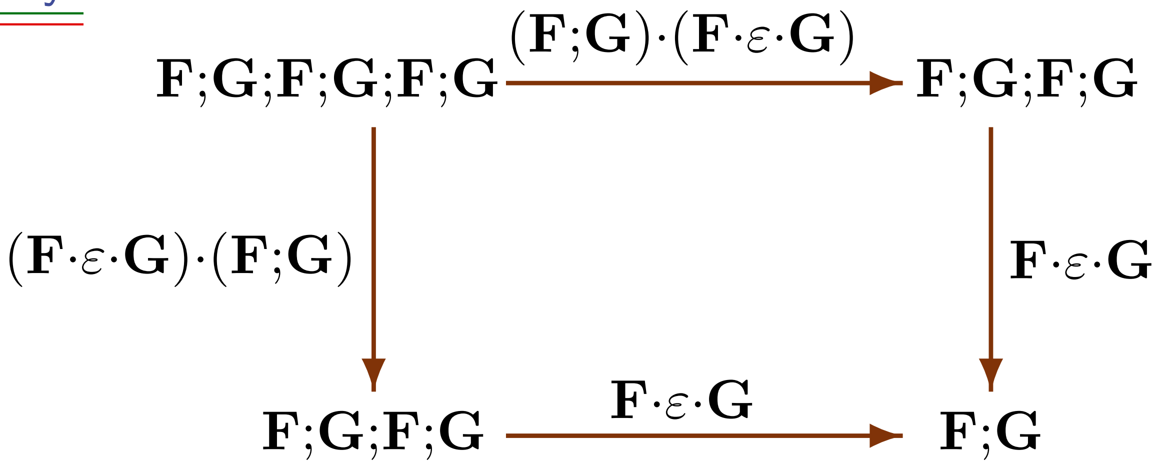

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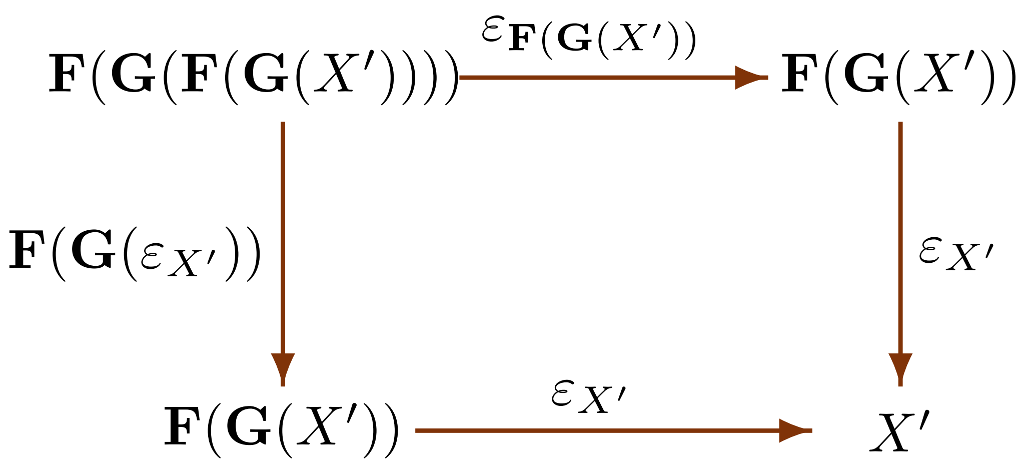

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$$\begin{array}{ccc}
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\Big\downarrow (\mathbf{F}\cdot\varepsilon\cdot\mathbf{G})\cdot(\mathbf{F};\mathbf{G}) & & \Big\downarrow \mathbf{F}\cdot\varepsilon\cdot\mathbf{G} \\
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\end{array}$$

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$$\begin{array}{ccc}
\mathbf{G};\mathbf{F};\mathbf{G};\mathbf{F} & \xrightarrow{\mathbf{G}\cdot(\mathbf{F}\cdot\varepsilon)} & \mathbf{G};\mathbf{F} \\
\Big\downarrow (\varepsilon\cdot\mathbf{G})\cdot\mathbf{F} & & \Big\downarrow \varepsilon \\
\mathbf{G};\mathbf{F} & \xrightarrow{\varepsilon} & \mathbf{Id_K}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{F}(\mathbf{G}(\mathbf{F}(\mathbf{G}(X')))) & \xrightarrow{\varepsilon_{\mathbf{F}(\mathbf{G}(X'))}} & \mathbf{F}(\mathbf{G}(X')) \\
\Big\downarrow \mathbf{F}(\mathbf{G}(\varepsilon_{X'})) & & \Big\downarrow \varepsilon_{X'} \\
\mathbf{F}(\mathbf{G}(X')) & \xrightarrow{\varepsilon_{X'}} & X'
\end{array}$$

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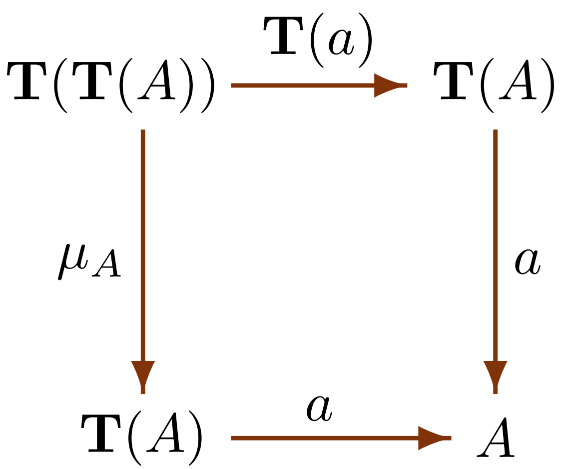

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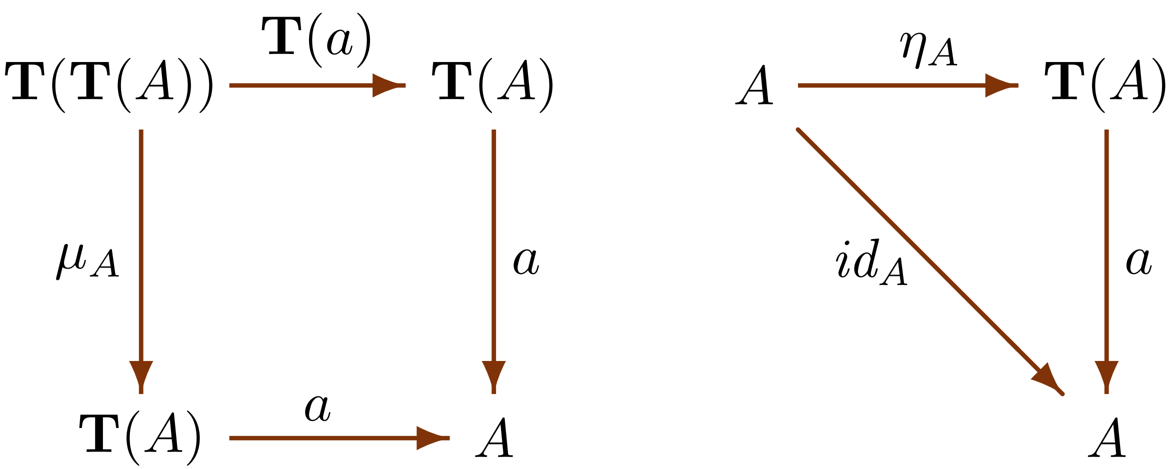

