Functors

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functors	$\rightsquigarrow$	*category morphisms*
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  for all $f\colon A \to B$ and $g\colon B \to C$ in $\mathbf{K}$.

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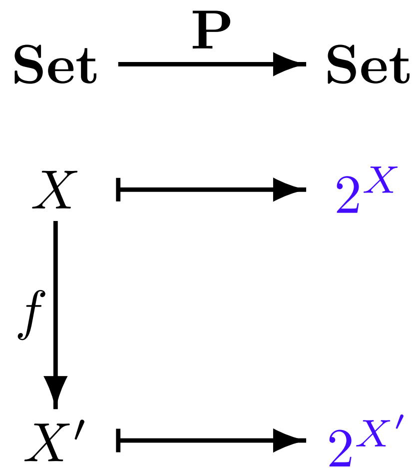

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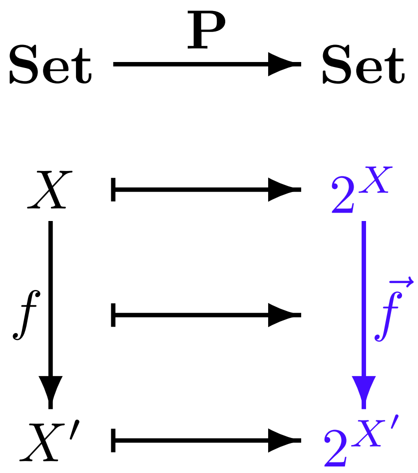

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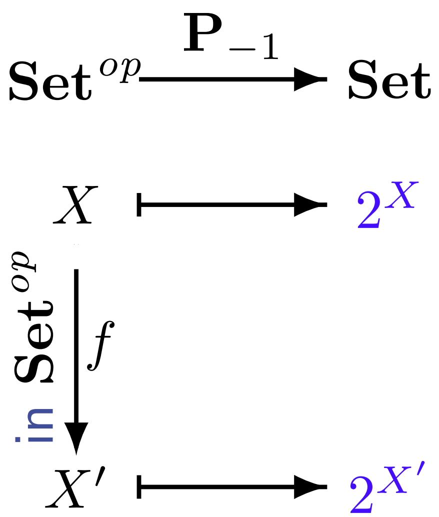

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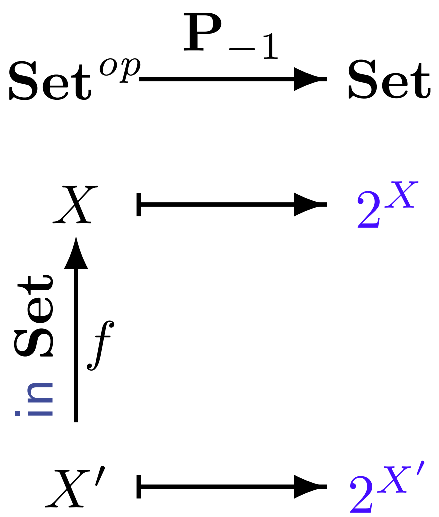

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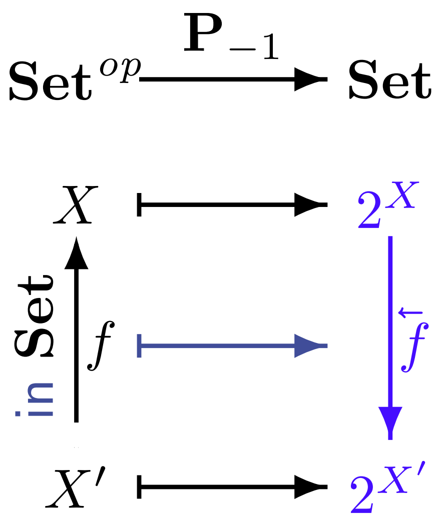

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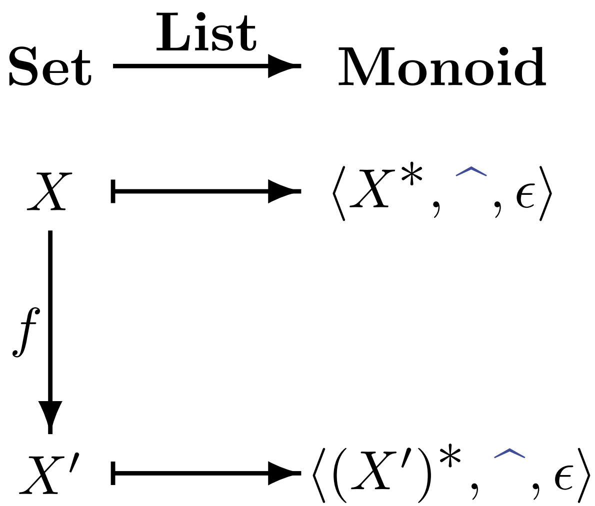

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  - $\mathbf{List}(X) = \langle X^*, {}^\frown, \epsilon \rangle$, for all $X \in |\mathbf{Set}|$, where X^* is the set of all finite lists of elements from X, ${}^\frown$ is the list concatenation, and ϵ is the empty list.
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Define $\mathbf{Set}_*$ as the category of algebras

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  - $\mathbf{Tot}(f)(x) = \begin{cases} f(x) & \text{if it is defined} \\ * & \text{otherwise} \end{cases}$

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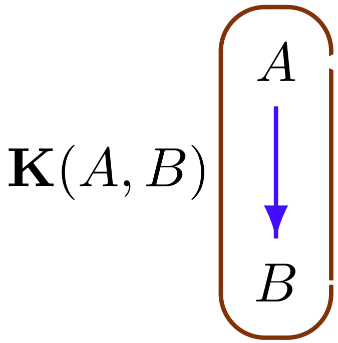

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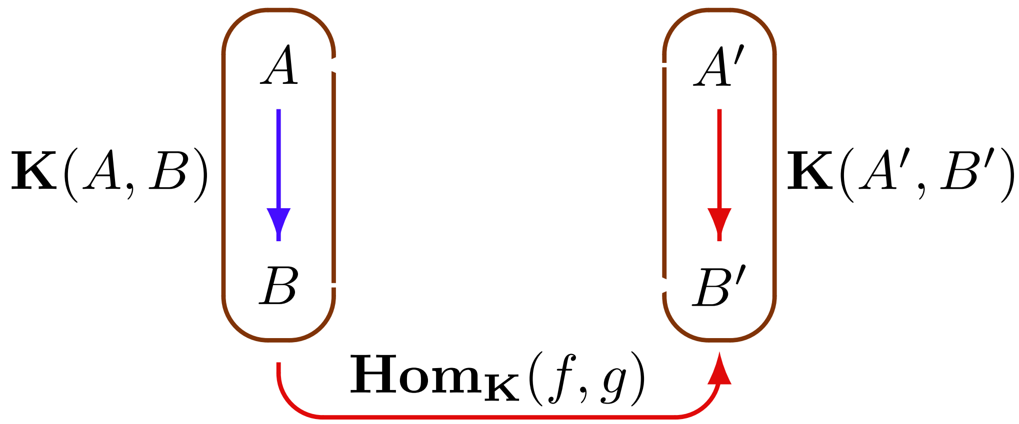

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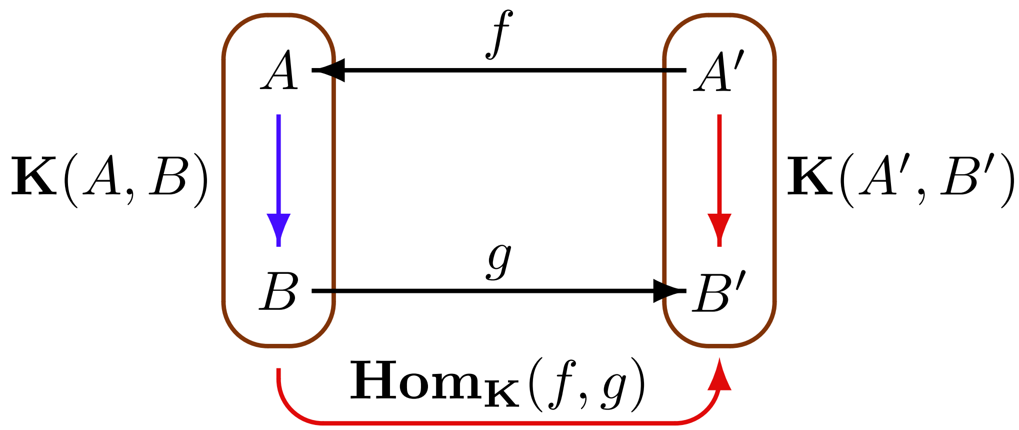

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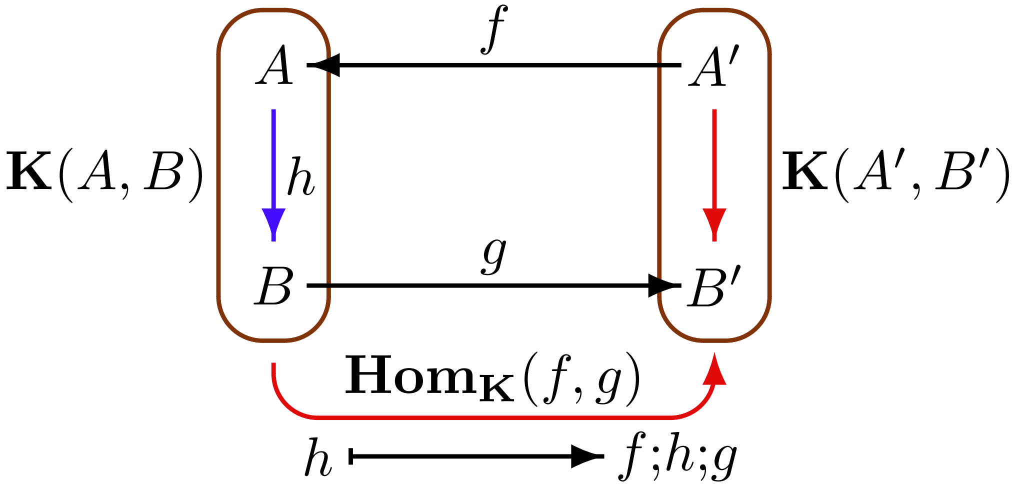

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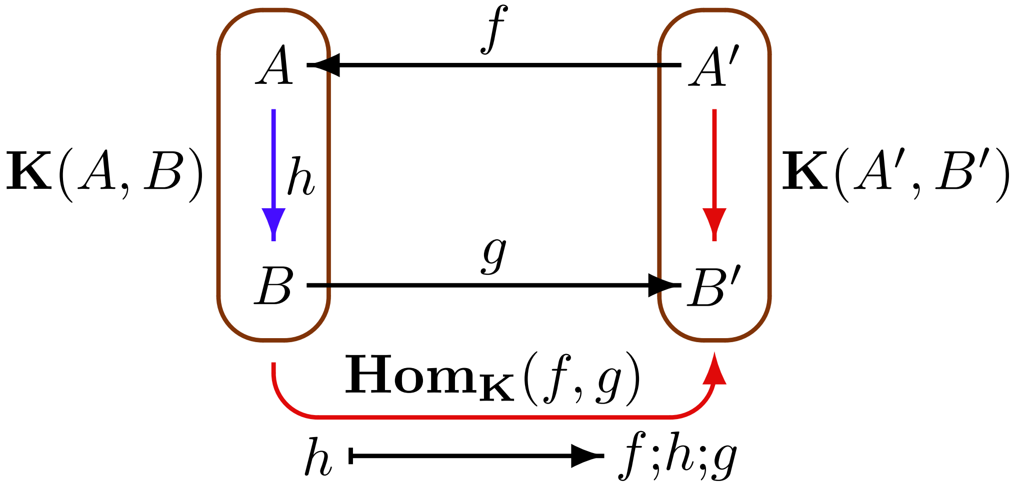

Also: $\mathbf{Hom_K}(A, \_) \colon \mathbf{K} \to \mathbf{Set}$
$\mathbf{Hom_K}(\_, B) \colon \mathbf{K}^{op} \to \mathbf{Set}$

Functors preserve...

