Universal constructions: limits and colimits

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Consider and arbitrary but fixed category $\mathbf{K}$ for a while.

Initial and terminal objects

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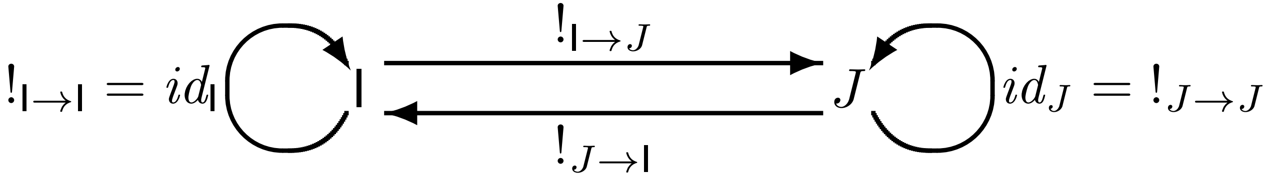

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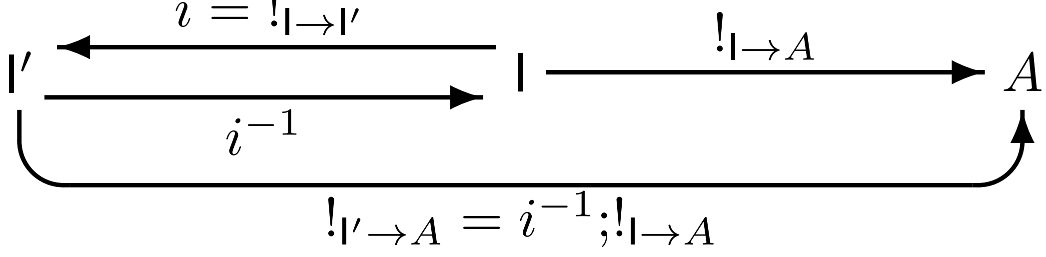

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- Show that terminal objects are unique to within an isomorphism.
- Look for categories where there is an object which is both initial and terminal.

Products

A *product* of two objects $A, B \in |\mathbf{K}|$

$$A \qquad\qquad\qquad B$$

Products

A *product* of two objects $A, B \in |\mathbf{K}|$ is any object $A \times B \in |\mathbf{K}|$ with two morphisms (*product projections*) $\pi_1 \colon A \times B \to A$ and $\pi_2 \colon A \times B \to B$

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

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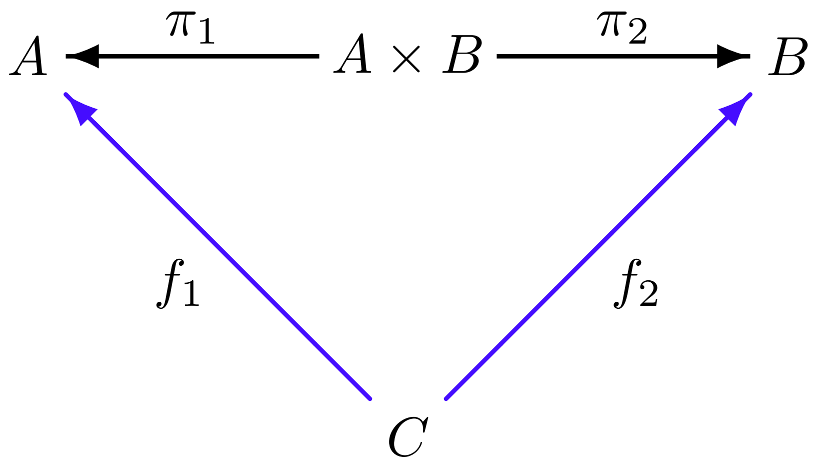

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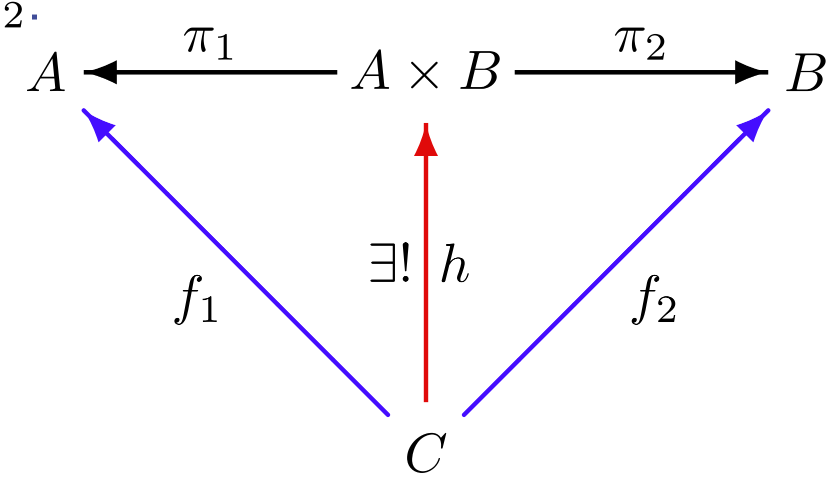

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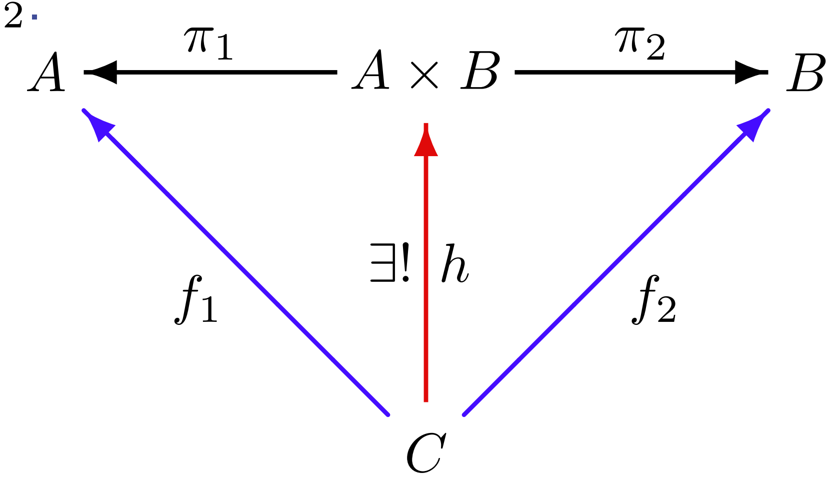

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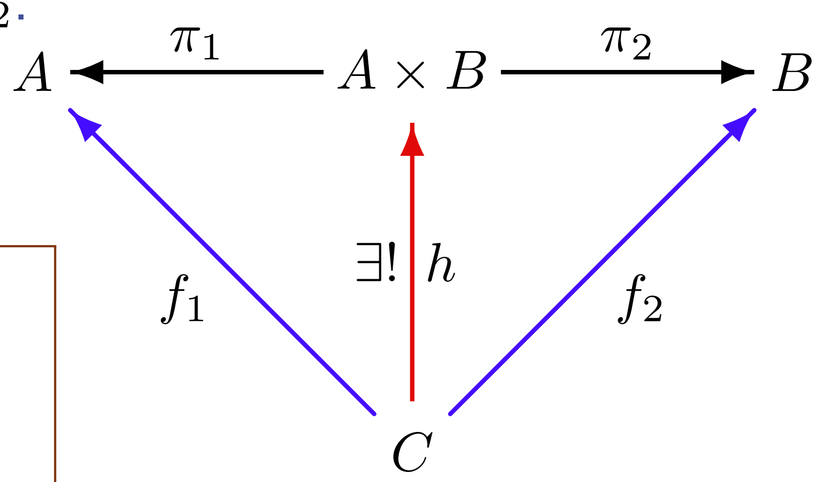

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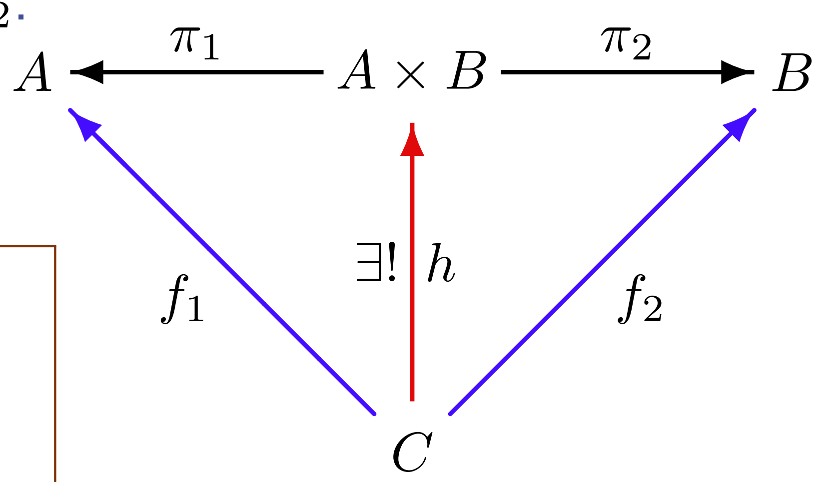

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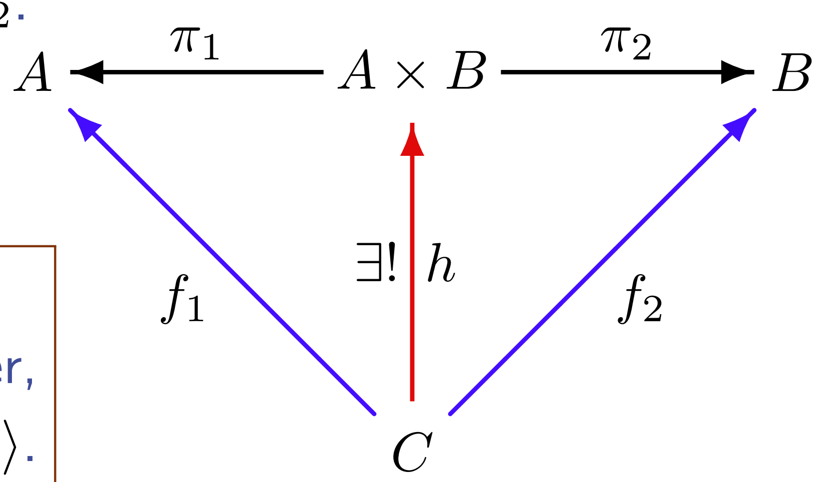

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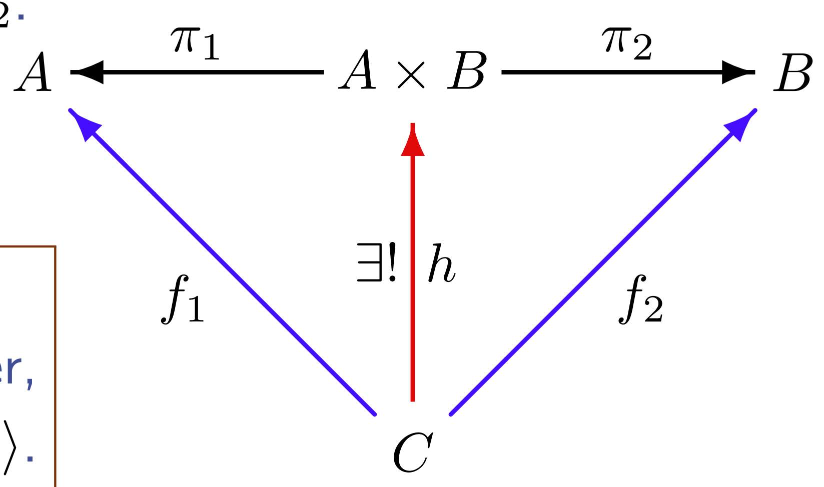

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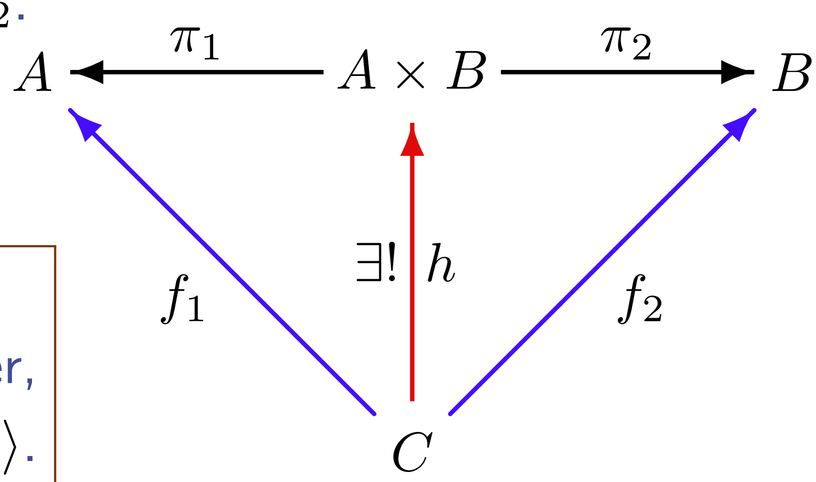

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Theorem: *Products are defined to within an isomorphism (which commutes with projections).*

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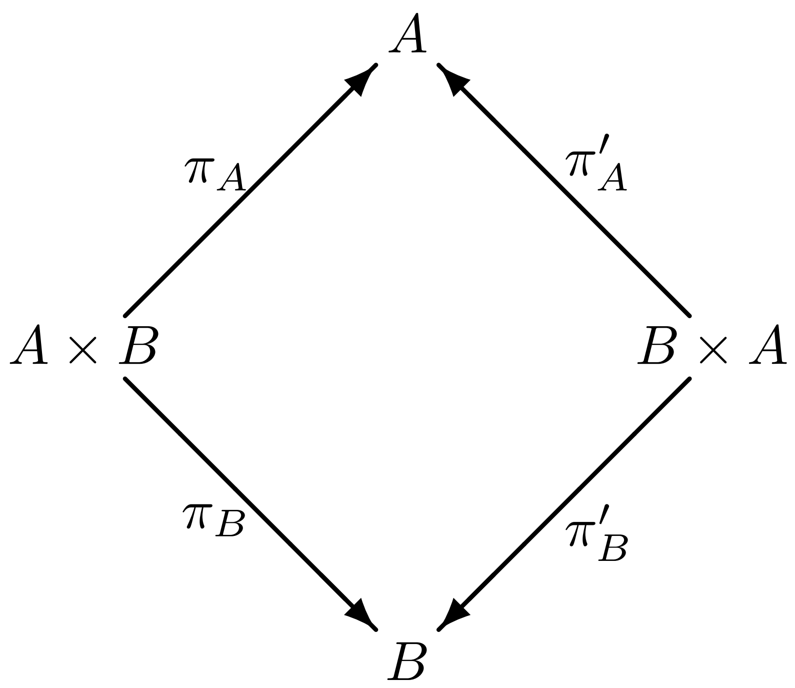

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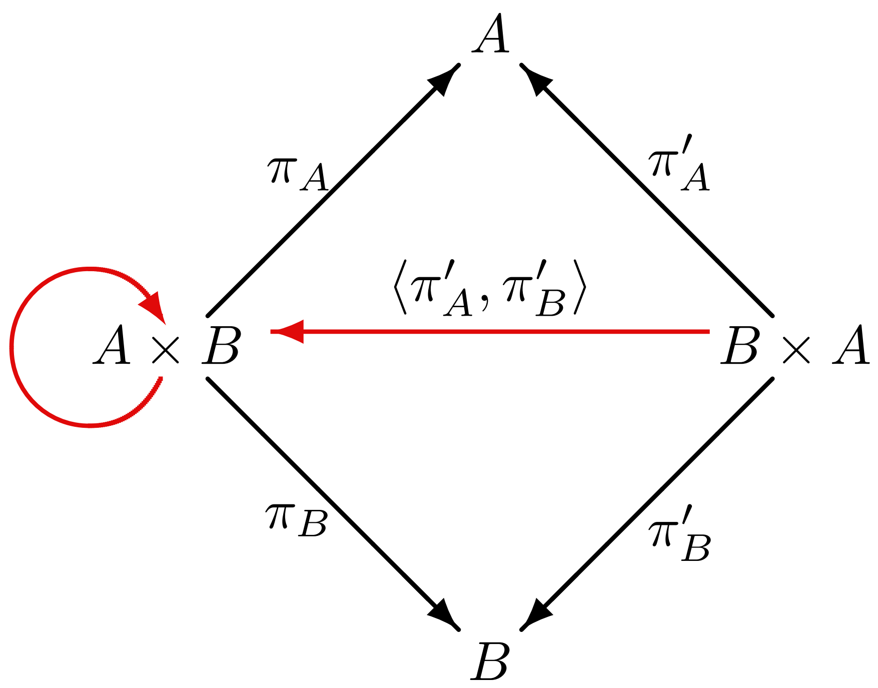
$$id_{A\times B} = \langle \pi_A, \pi_B \rangle$$



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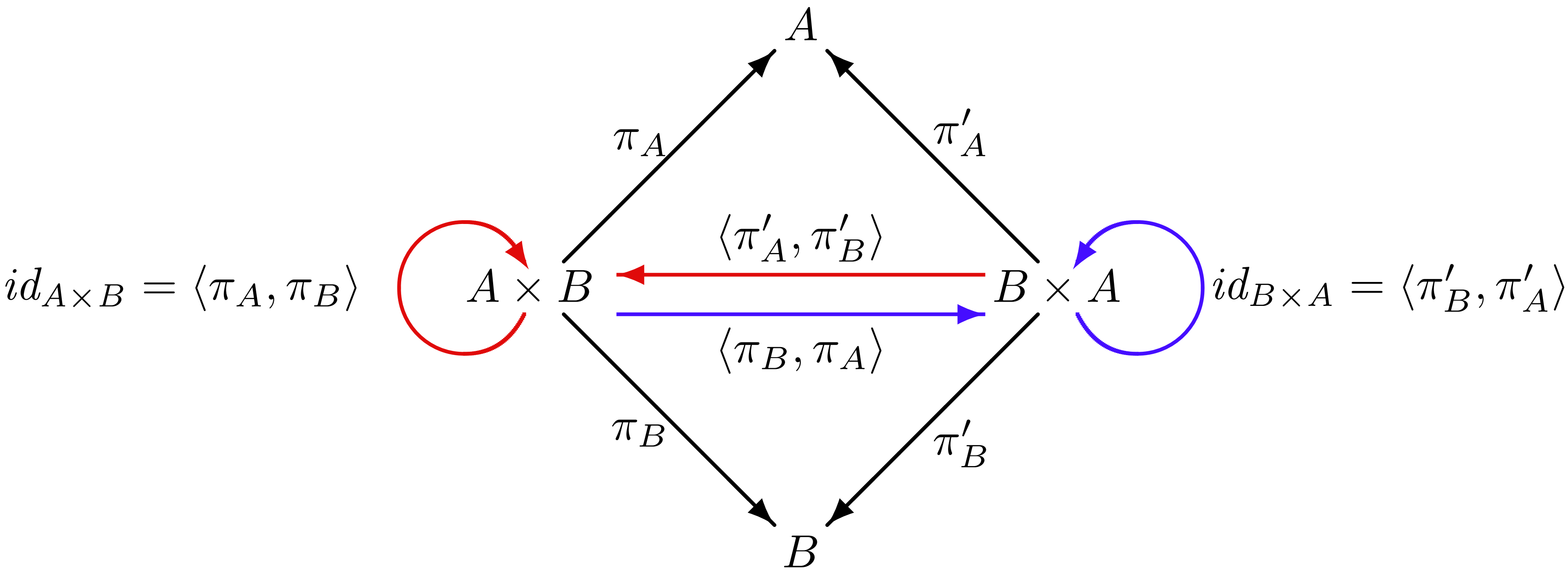

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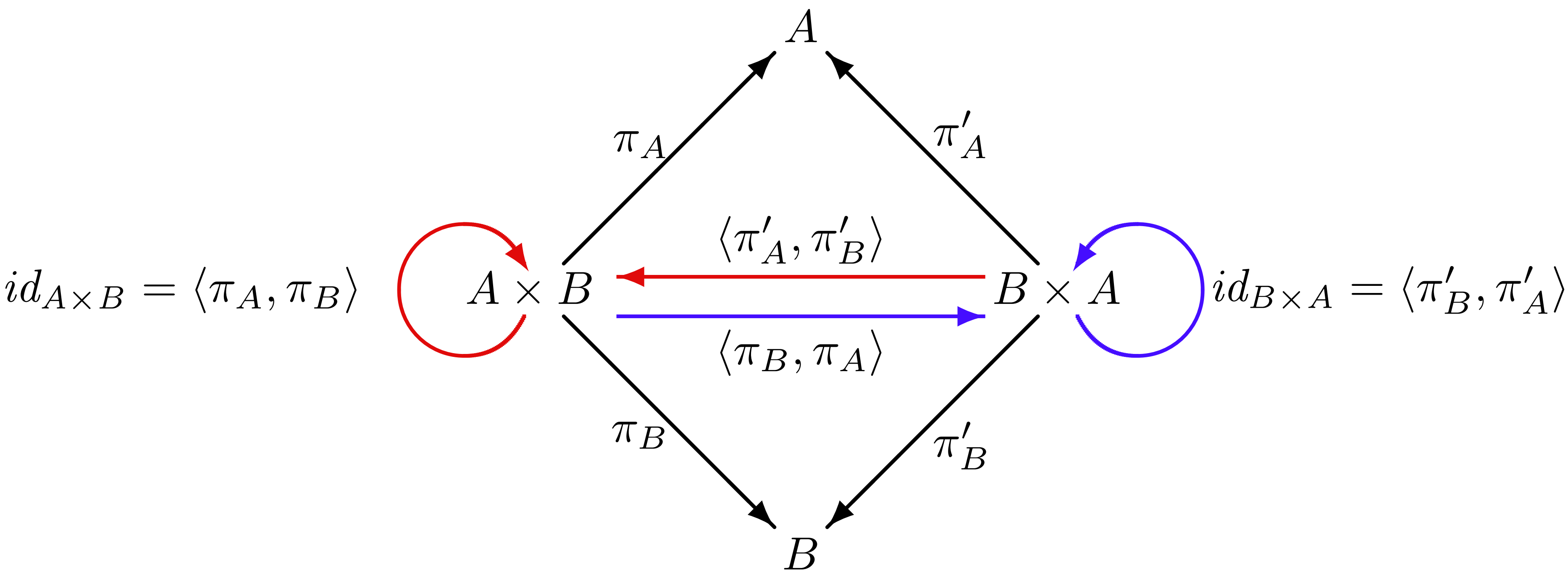

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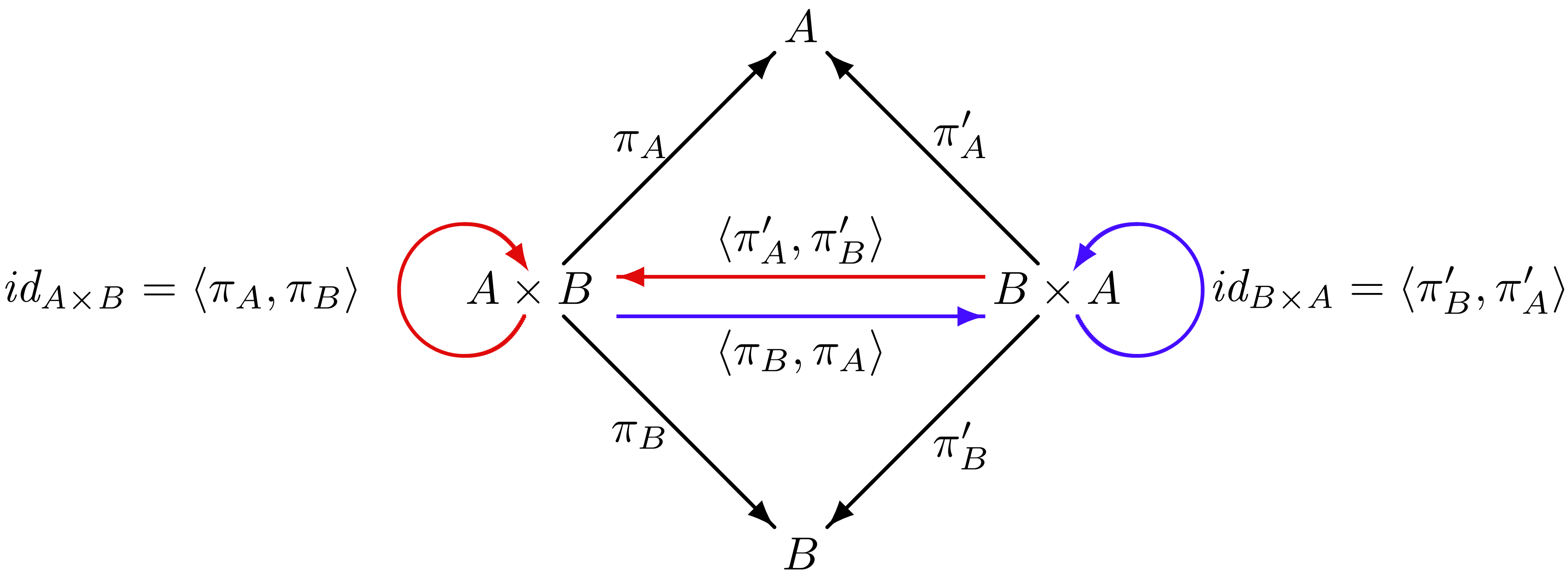

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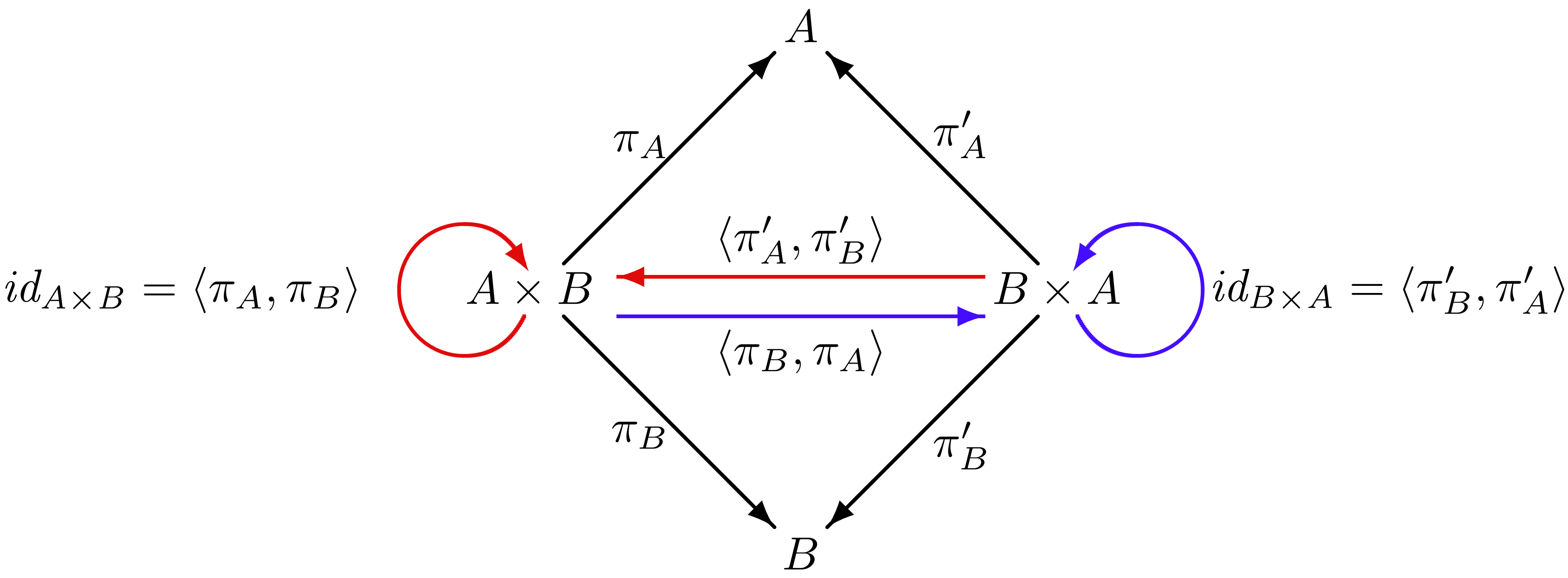

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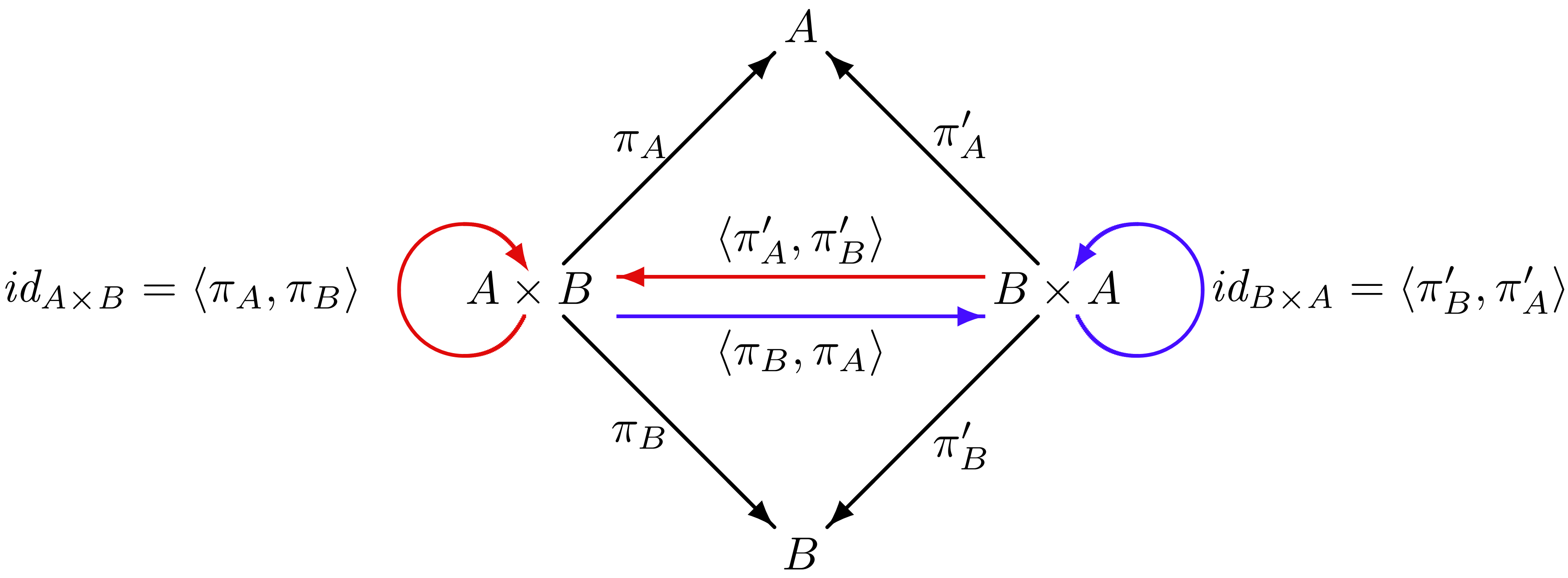