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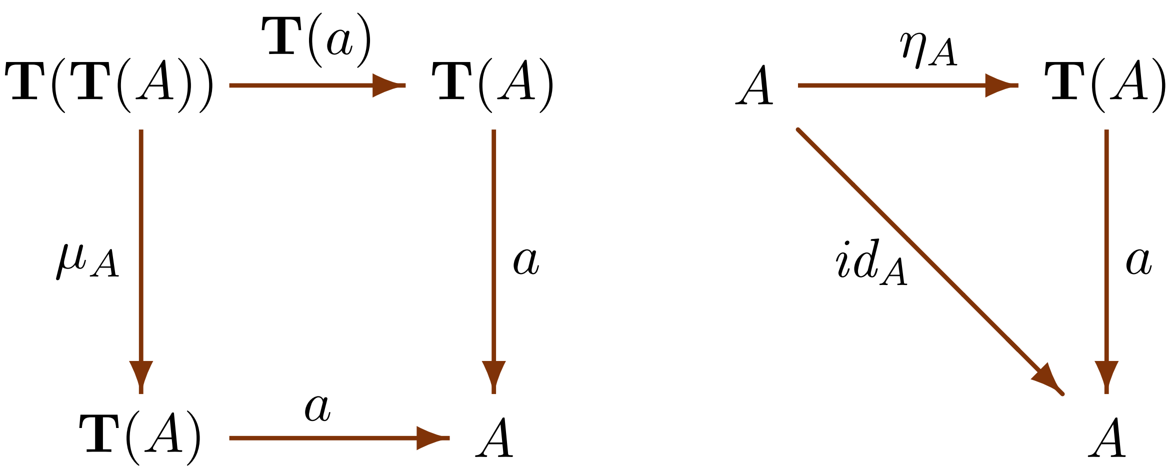
- $\mathbf{T}$-*homomorphism* from $\langle A, a \colon \mathbf{T}(A) \to A \rangle$ to $\langle B, b \colon \mathbf{T}(B) \to B \rangle$:



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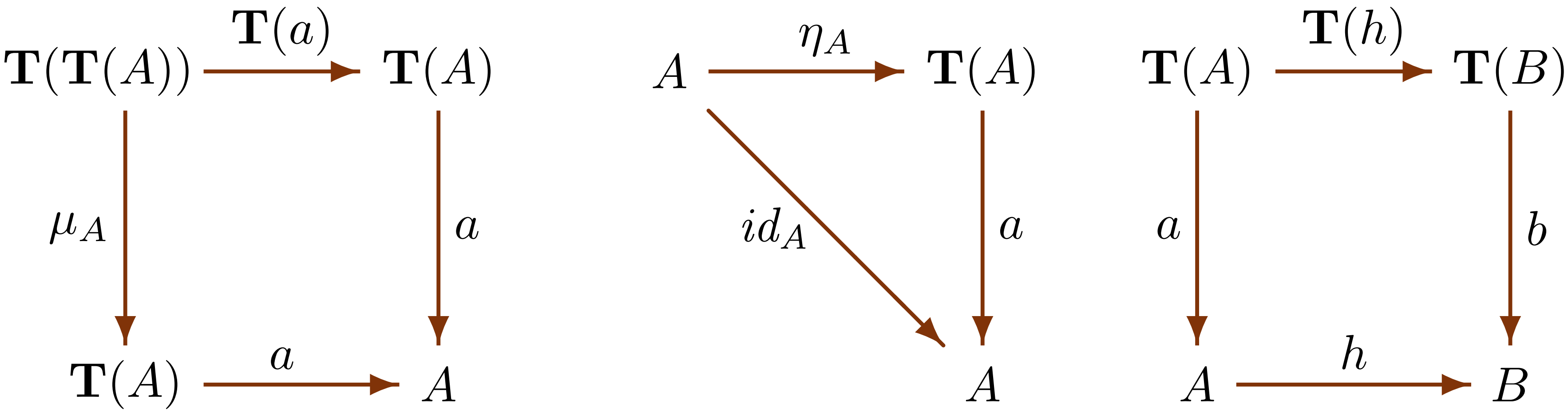
- $\mathbf{T}$-*algebras*:
  $\langle A \in |\mathbf{K}|, a\colon \mathbf{T}(A) \to A \rangle$ such that $\mathbf{T}(a);a = \mu_A;a$ and $\eta_A;a = id_A$
- $\mathbf{T}$-*homomorphism* from $\langle A, a\colon \mathbf{T}(A) \to A \rangle$ to $\langle B, b\colon \mathbf{T}(B) \to B \rangle$:
  $h\colon A \to B$ such that $\mathbf{T}(h);b = a;h$



Monadic adjunction

Monadic adjunction

Let $\mathbf{G}^{\mathbf{T}} \colon \mathbf{Alg}(\mathbf{T}) \to \mathbf{K}$ be the obvious projection: $\mathbf{G}^{\mathbf{T}}(\langle A, a\rangle) = A$, ...

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For $X \in |\mathbf{K}|$, $\mathbf{F^T}(X) = \langle \mathbf{T}(X), \mu_X \colon \mathbf{T}(\mathbf{T}(X)) \to \mathbf{T}(X)\rangle$ with unit $\eta_X \colon X \to \mathbf{G^T}(\mathbf{F^T}(X))$ is free over X w.r.t. $\mathbf{G^T}$:

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$$X \xrightarrow{\eta_X} \mathbf{T}(X) \qquad\qquad \mathbf{T}(\mathbf{T}(X)) \xrightarrow{\mu_X} \mathbf{T}(X)$$

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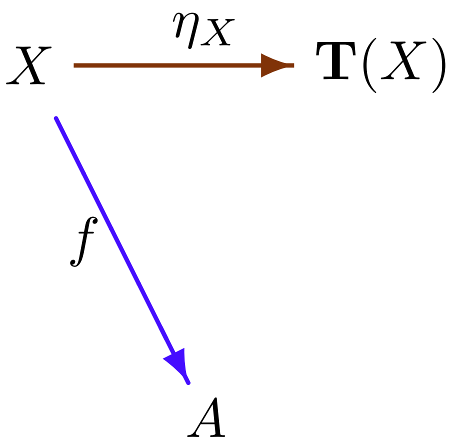