- Check whether functors preserve:

Functors preserve. . .

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$\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$

If $f\colon A \to B$ is mono in $\mathbf{K}$ then
$\mathbf{F}(f)\colon \mathbf{F}(A) \to \mathbf{F}(B)$ is mono in $\mathbf{K}'$??

Functors preserve. . .

- Check whether functors preserve:
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  - epimorphisms

$\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$

If $f\colon A \to B$ is epi in $\mathbf{K}$ then
$\mathbf{F}(f)\colon \mathbf{F}(A) \to \mathbf{F}(B)$ is epi in $\mathbf{K}'$??

Functors preserve...

- Check whether functors preserve:
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  - epimorphisms
  - (co)retractions

$$\mathbf{F} \colon \mathbf{K} \to \mathbf{K}'$$

If $f \colon A \to B$ is a retraction in $\mathbf{K}$ then $\mathbf{F}(f) \colon \mathbf{F}(A) \to \mathbf{F}(B)$ is a retraction in $\mathbf{K}'$??

Functors preserve. . .

- Check whether functors preserve:
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$$\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$$

If $f\colon A \to B$ is iso in $\mathbf{K}$ then

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$$\mathbf{F}\colon \mathbf{K} \to \mathbf{K}'$$

If $\alpha\colon X \to D$ is a cone on diagram D in $\mathbf{K}$ then $\mathbf{F}(\alpha)\colon \mathbf{F}(X) \to \mathbf{F}(D)$ is a cone on diagram $\mathbf{F}(D)$ in $\mathbf{K}'$??

BTW:

- $\mathbf{F}(D)$ has the same shape as D, i.e. $\mathcal{G}(\mathbf{F}(D)) = \mathcal{G}(D)$ (with nodes N and edges E)
  - $(\mathbf{F}(D))_n = \mathbf{F}(D_n)$ for $n \in N$
  - $(\mathbf{F}(D))_e = \mathbf{F}(D_e)$ for $e \in E$
- $\mathbf{F}(\alpha) = \langle \mathbf{F}(\alpha_n)\colon \mathbf{F}(X) \to (\mathbf{F}(D))_n \rangle_{n \in N}$

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Dualise!

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Characterise isomorphisms in **Cat**

Define products, terminal objects, equalisers and pullback in **Cat**

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Cat, *the category of (sm)all categories*

- objects: (sm)all categories
- morphisms: functors between them
- composition: as above

Characterise isomorphisms in **Cat**

Define products, terminal objects, equalisers and pullback in **Cat**

Try to define their duals

Comma categories

Given two functors with a common target, $\mathbf{F}\colon \mathbf{K1} \to \mathbf{K}$ and $\mathbf{G}\colon \mathbf{K2} \to \mathbf{K}$, define their *comma category*

$$\boxed{(\mathbf{F}, \mathbf{G})}$$

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$\mathbf{K1}$: A_1 $\quad$ $\mathbf{K}$: $\mathbf{F}(A_1) \xrightarrow{f} \mathbf{G}(A_2)$ $\quad$ $\mathbf{K2}$: A_2

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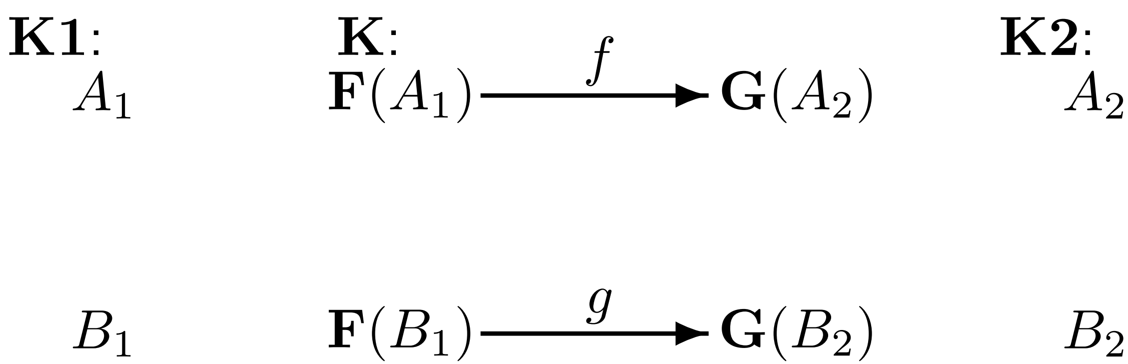

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$$\begin{array}{ccccc}
\mathbf{K1}: & \mathbf{K}: & & & \mathbf{K2}: \\
A_1 & \mathbf{F}(A_1) & \xrightarrow{f} & \mathbf{G}(A_2) & A_2 \\
\downarrow h_1 & & & & \downarrow h_2 \\
B_1 & \mathbf{F}(B_1) & \xrightarrow{g} & \mathbf{G}(B_2) & B_2
\end{array}$$

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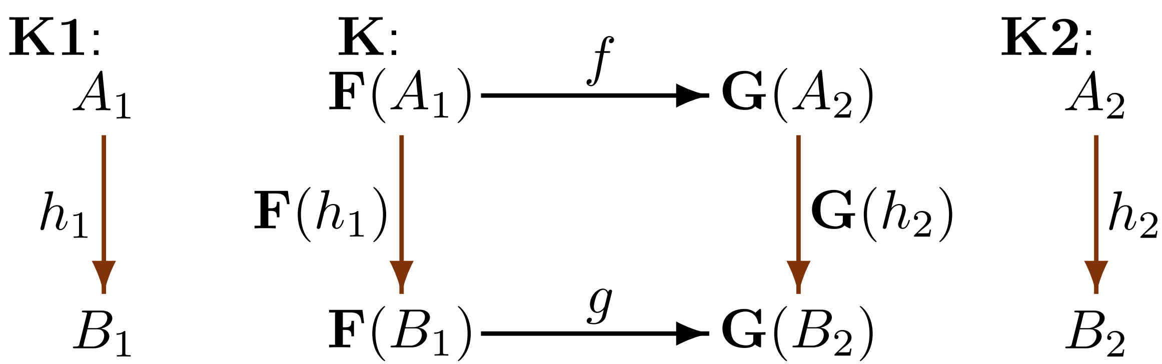

Comma categories

– composition:

$$\mathbf{K1}: \qquad \mathbf{K}: \qquad \qquad \mathbf{K2}:$$

$$A_1 \qquad \mathbf{F}(A_1) \xrightarrow{f} \mathbf{G}(A_2) \qquad A_2$$

Comma categories

– composition:

$\mathbf{K1}$: $\mathbf{K}$: $\mathbf{K2}$:

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$$A_1' \qquad \mathbf{F}(A_1') \xrightarrow{f'} \mathbf{G}(A_2') \qquad A_2'$$

Comma categories

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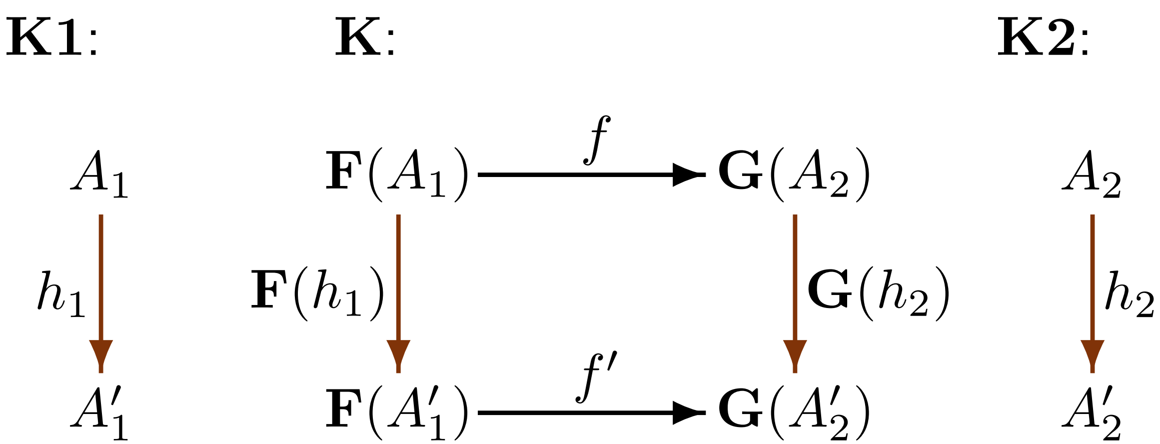

Comma categories

– composition:

$$
\begin{array}{ccccc}
\mathbf{K1}: & \mathbf{K}: & & & \mathbf{K2}: \\
A_1 & \mathbf{F}(A_1) & \xrightarrow{\;f\;} & \mathbf{G}(A_2) & A_2 \\
h_1 \downarrow & \mathbf{F}(h_1) \downarrow & & \downarrow \mathbf{G}(h_2) & \downarrow h_2 \\
A_1' & \mathbf{F}(A_1') & \xrightarrow{\;f'\;} & \mathbf{G}(A_2') & A_2' \\
 & & & & \\
A_1'' & \mathbf{F}(A_1'') & \xrightarrow{\;f''\;} & \mathbf{G}(A_2'') & A_2''
\end{array}
$$

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$\mathbf{K1}$: $\mathbf{K}$: $\mathbf{K2}$:

$$
\begin{array}{ccccccc}
A_1 & \quad & \mathbf{F}(A_1) & \xrightarrow{f} & \mathbf{G}(A_2) & \quad & A_2 \\
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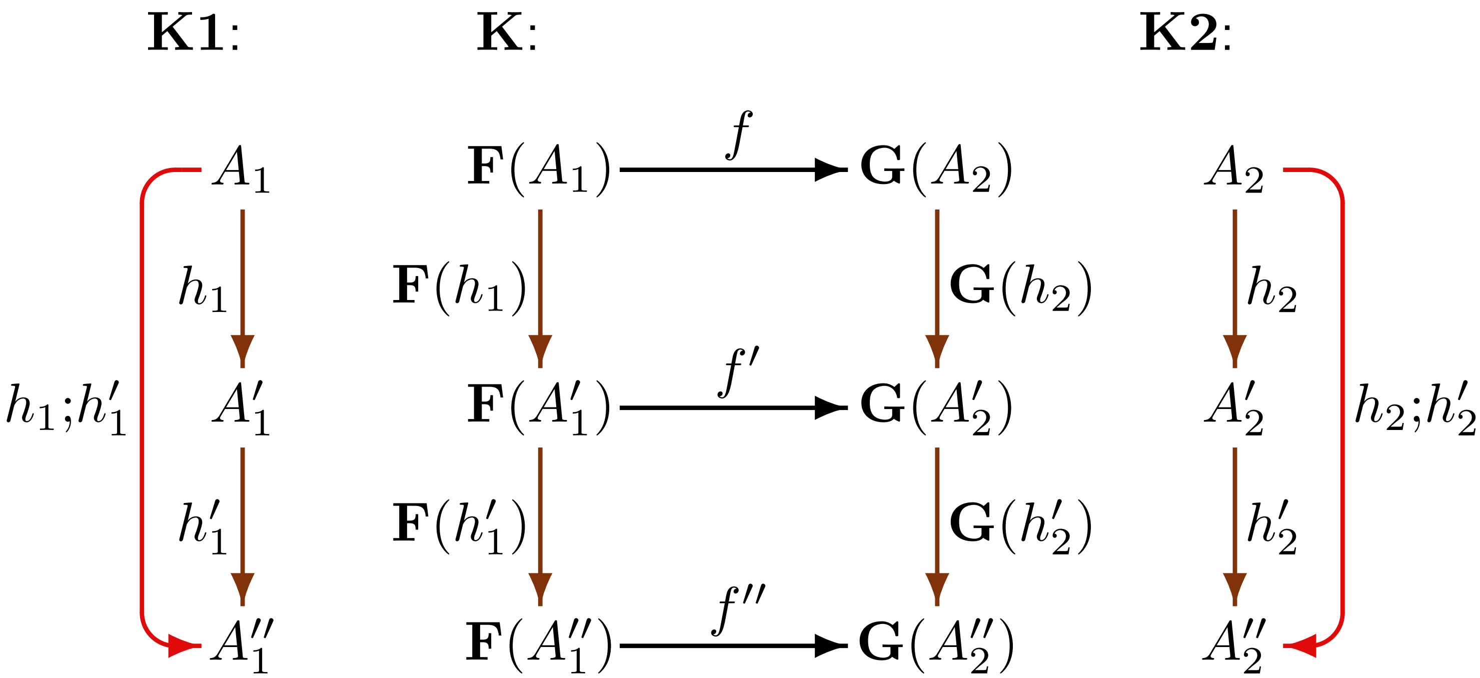
A_1' & & \mathbf{F}(A_1') & \xrightarrow{f'} & \mathbf{G}(A_2') & & A_2' \\
{\scriptstyle h_1'}\downarrow & & {\scriptstyle \mathbf{F}(h_1')}\downarrow & & \downarrow{\scriptstyle \mathbf{G}(h_2')} & & \downarrow{\scriptstyle h_2'} \\
A_1'' & & \mathbf{F}(A_1'') & \xrightarrow{f''} & \mathbf{G}(A_2'') & & A_2''
\end{array}
$$

Comma categories

— composition: component-wise



$$\langle h_1, h_2\rangle;\langle h'_1, h'_2\rangle = \langle h_1;h'_1, h_2;h'_2\rangle$$

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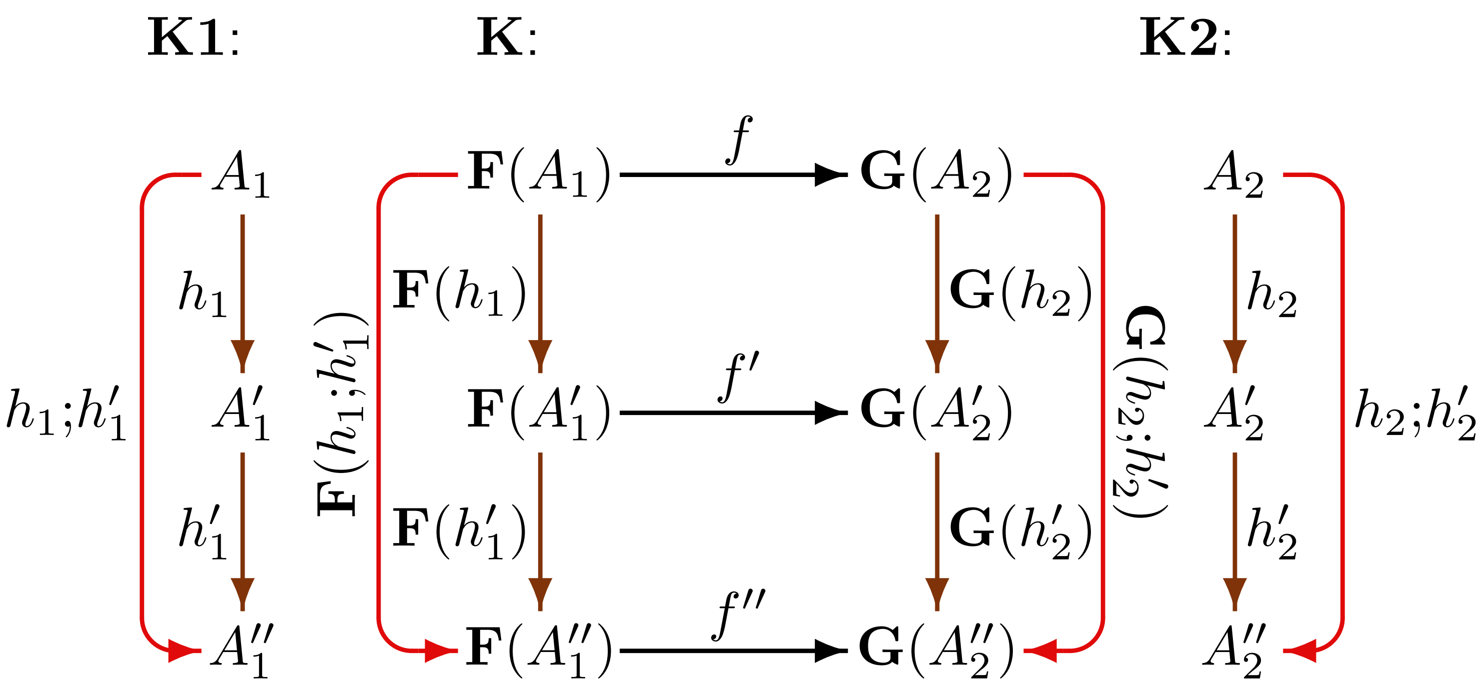

$$\langle h_1, h_2\rangle;\langle h'_1, h'_2\rangle = \langle h_1;h'_1, h_2;h'_2\rangle$$

$$\mathbf{F}(h_1;h'_1);f'' = \mathbf{F}(h_1);\mathbf{F}(h'_1);f'' = \mathbf{F}(h_1);f';\mathbf{G}(h'_2) = f;\mathbf{G}(h_2);\mathbf{G}(h'_2) = f;\mathbf{G}(h_2;h'_2)$$

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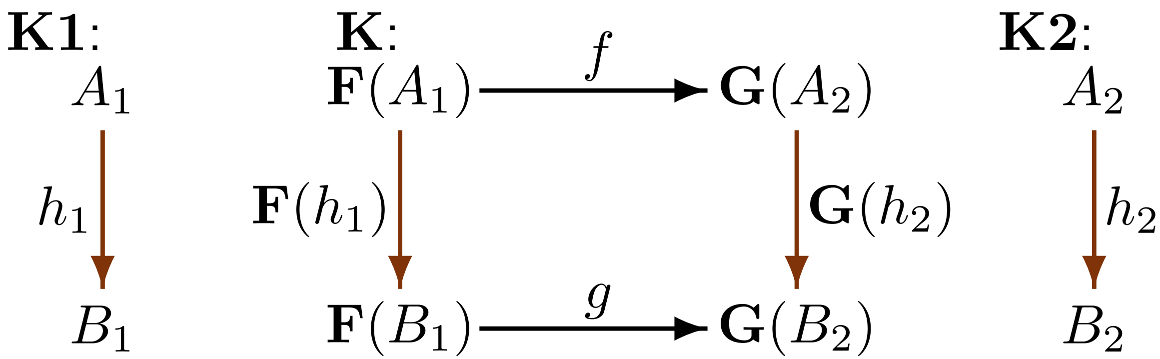

Examples

Examples

- The category of graphs as a comma category:

$$\boxed{\mathbf{Graph} = (\mathbf{Id}_{\mathbf{Set}}, \mathbf{CP})}$$

where $\mathbf{CP} \colon \mathbf{Set} \to \mathbf{Set}$ is the (Cartesian) product functor, i.e. $\mathbf{CP}(X) = X \times X$ and $\mathbf{CP}(f)(\langle x, x' \rangle) = \langle f(x), f(x') \rangle$.

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Theorem: *If* $\mathbf{K1}$ *and* $\mathbf{K2}$ *are (finitely) cocomplete categories,* $\mathbf{F}\colon \mathbf{K1} \to \mathbf{K}$ *is a (finitely) cocontinuous functor, and* $\mathbf{G}\colon \mathbf{K2} \to \mathbf{K}$ *is a functor then the comma category* $(\mathbf{F}, \mathbf{G})$ *is (finitely) cocomplete.*

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> *State and prove the dual fact,*
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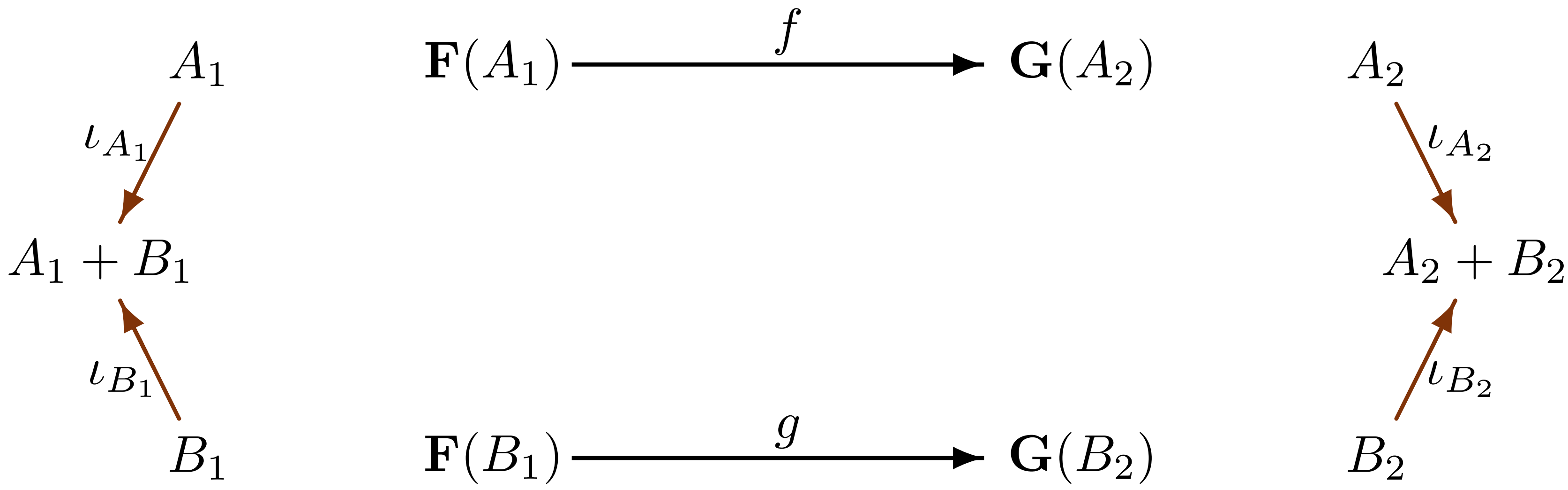
Theorem: *If* $\mathbf{K1}$ *and* $\mathbf{K2}$ *are (finitely) complete categories,* $\mathbf{F}\colon \mathbf{K1} \to \mathbf{K}$ *is a functor, and* $\mathbf{G}\colon \mathbf{K2} \to \mathbf{K}$ *is a (finitely) continuous functor then the comma category* $(\mathbf{F}, \mathbf{G})$ *is (finitely) complete.*