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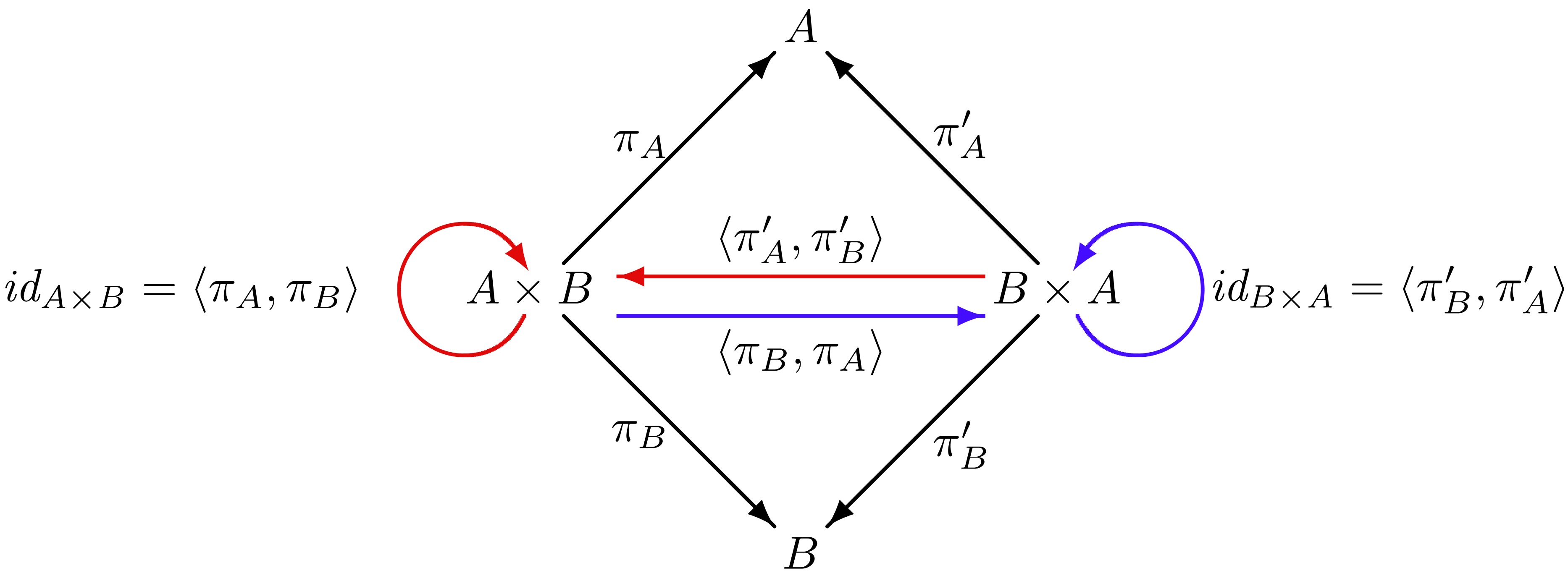

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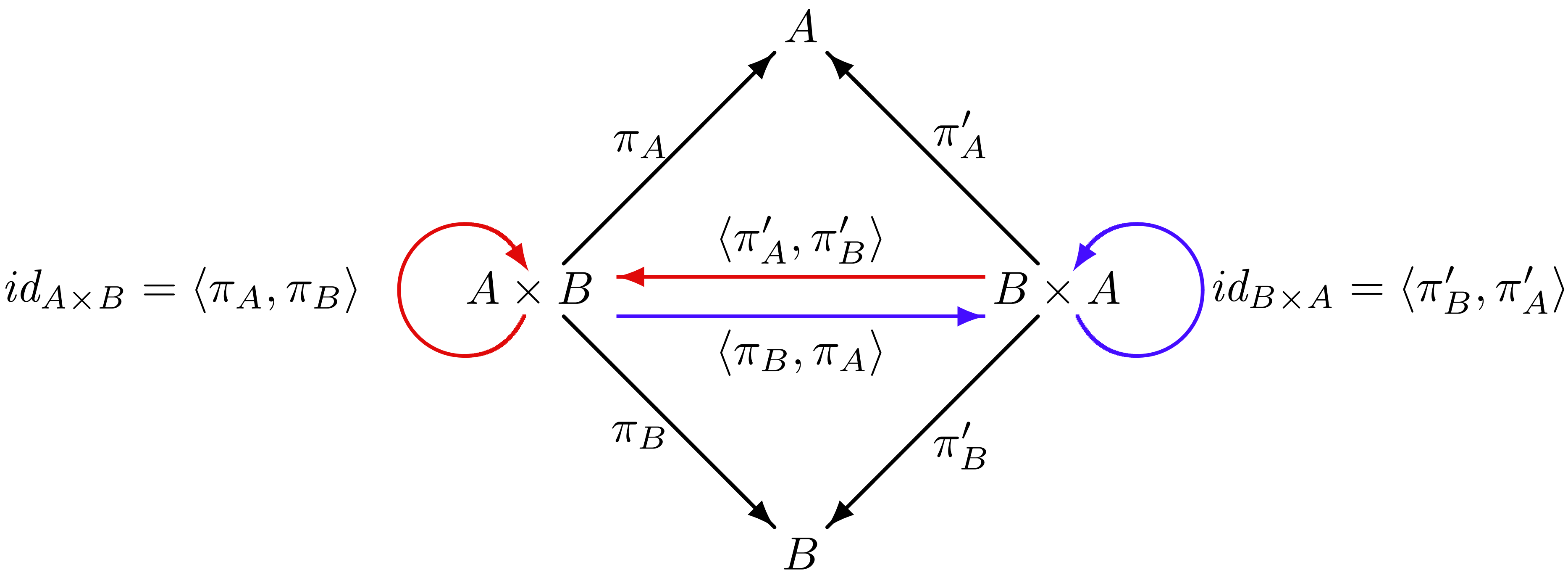

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  – Thus: $\langle \pi_B, \pi_A \rangle;\langle \pi'_A, \pi'_B \rangle = \langle \pi_A, \pi_B \rangle = id_{A\times B}$

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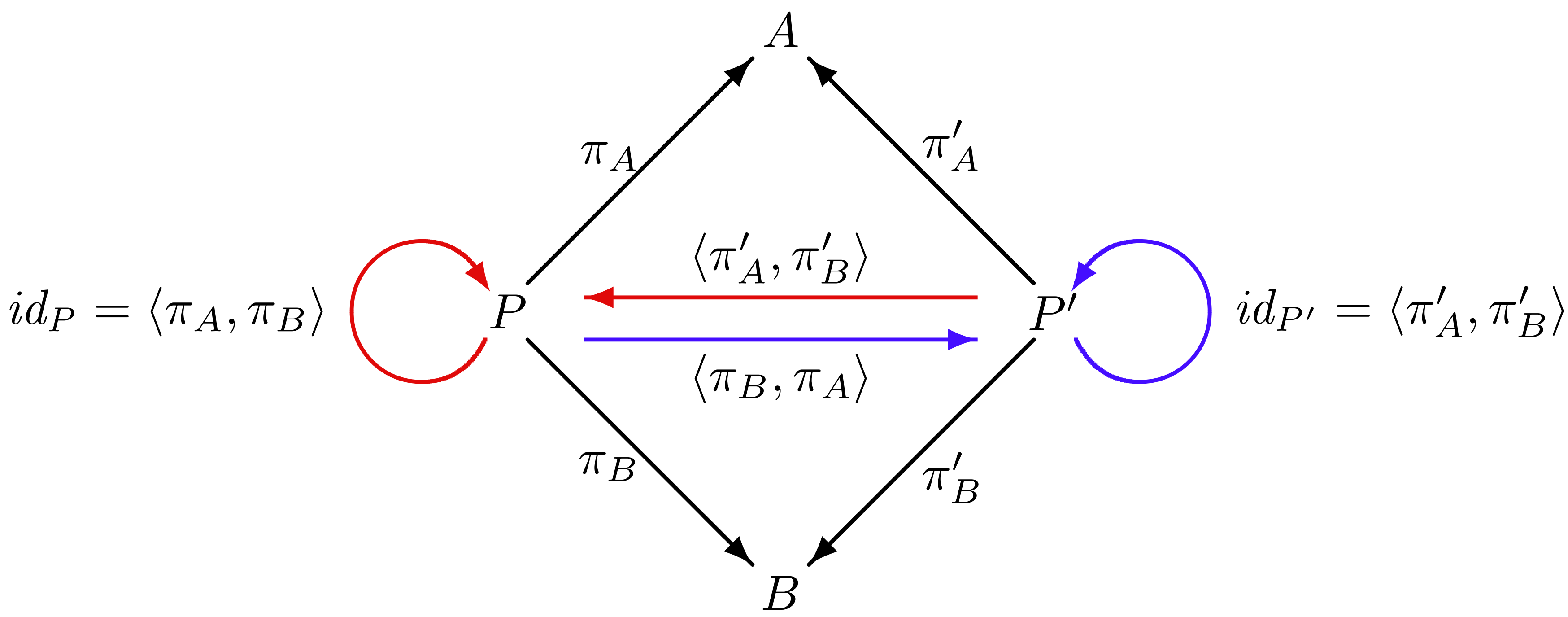

  – By much the same argument, any two products of A and B are isomorphic.

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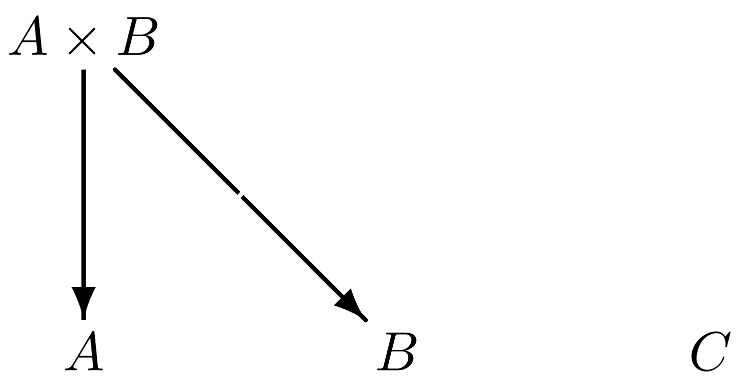

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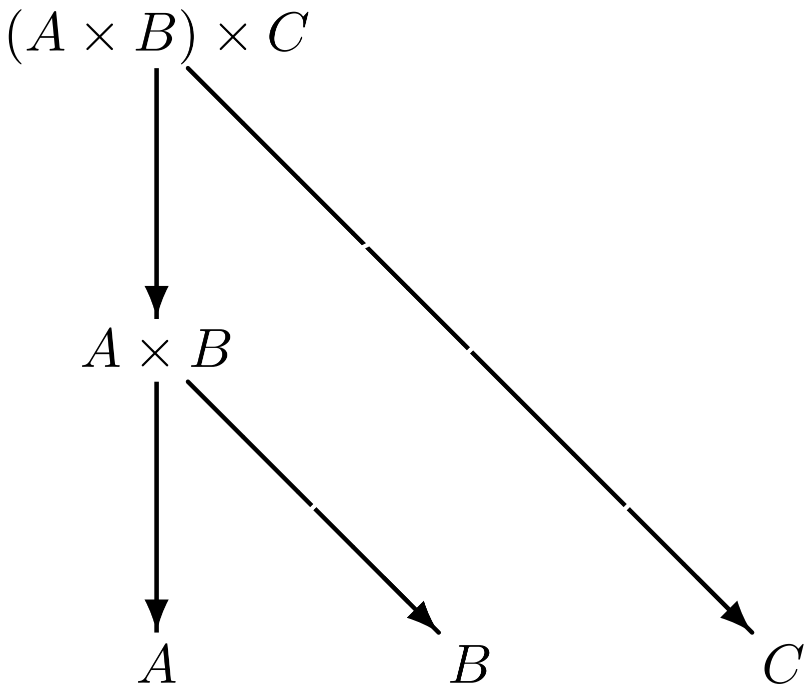

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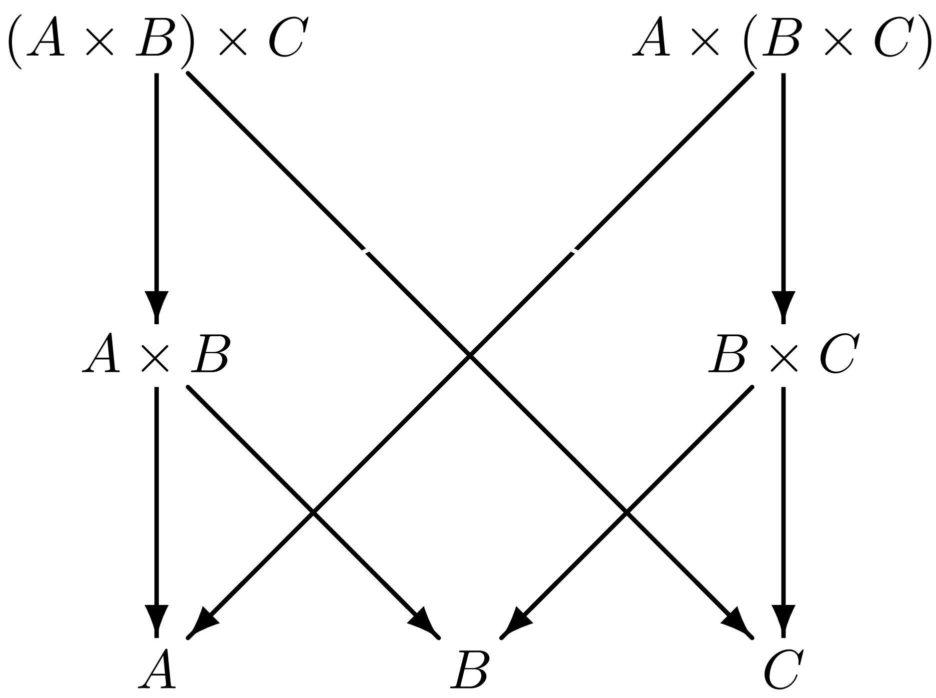

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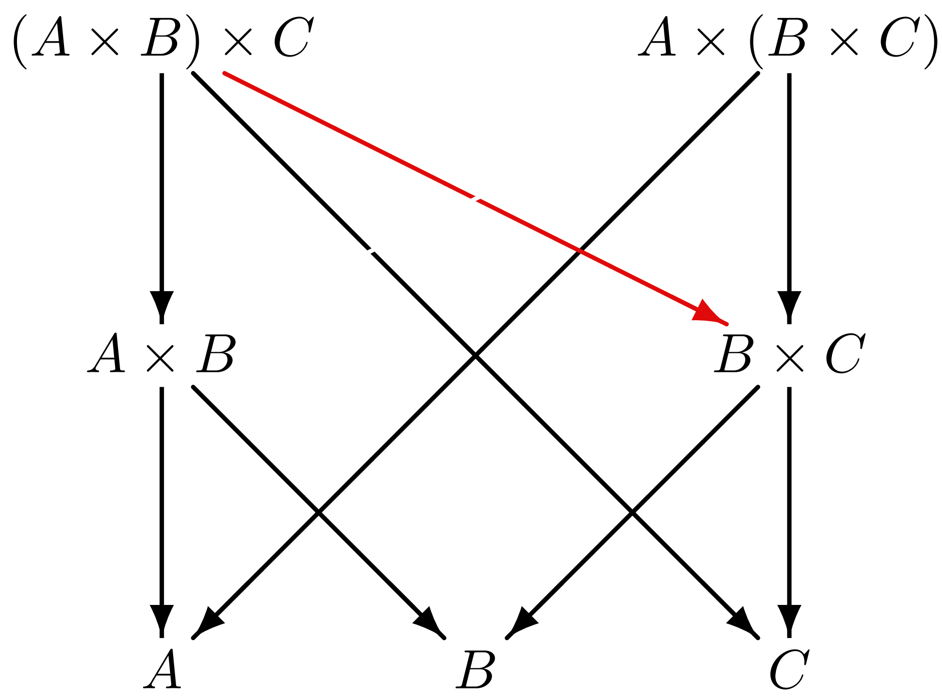

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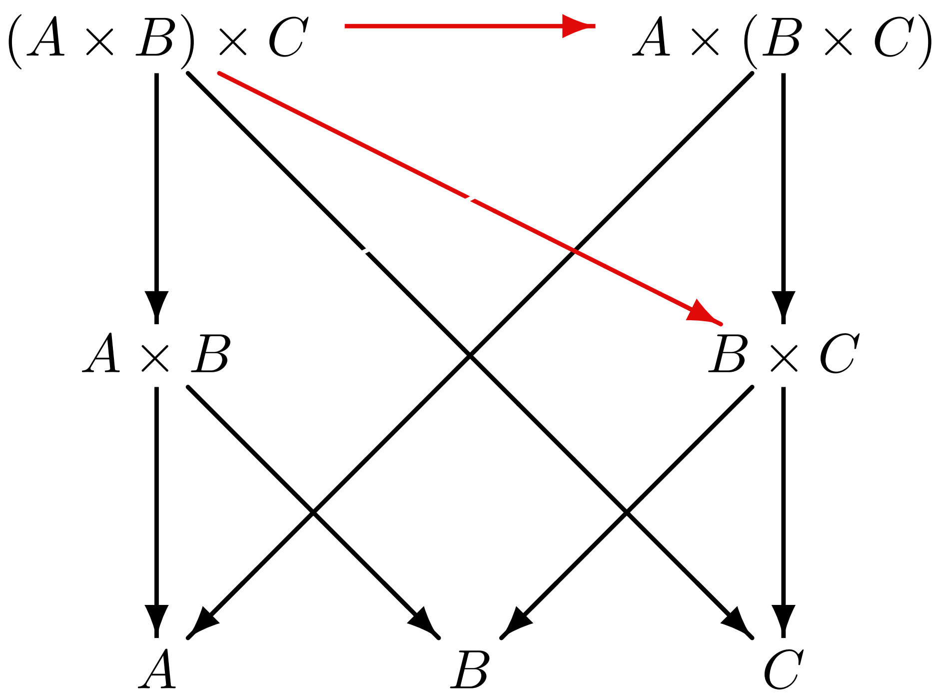

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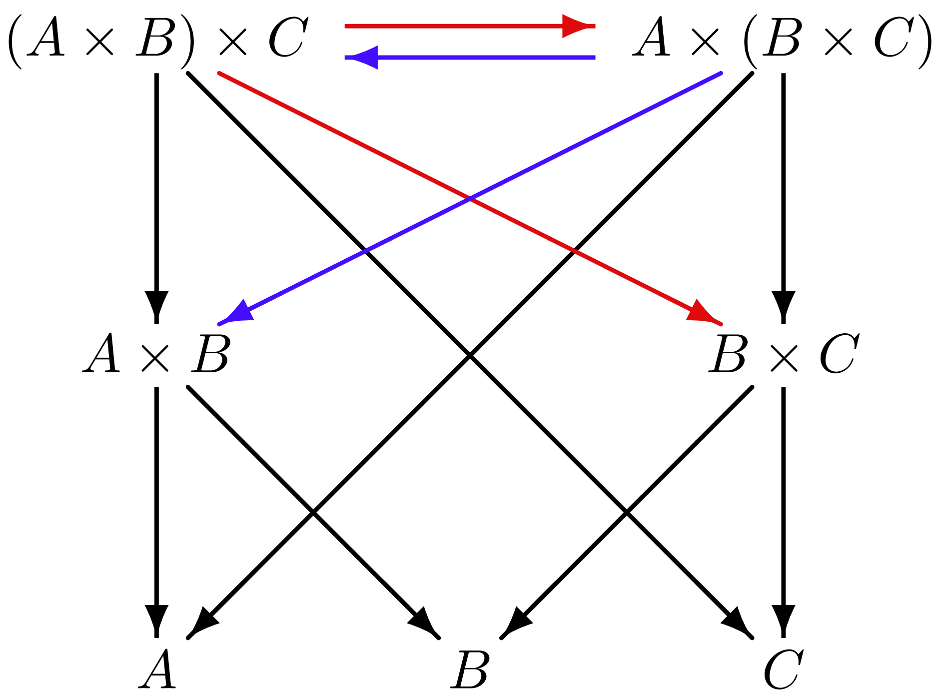

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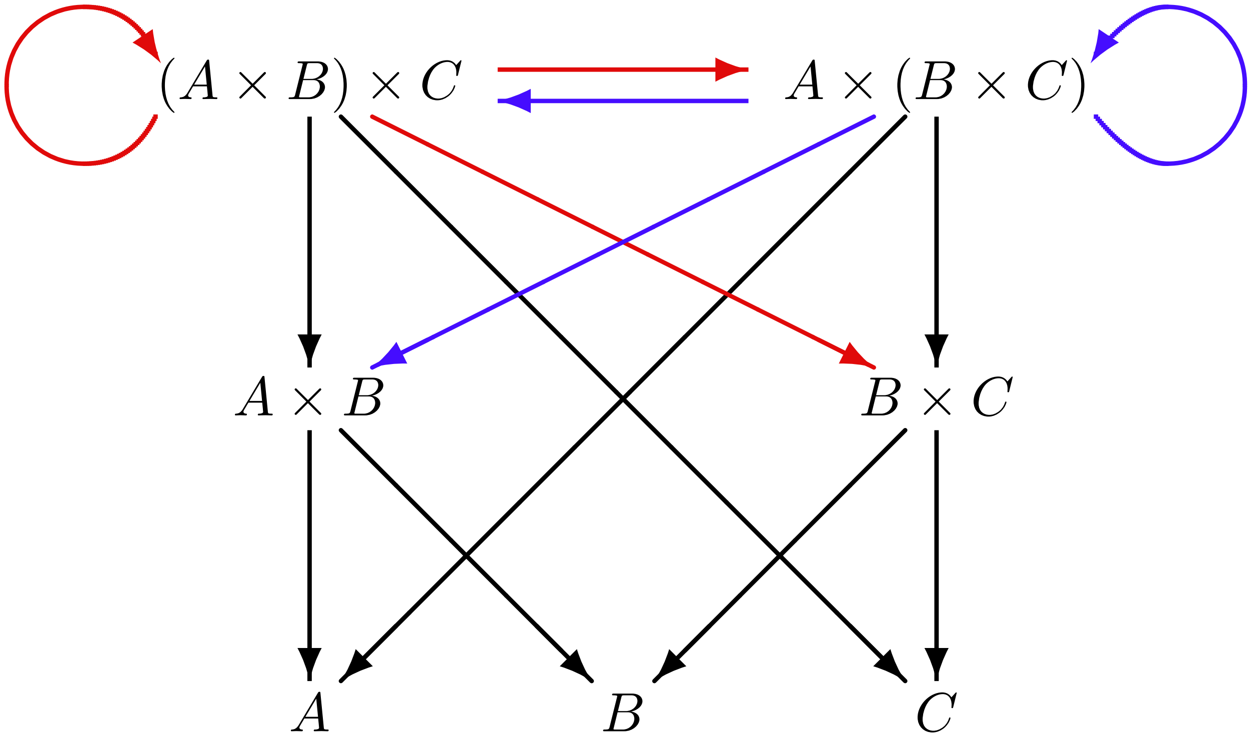

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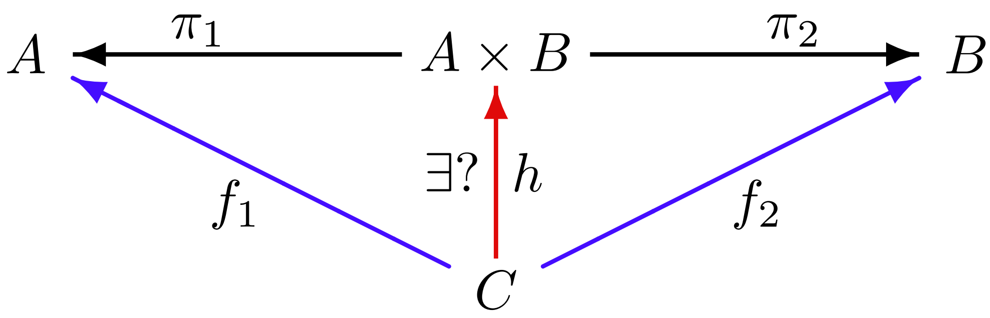

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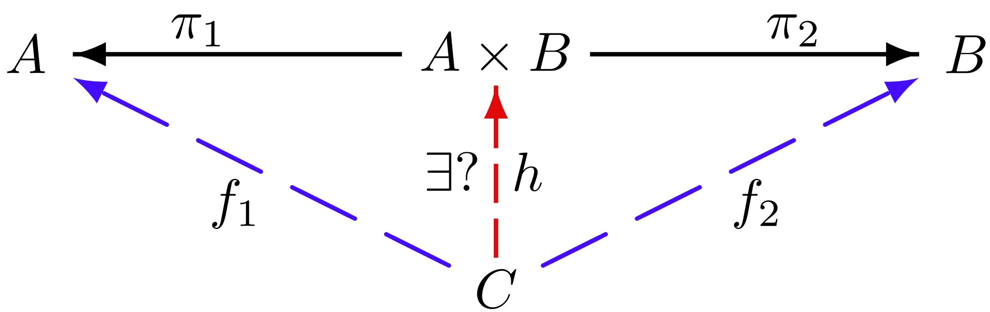

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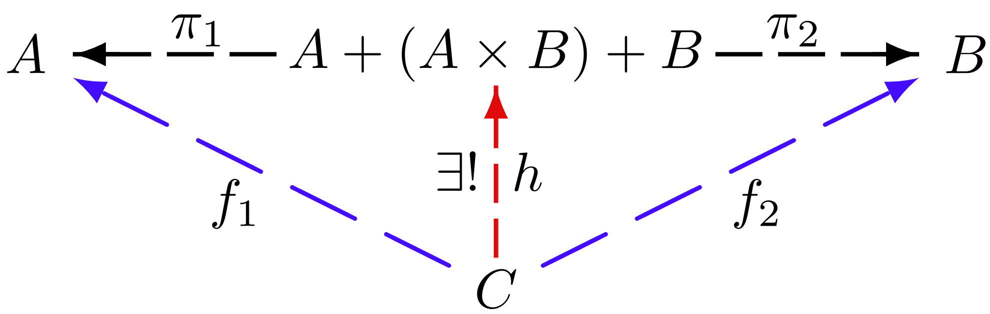

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Coproducts

coproduct = *co*-product

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A B

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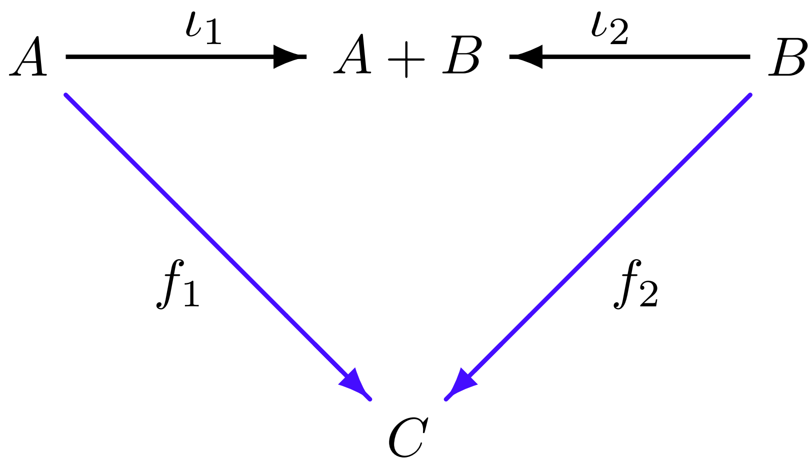

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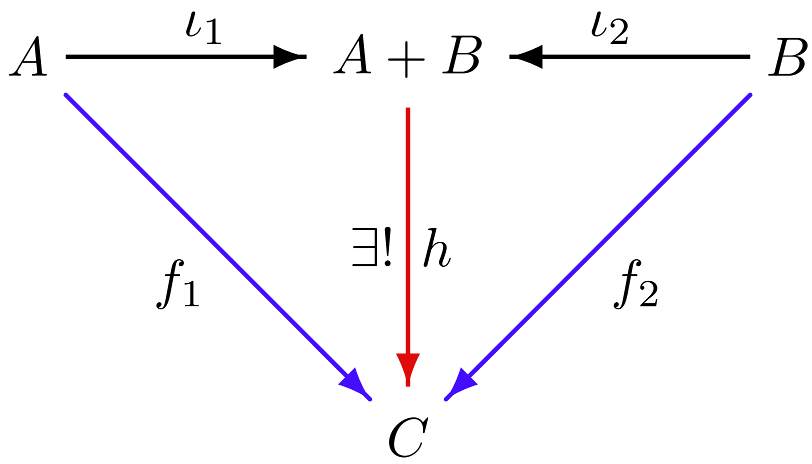

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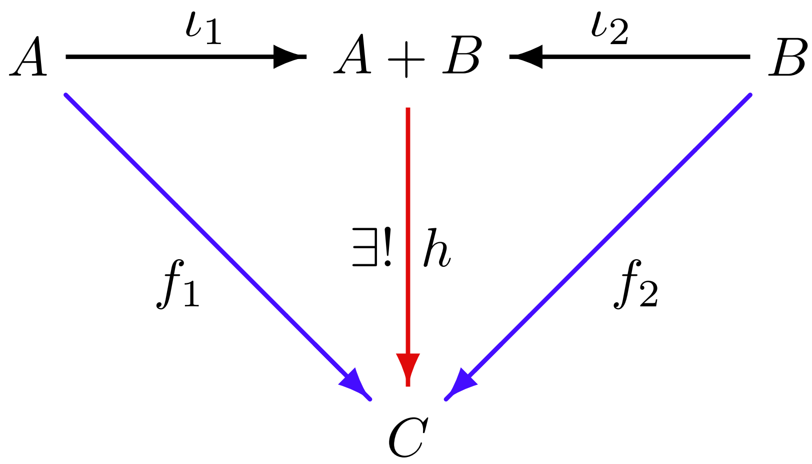

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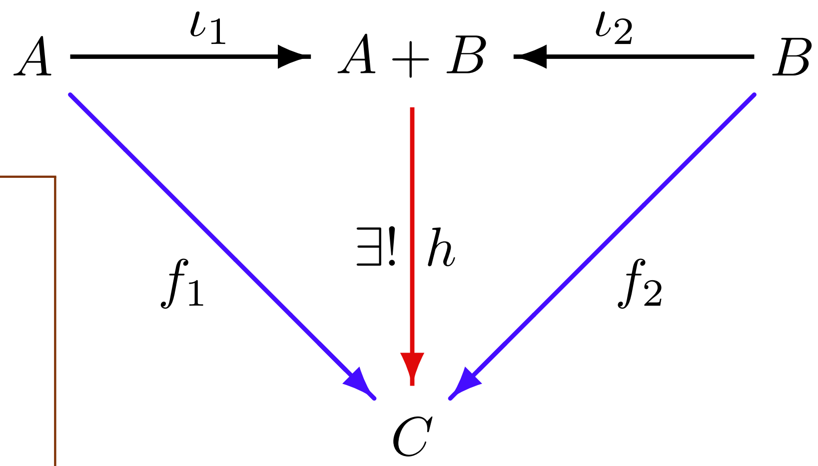