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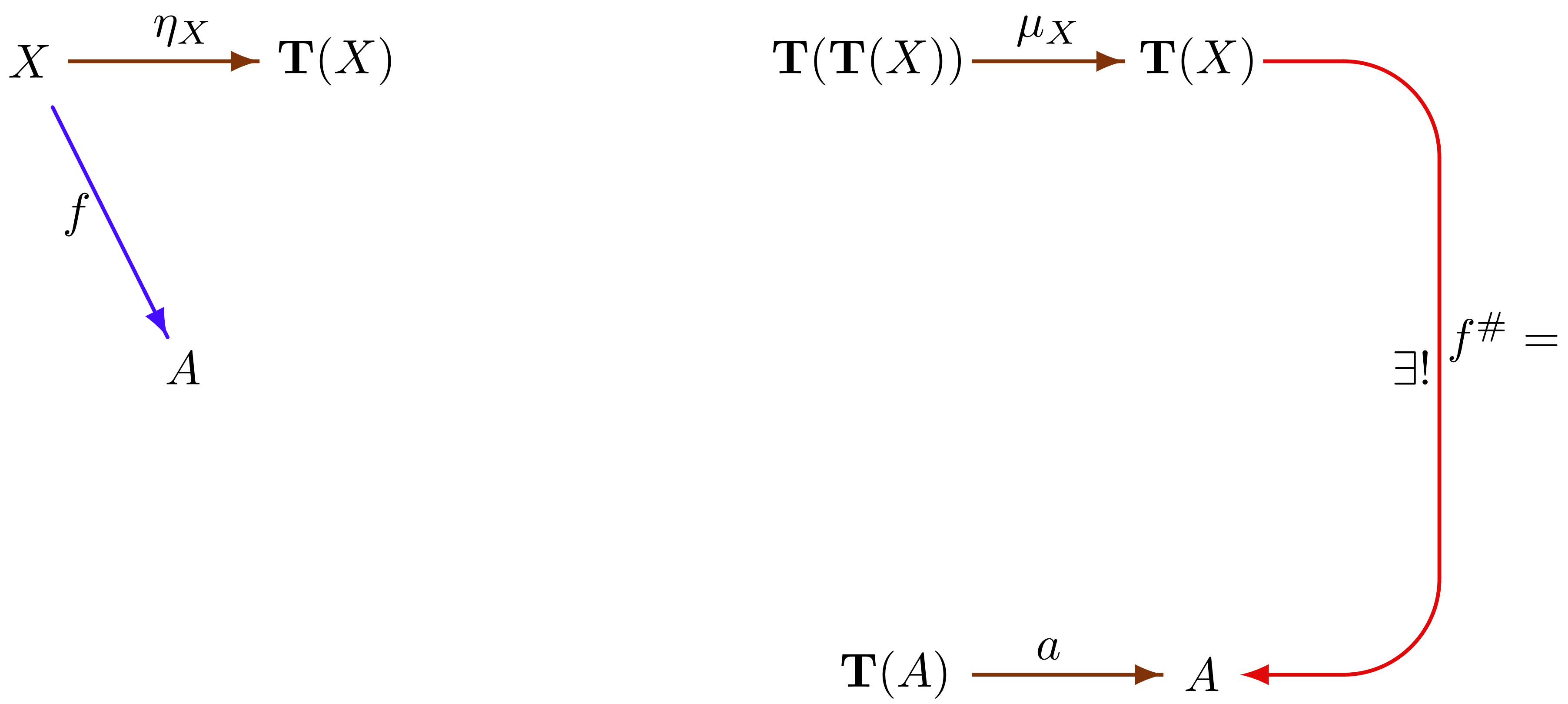

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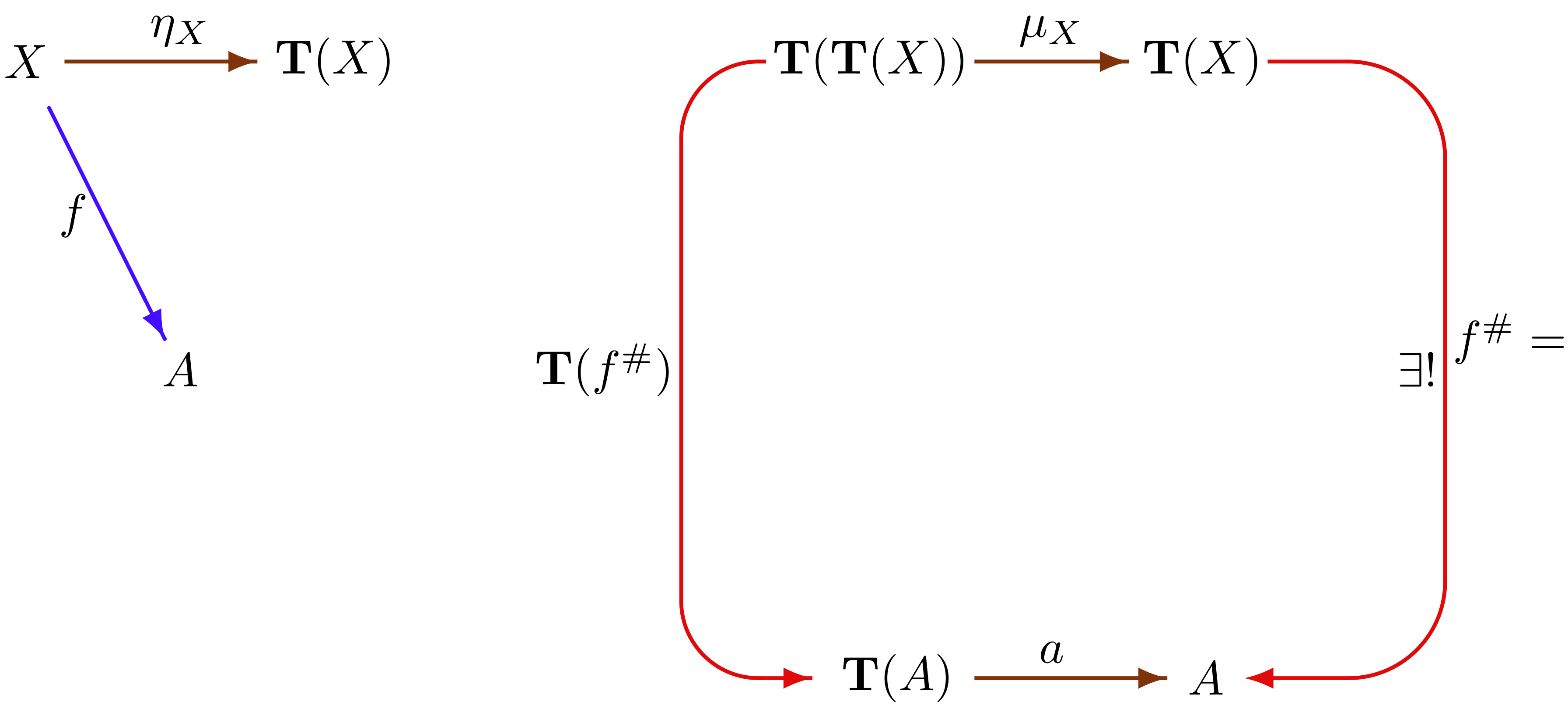

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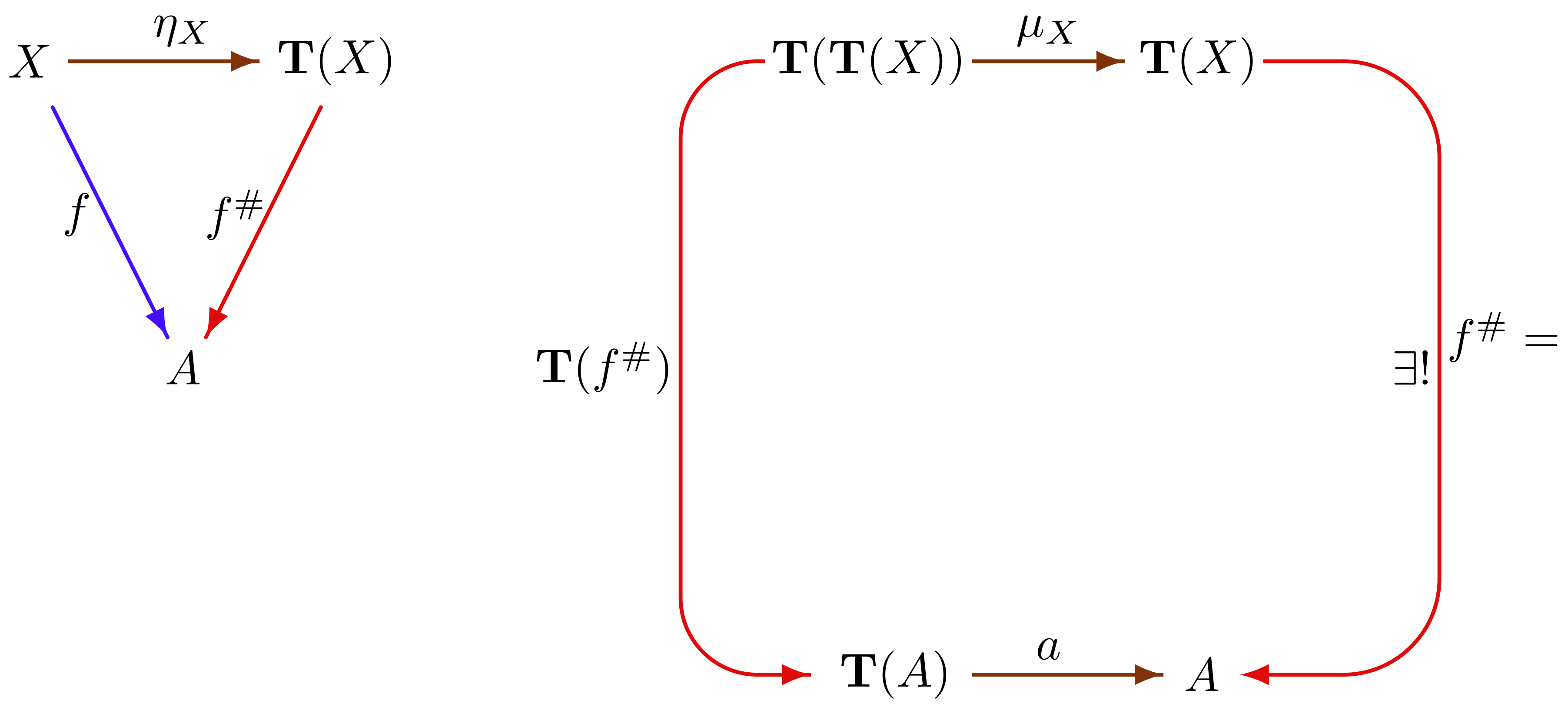