Coproducts:

$$A_1 \qquad \mathbf{F}(A_1) \xrightarrow{\;f\;} \mathbf{G}(A_2) \qquad A_2$$

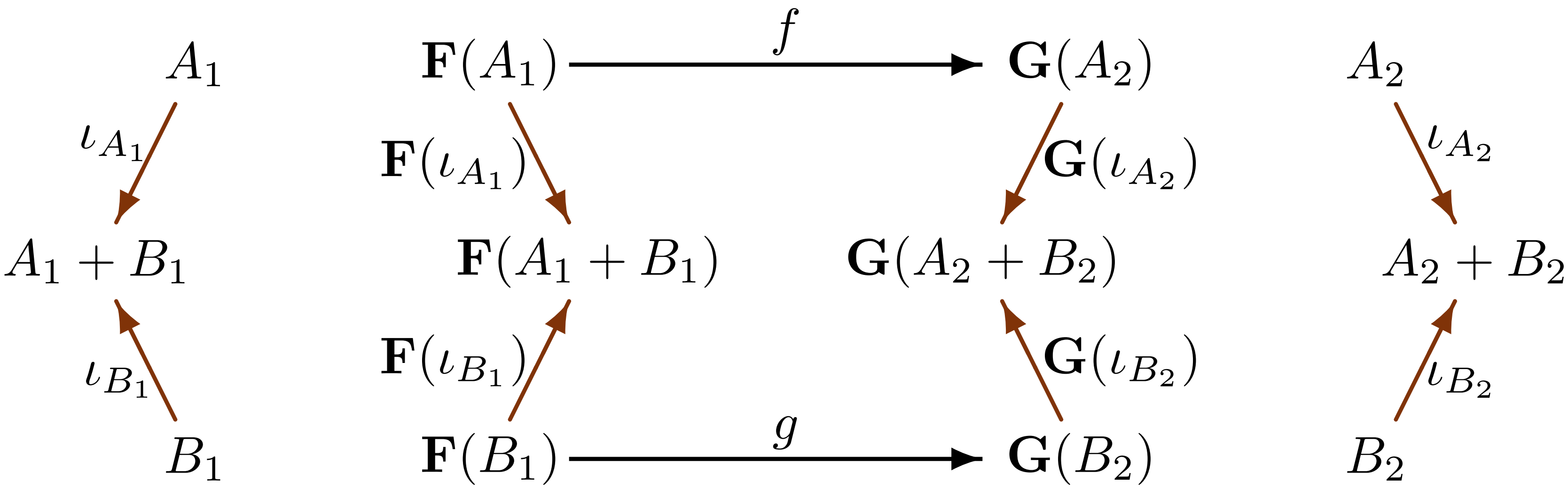
$$B_1 \qquad \mathbf{F}(B_1) \xrightarrow{\;g\;} \mathbf{G}(B_2) \qquad B_2$$

Coproducts:



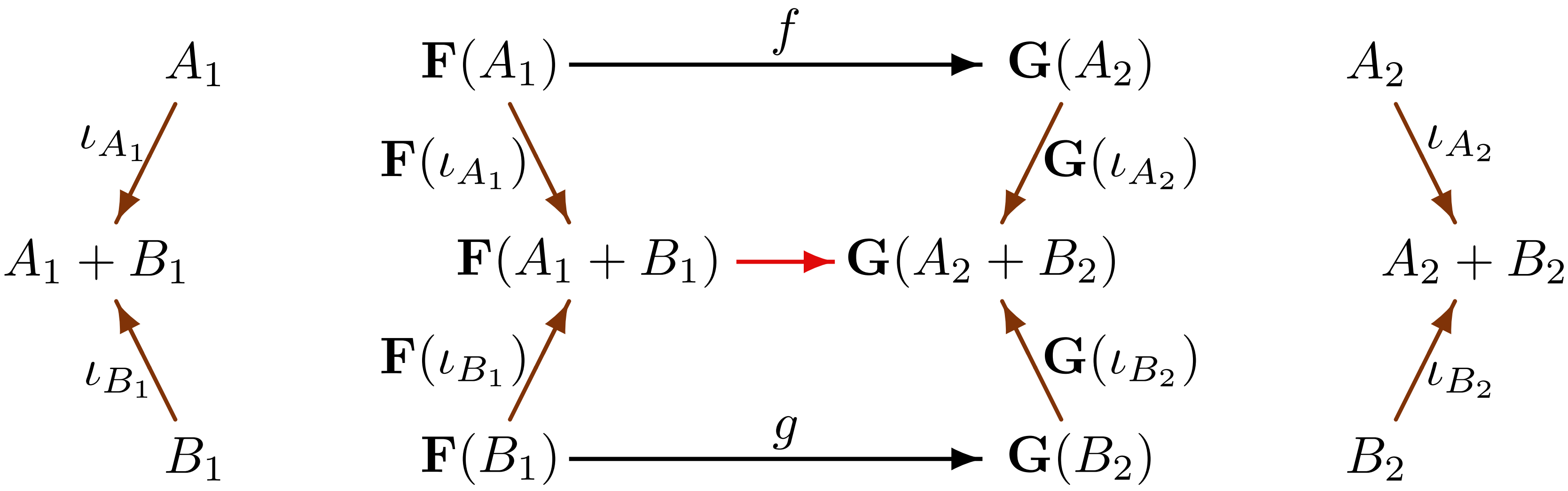
A_1
ι_{A_1}
$A_1 + B_1$
ι_{B_1}
B_1
$\mathbf{F}(A_1)$
f
$\mathbf{G}(A_2)$
$\mathbf{F}(B_1)$
g
$\mathbf{G}(B_2)$
A_2
ι_{A_2}
$A_2 + B_2$
ι_{B_2}
B_2

Coproducts:



A_1
ι_{A_1}
$A_1 + B_1$
ι_{B_1}
B_1
$\mathbf{F}(A_1)$
f
$\mathbf{G}(A_2)$
$\mathbf{F}(\iota_{A_1})$
$\mathbf{G}(\iota_{A_2})$
$\mathbf{F}(A_1 + B_1)$
$\mathbf{G}(A_2 + B_2)$
$\mathbf{F}(\iota_{B_1})$
$\mathbf{G}(\iota_{B_2})$
$\mathbf{F}(B_1)$
g
$\mathbf{G}(B_2)$
A_2
ι_{A_2}
$A_2 + B_2$
ι_{B_2}
B_2

Coproducts:


A_1
ι_{A_1}
$A_1 + B_1$
ι_{B_1}
B_1
$\mathbf{F}(A_1)$
f
$\mathbf{G}(A_2)$
$\mathbf{F}(\iota_{A_1})$
$\mathbf{G}(\iota_{A_2})$
$\mathbf{F}(A_1 + B_1)$
$\mathbf{G}(A_2 + B_2)$
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A_2
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$A_2 + B_2$
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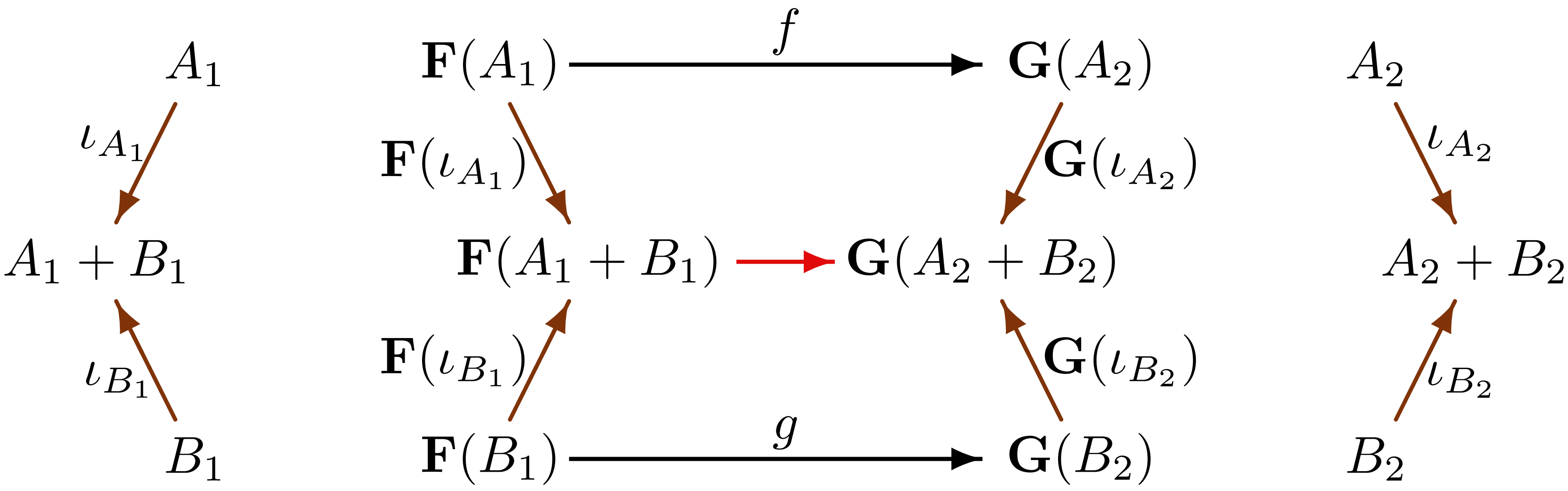
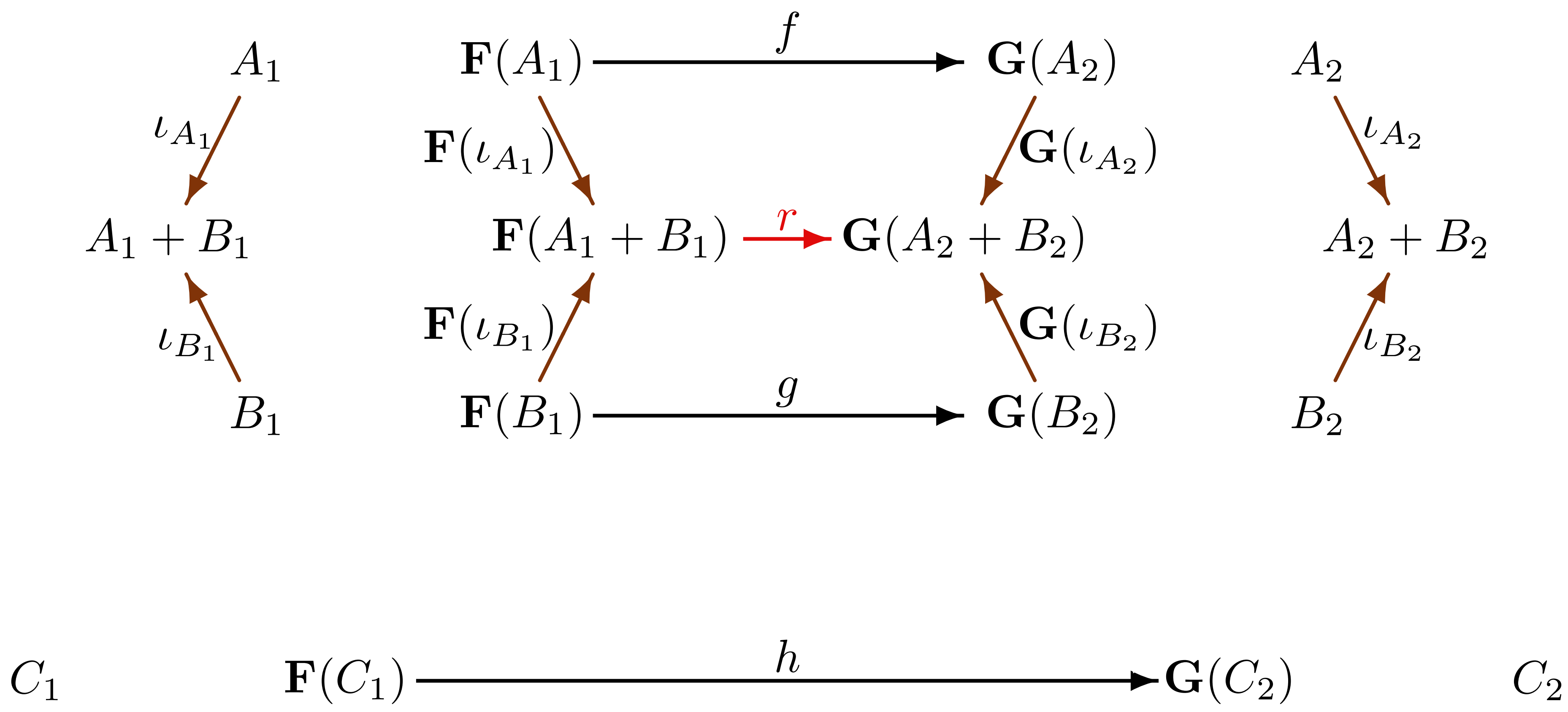