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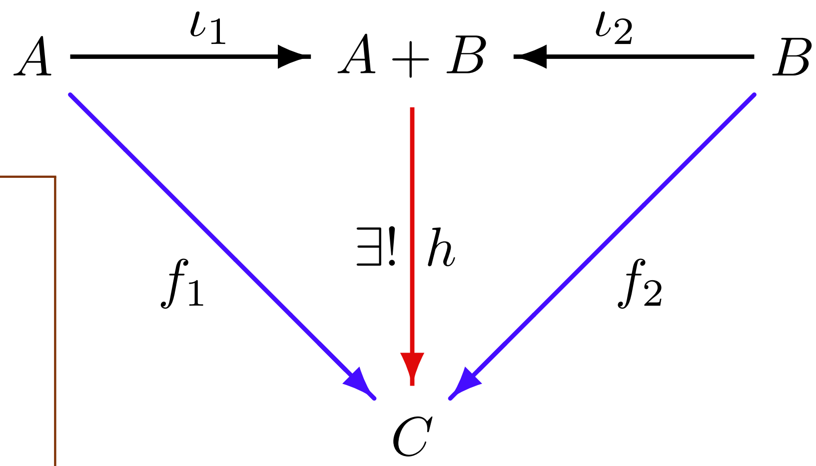

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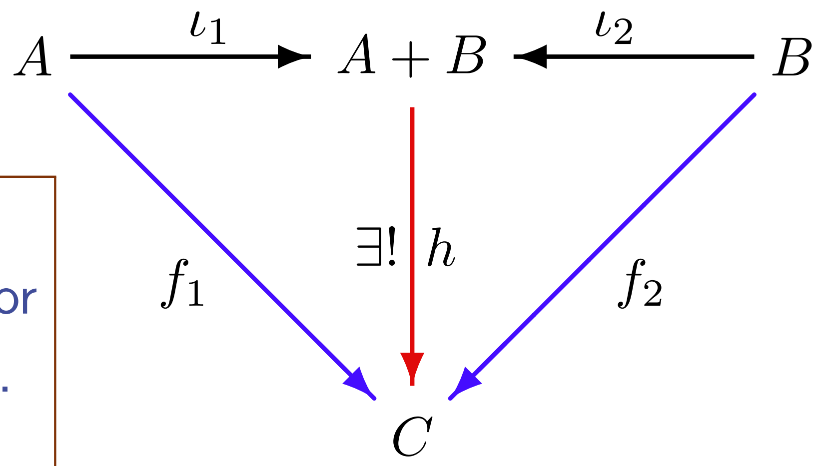

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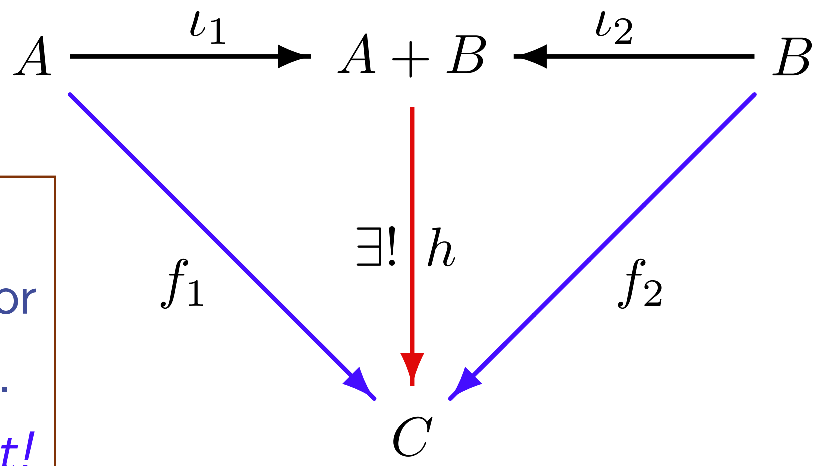

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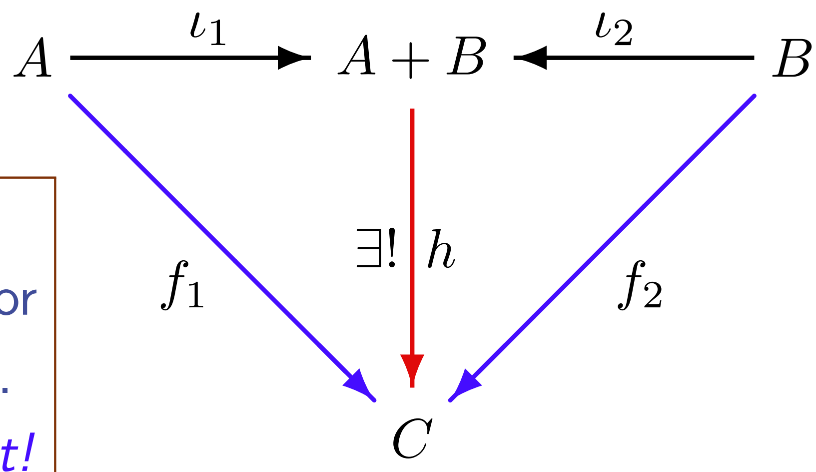

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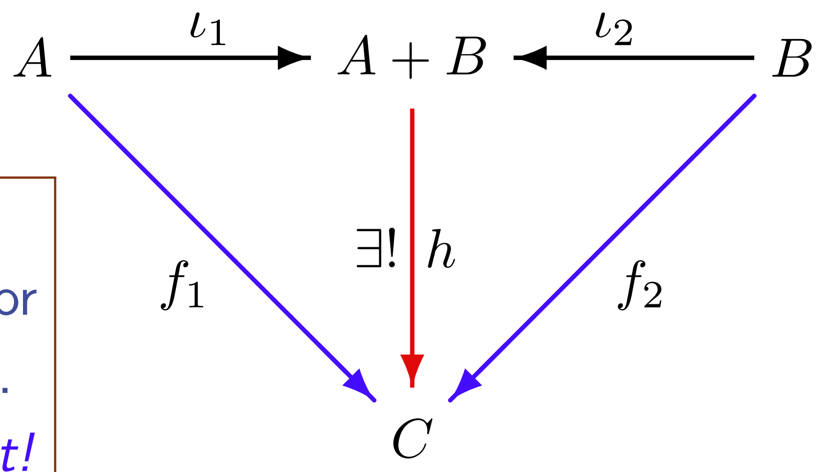

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Exercises: Dualise!

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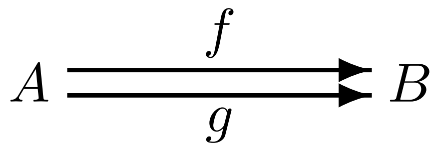
An *equaliser* of two "parallel" morphisms $f, g \colon A \to B$



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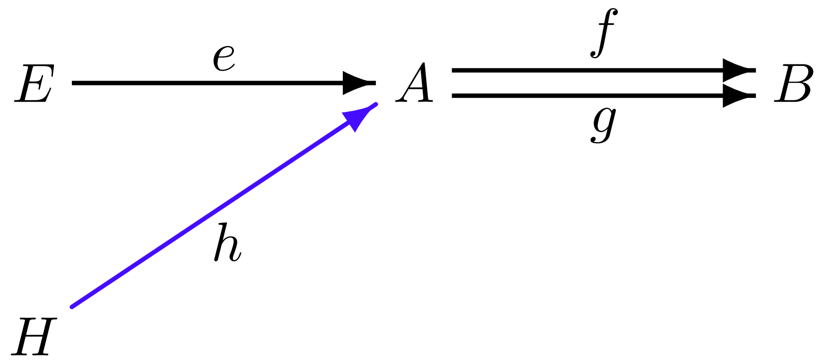

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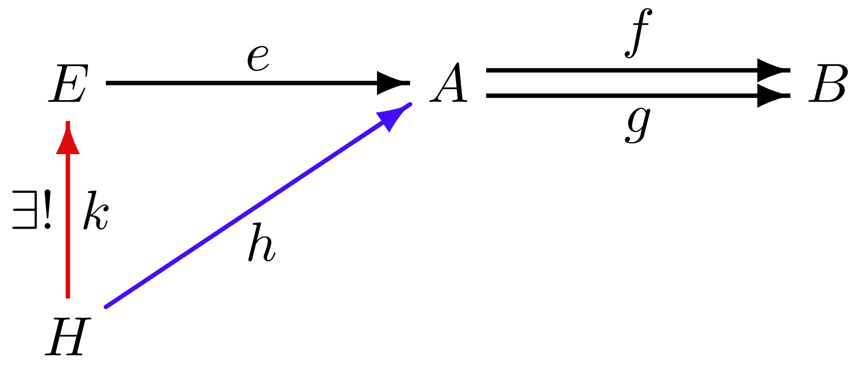

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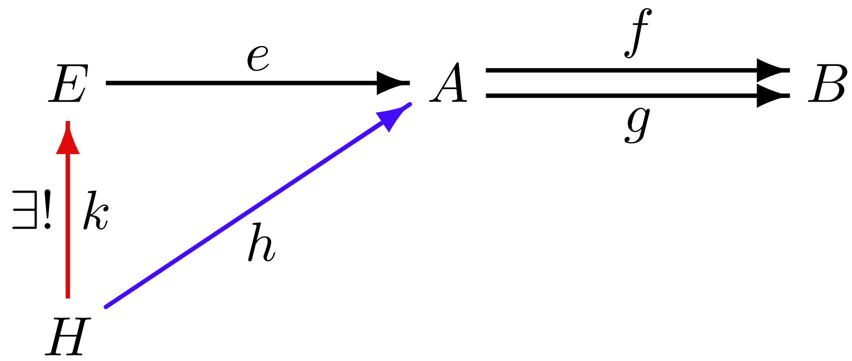

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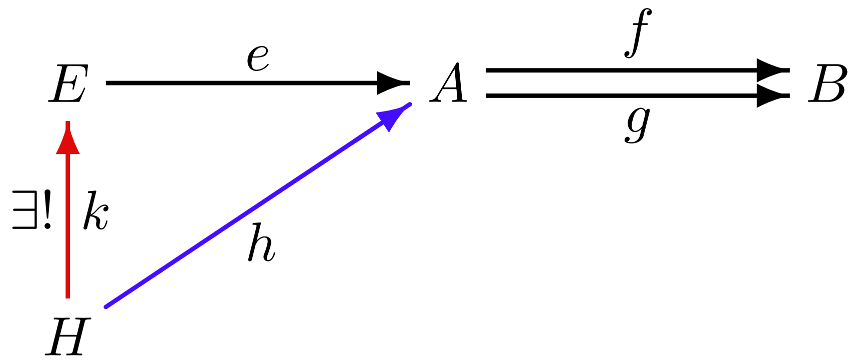

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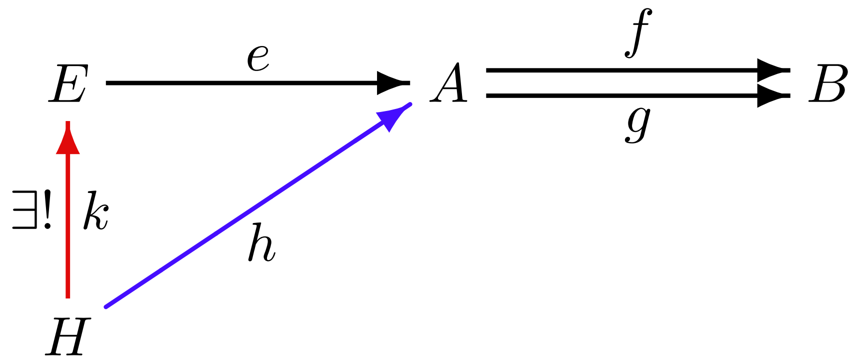

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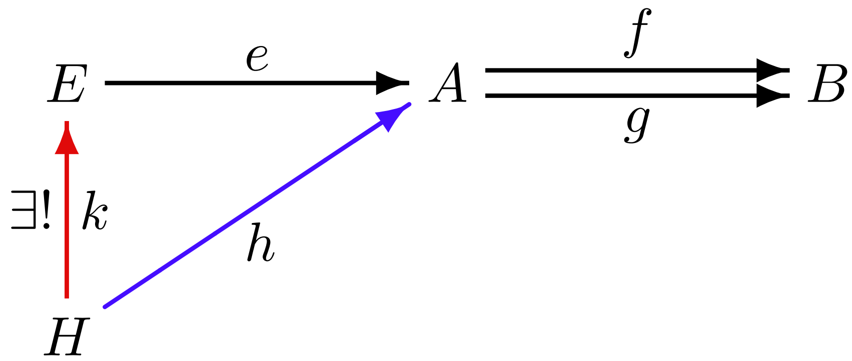

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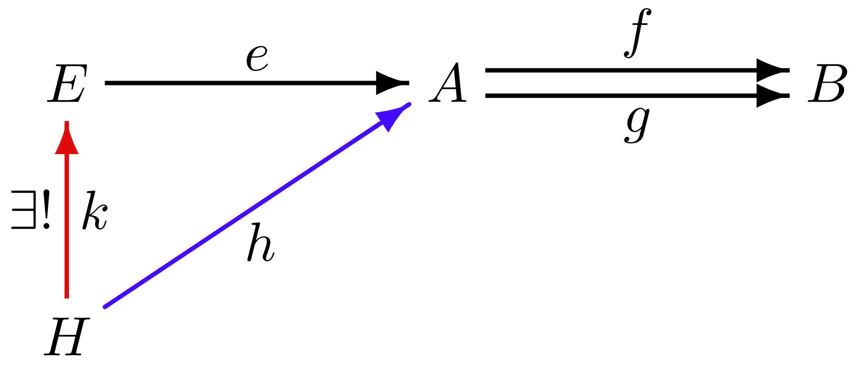

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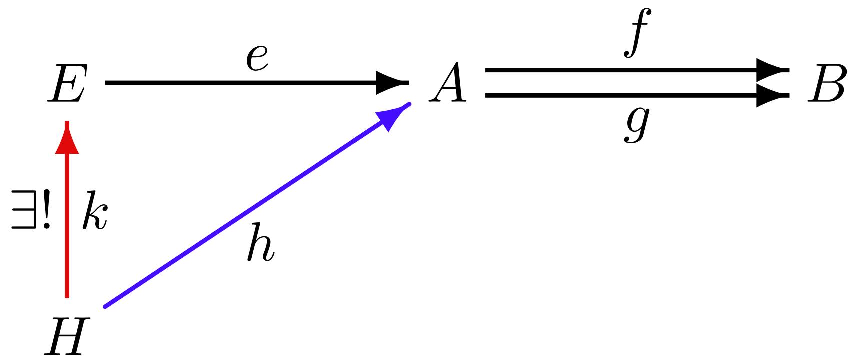

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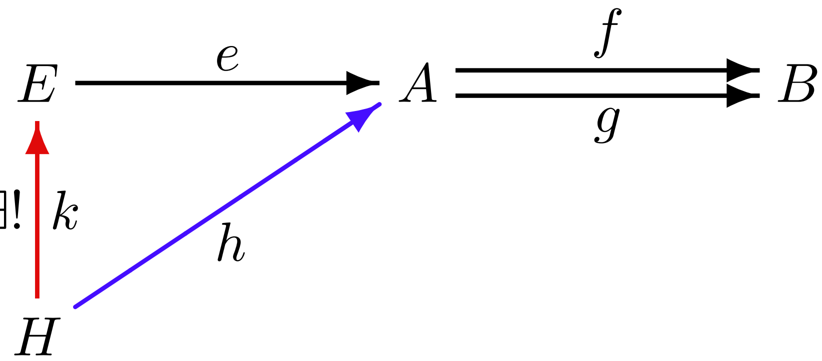

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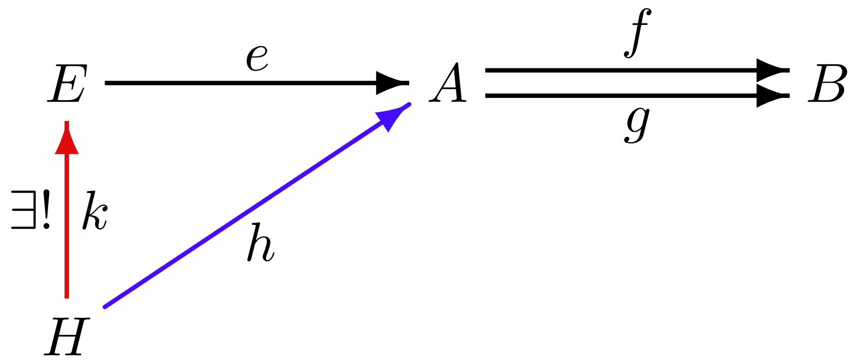

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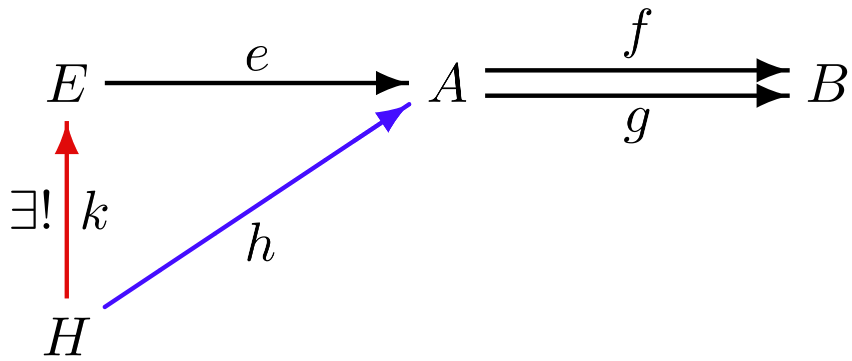

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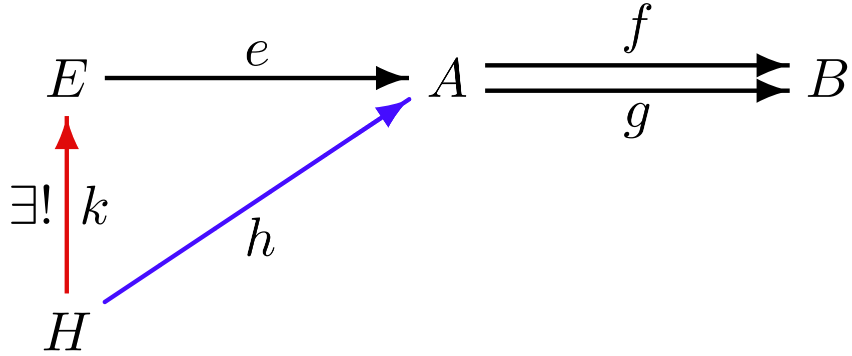

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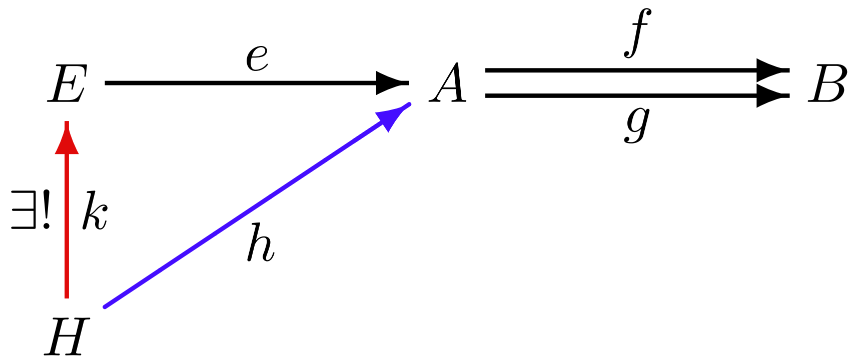

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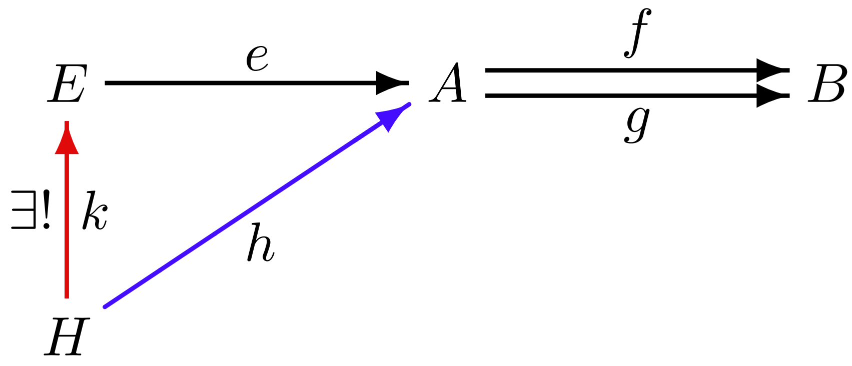

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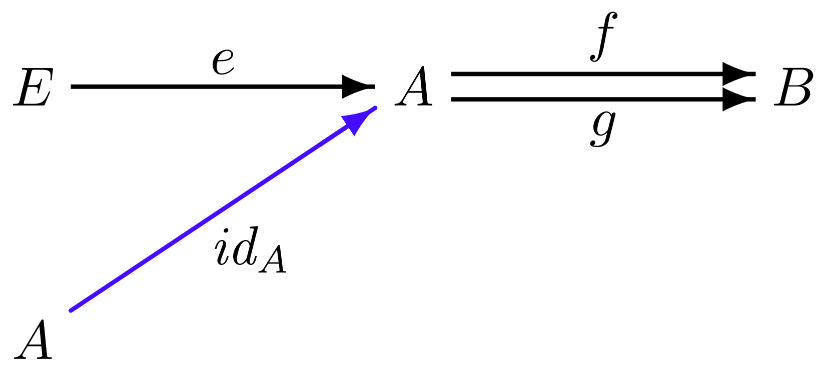

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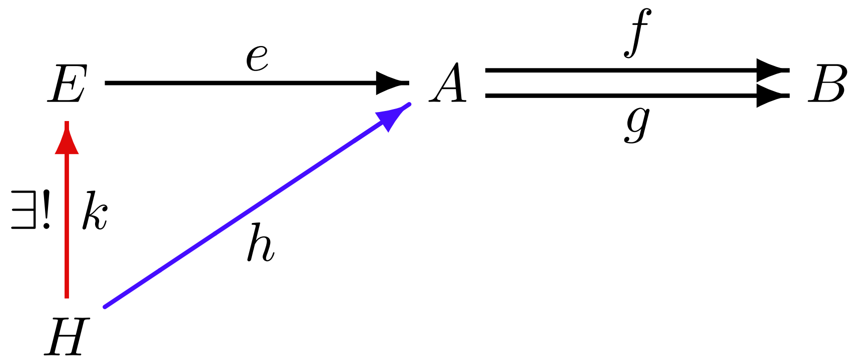

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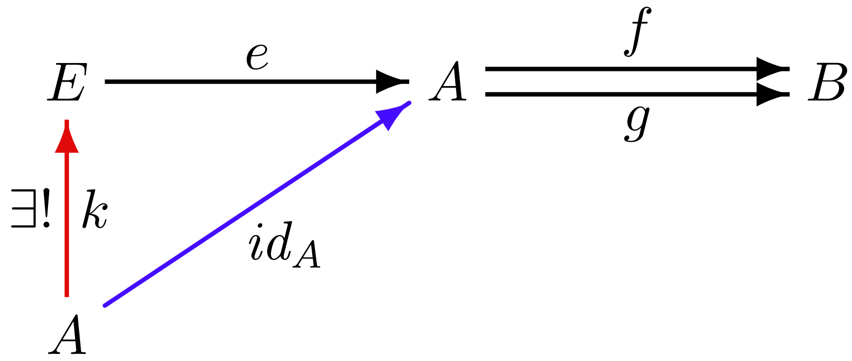

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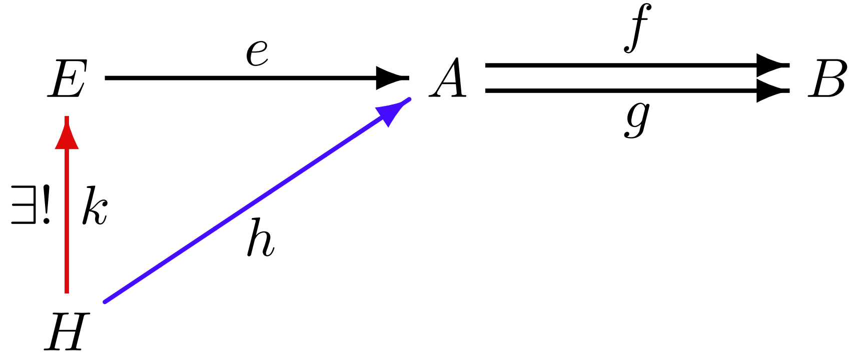

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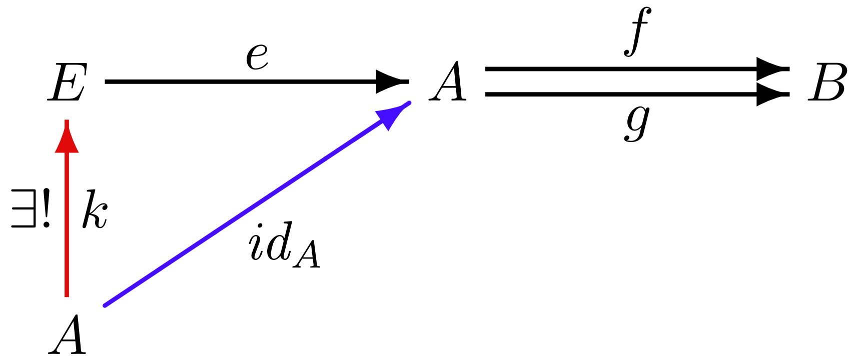

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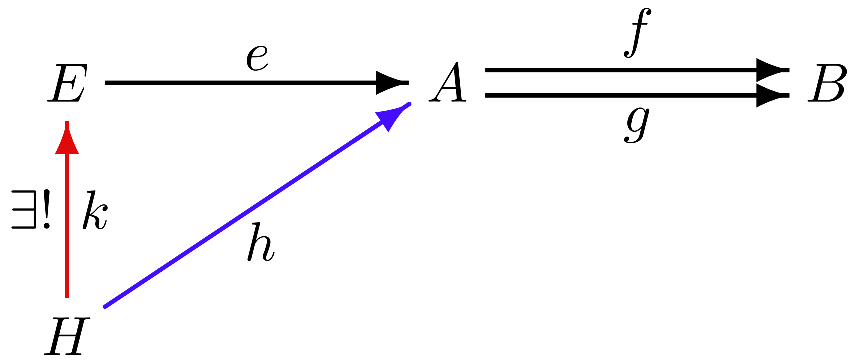

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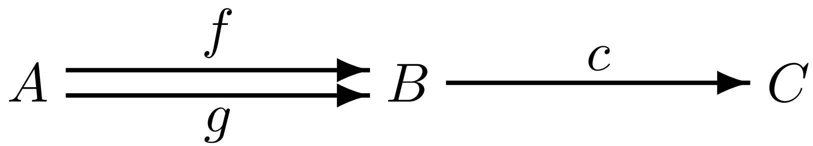

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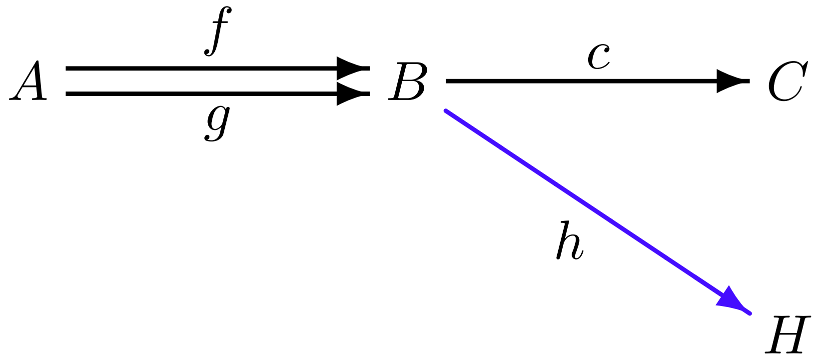

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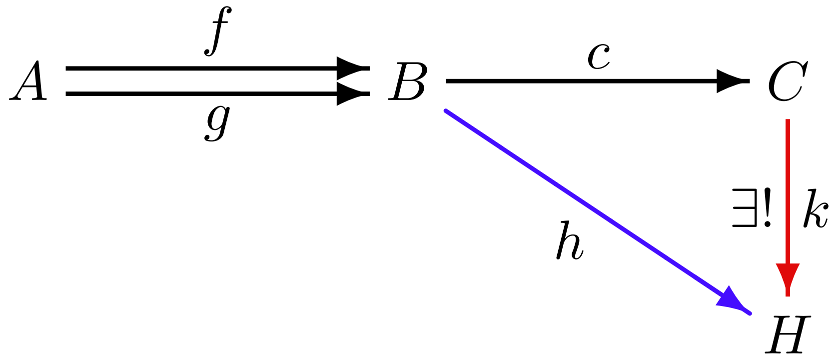

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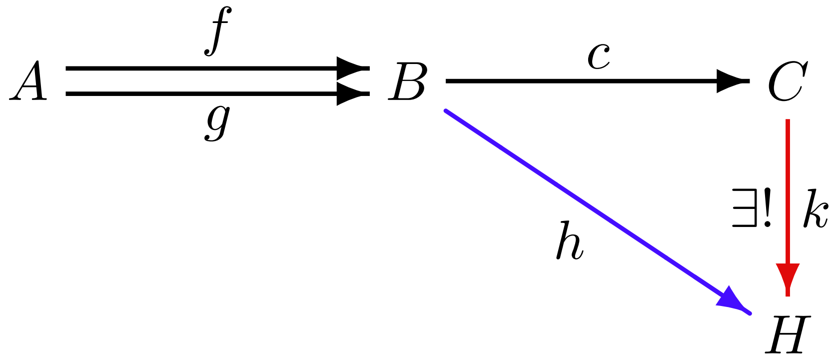

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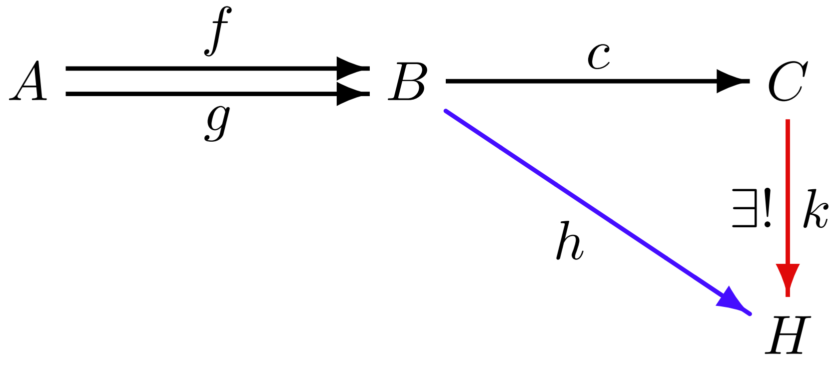