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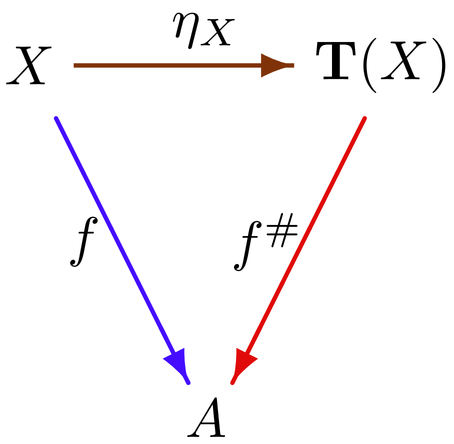

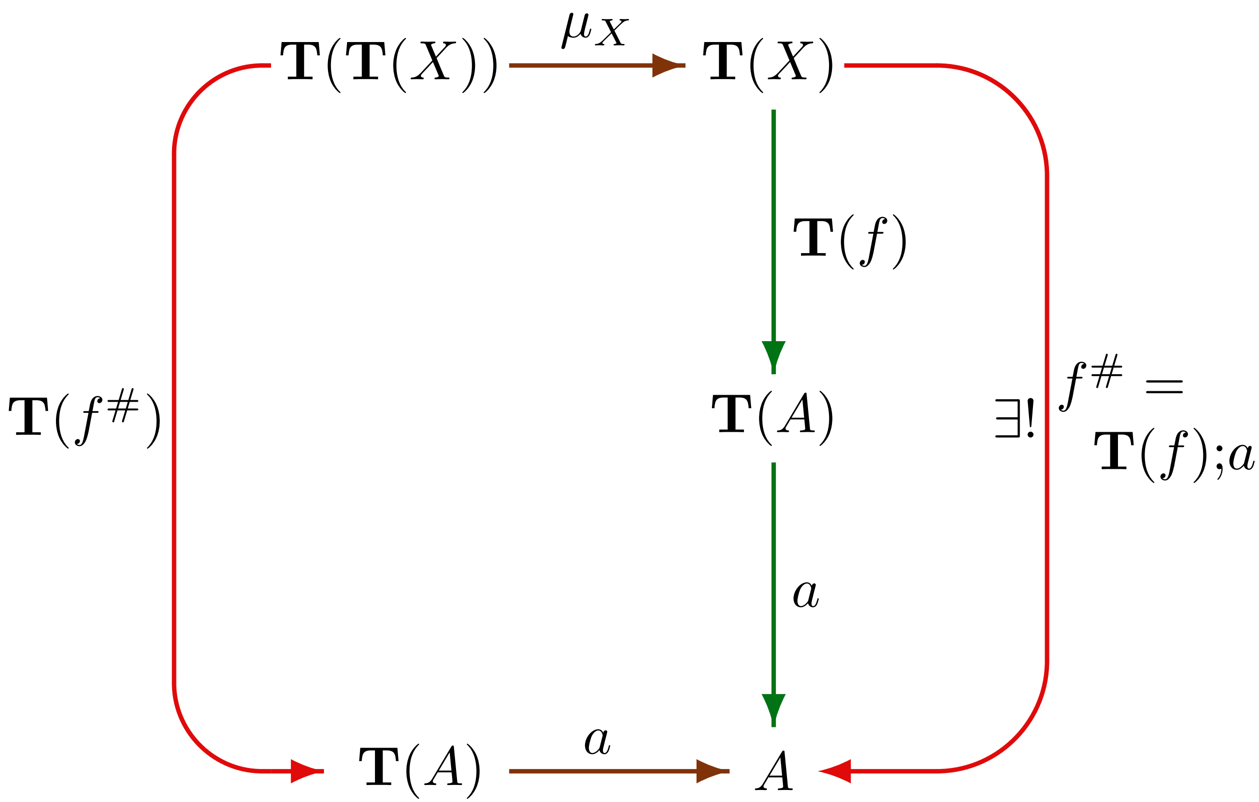

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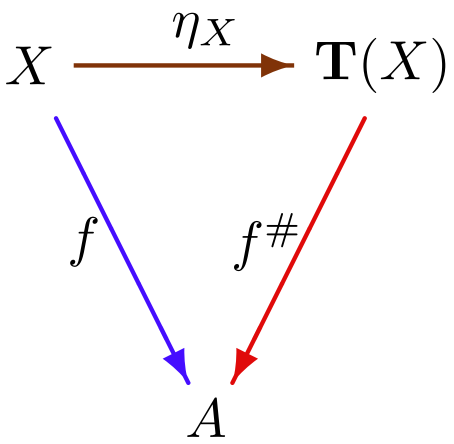