Fact: $\langle A_1 + B_1, [f;\mathbf{G}(\iota_{A_2}), g;\mathbf{G}(\iota_{B_2})] \colon \mathbf{F}(A_1 + B_1) \to \mathbf{G}(A_2 + B_2), A_2 + B_2\rangle$ *with injections* $\langle \iota_{A_1}, \iota_{A_2}\rangle$ *and* $\langle \iota_{B_1}, \iota_{B_2}\rangle$ *is a coproduct of* $\langle A_1, f \colon \mathbf{F}(A_1) \to \mathbf{G}(A_2), A_2\rangle$ *and* $\langle B_1, g \colon \mathbf{F}(B_1) \to \mathbf{G}(B_2), B_2\rangle$ *in* $(\mathbf{F}, \mathbf{G})$.

Coproducts:



where $r = [f;\mathbf{G}(\iota_{A_2}), g;\mathbf{G}(\iota_{B_2})]$,

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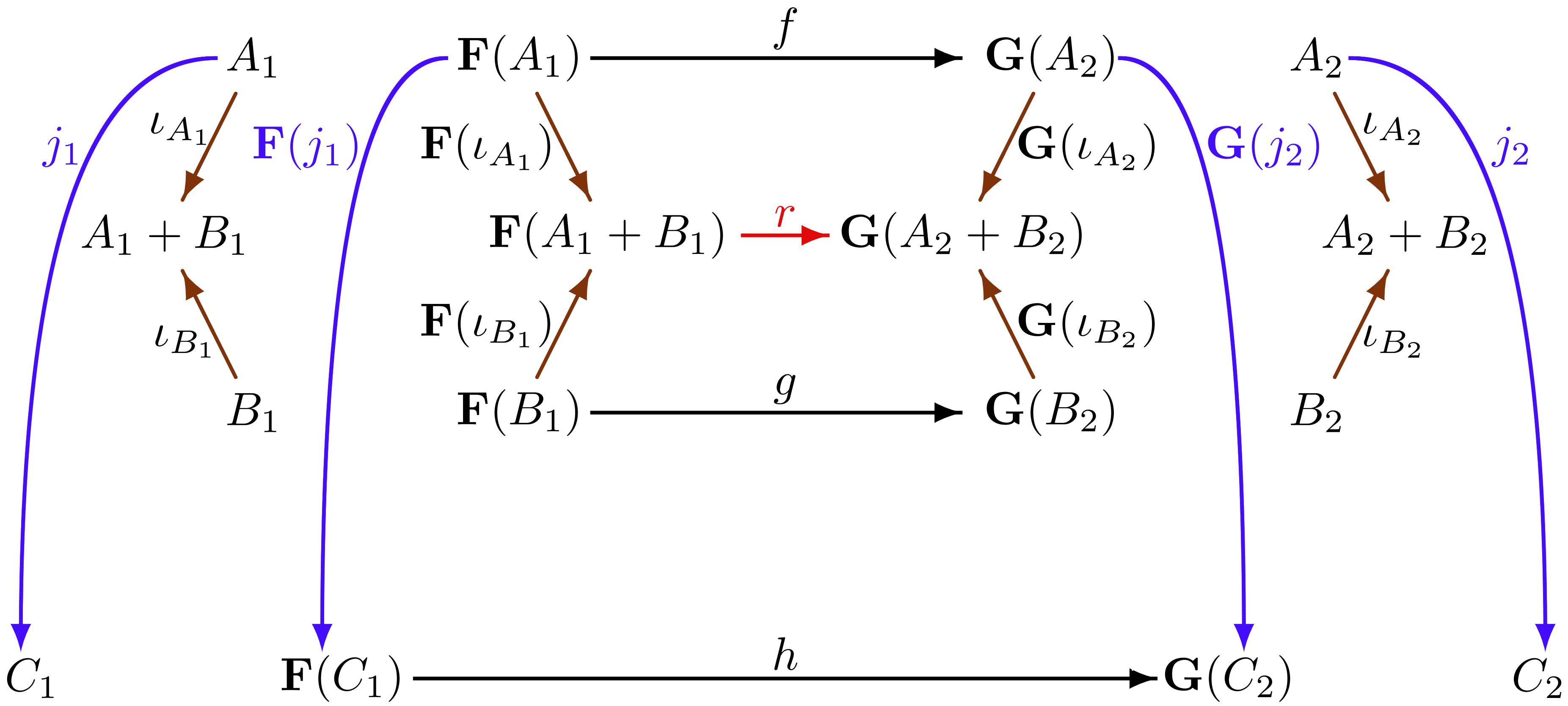

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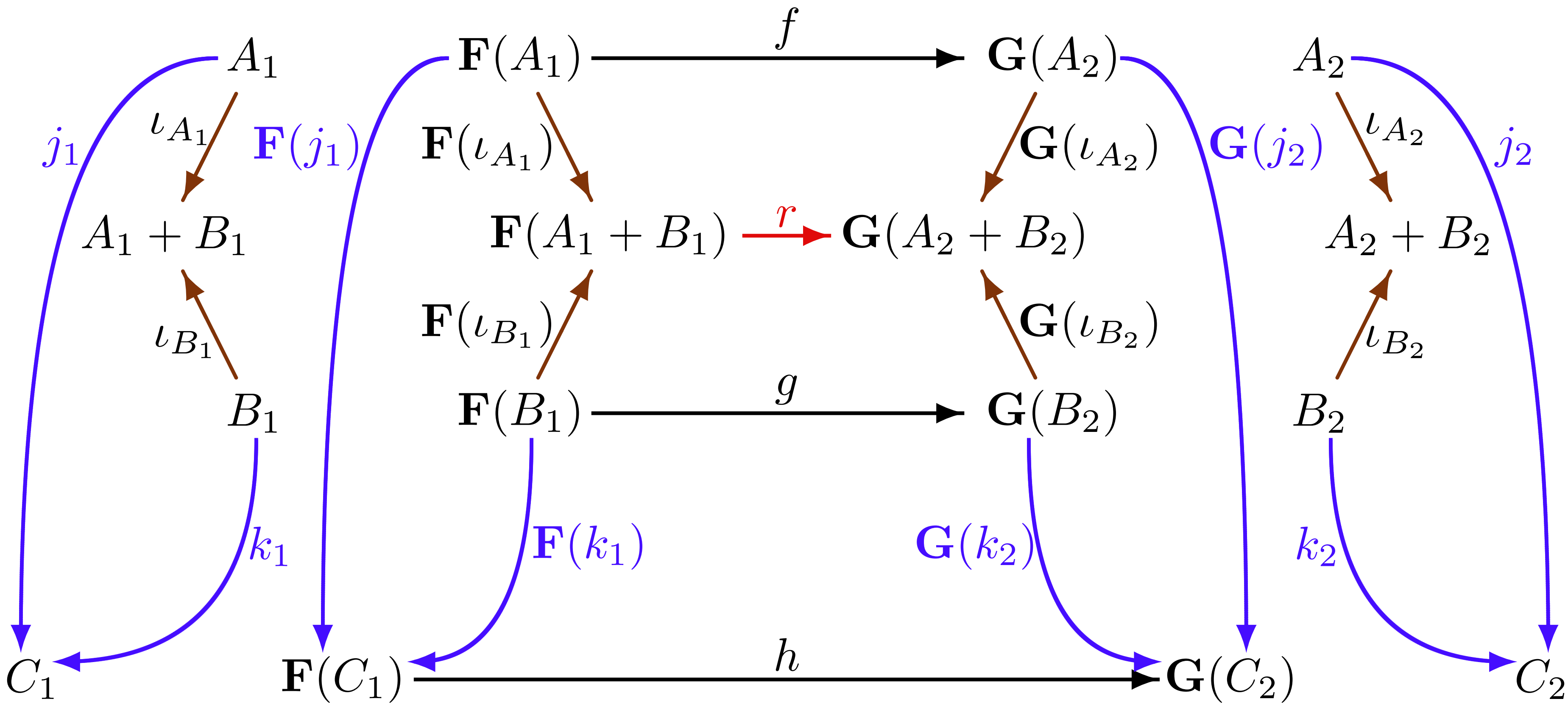

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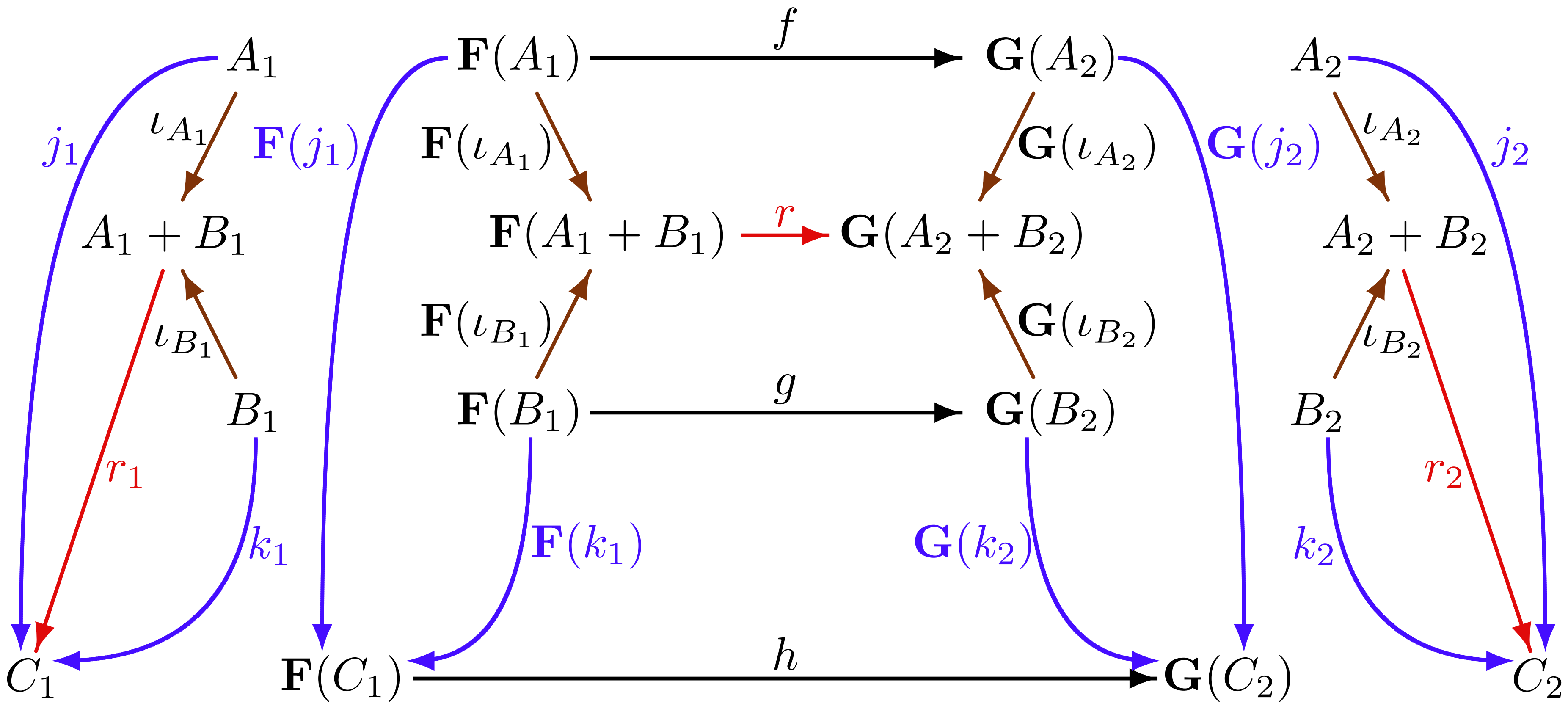