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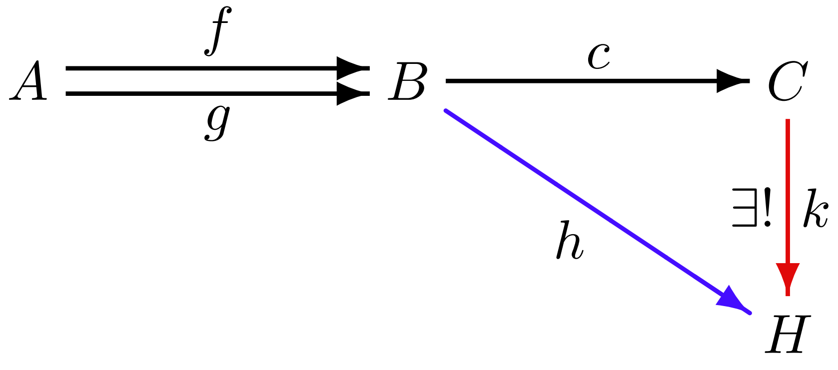

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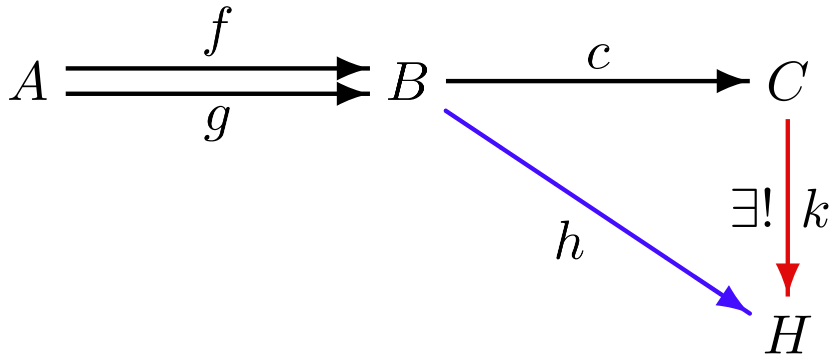

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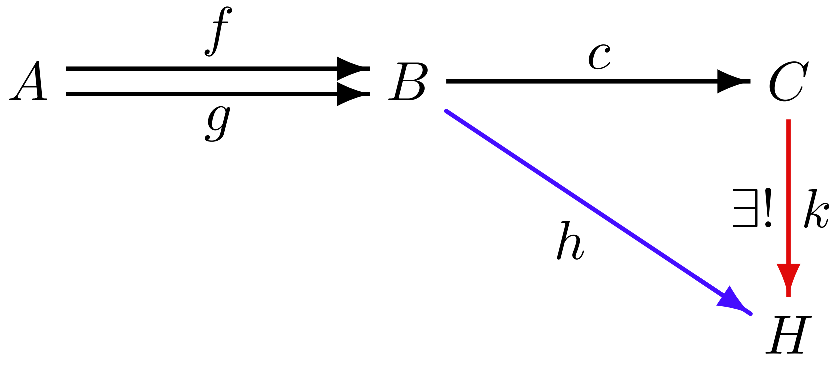

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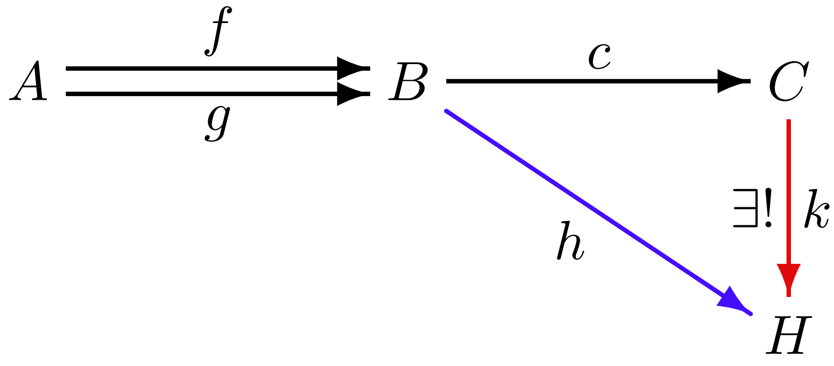

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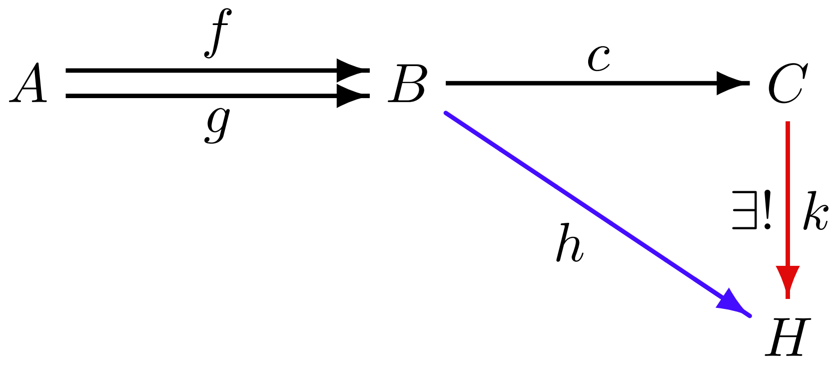

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Pullbacks

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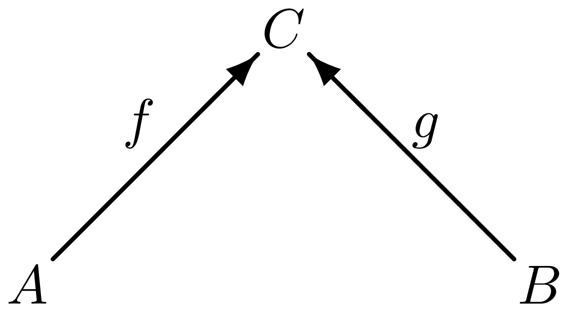

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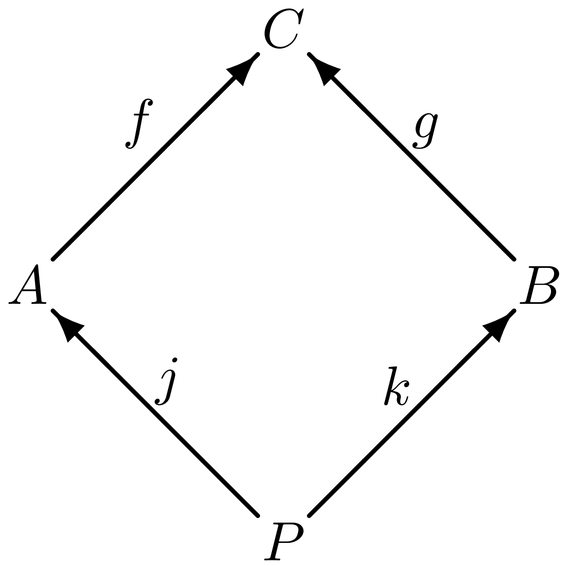

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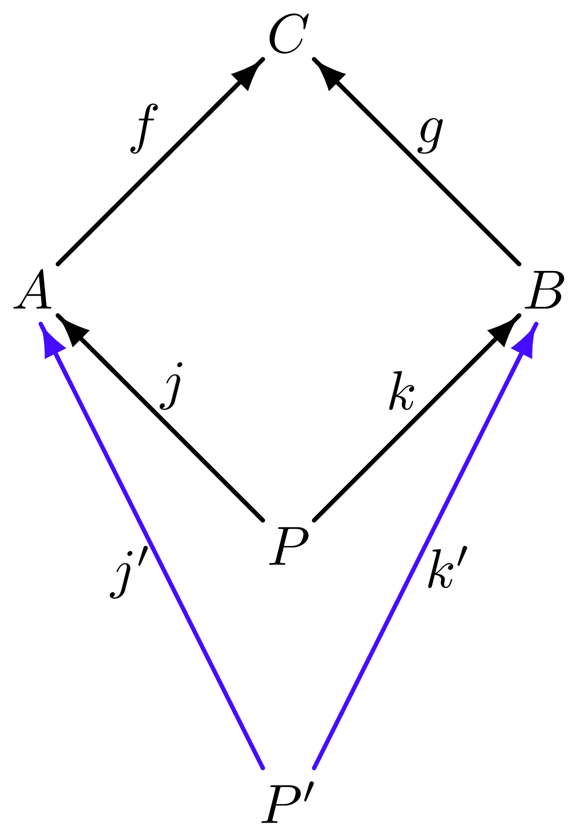

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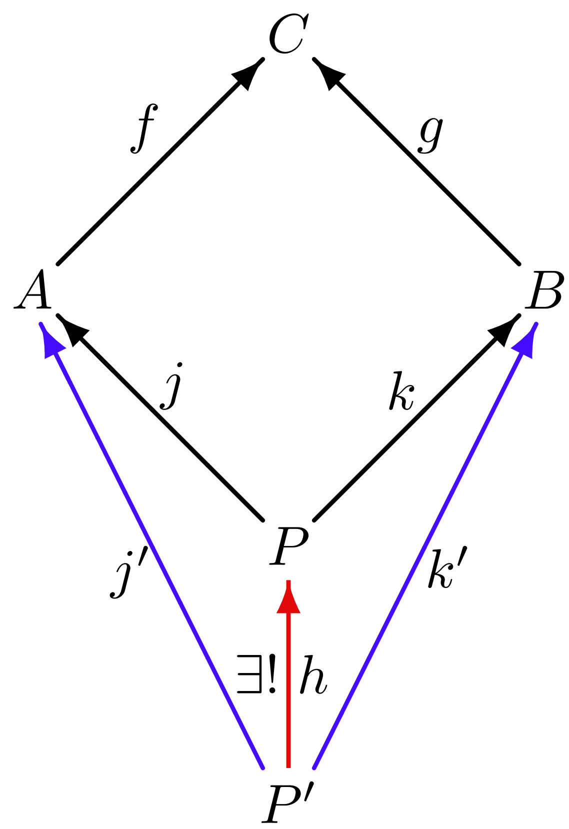

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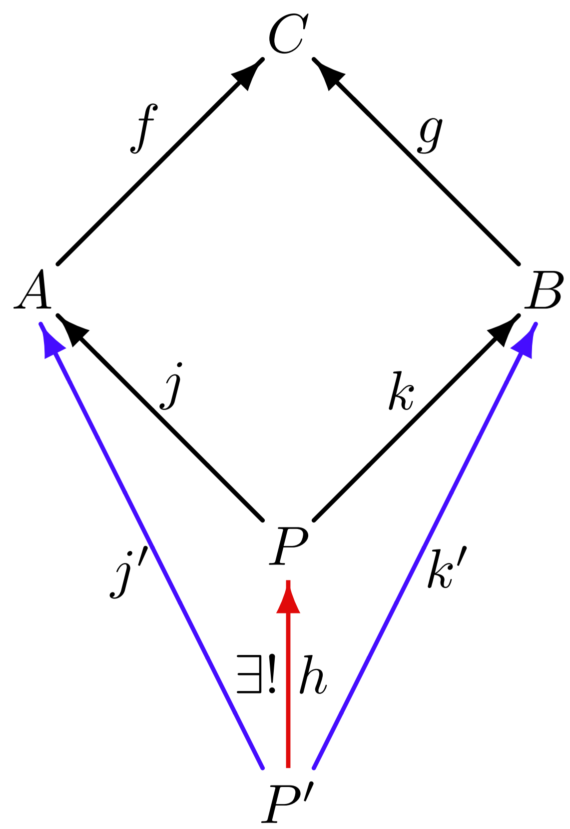

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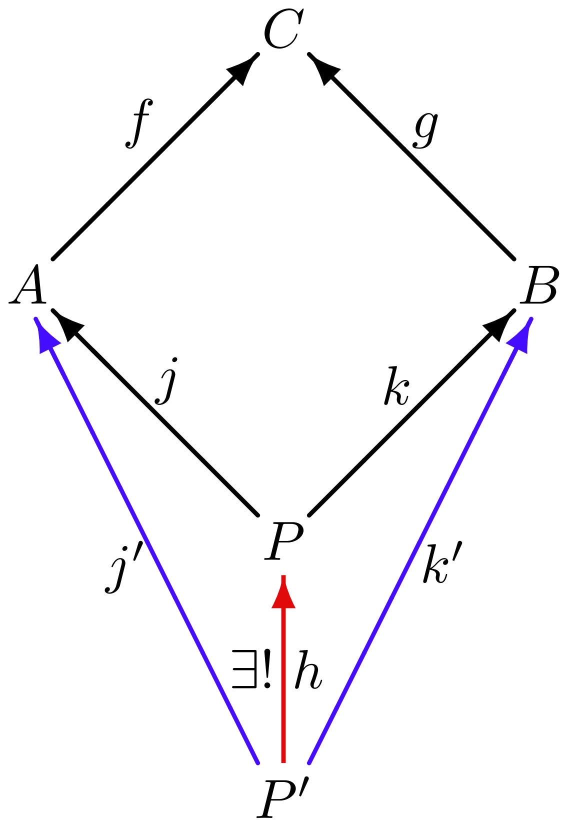

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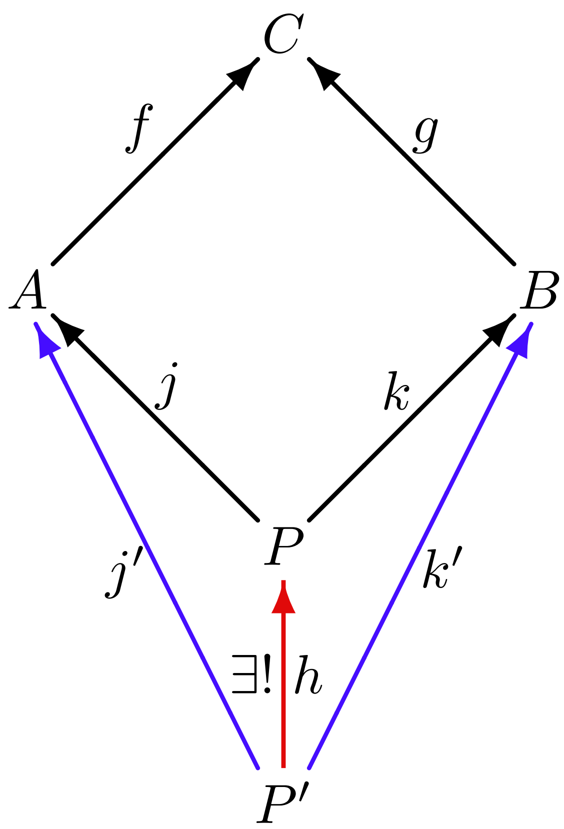

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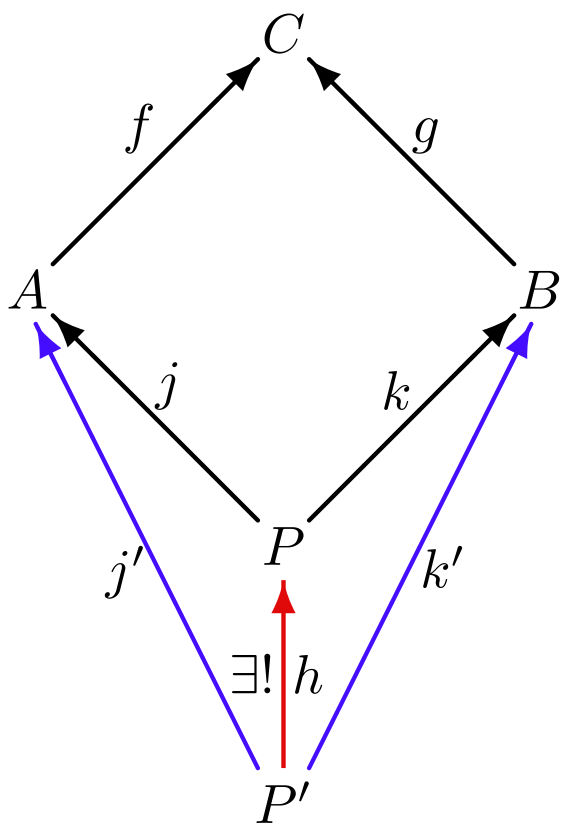

Define pullbacks in $\mathbf{Alg}(\Sigma)$.

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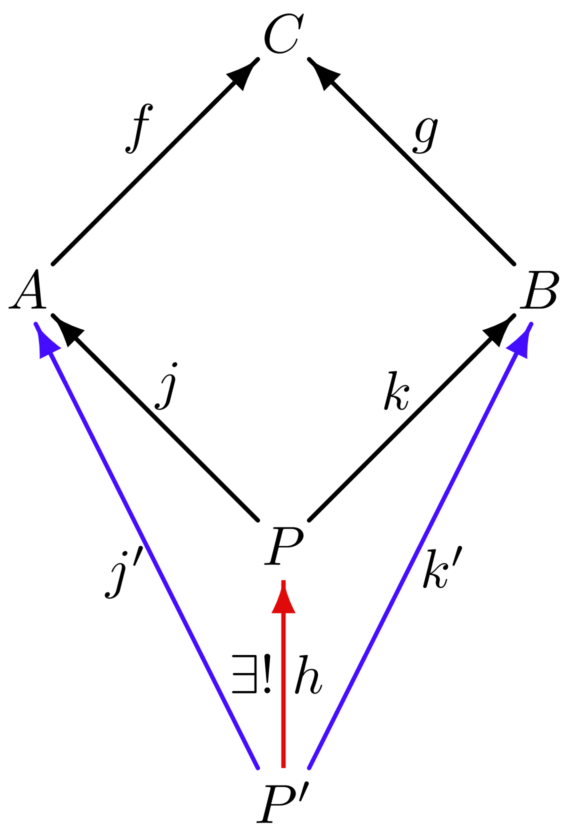

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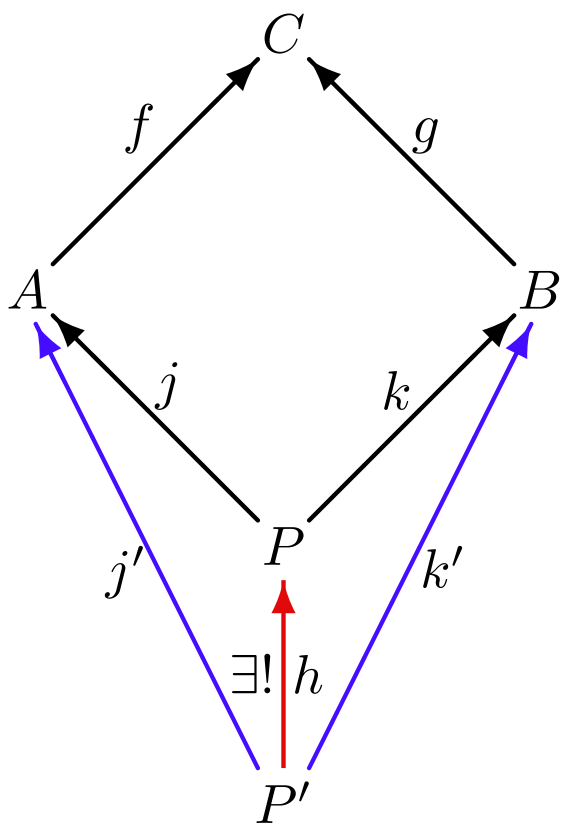

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Wait for a hint to come...

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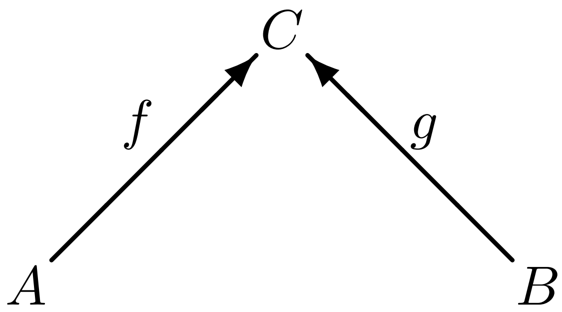

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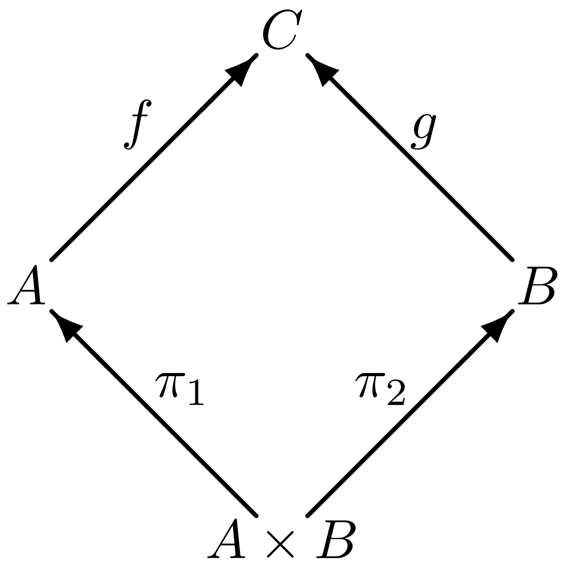

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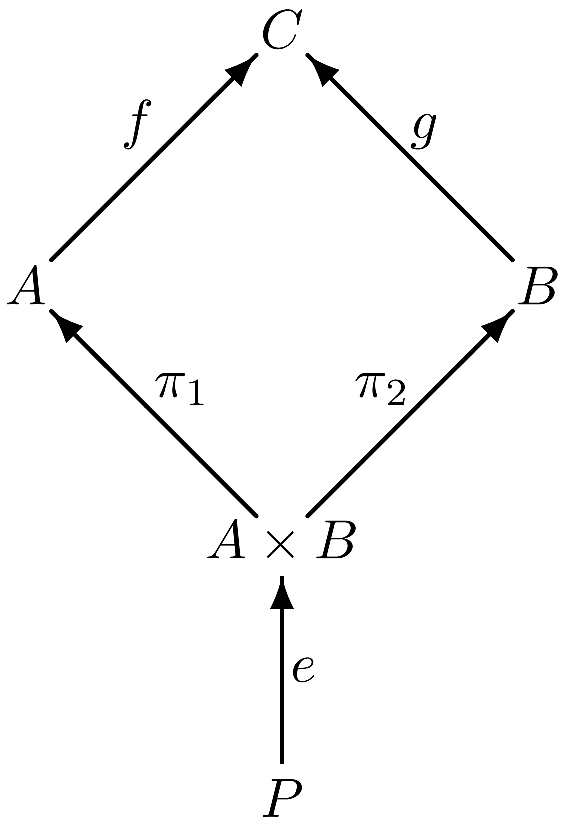

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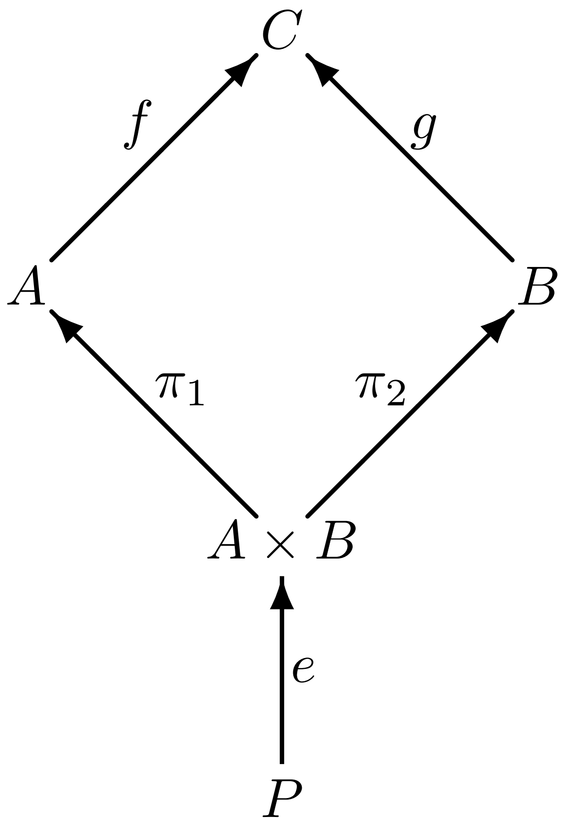

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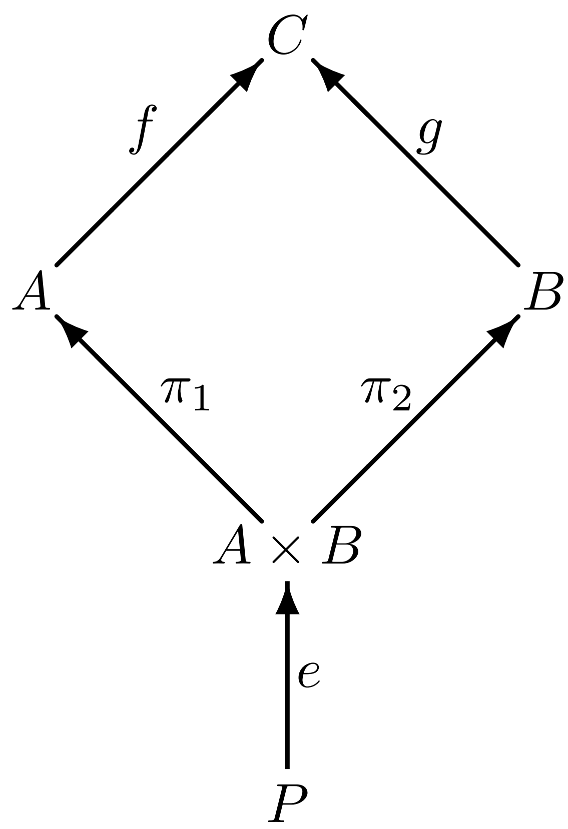

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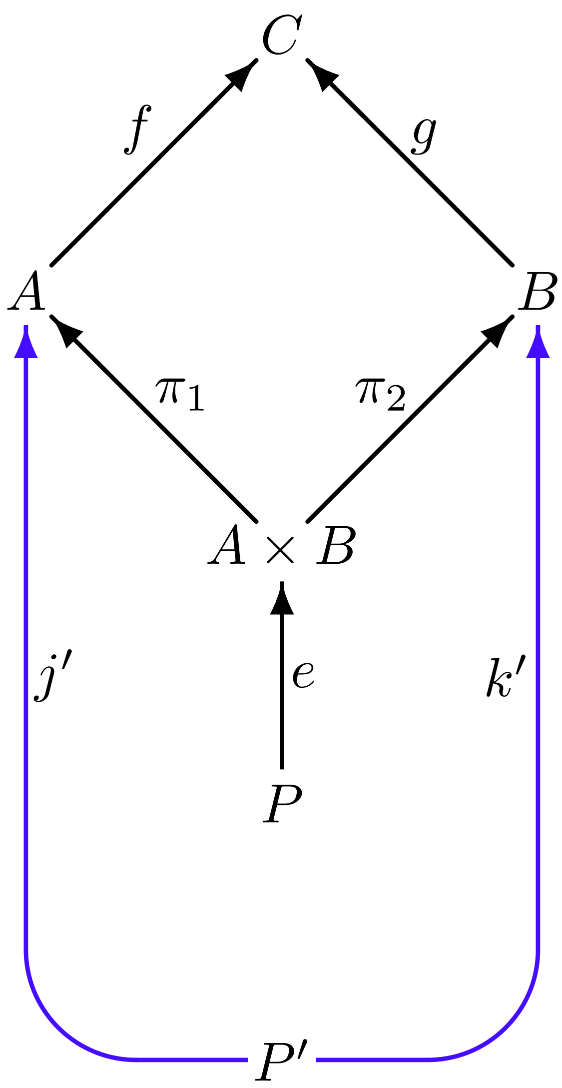

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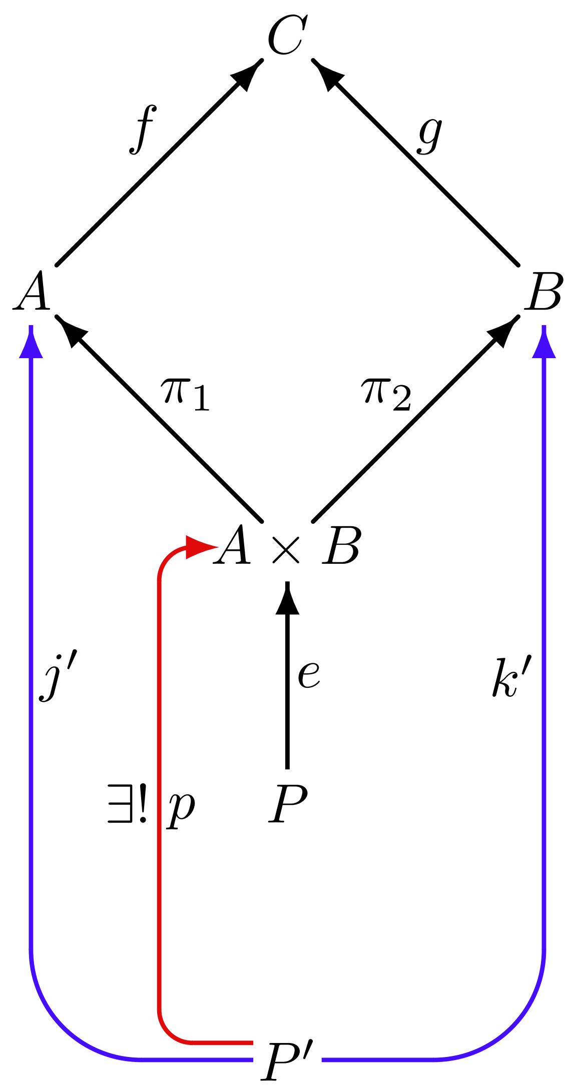

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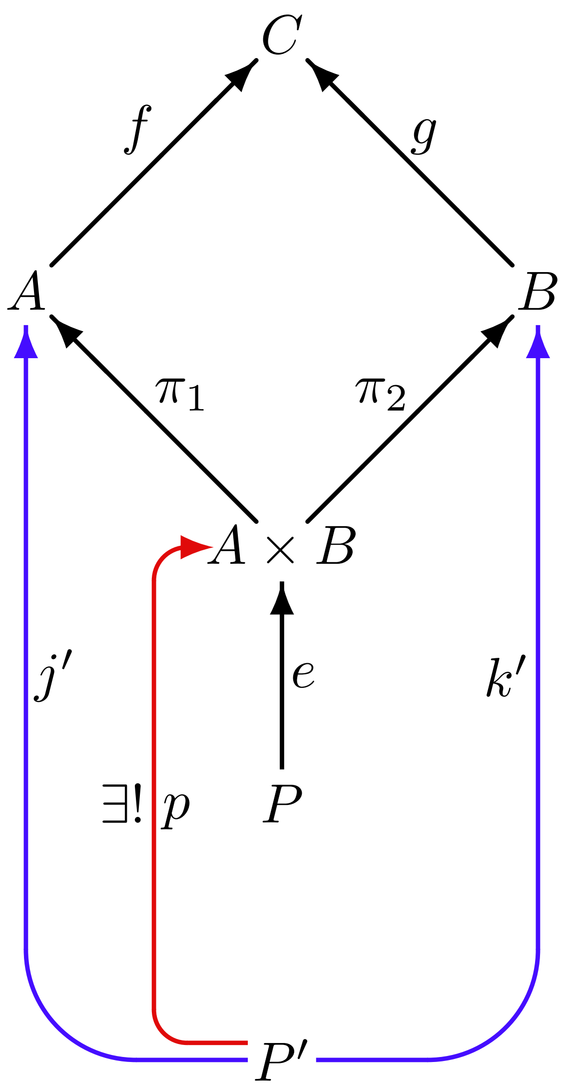