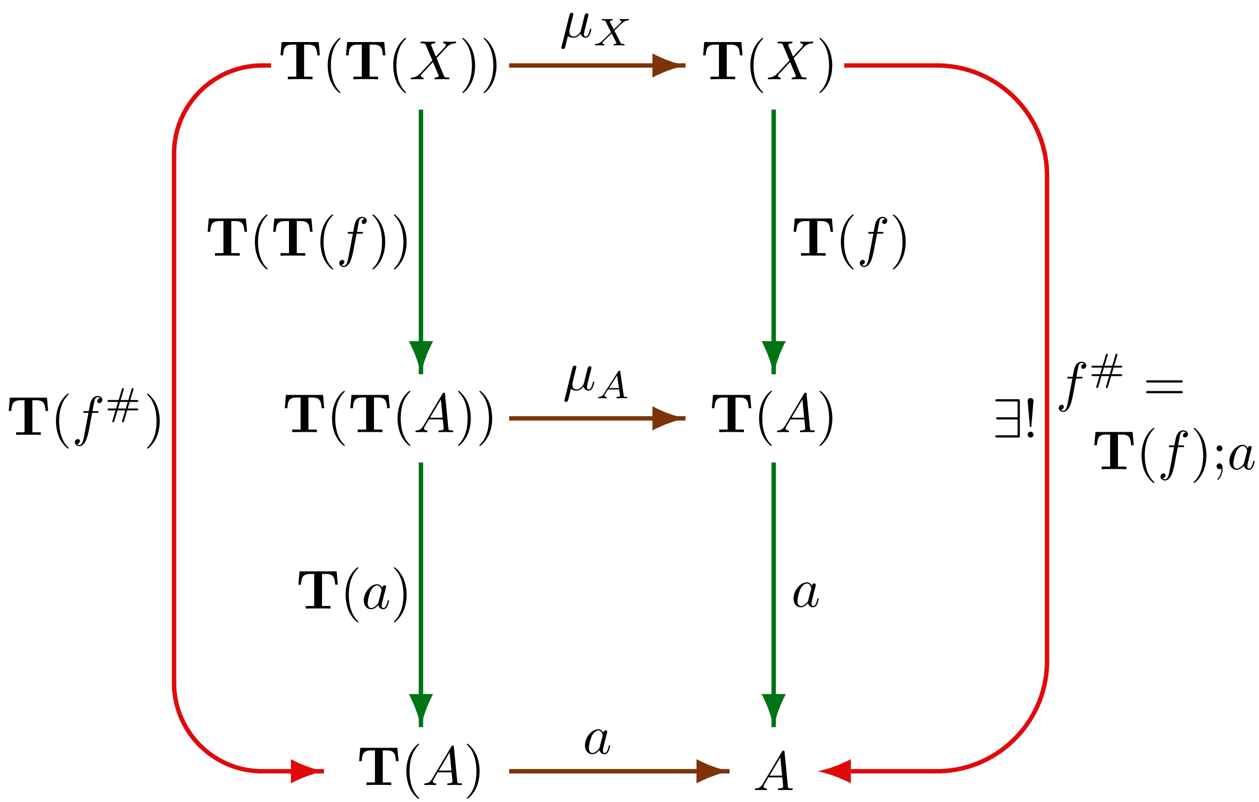

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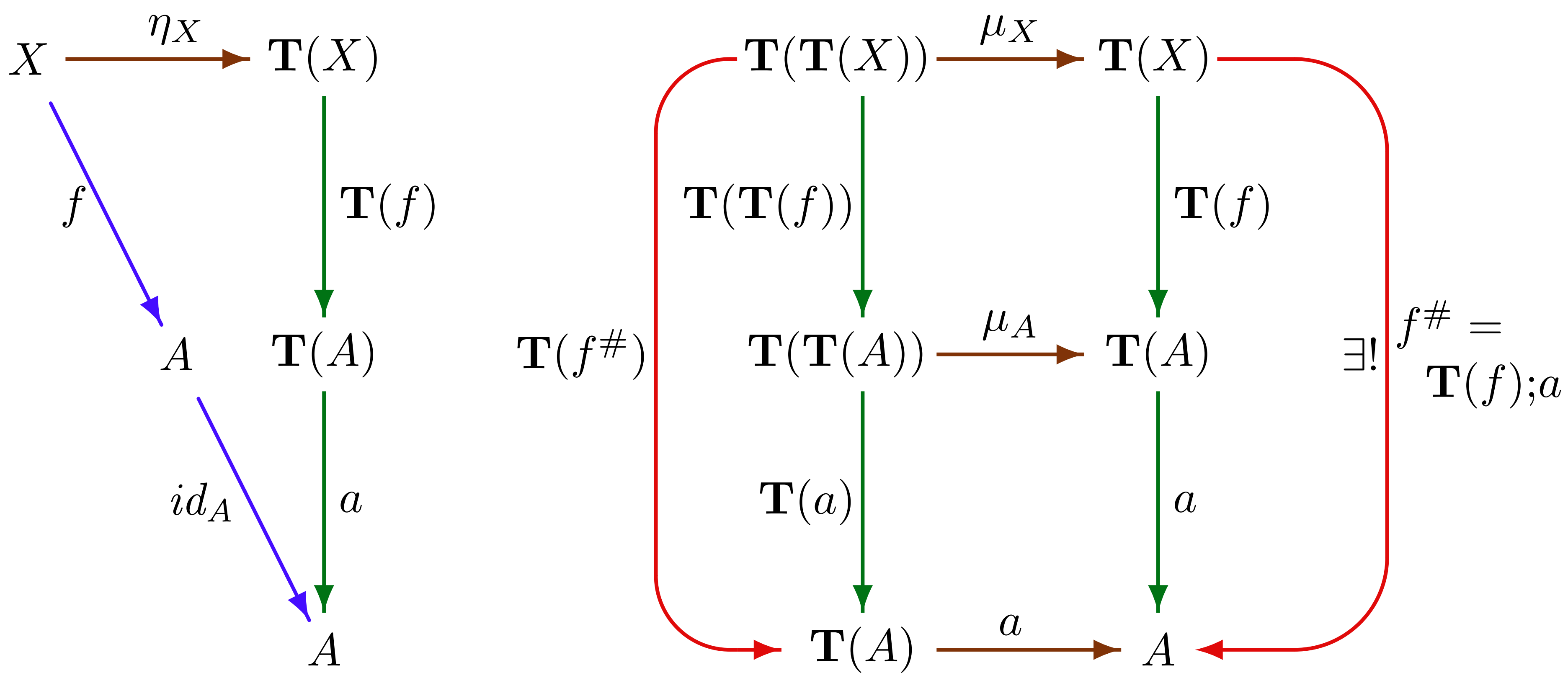

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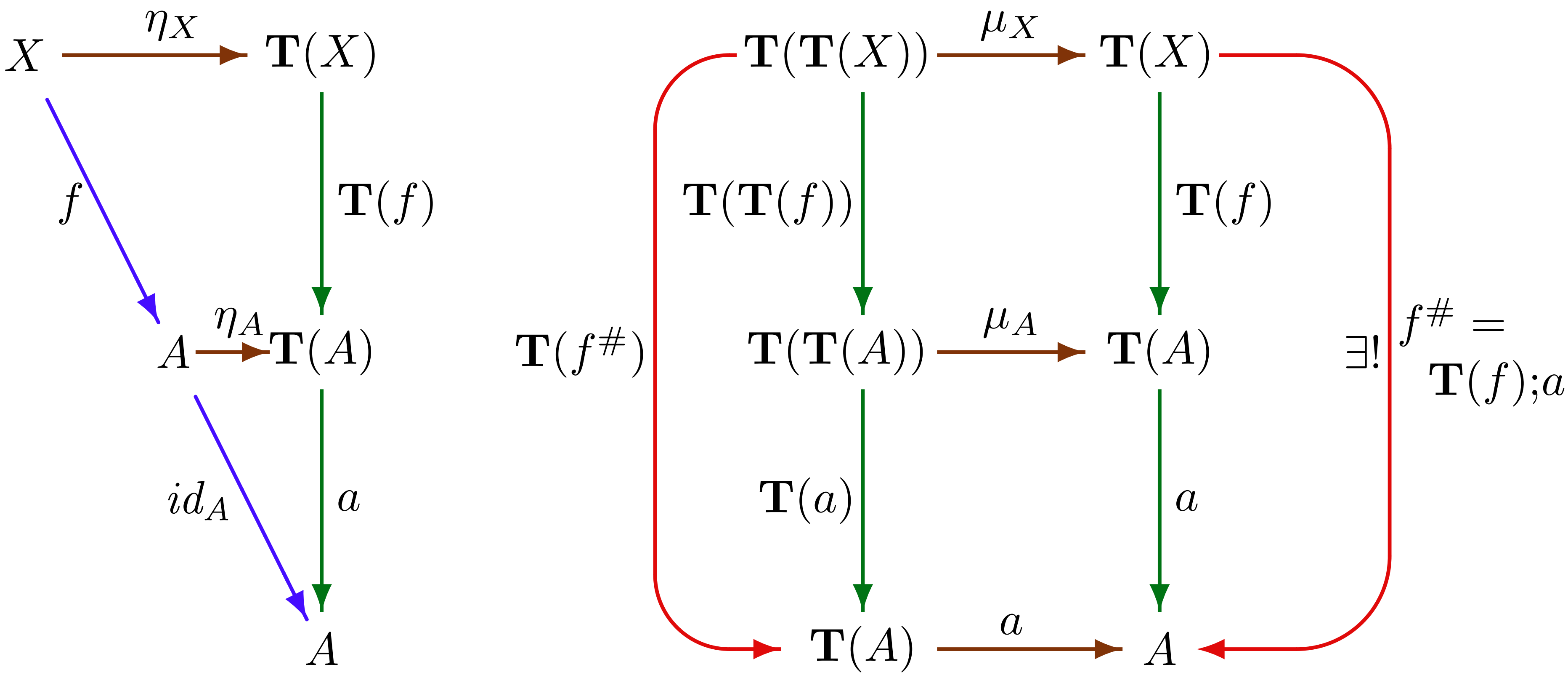