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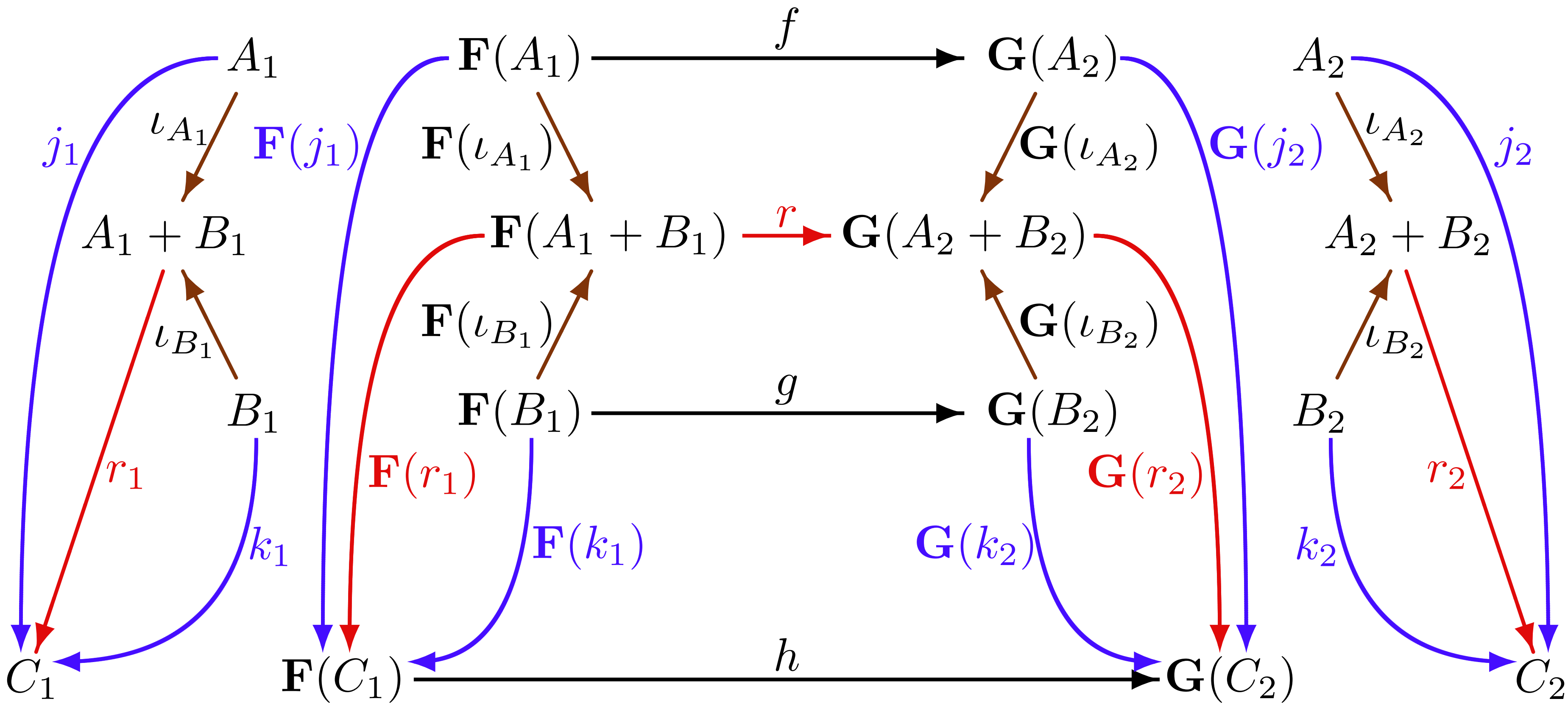

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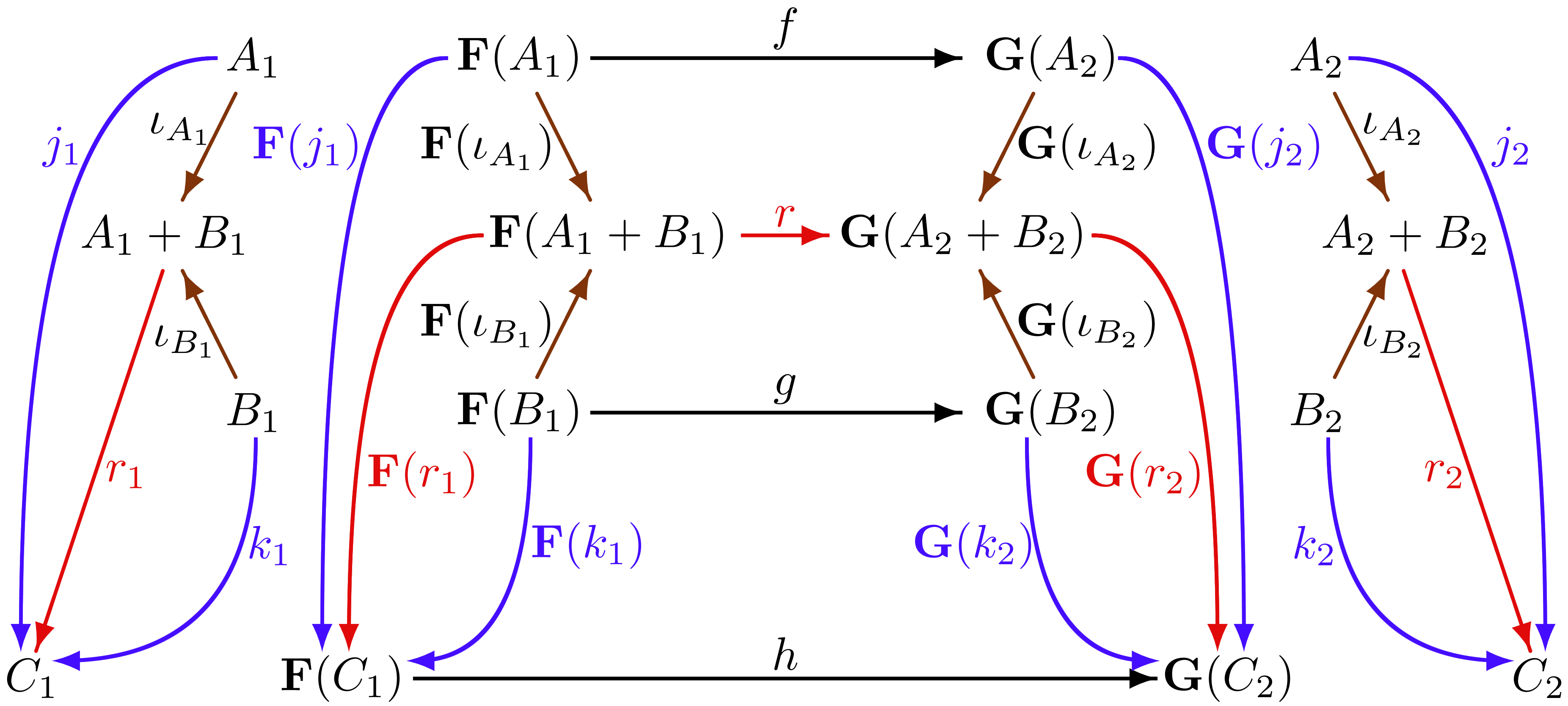

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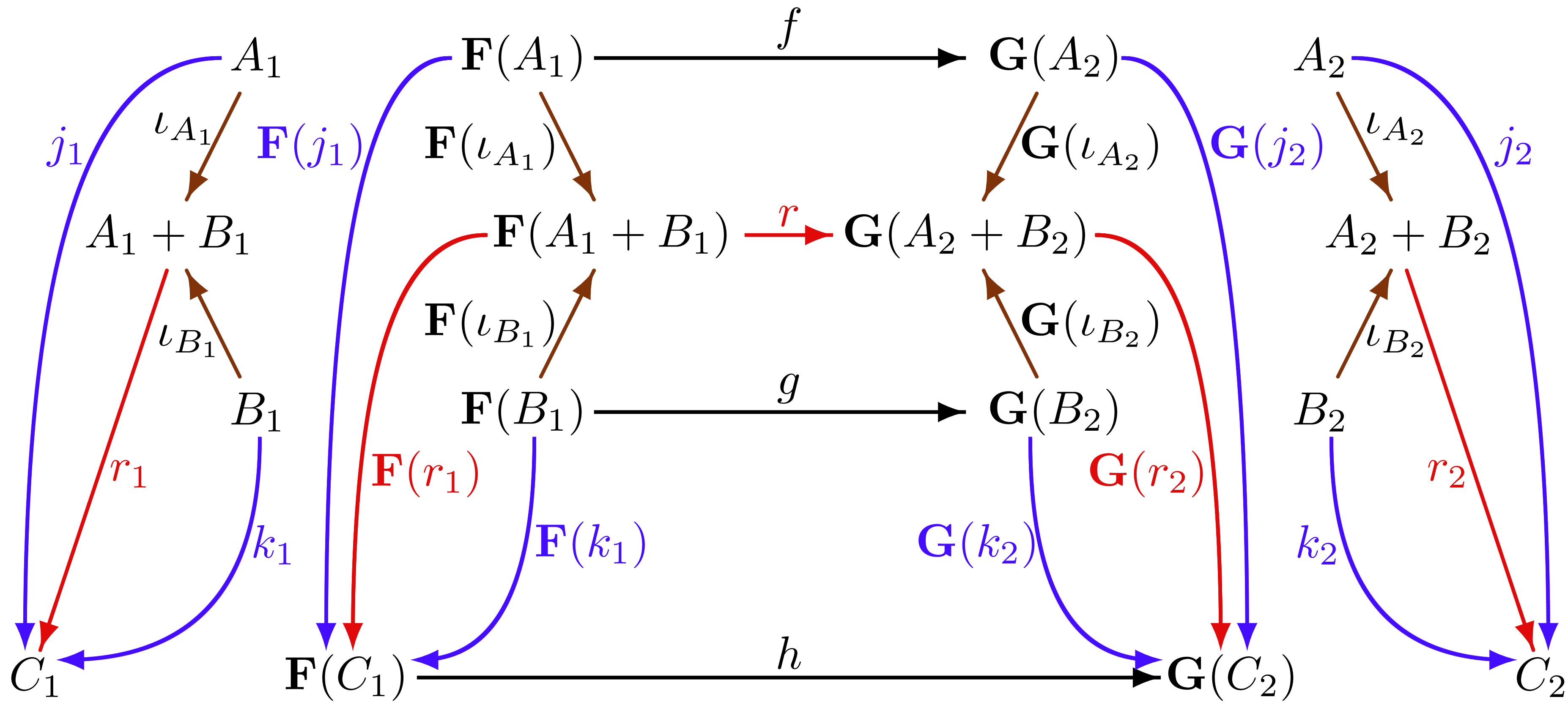
$r_1 = [j_1, k_1]$, $r_2 = [j_2, k_2]$. We need $\boxed{\mathbf{F}(r_1);h = r;\mathbf{G}(r_2)}$