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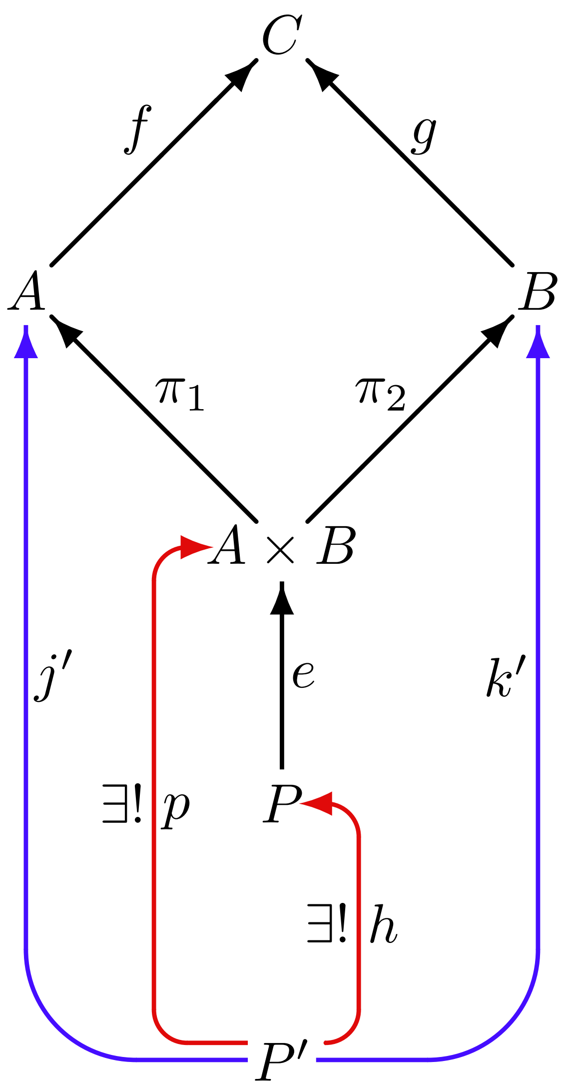

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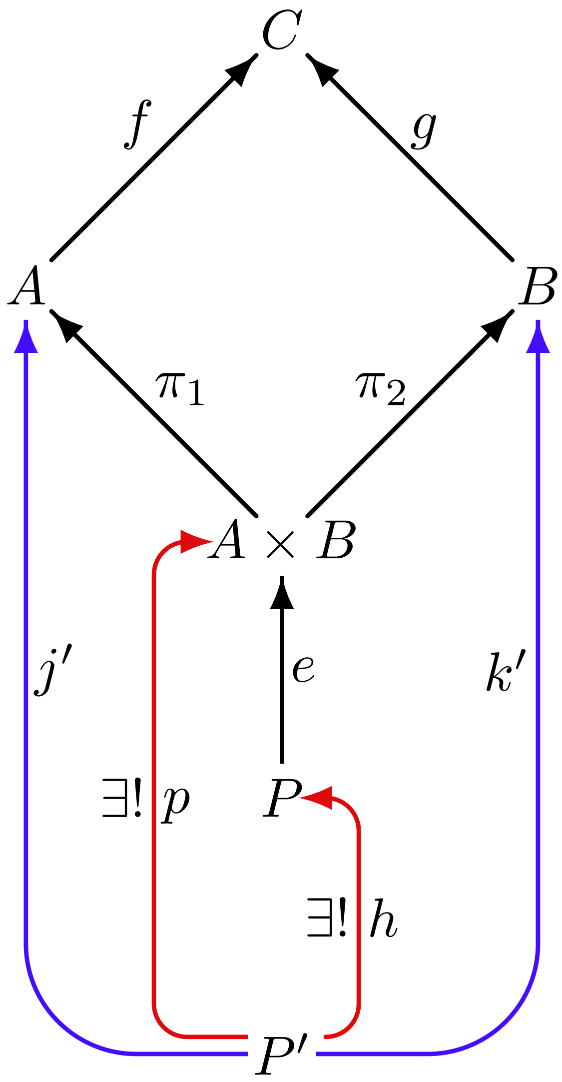

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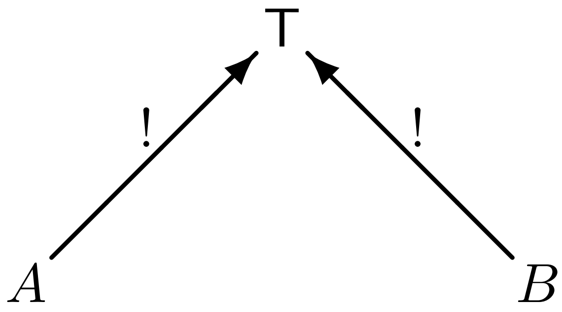

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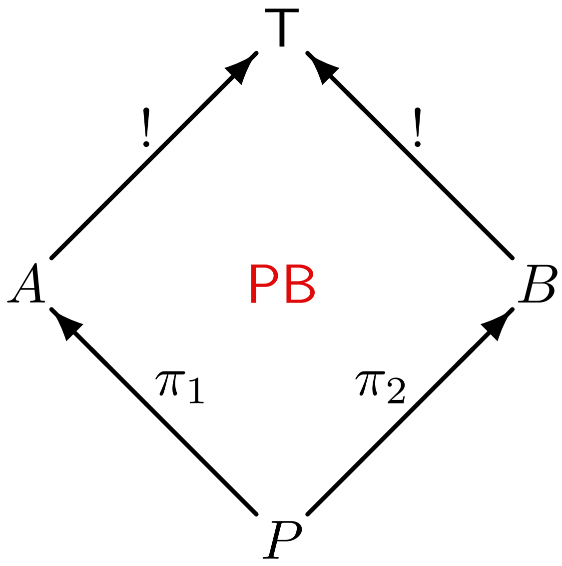

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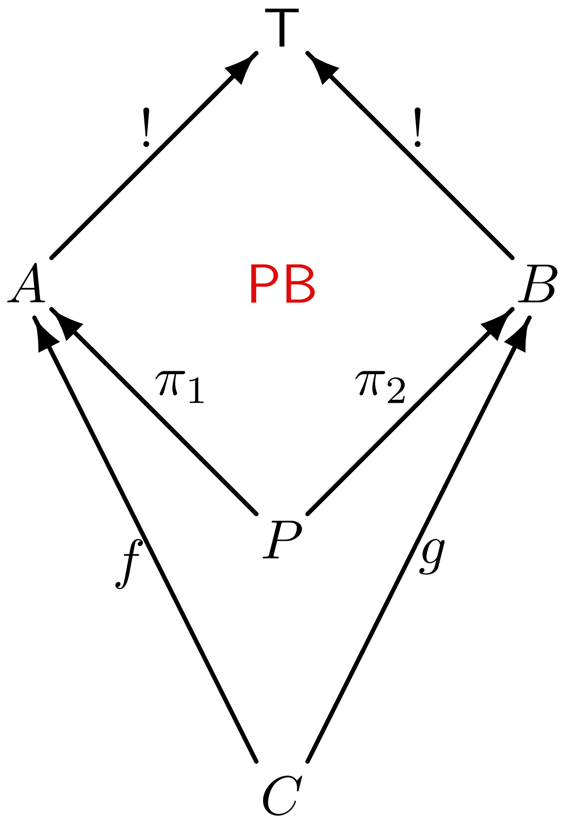

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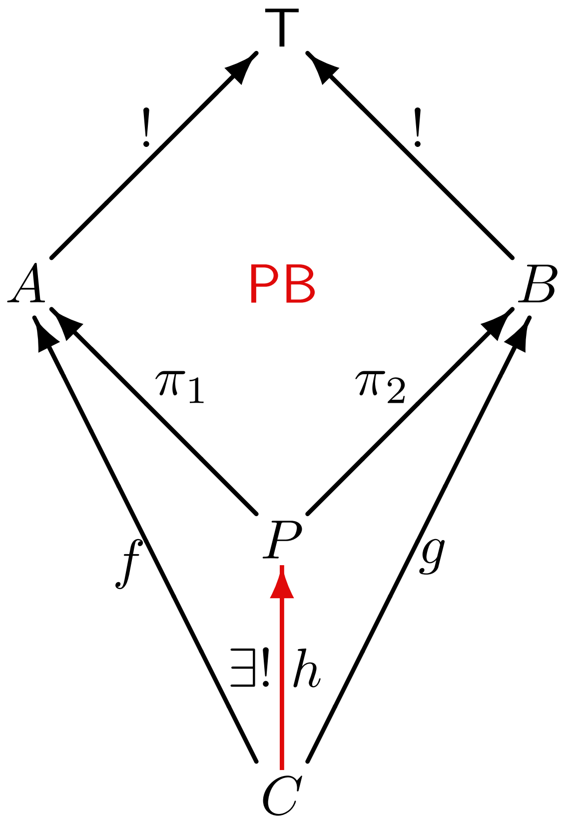

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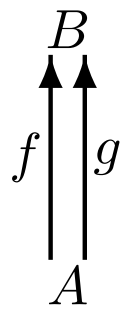

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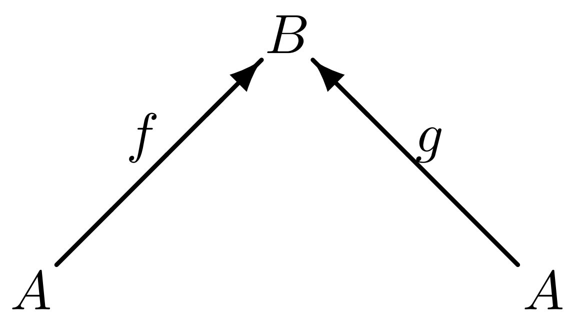

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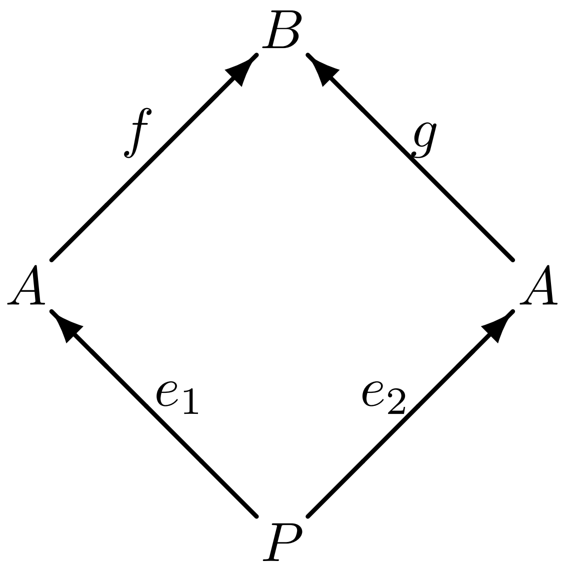

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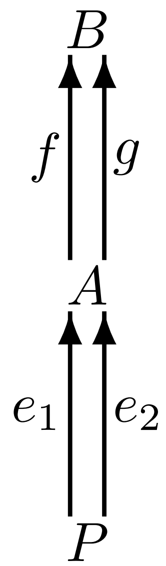

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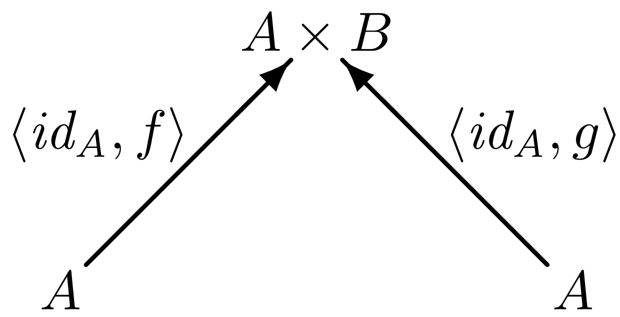

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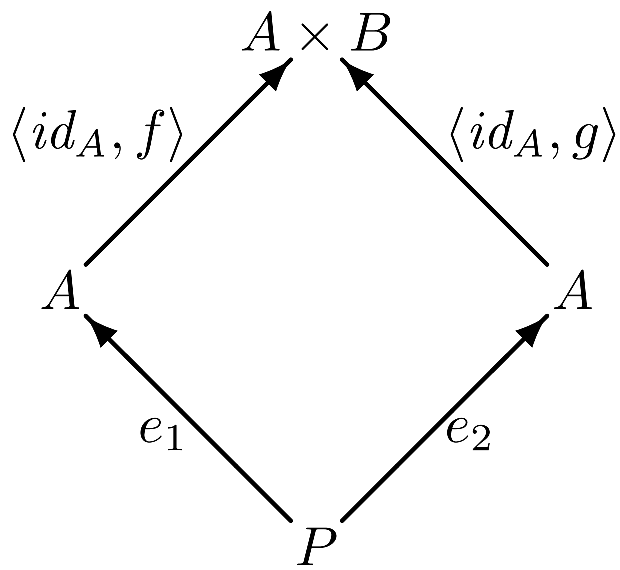

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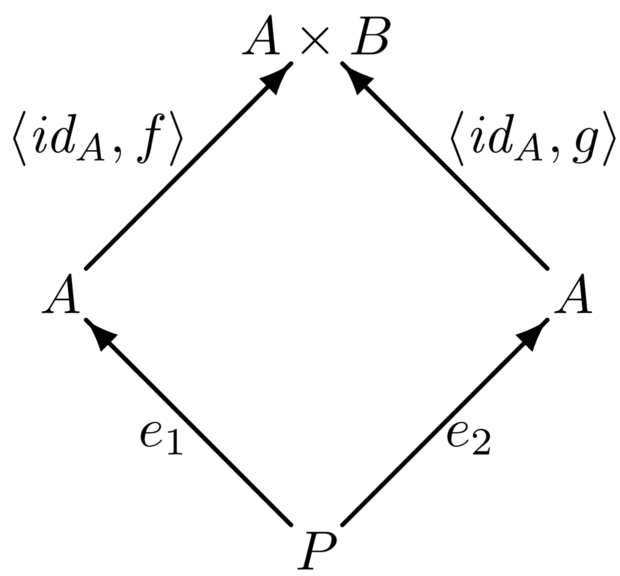

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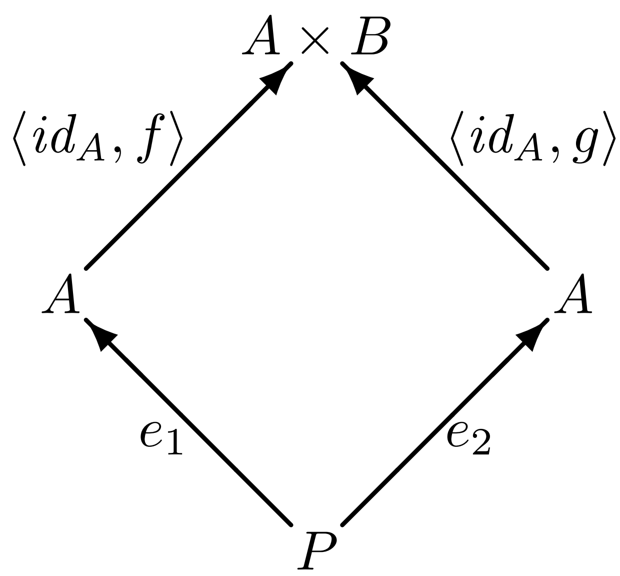

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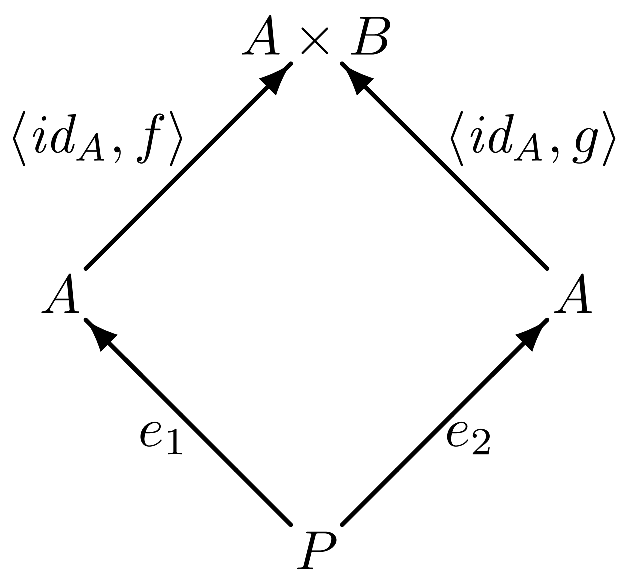

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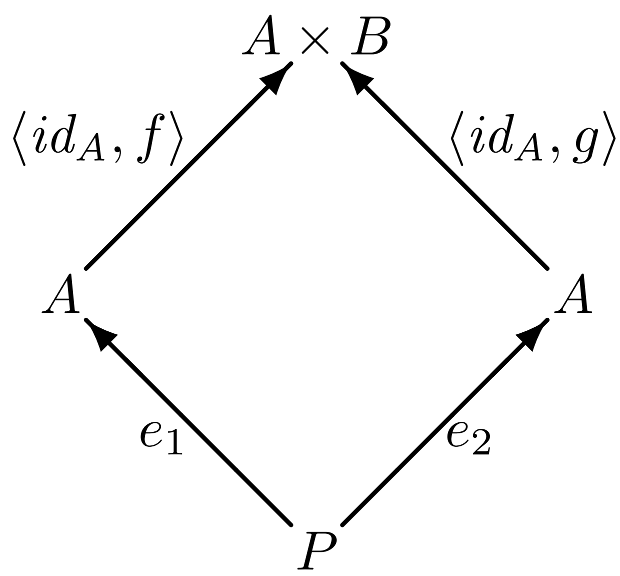

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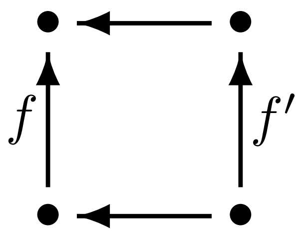
- Pullbacks translate monos to monos: if the following is a pullback square and f is mono then f' is mono as well.



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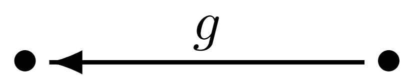

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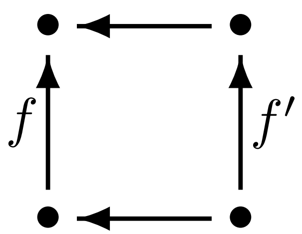

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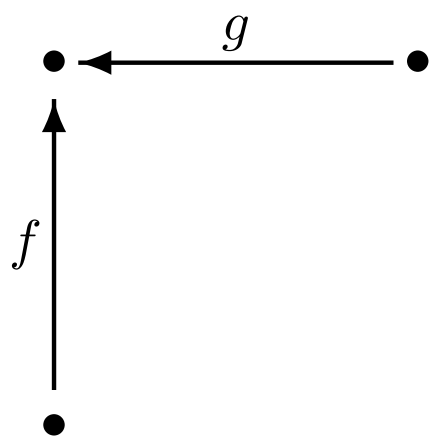

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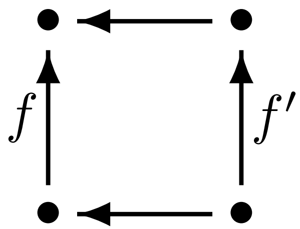

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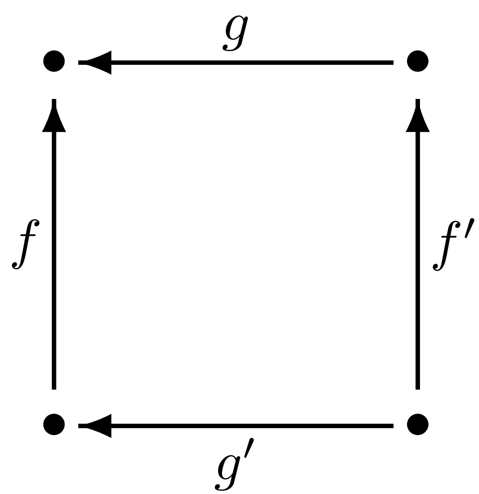

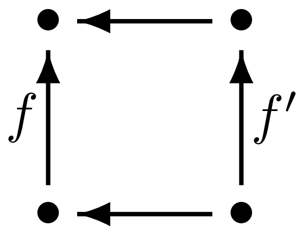

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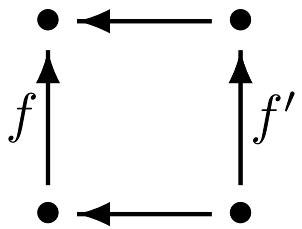

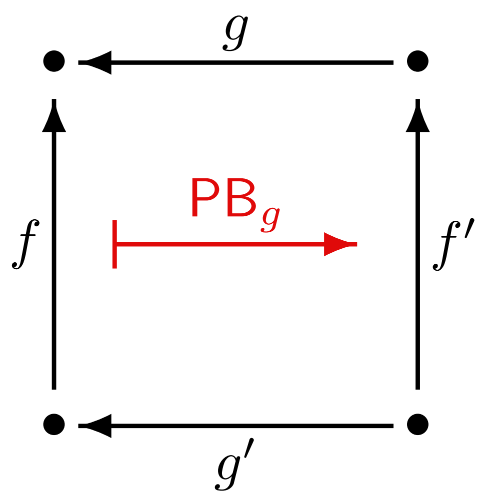

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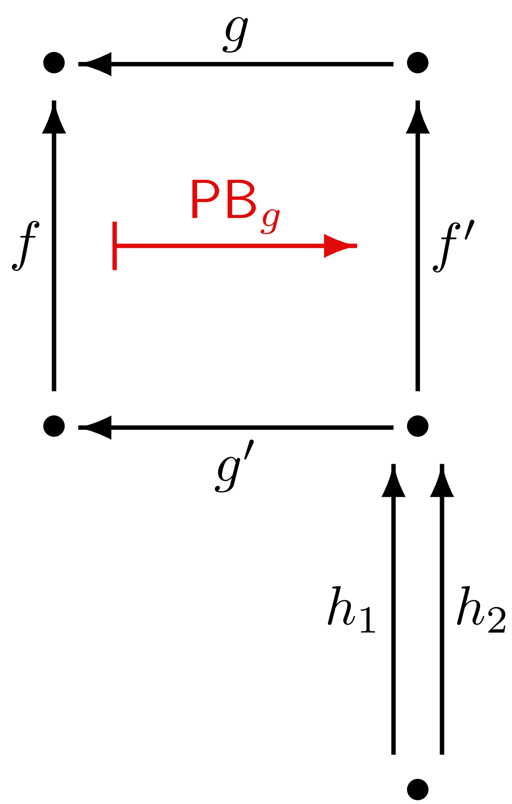

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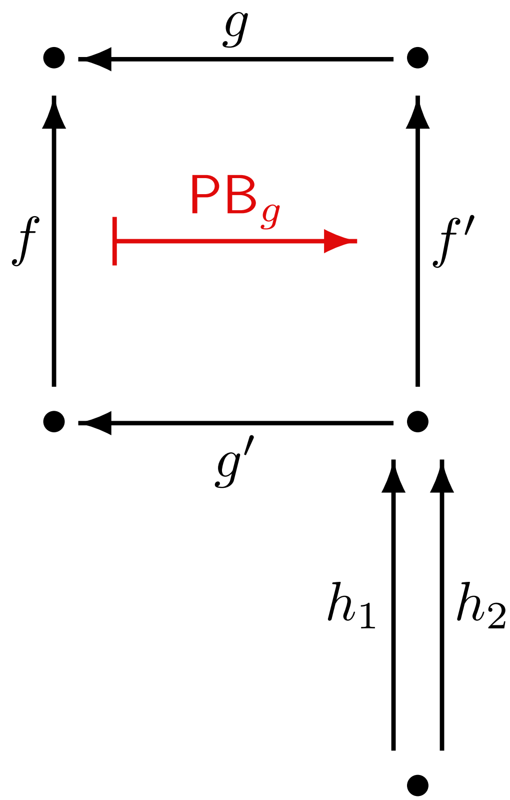

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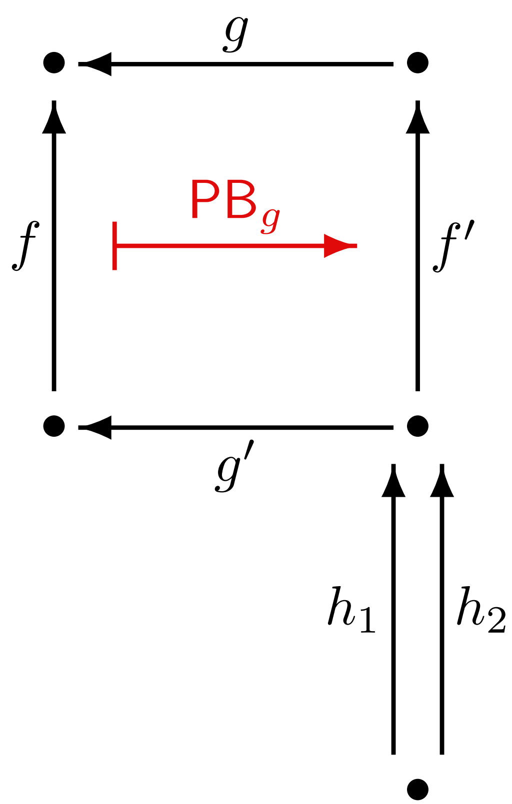

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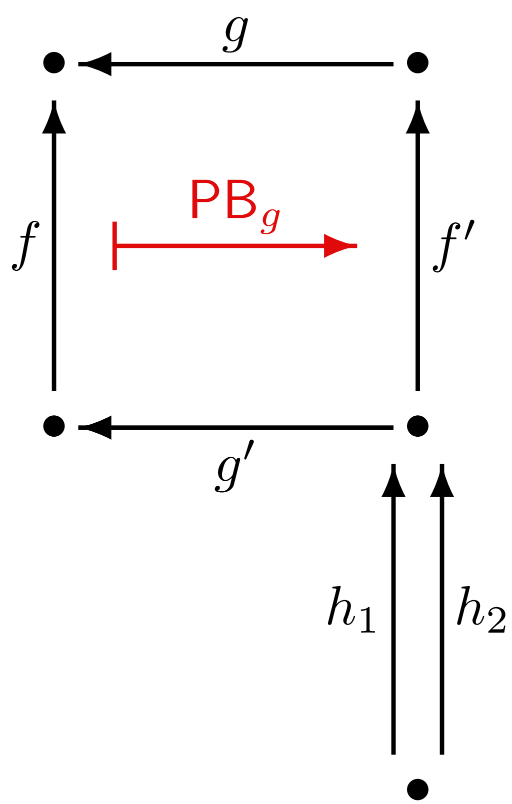

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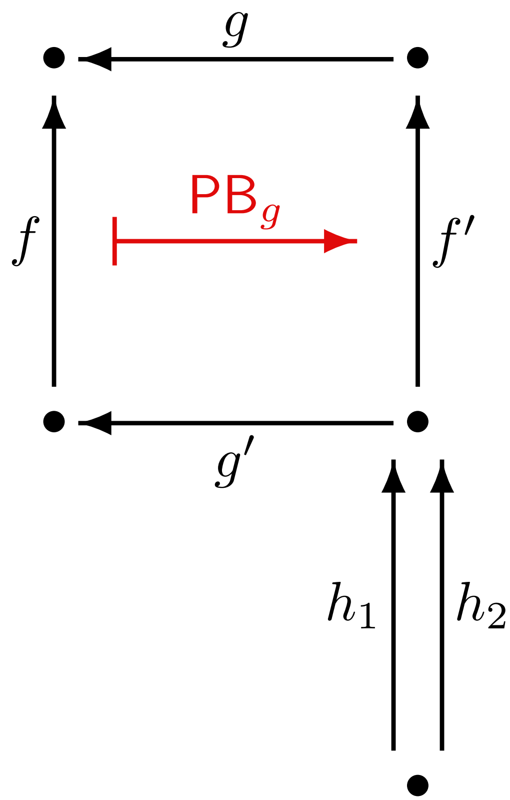

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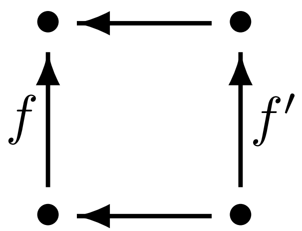

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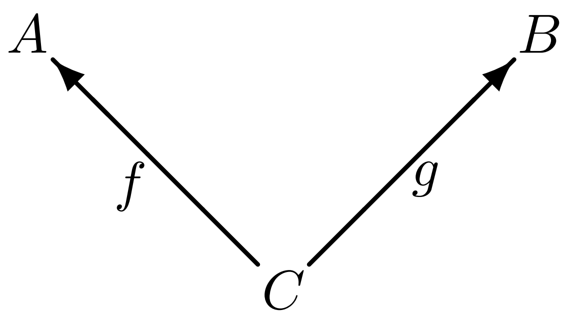

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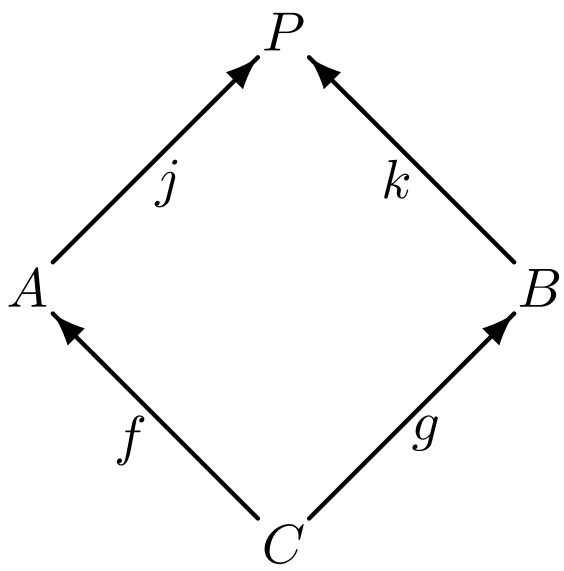