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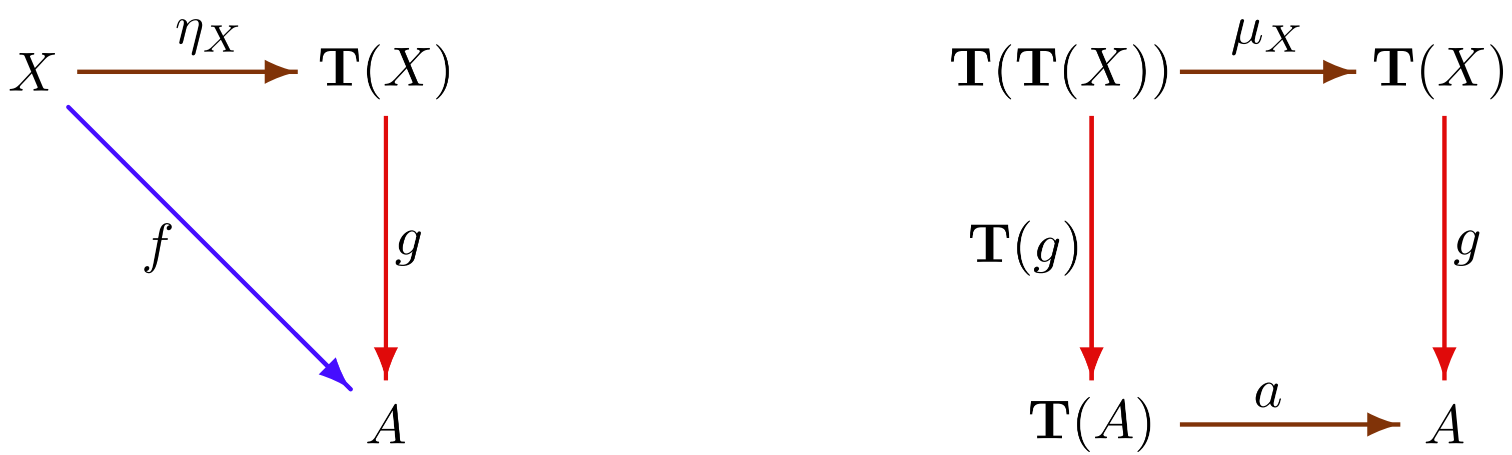

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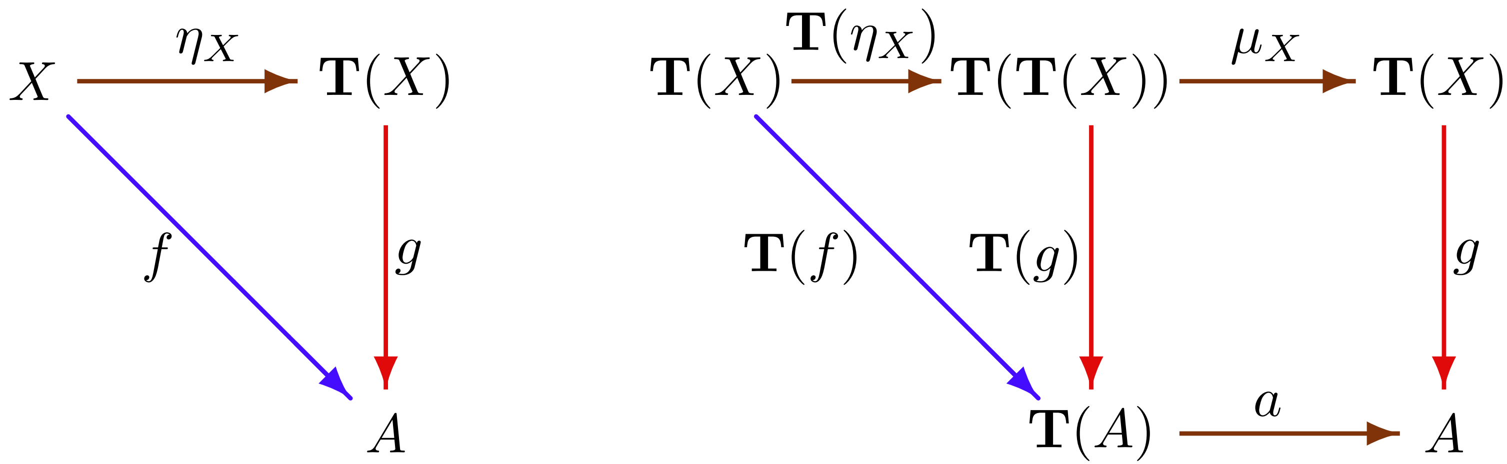

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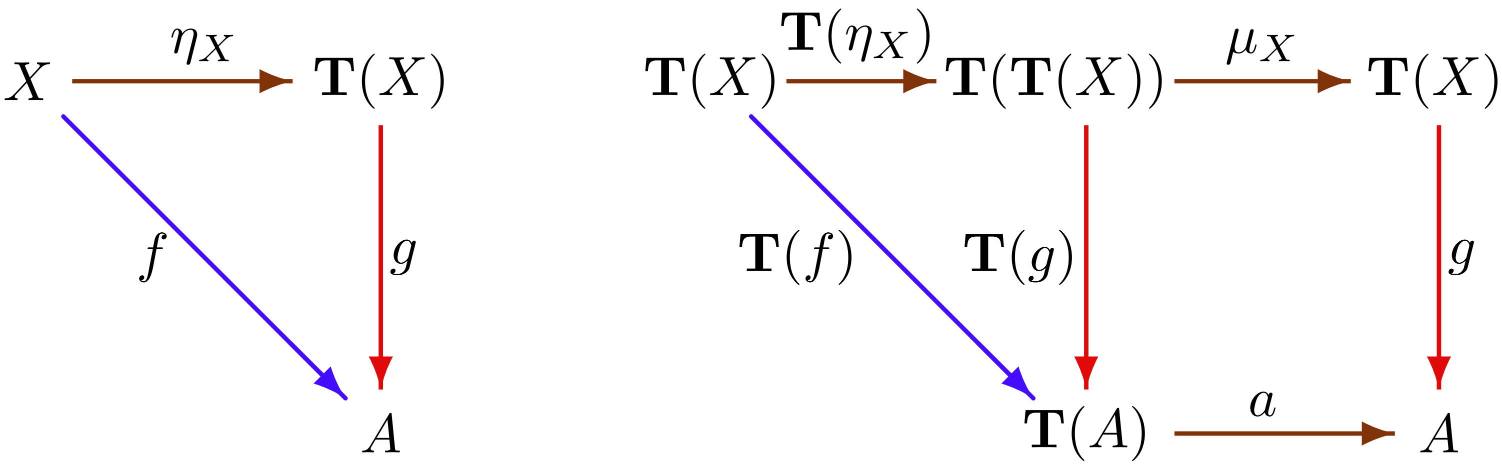