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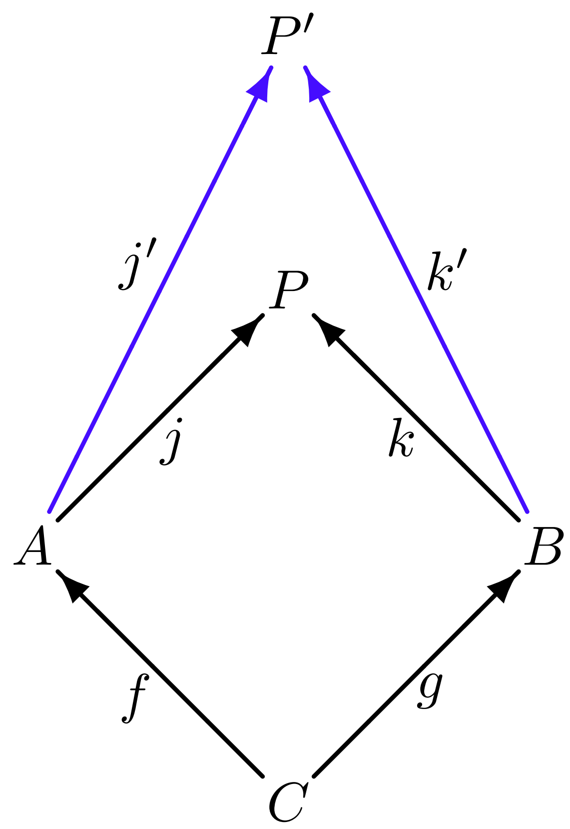

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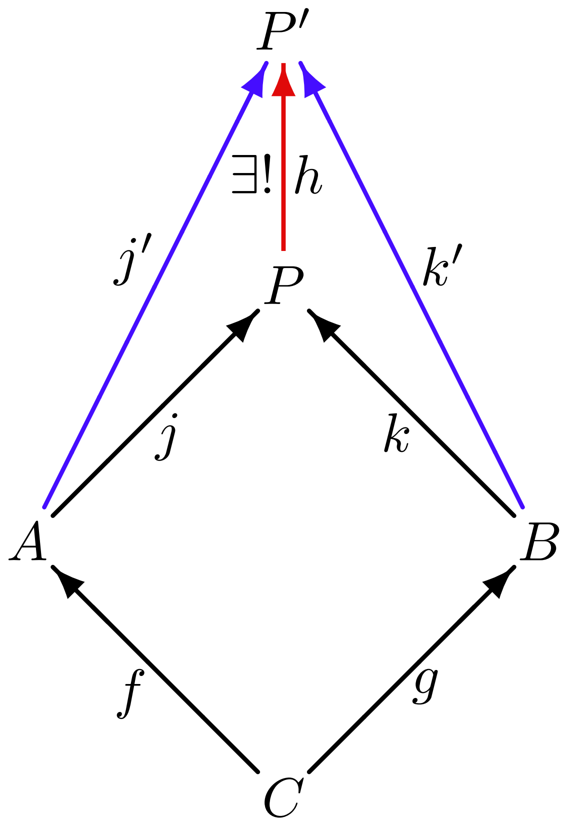

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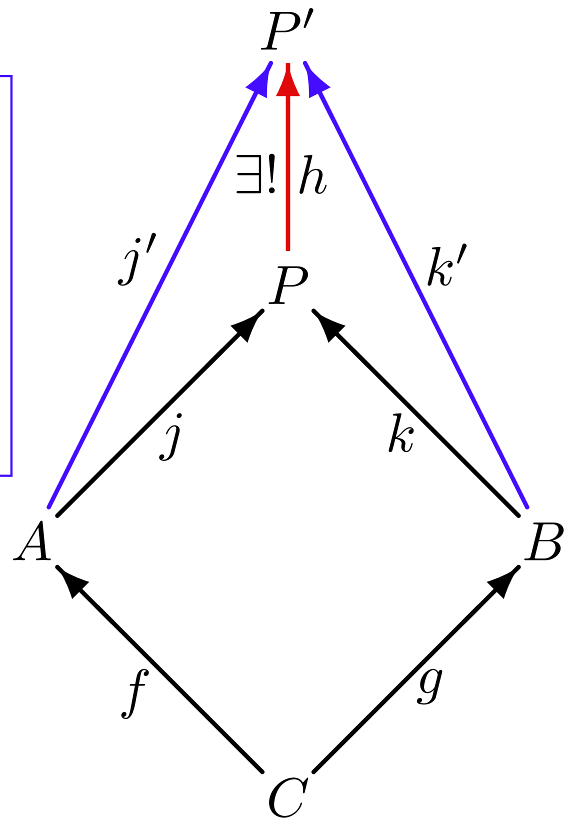

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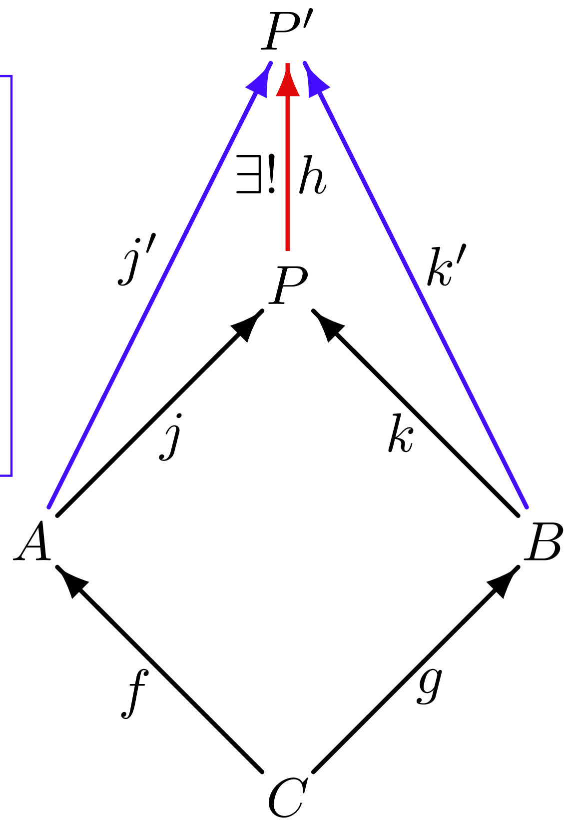

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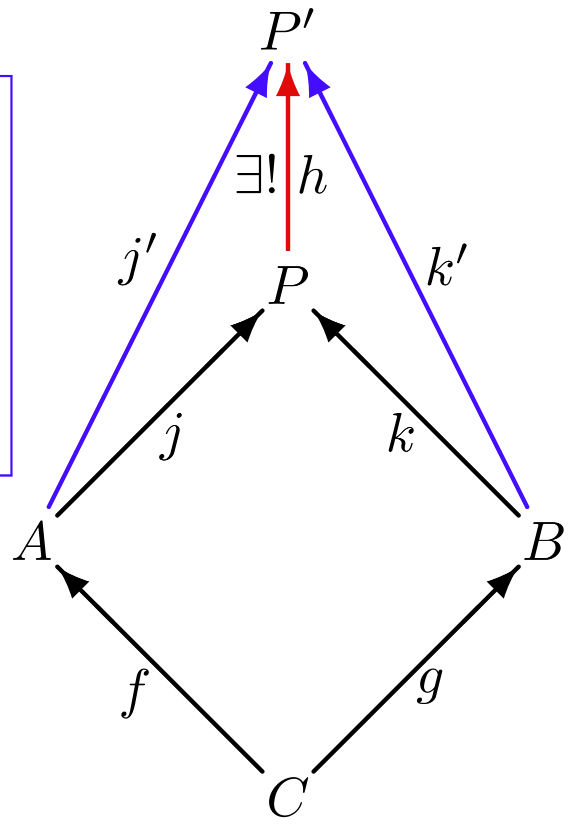

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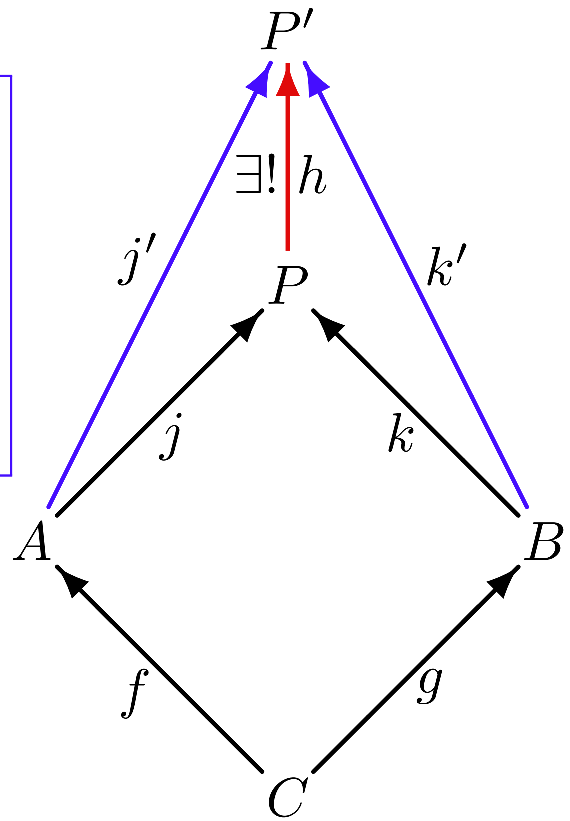

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Dualise facts for pullbacks!

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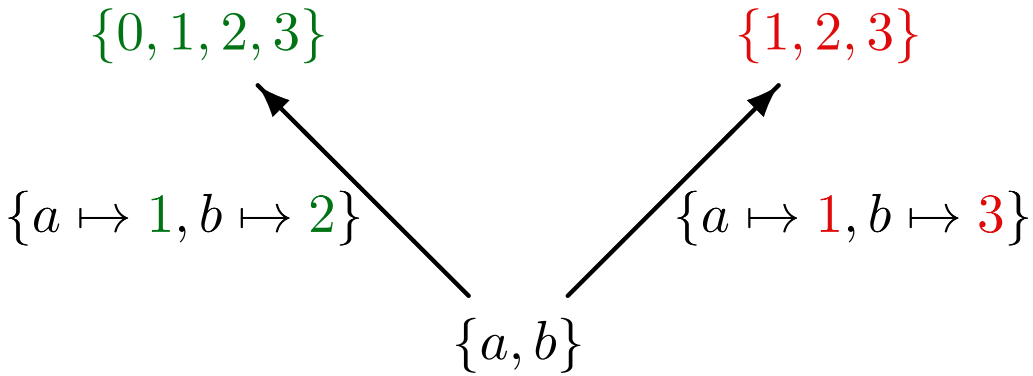

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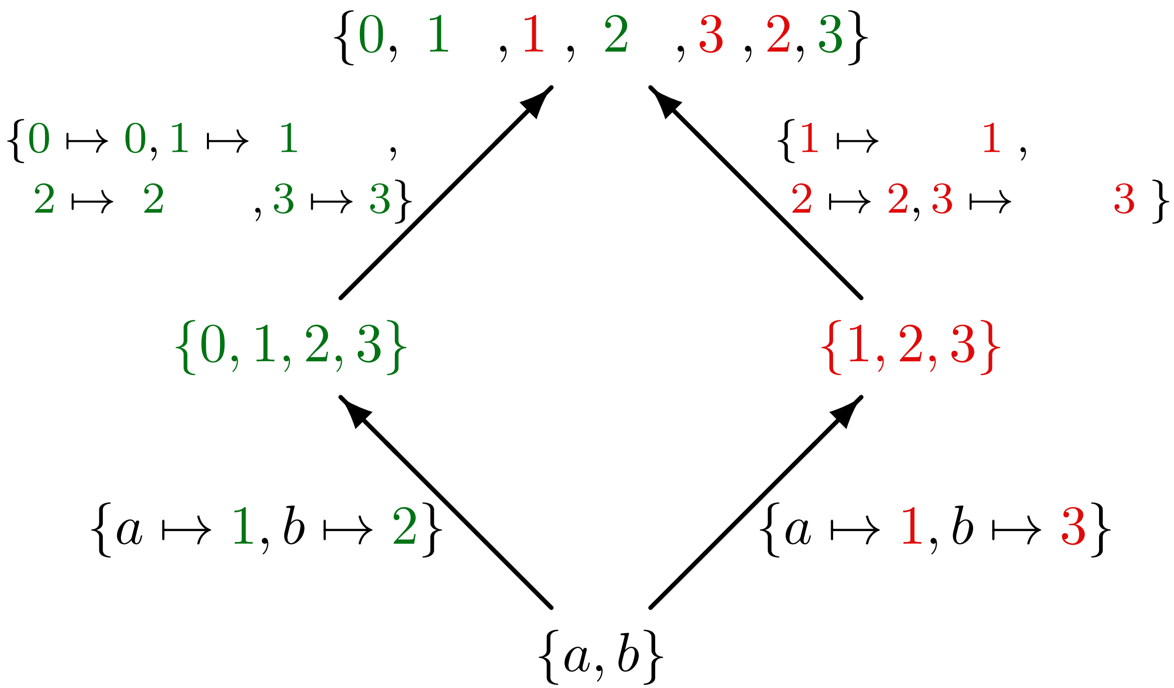
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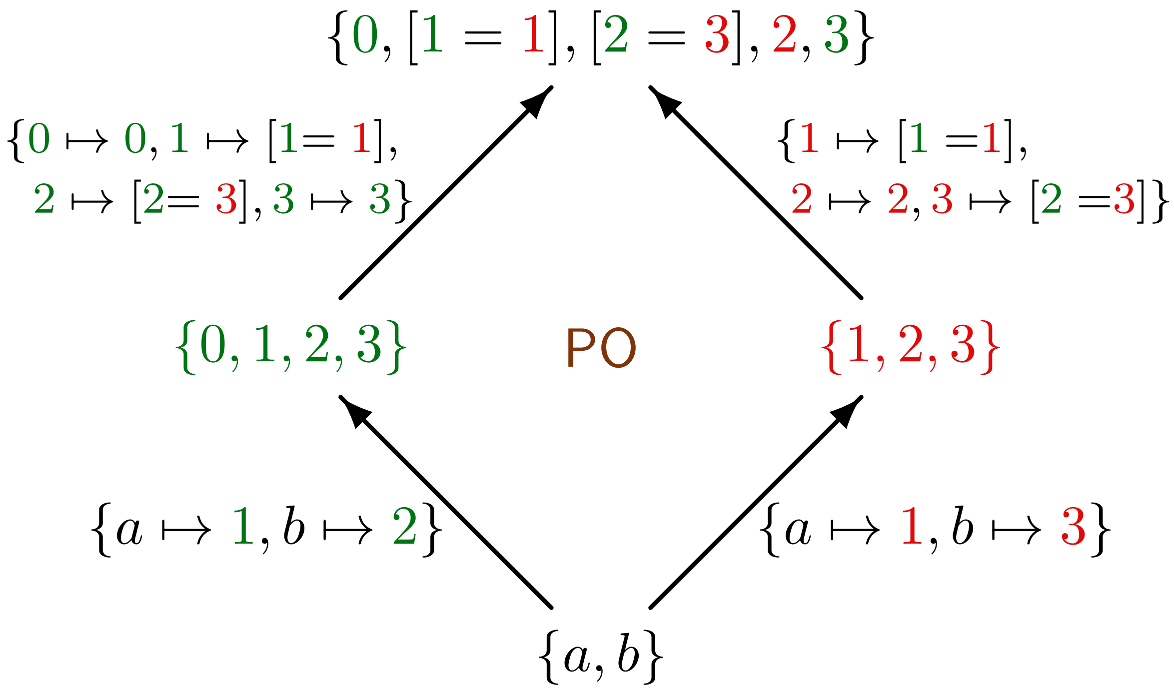

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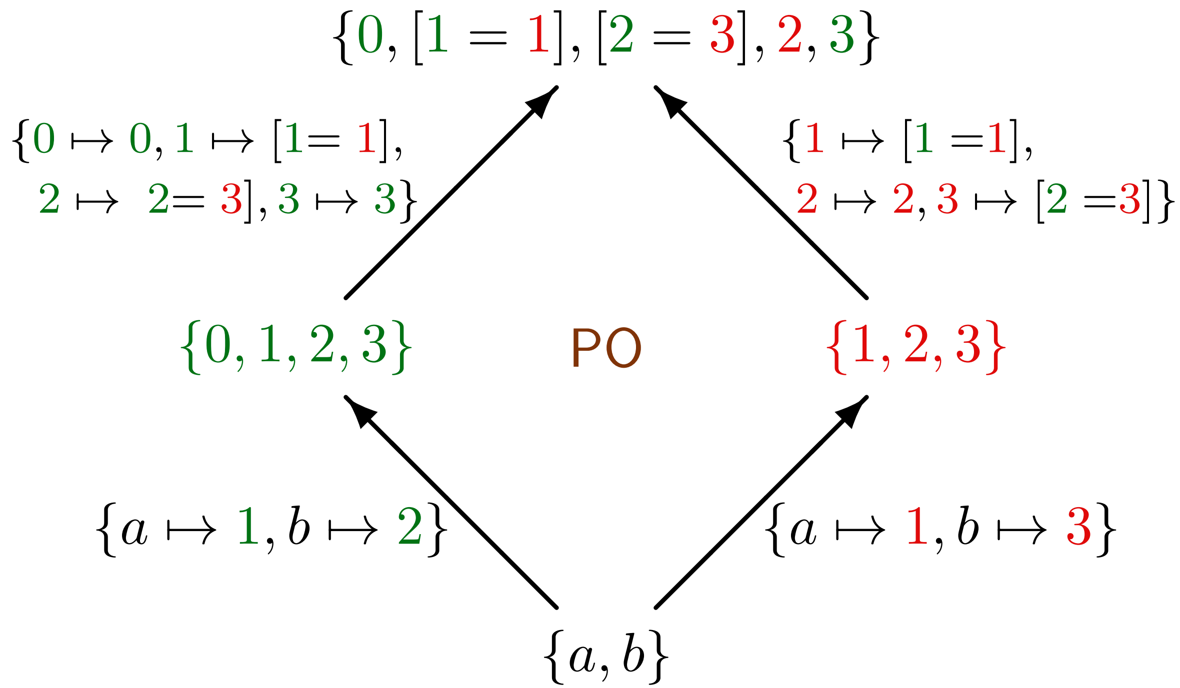

Pushouts put objects together taking account of the indicated sharing.

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sort $Elem$

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ops $a, \ldots, z : String$;
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$\uparrow$

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sorts $\mathit{Elem}, \mathit{Nat}, \mathit{Array}[\mathit{Elem}]$
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- objects: $|G|_{nodes}$
- morphisms: *paths* in G, i.e., sequences $n_0 e_1 n_1 \ldots n_{k-1} e_k n_k$ of nodes $n_0, \ldots, n_k \in |G|_{nodes}$ and edges $e_1, \ldots, e_k \in |G|_{edges}$ such that $source(e_i) = n_{i-1}$ and $target(e_i) = n_i$ for $i = 1, \ldots, k$.

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BTW: A diagram D *commutes* (or is *commutative*) if for any two paths in $\mathcal{G}(D)$ with common source and target, the compositions of morphisms that label the edges of each of them coincide.

Diagram categories

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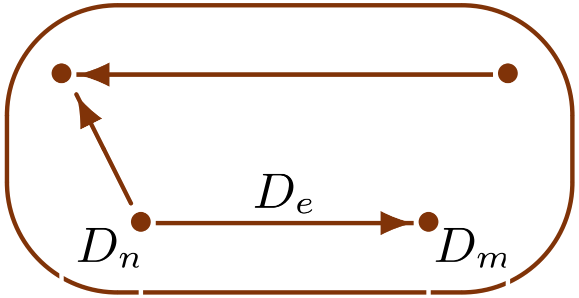

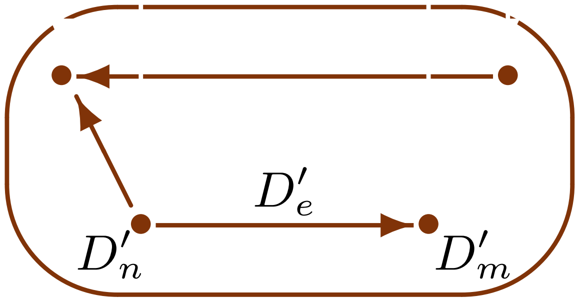

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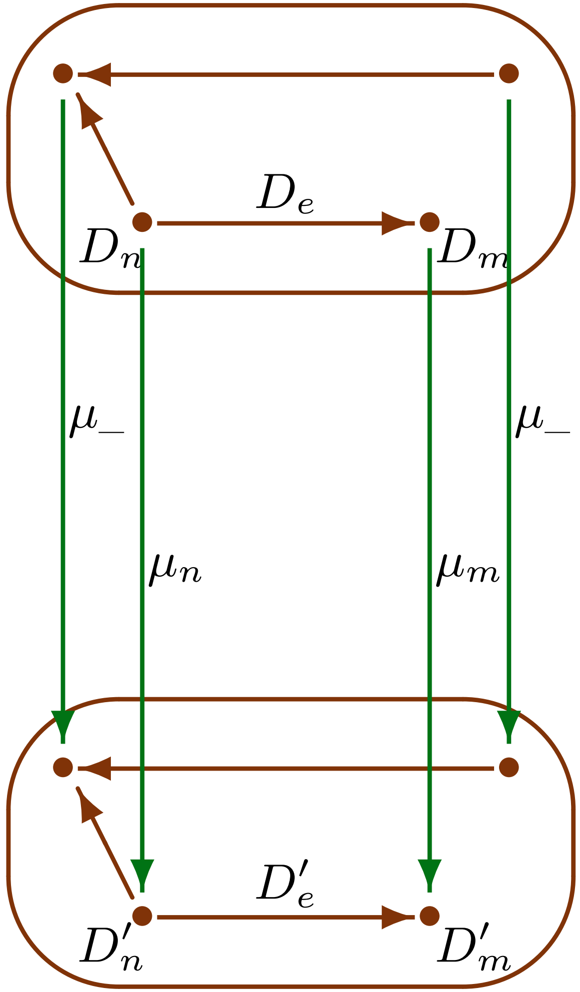

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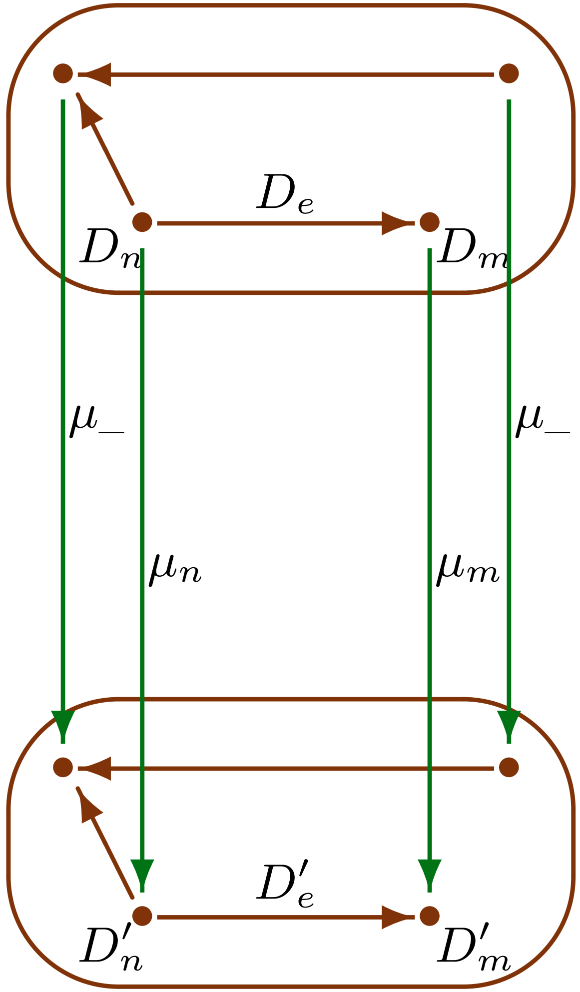
$$\mu_n;D'_e = D_e;\mu_m$$

.



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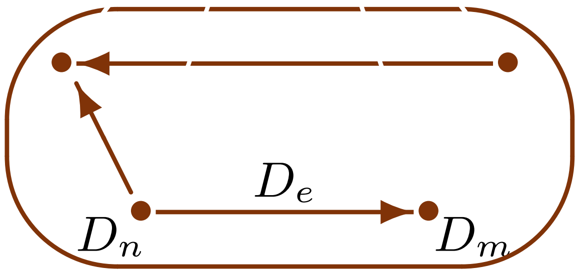

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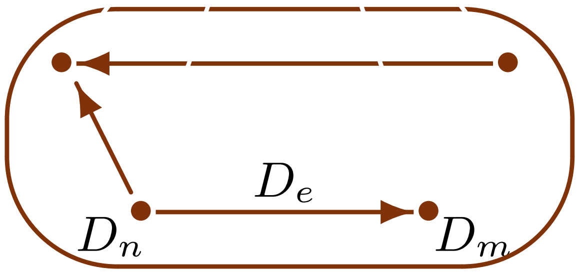
Cones and cocones



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Cones and cocones

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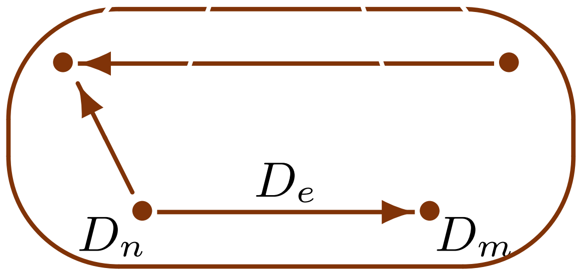

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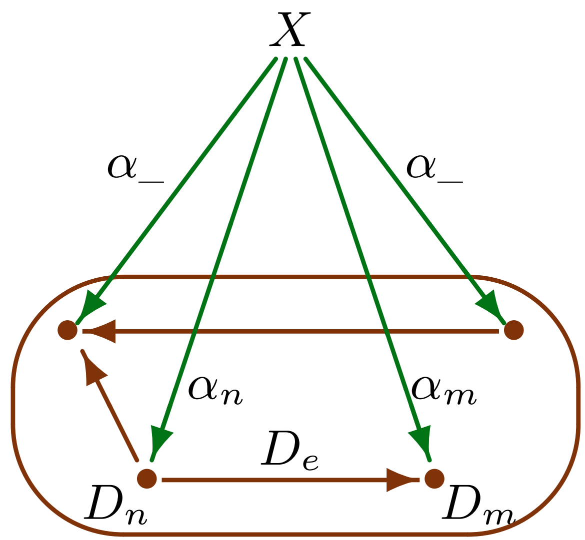

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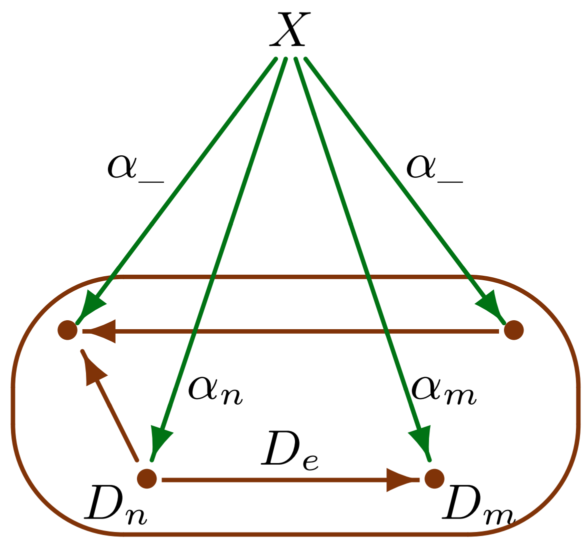

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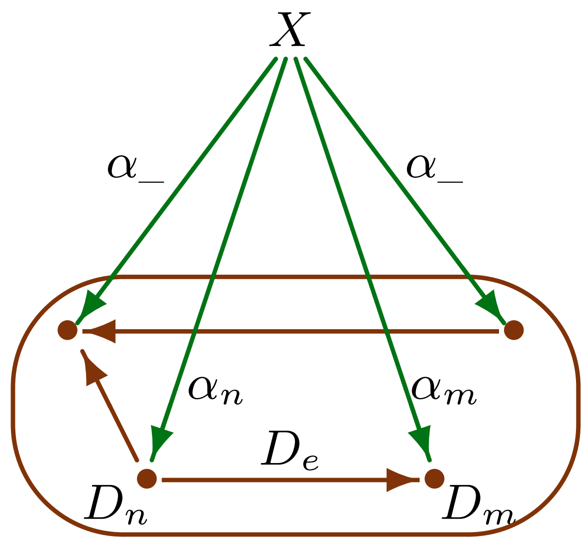

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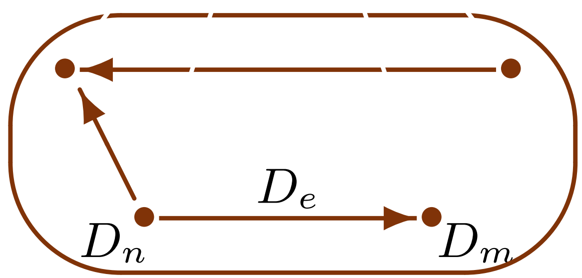

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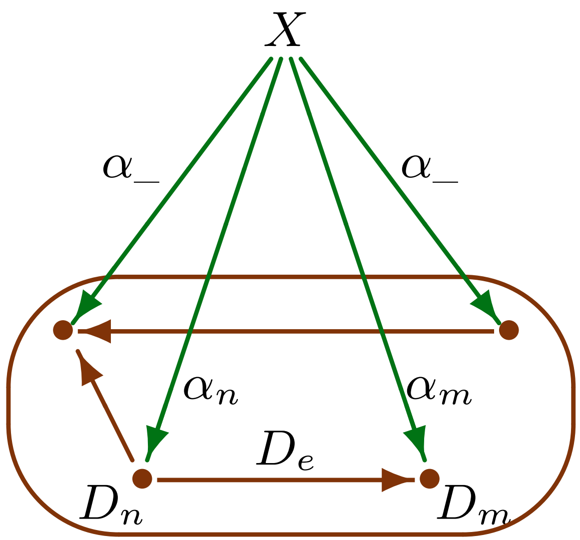

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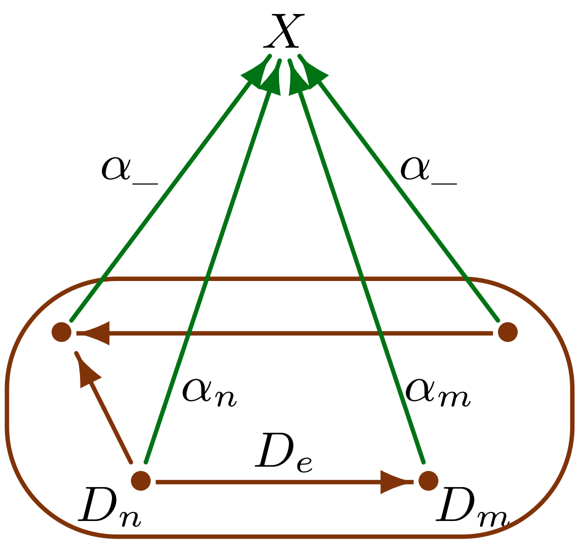