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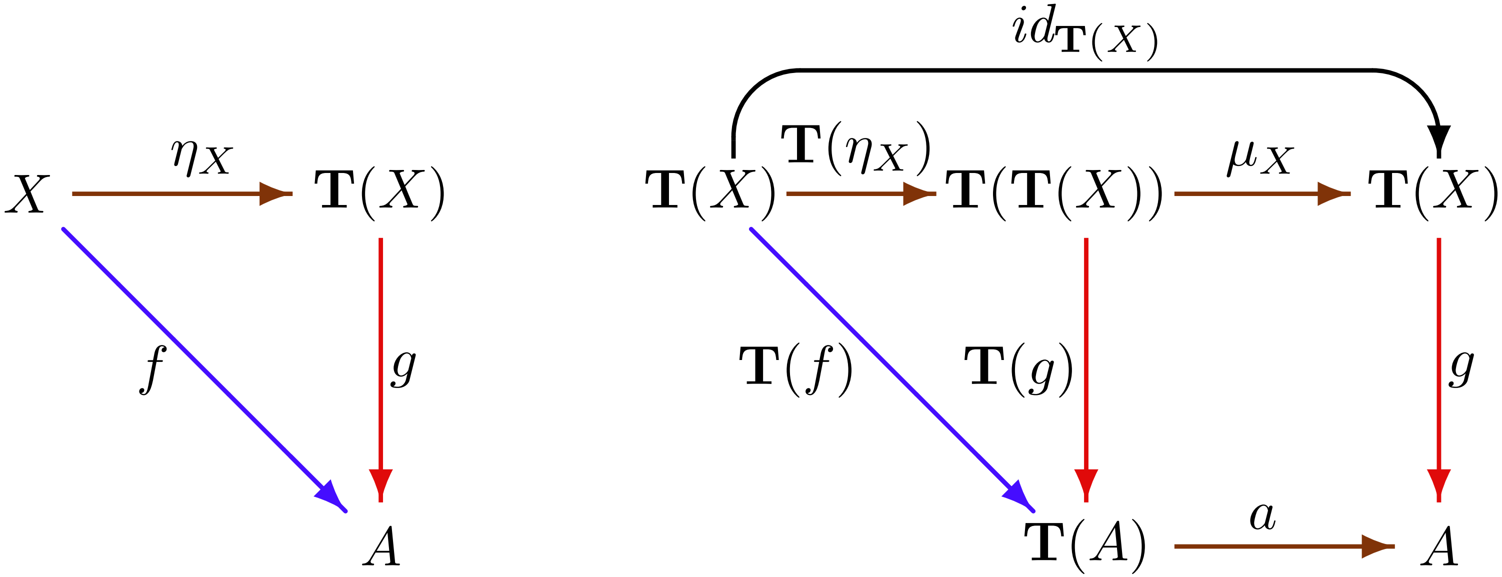

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BTW:
$$\begin{aligned}
\mathbf{F^T}(f\colon X \to Y) &= (f;\eta_Y)^{\#} \\
&= \mathbf{T}(f;\eta_Y);\mu_Y \\
&= \mathbf{T}(f);\mathbf{T}(\eta_Y);\mu_Y \\
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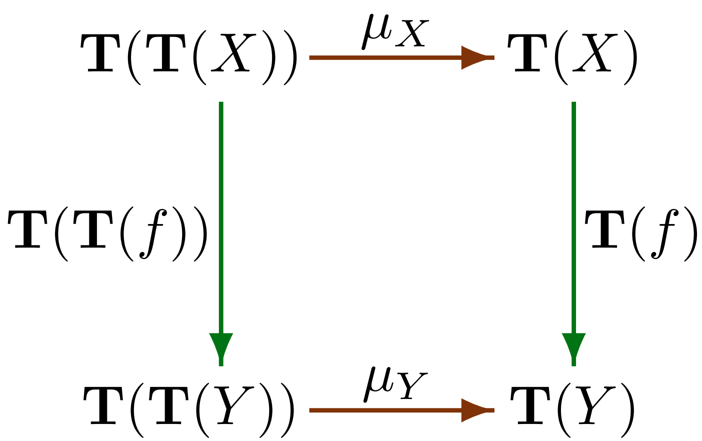

All monads arise from adjunctions

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Given a monad $\langle \mathbf{T}, \eta, \mu \rangle$ in $\mathbf{K}$ we have an adjunction

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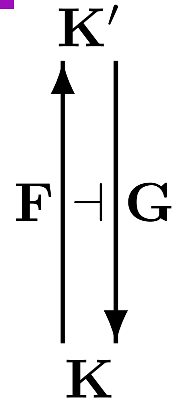

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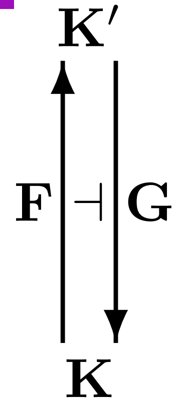

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Theorem: *Given an adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle : \mathbf{K} \to \mathbf{K}'$, *let* $\langle \mathbf{T} = \mathbf{F};\mathbf{G}, \eta, \mu^{\mathbf{T}} = \mathbf{F}{\cdot}\varepsilon{\cdot}\mathbf{G} \rangle$ *be the monad it yields.*

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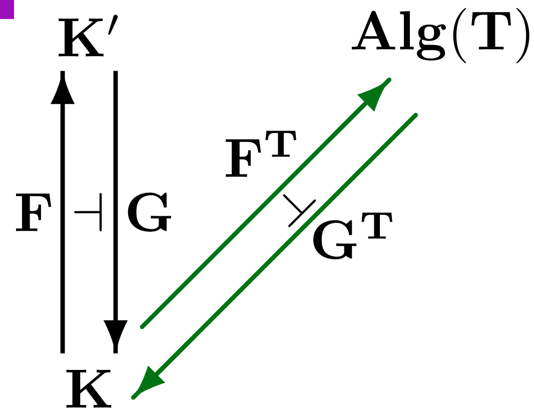

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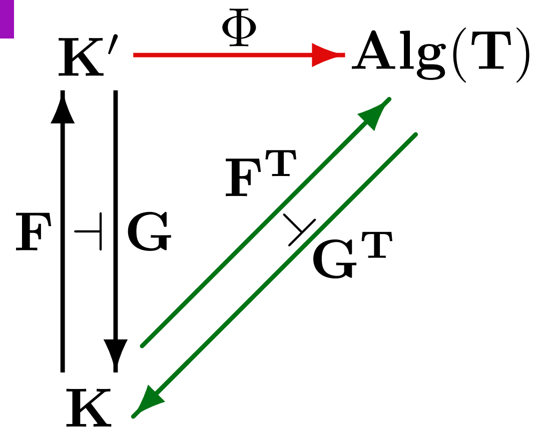

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Theorem: $\langle \mathbf{T}, \eta, \mu \rangle$ *is the monad associated with* $\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle$.

Theorem: *Given an adjunction* $\langle \mathbf{F}, \mathbf{G}, \eta, \varepsilon \rangle \colon \mathbf{K} \to \mathbf{K}'$, *let* $\langle \mathbf{T} = \mathbf{F};\mathbf{G}, \eta, \mu^{\mathbf{T}} = \mathbf{F} \cdot \varepsilon \cdot \mathbf{G} \rangle$ *be the monad it yields.* *Let then* $\langle \mathbf{F}^{\mathbf{T}}, \mathbf{G}^{\mathbf{T}}, \eta, \varepsilon^{\mathbf{T}} \rangle \colon \mathbf{K} \to \mathbf{Alg}(\mathbf{T})$ *be the adjunction for* $\mathbf{T}$. *Then there is a unique comparison functor* $\Phi \colon \mathbf{K}' \to \mathbf{Alg}(\mathbf{T})$ *such that* $\Phi;\mathbf{G}^{\mathbf{T}} = \mathbf{G}$ *and* $\mathbf{F};\Phi = \mathbf{F}^{\mathbf{T}}$.

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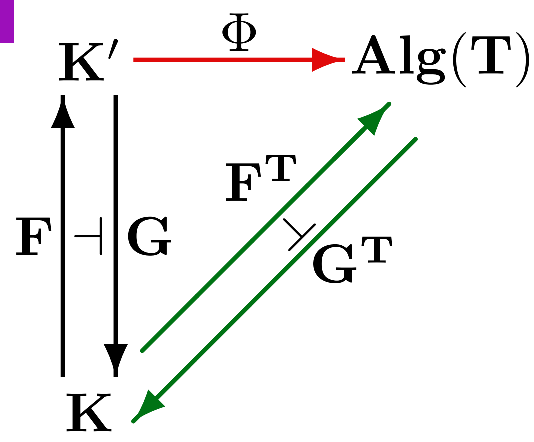

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Free algebras

Kleisli '65

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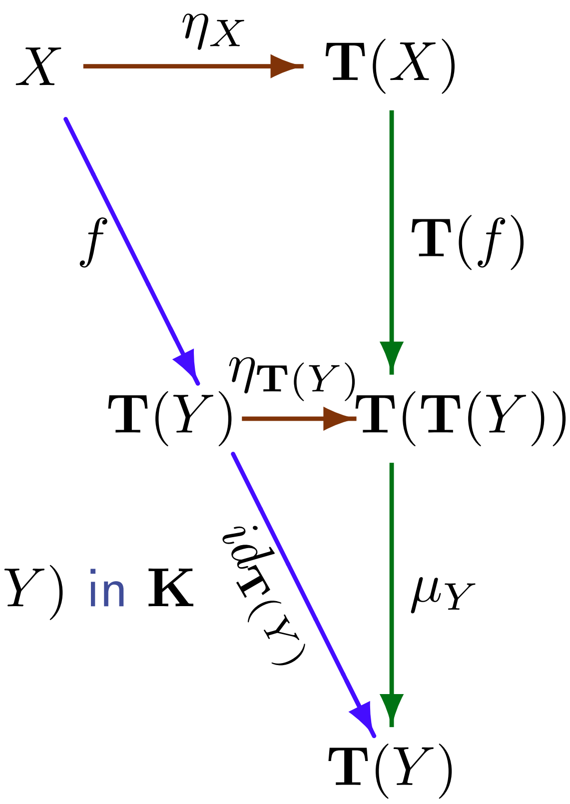
Free algebras



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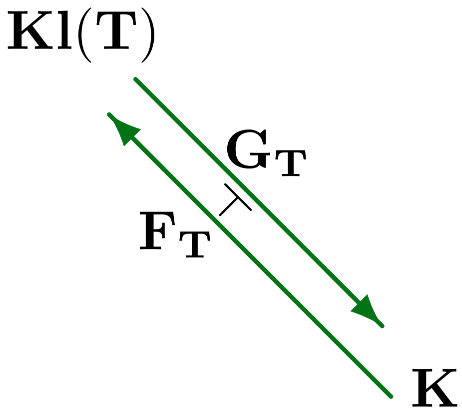

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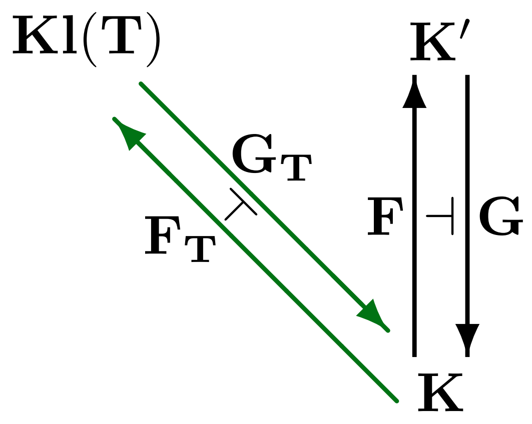

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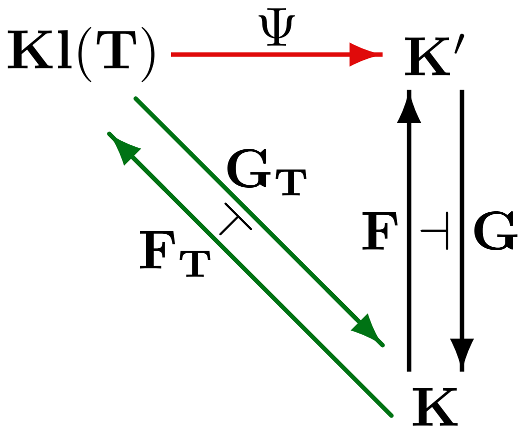

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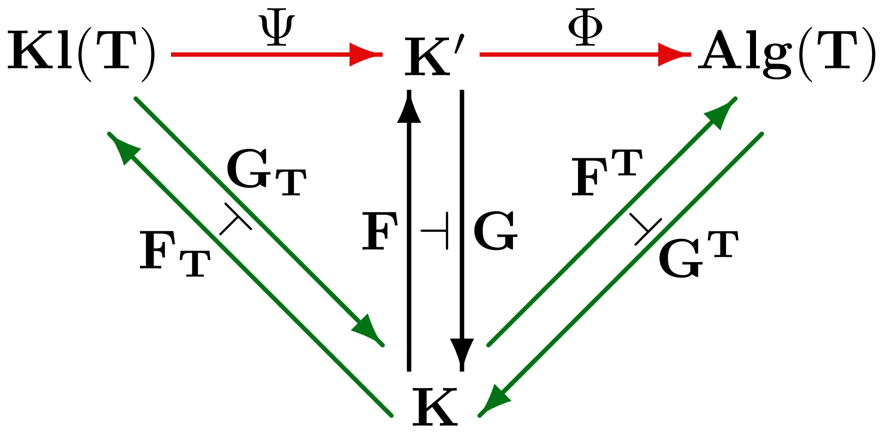

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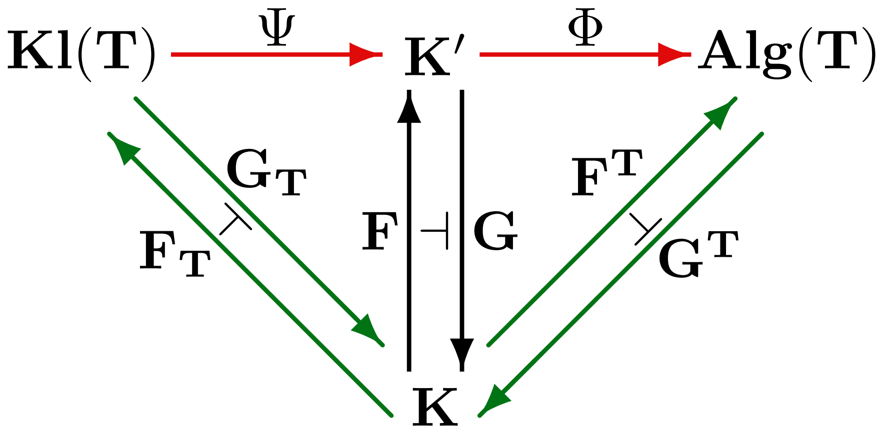

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View $\mathbf{Kl}(\mathbf{T})$ as the image of $\mathbf{F^T}$ in $\mathbf{Alg}(\mathbf{T})$

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Triples and monads are the same concepts

Triples as monads, monads as triples

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