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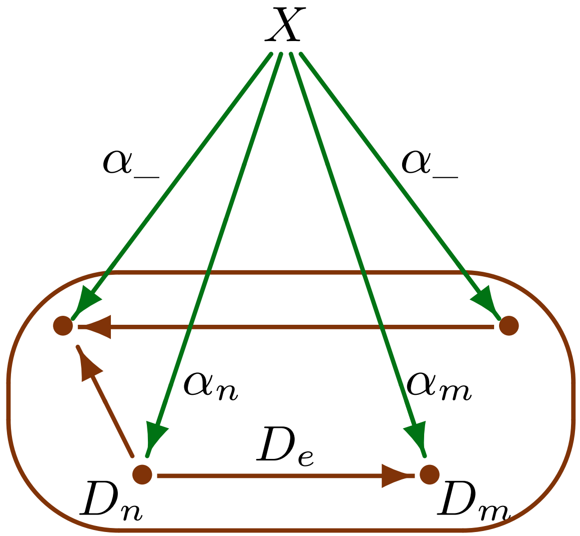

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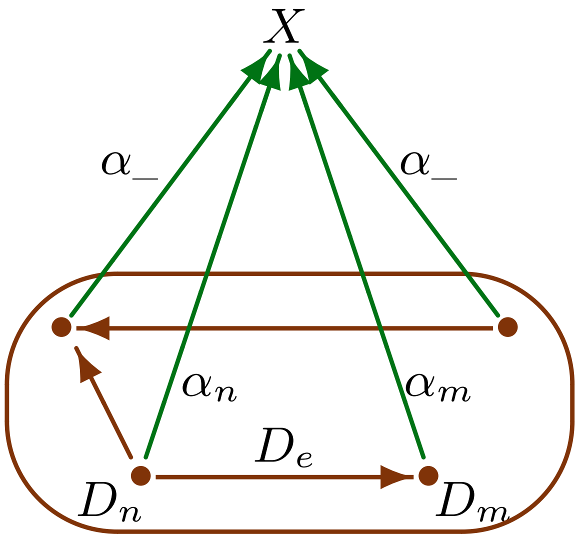

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Limits and colimits

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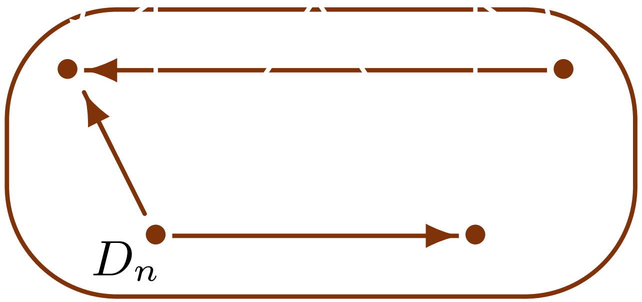
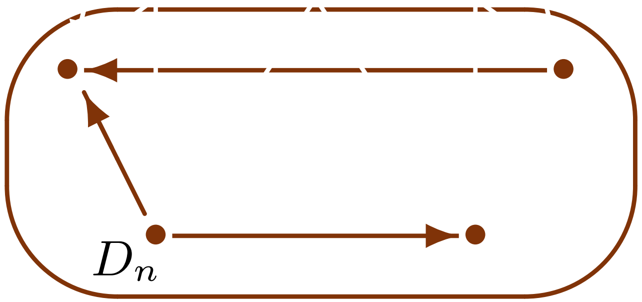


Limits and colimits

A *limit* of D (in $\mathbf{K}$)



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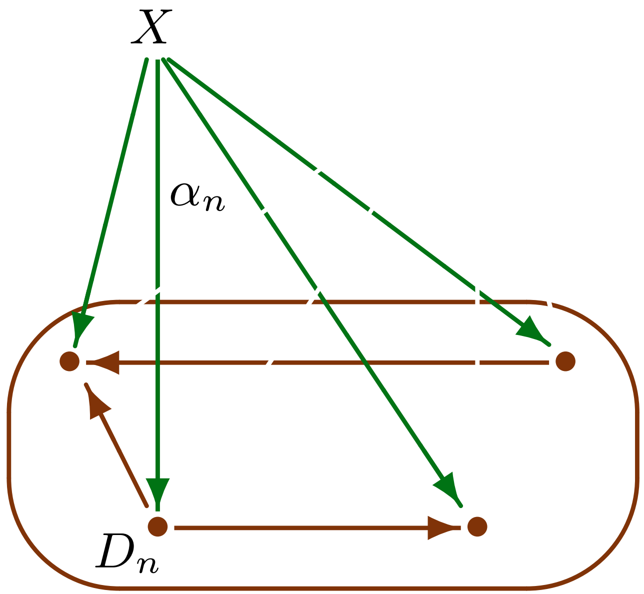
A *limit* of D (in $\mathbf{K}$) is a cone $\langle \alpha_n : X \to D_n \rangle_{n \in N}$ on D



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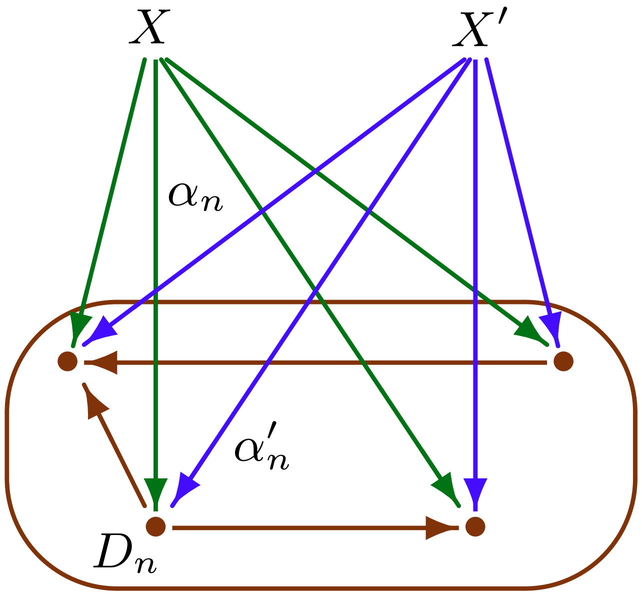

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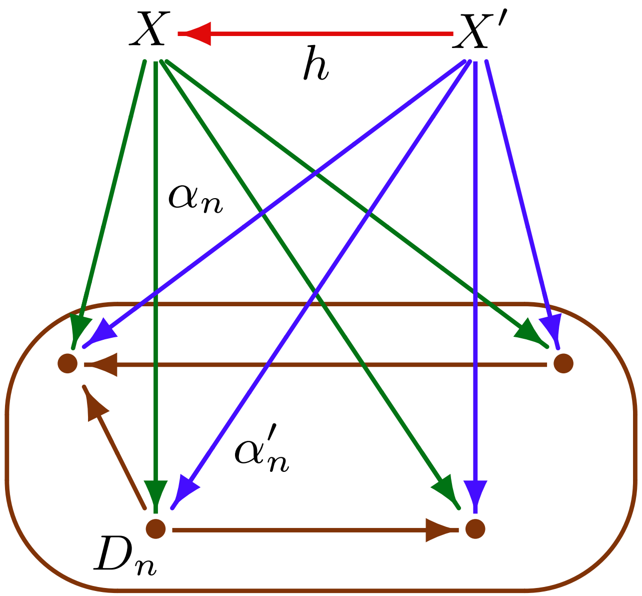

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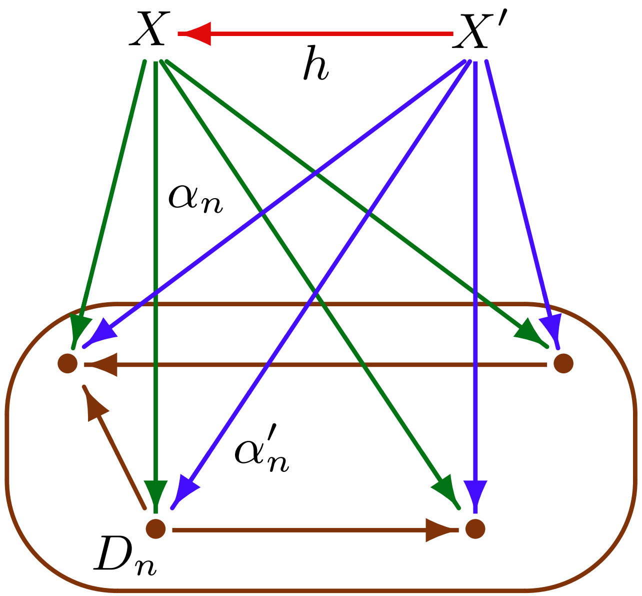

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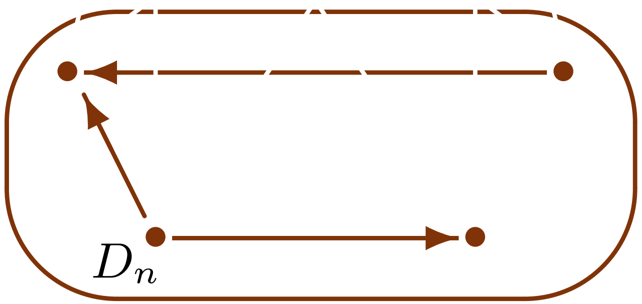

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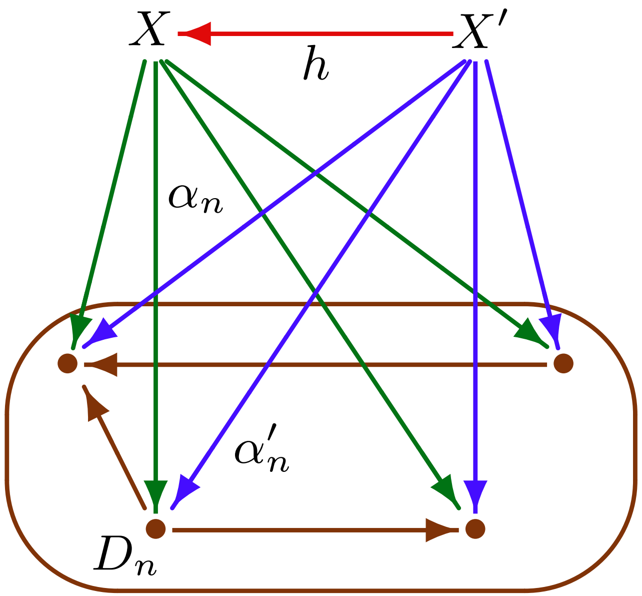
Limits and colimits



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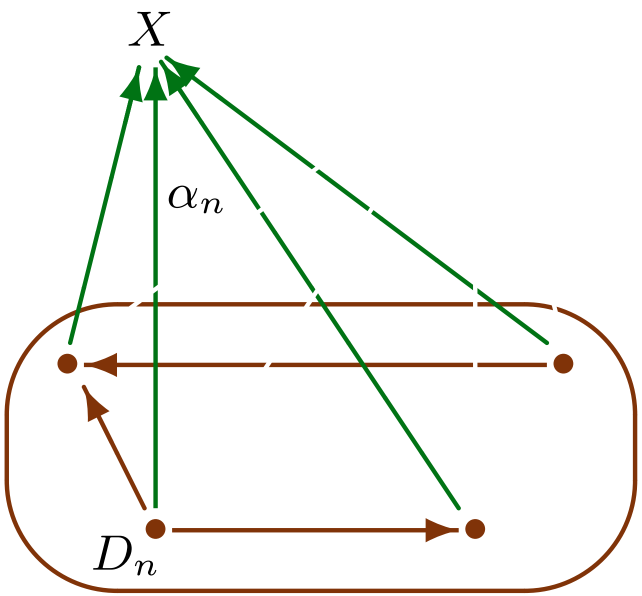

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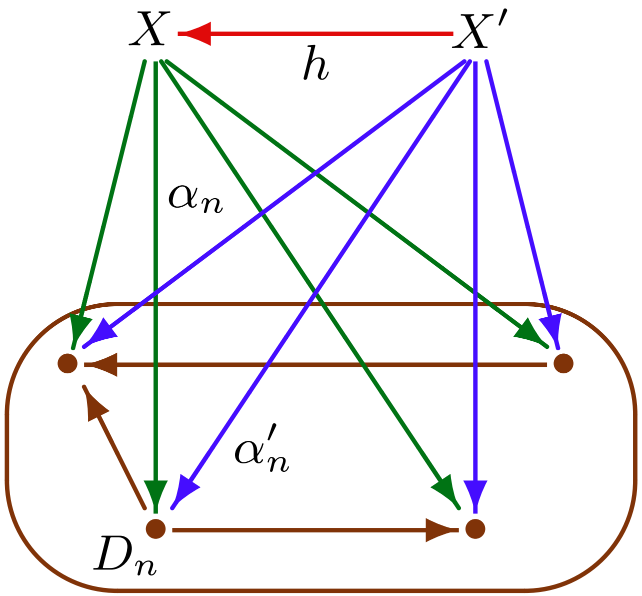

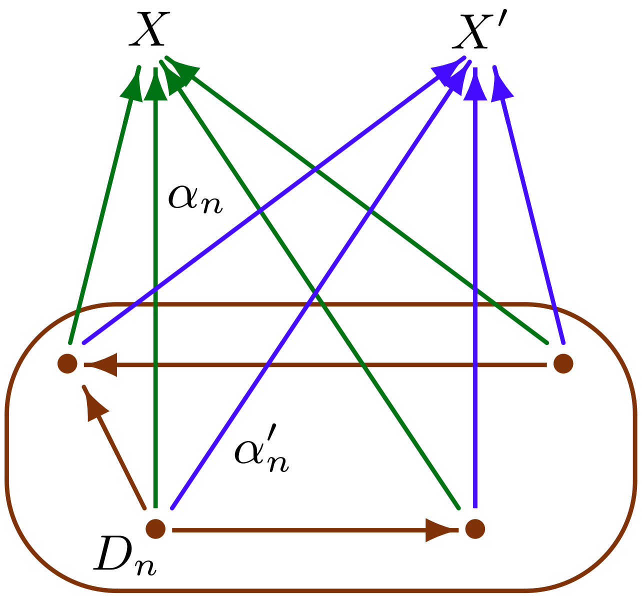

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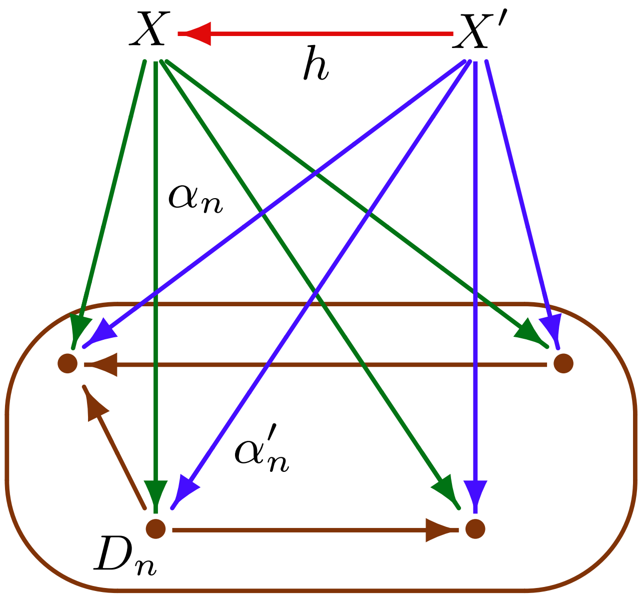

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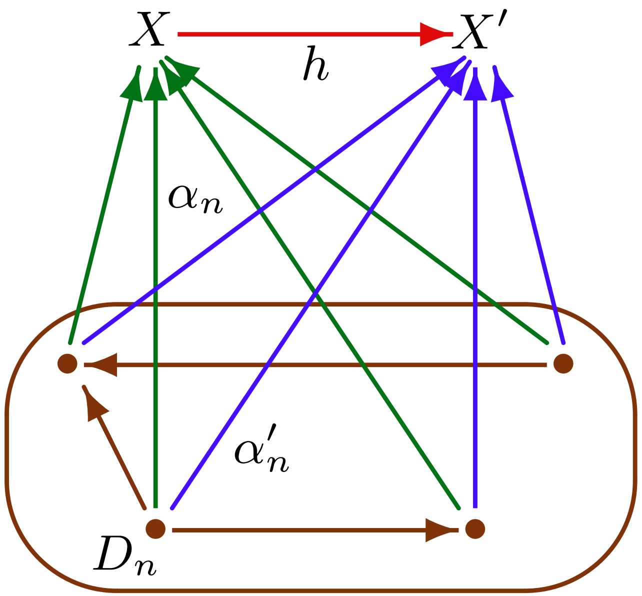

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Some limits

diagram	limit	in **Set**

Some limits

diagram	limit	in $\mathbf{Set}$
(empty)	*terminal object*	$\{*\}$

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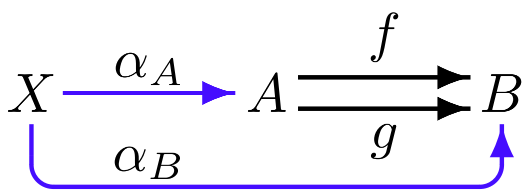
Cones 

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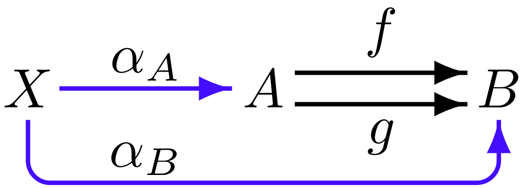
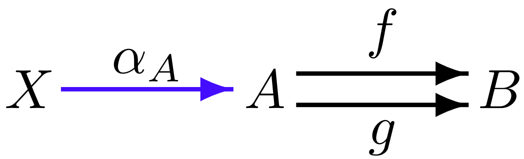
Cones  where $\alpha_A;f = \alpha_B$ and $\alpha_A;g = \alpha_B$

coincide with morphisms  where $\alpha_A;f = \alpha_A;g$.

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$A \xrightarrow{f} C \xleftarrow{g} B$	*pullback*	$\{(a, b) \in A \times B \mid f(a) = g(b)\}$

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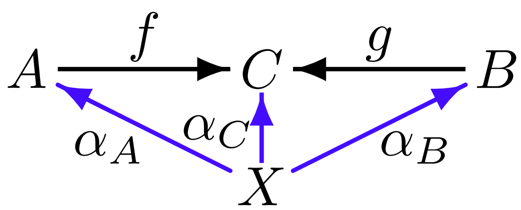
Cones  where $\alpha_A;f = \alpha_C$ and $\alpha_B;g = \alpha_C$

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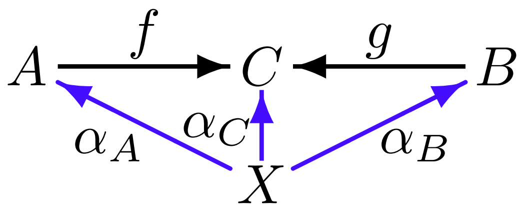
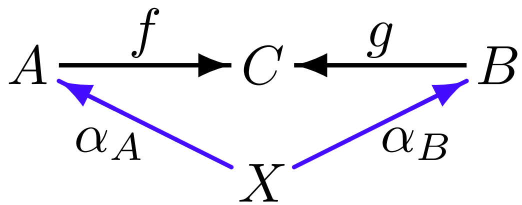
Cones  where $\alpha_A;f = \alpha_C$ and $\alpha_B;g = \alpha_C$

coincide with pairs of morphisms  where $\alpha_A;f = \alpha_B;g$.

... & colimits

diagram	colimit	in **Set**
(empty)	*initial object*	$\emptyset$
$A \qquad B$	*coproduct*	$A \uplus B$
$A \overset{f}{\underset{g}{\rightrightarrows}} B$	*coequaliser*	$B \longrightarrow B/{\equiv}$ where $f(a) \equiv g(a)$ for all $a \in A$
$A \xleftarrow{f} C \xrightarrow{g} B$	*pushout*	$(A \uplus B)/{\equiv}$ where $f(c) \equiv g(c)$ for all $c \in C$

Exercises

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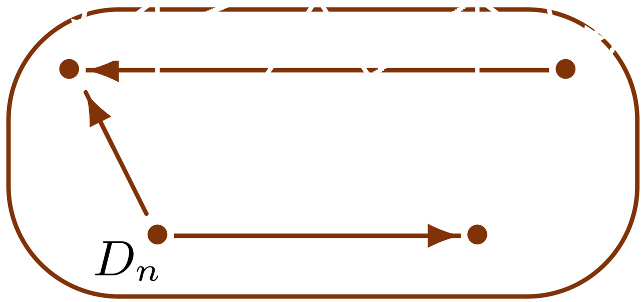
- For any diagram D, define the *category of cones over* D, $\mathbf{Cone}(D)$:



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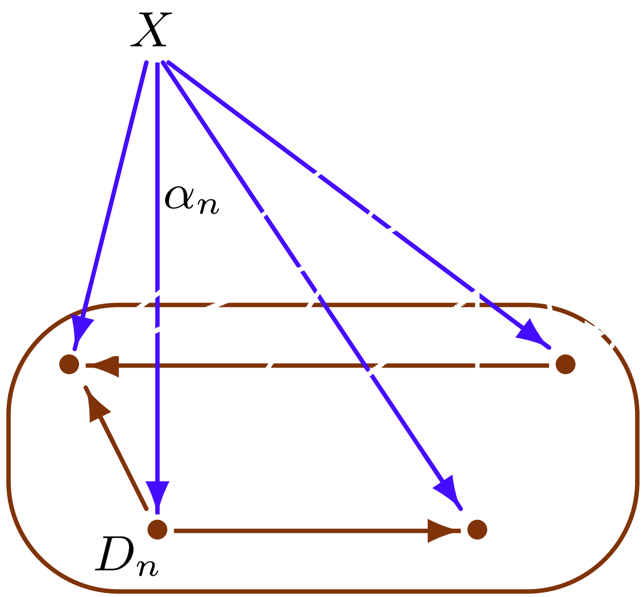

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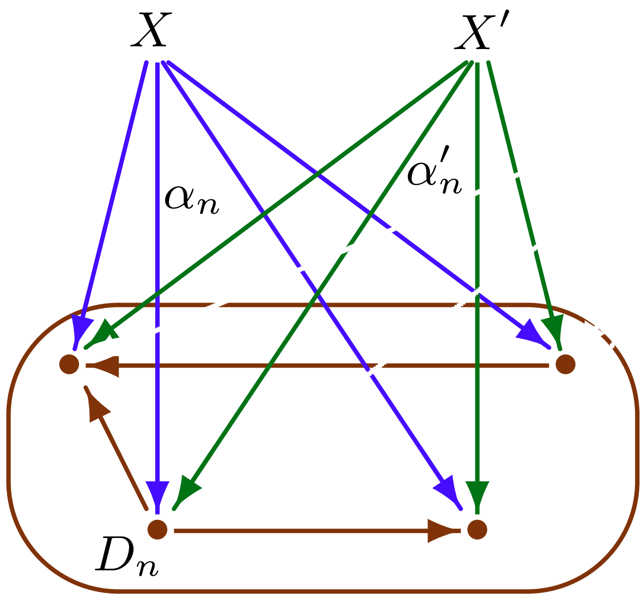

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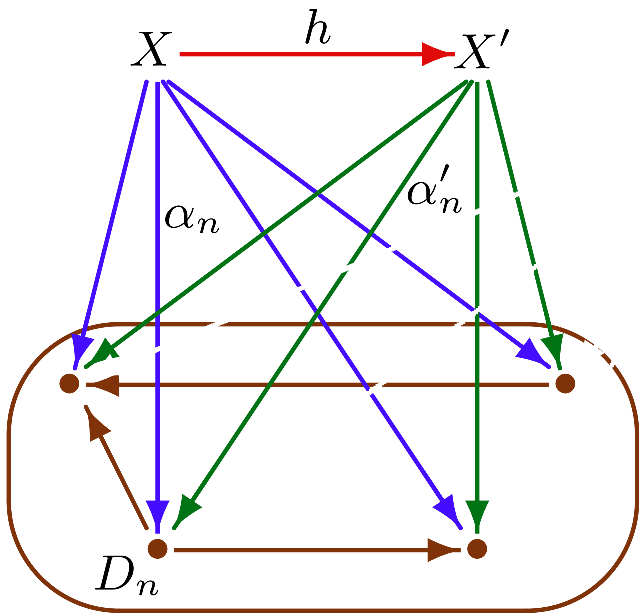

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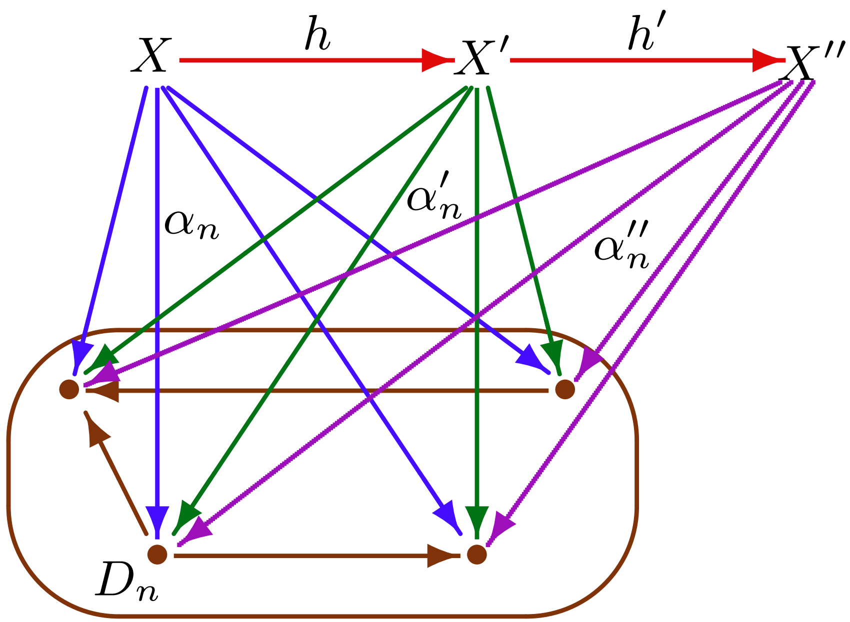

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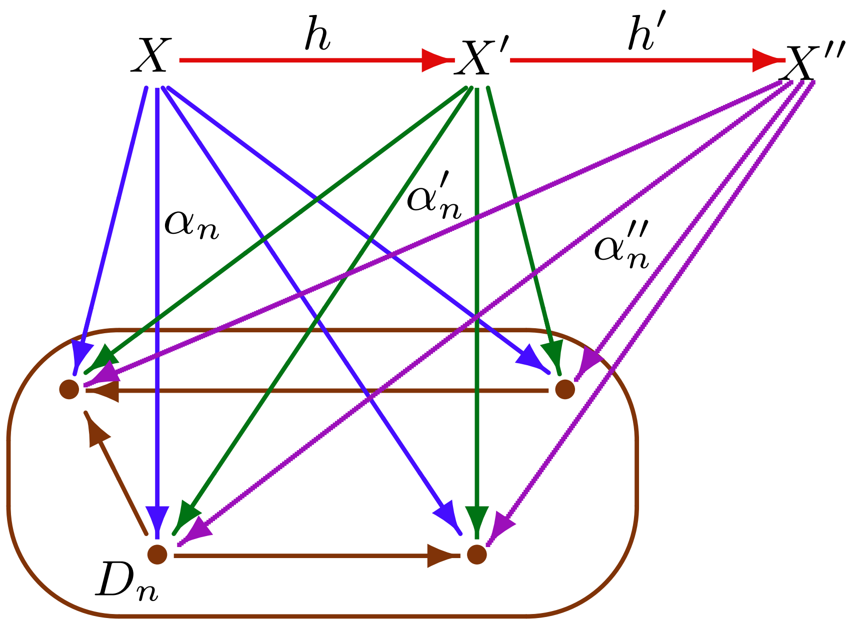

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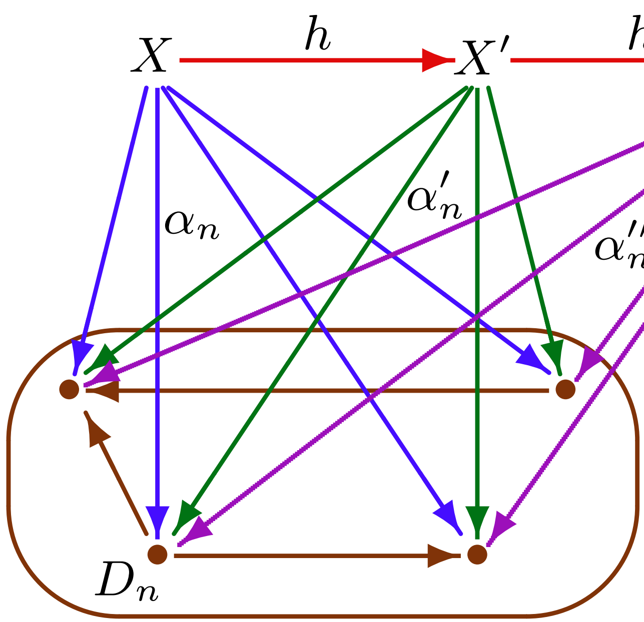

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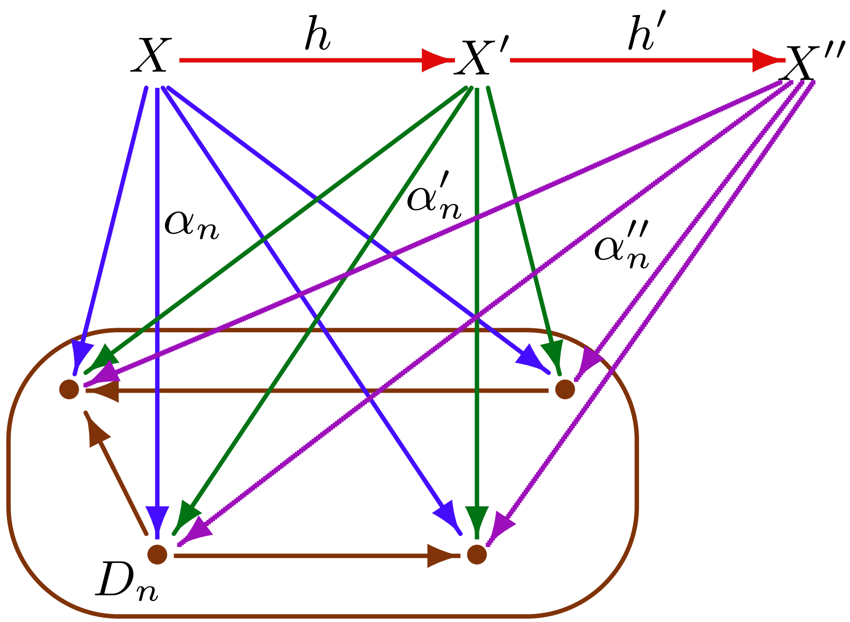

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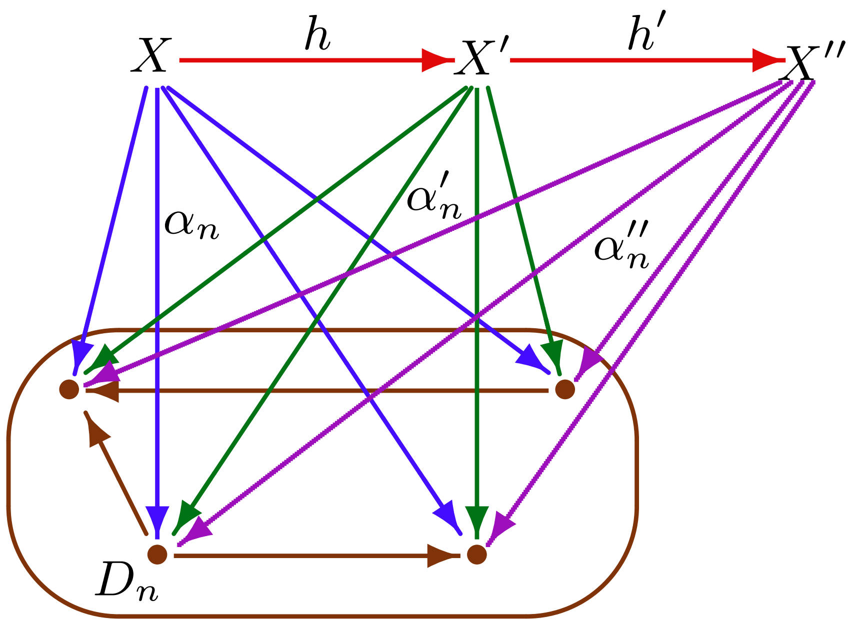

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Hint: $\{\langle a_i \rangle_{i \geq 0} \mid$ for $i \geq 0, a_i \in A_i$ and $f_i(a_{i+1}) = a_i\}$

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Dualise all the exercises above!

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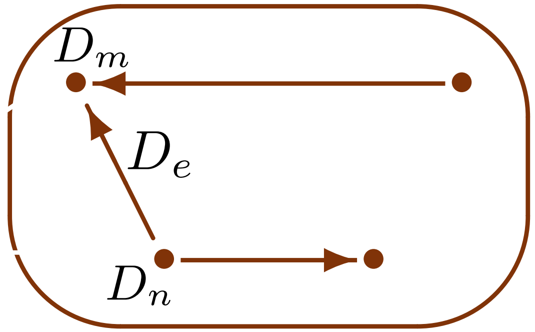

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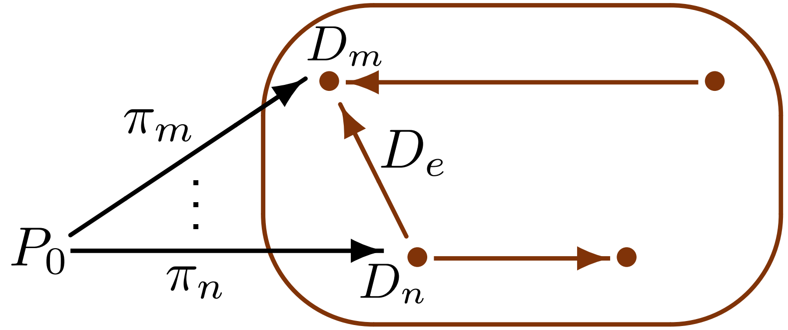

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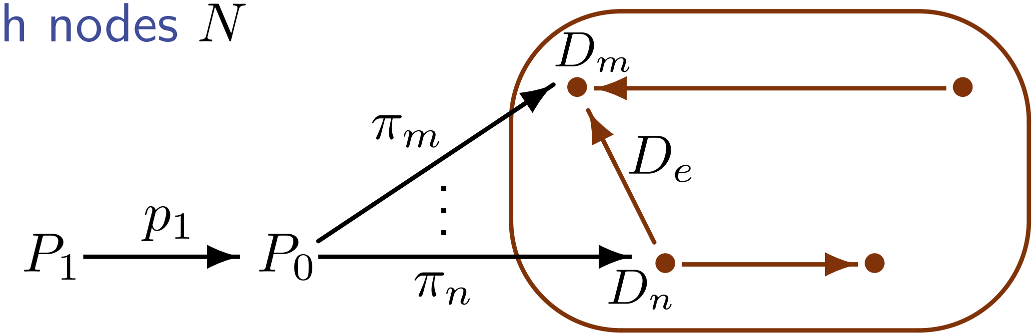

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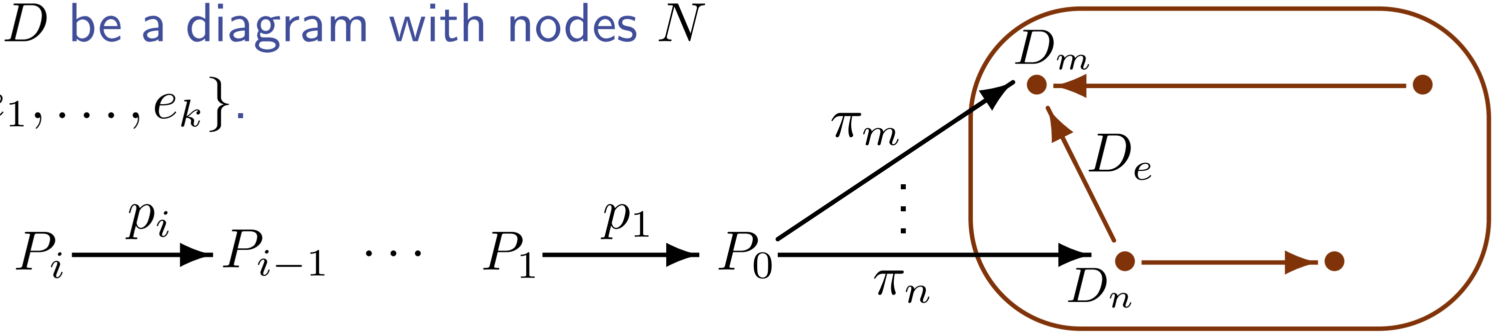

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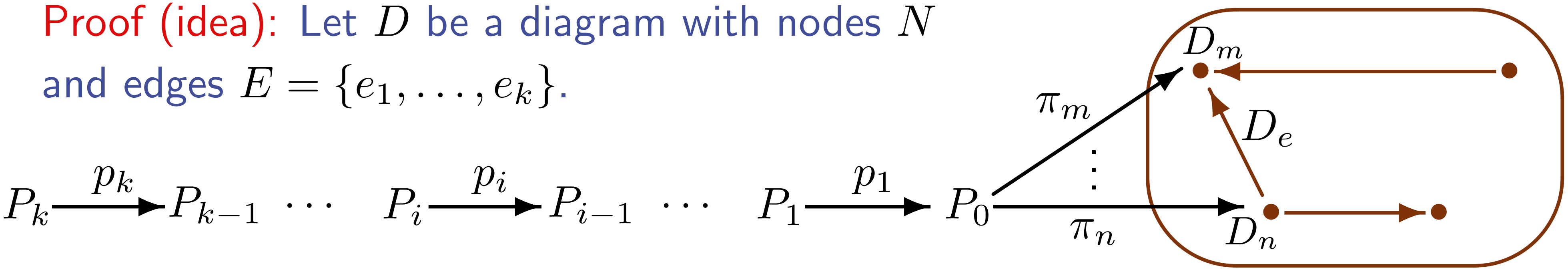

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  – P_k with projections $p_k;\cdots;p_1;\pi_n : P_k \to D_n$, $n \in N$, is the limit of D.

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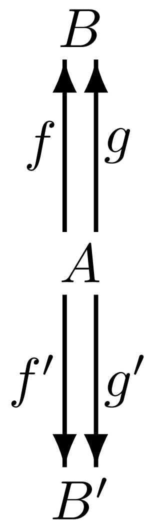

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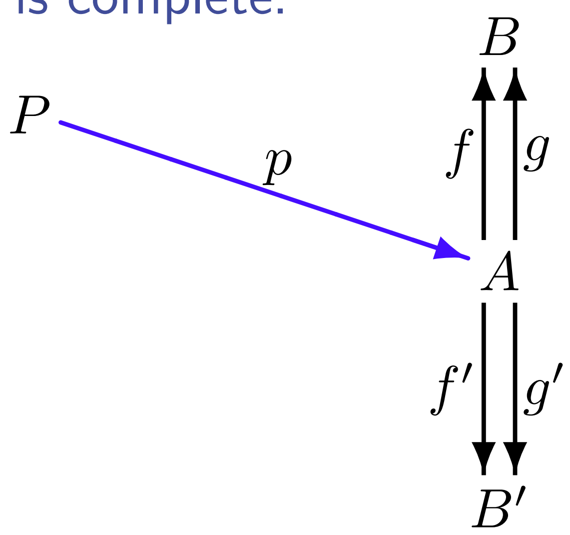

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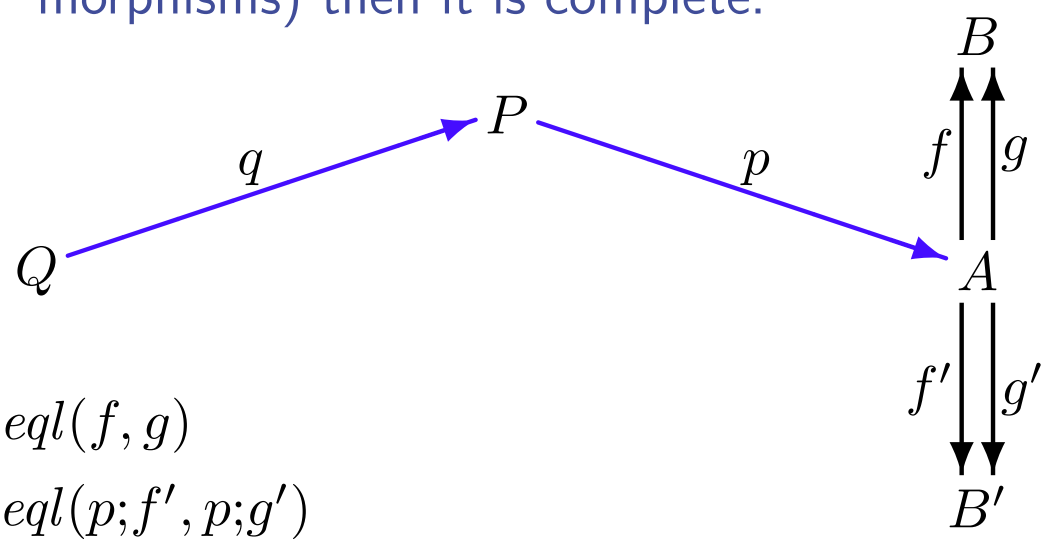

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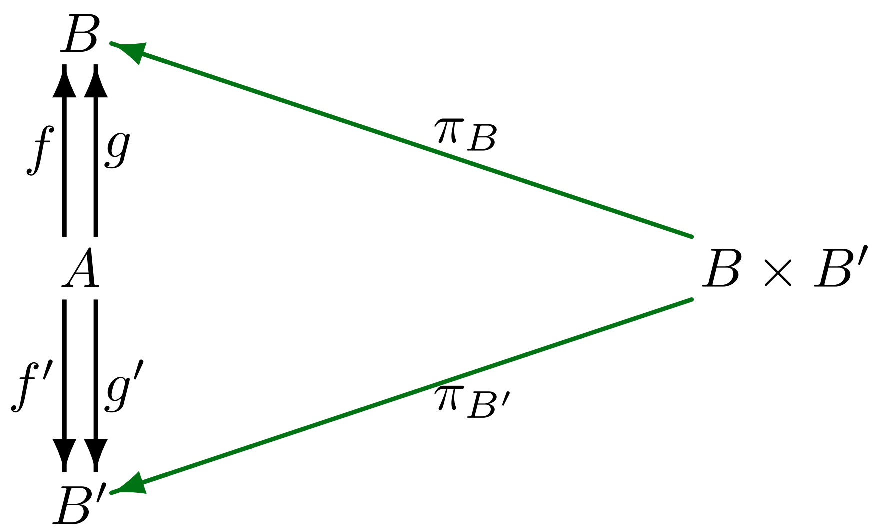

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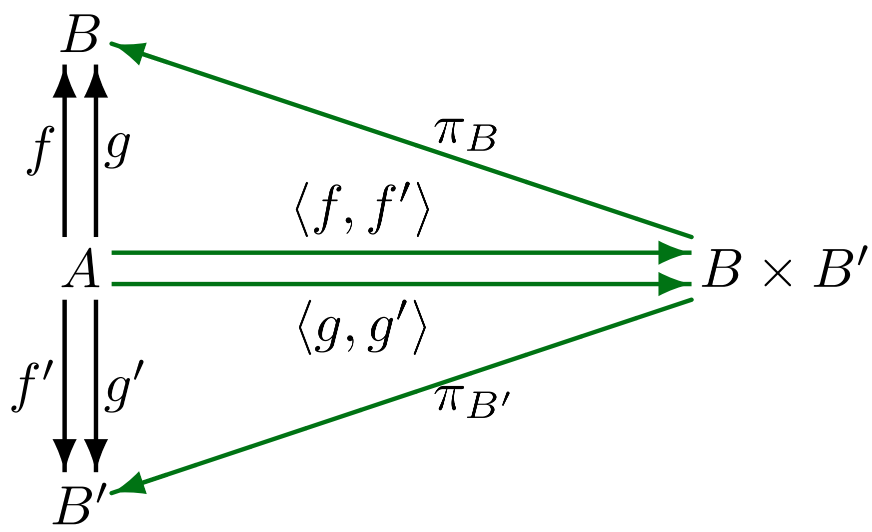

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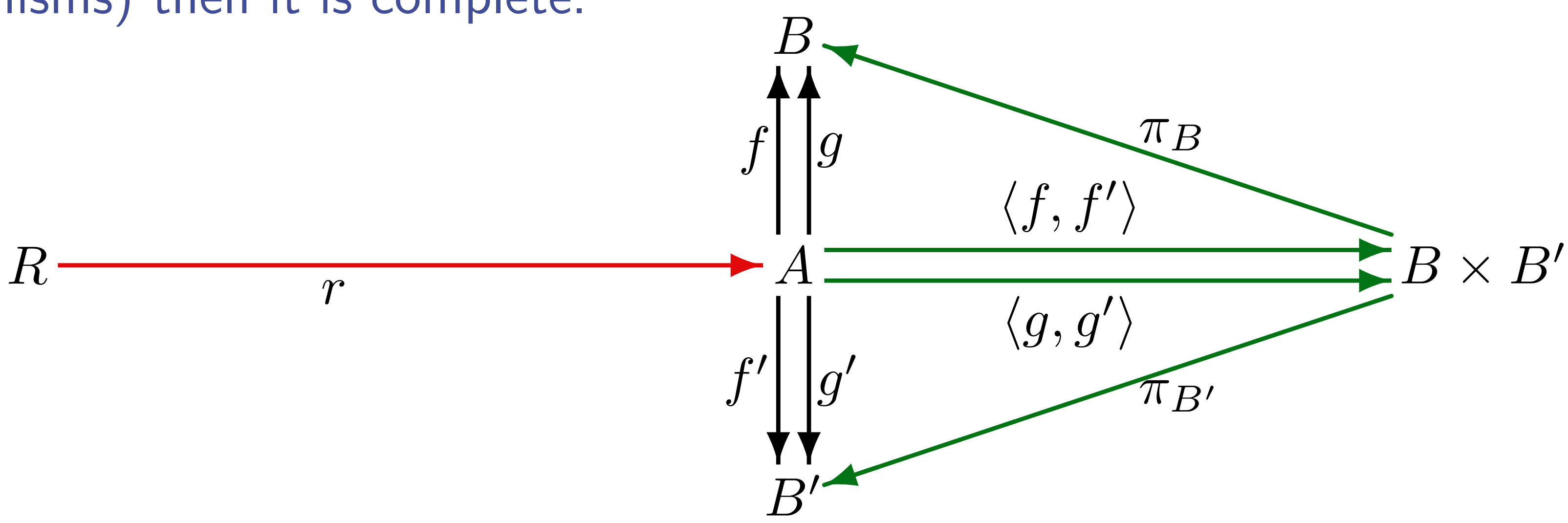

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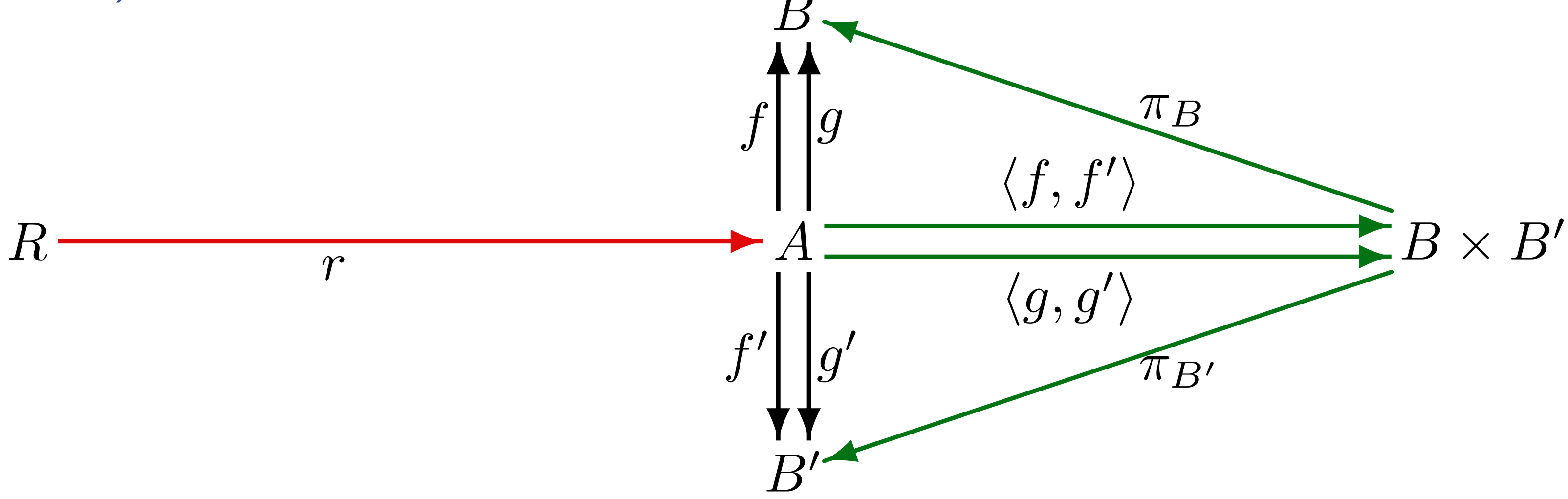

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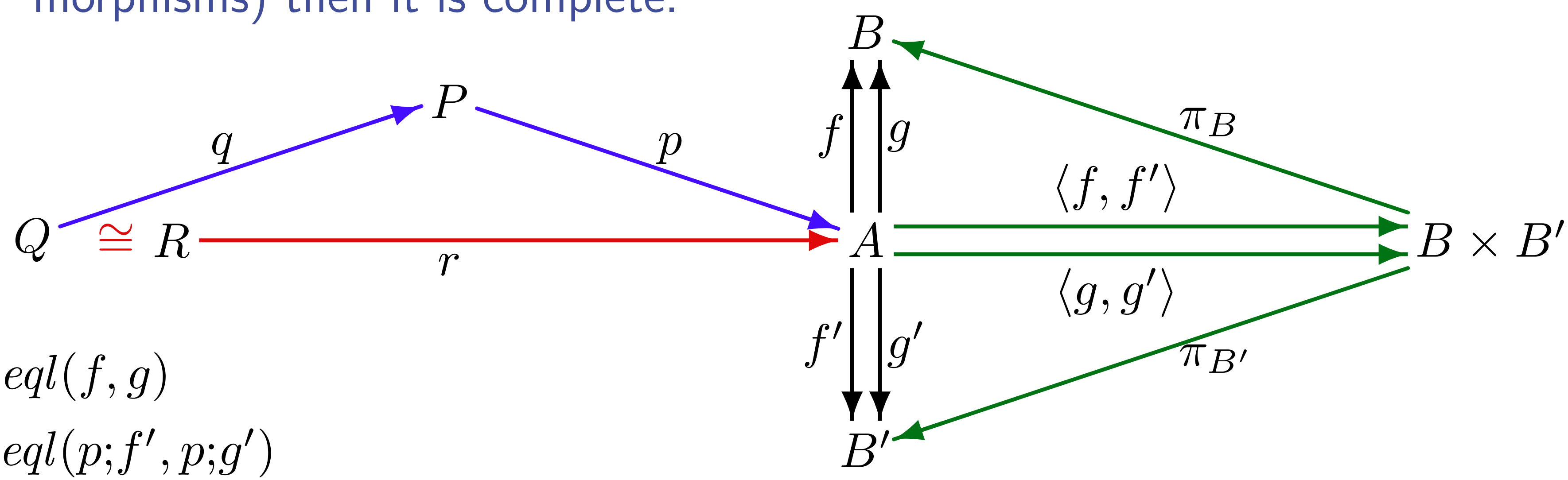

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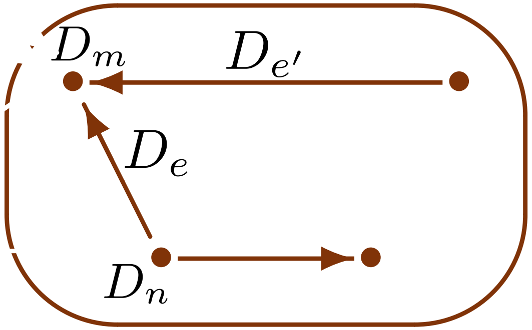

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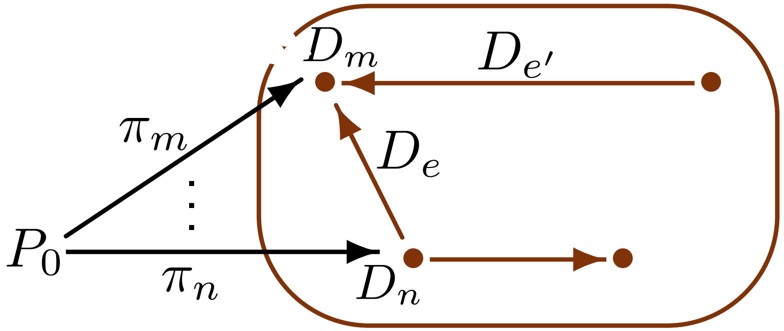

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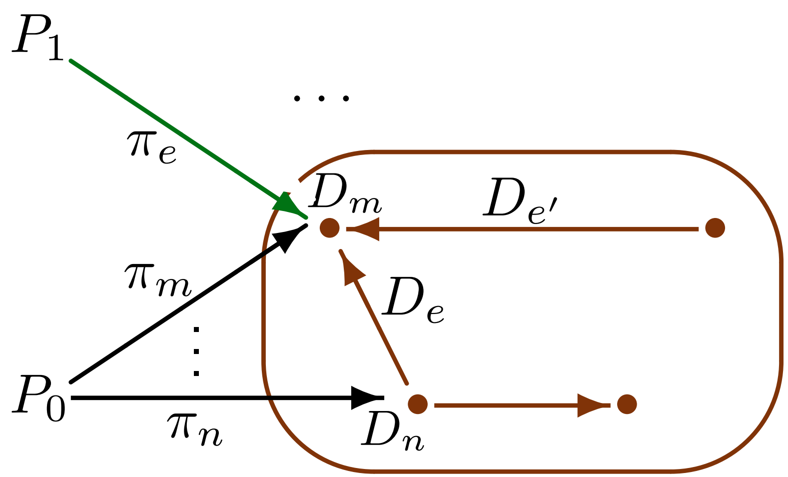

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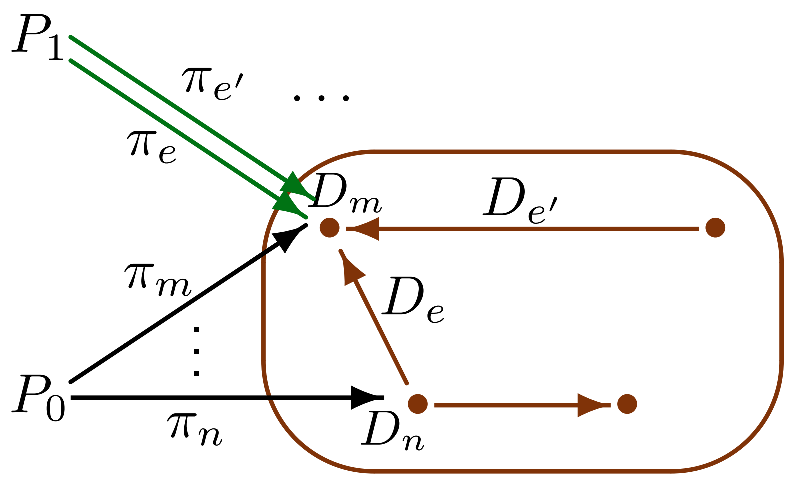

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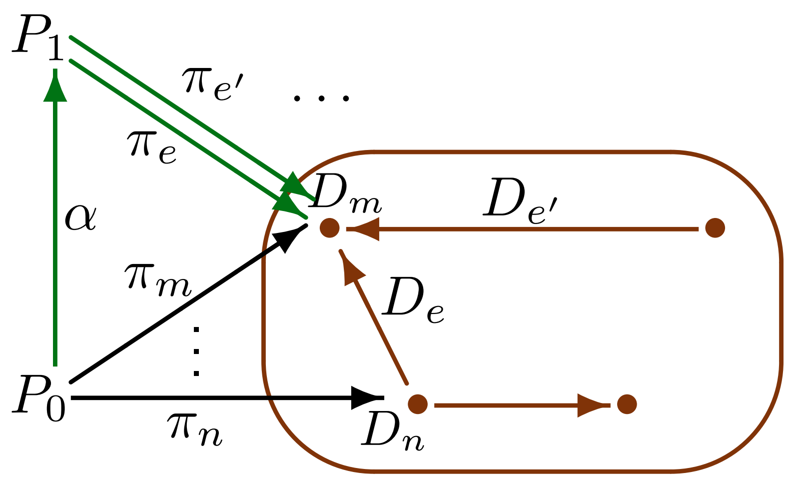

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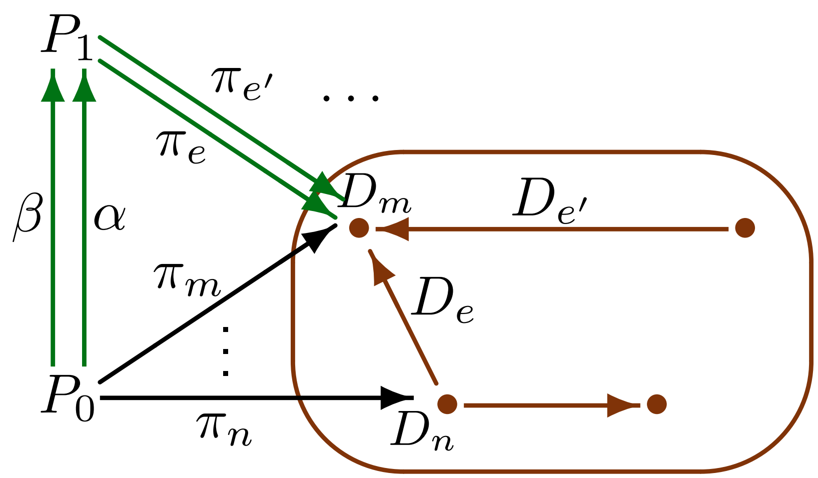

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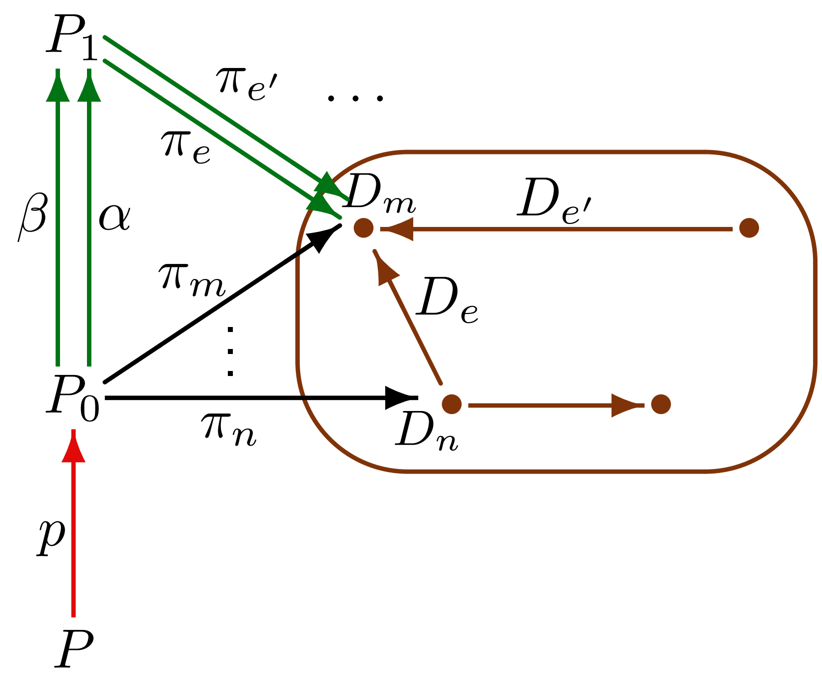

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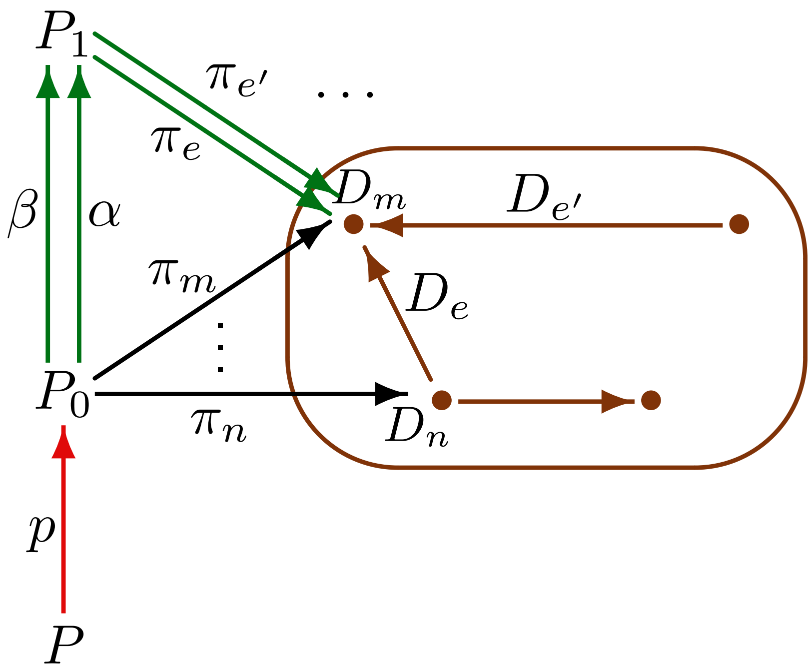

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