Coproducts:



where $r = [f;\mathbf{G}(\iota_{A_2}), g;\mathbf{G}(\iota_{B_2})]$, $\mathbf{F}(j_1);h = f;\mathbf{G}(j_2)$, $\mathbf{F}(k_1);h = g;\mathbf{G}(k_2)$, $r_1 = [j_1, k_1]$, $r_2 = [j_2, k_2]$. We need $\boxed{\mathbf{F}(r_1);h = r;\mathbf{G}(r_2)}$ This follows from $\mathbf{F}(\iota_{A_1});\mathbf{F}(r_1);h = \mathbf{F}(\iota_{A_1});r;\mathbf{G}(r_2)$ and $\mathbf{F}(\iota_{B_1});\mathbf{F}(r_1);h = \mathbf{F}(\iota_{B_1});r;\mathbf{G}(r_2)$.

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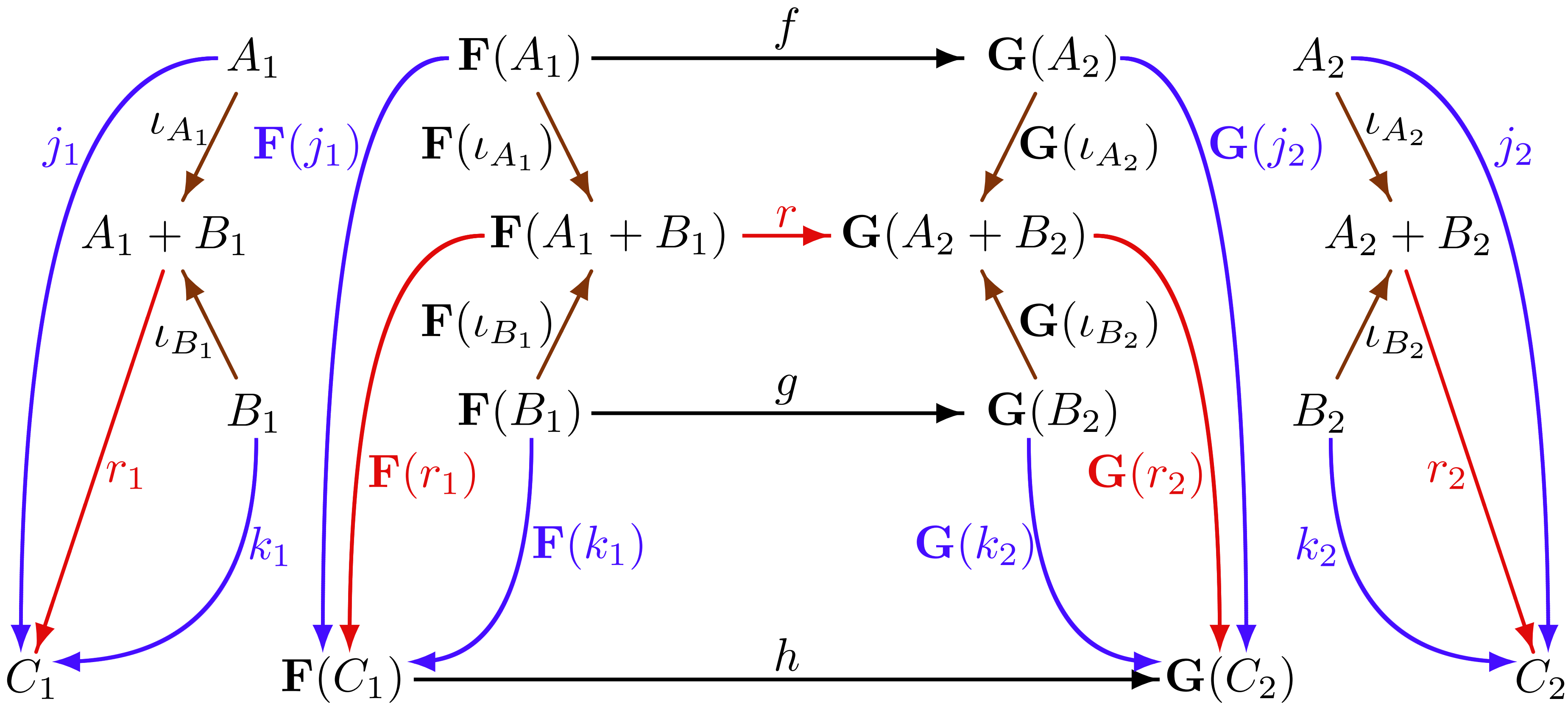

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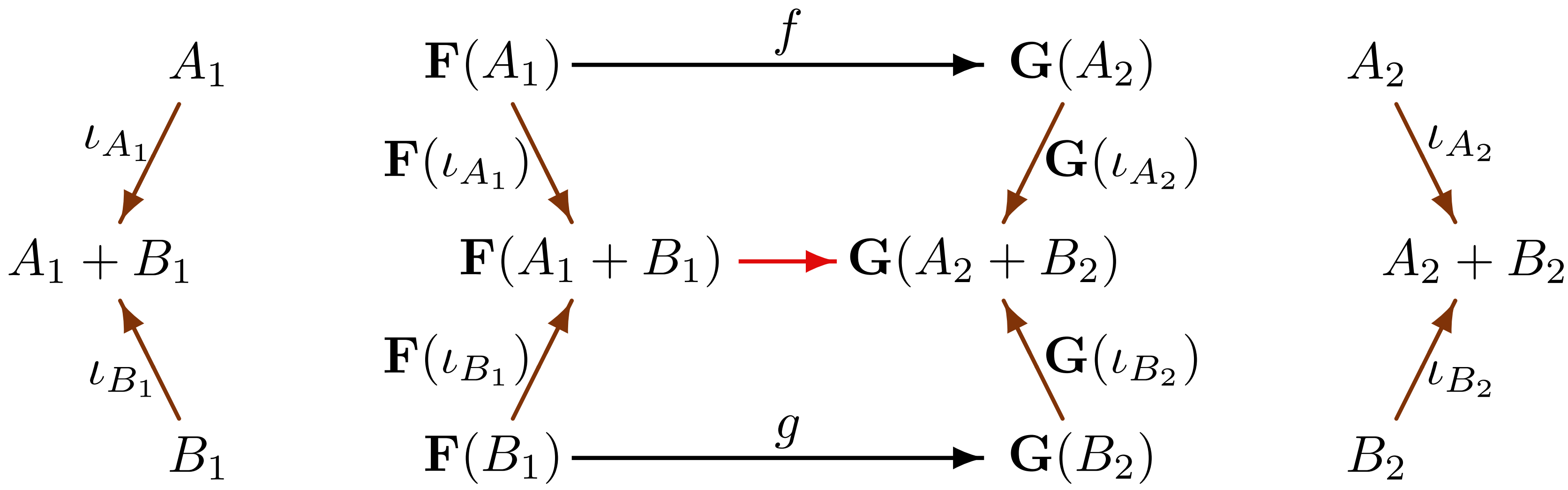
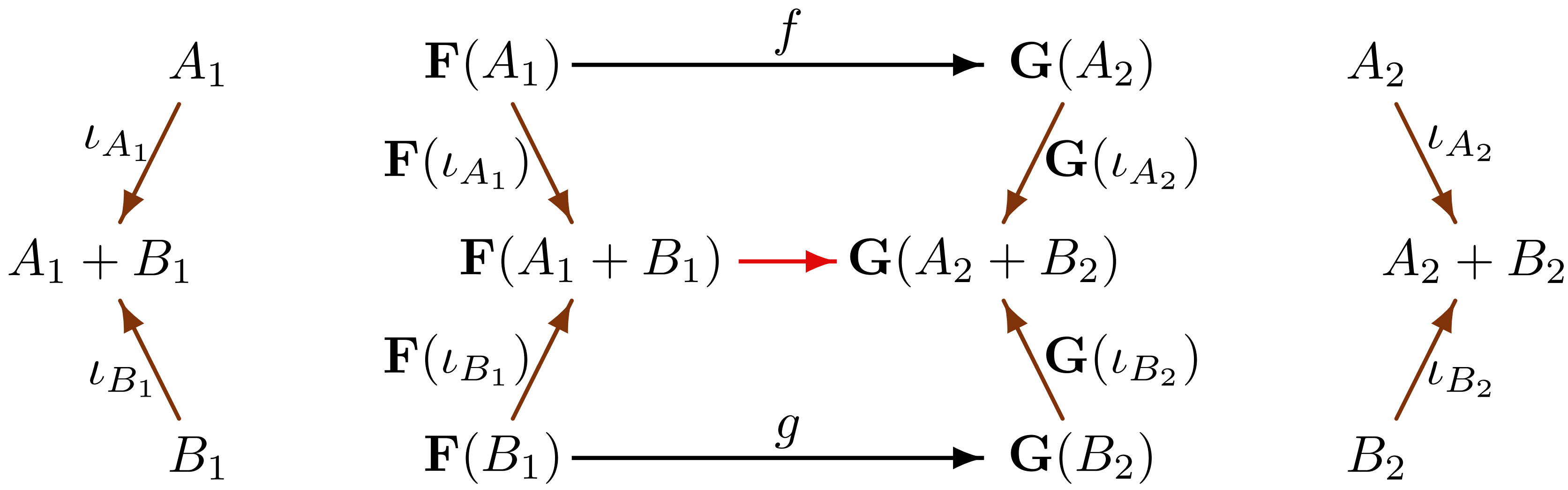
– $\mathbf{F}(\iota_{A_1});\mathbf{F}(r_1);h = \mathbf{F}(j_1);h = f;\mathbf{G}(j_2) = f;\mathbf{G}(\iota_{A_2});\mathbf{G}(r_2) = \mathbf{F}(\iota_{A_1});r;\mathbf{G}(r_2)$

Coproducts:

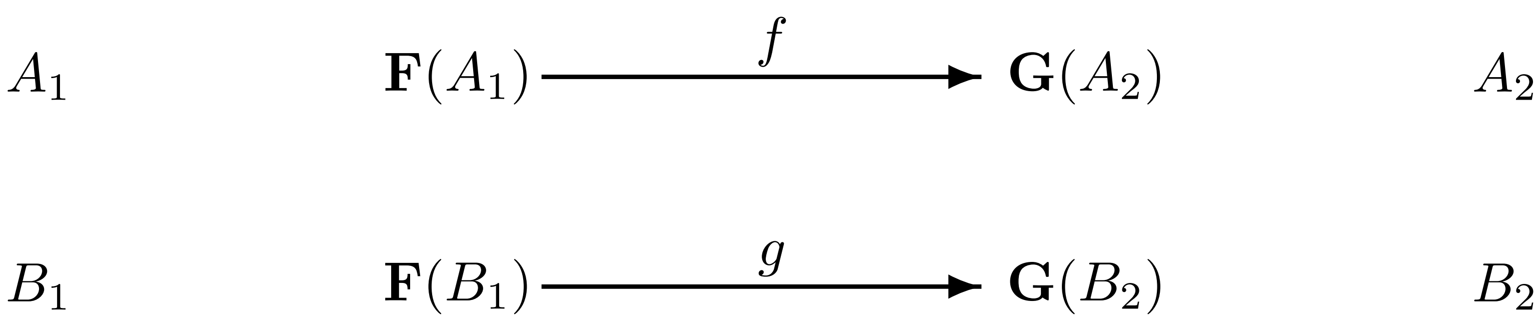


Coequalisers:

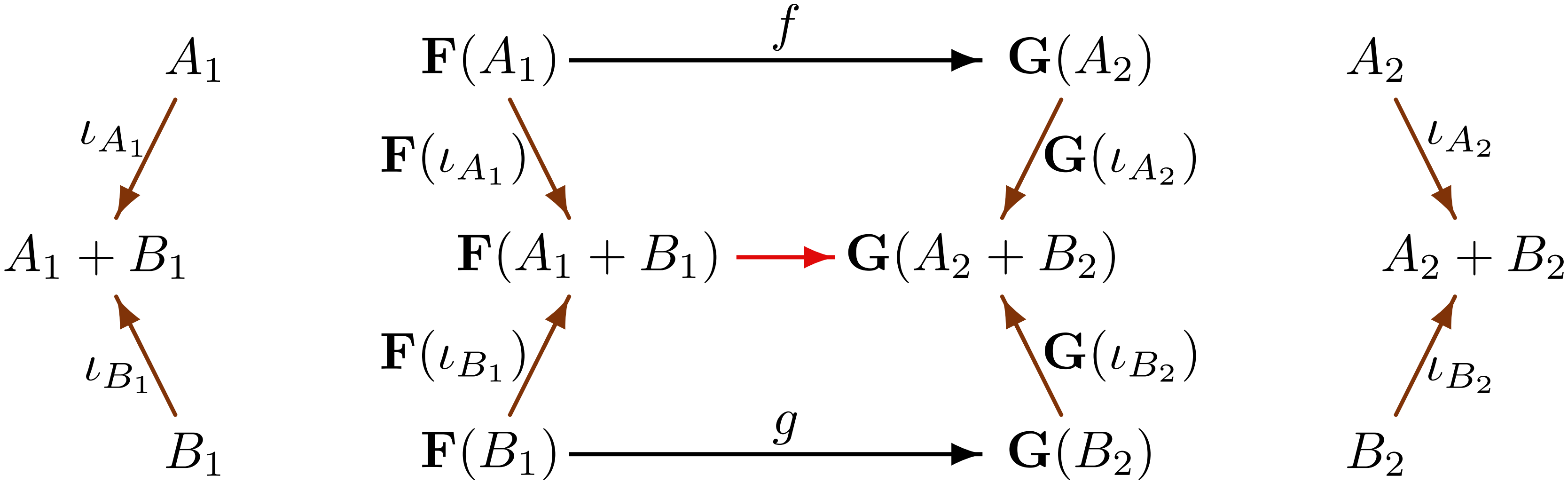
Coproducts:



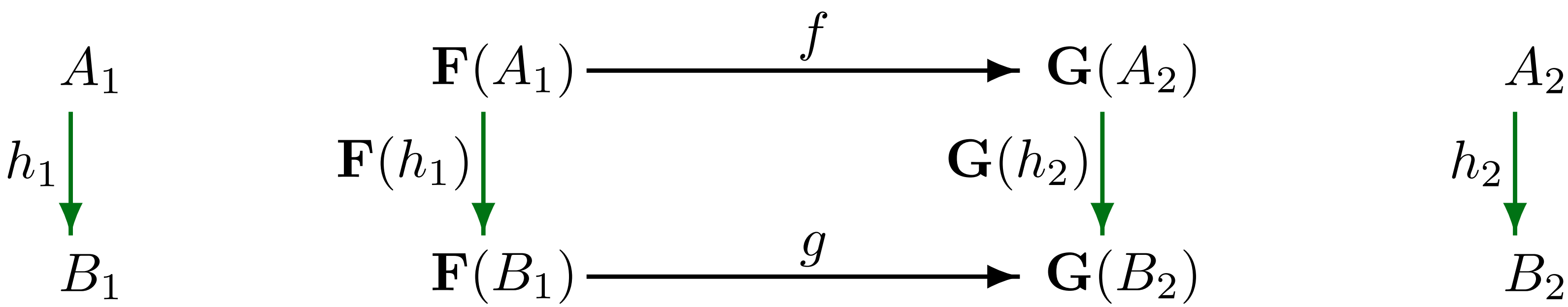
Coequalisers:



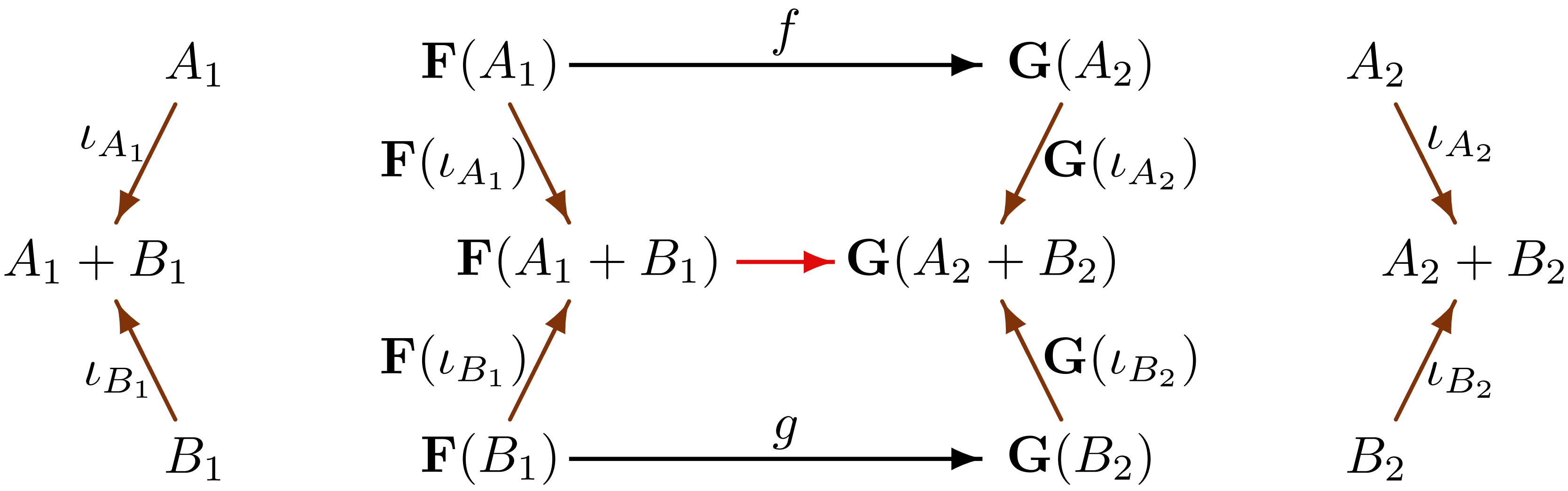
Coproducts:



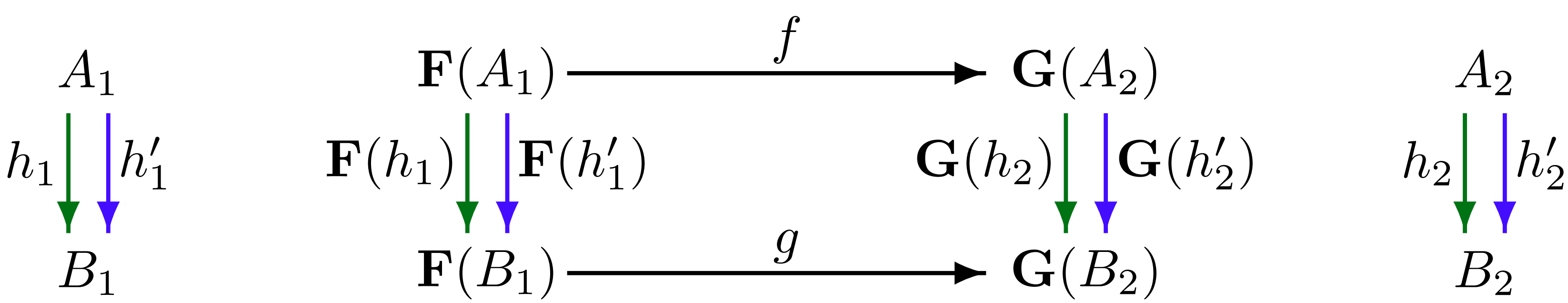
Coequalisers:



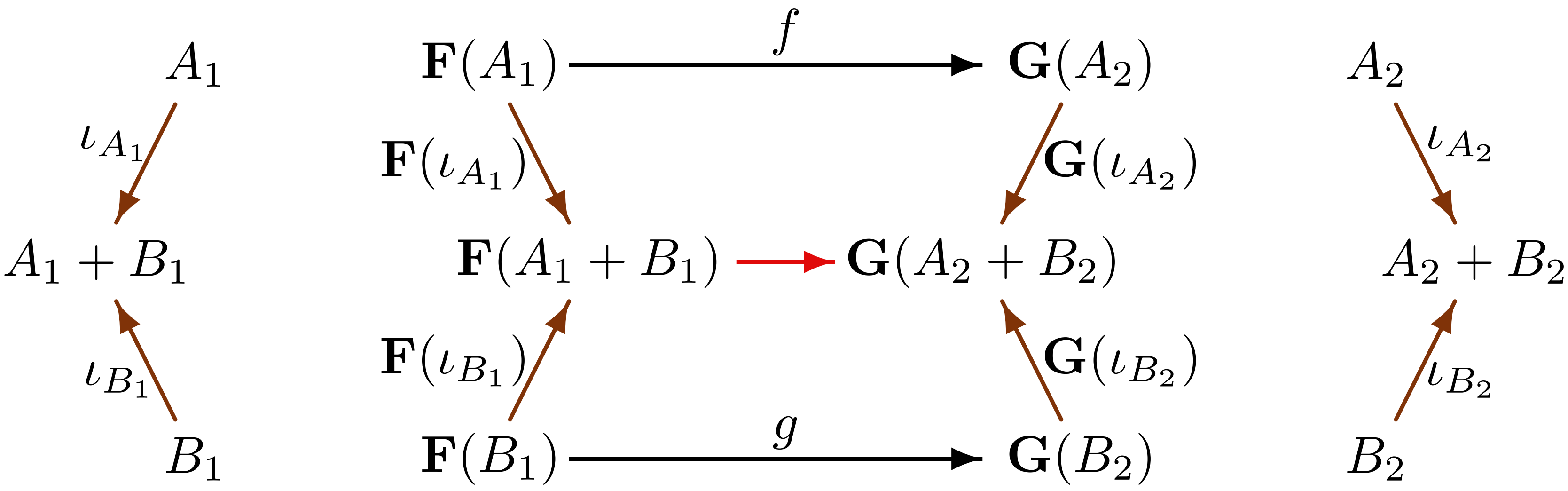
Coproducts:



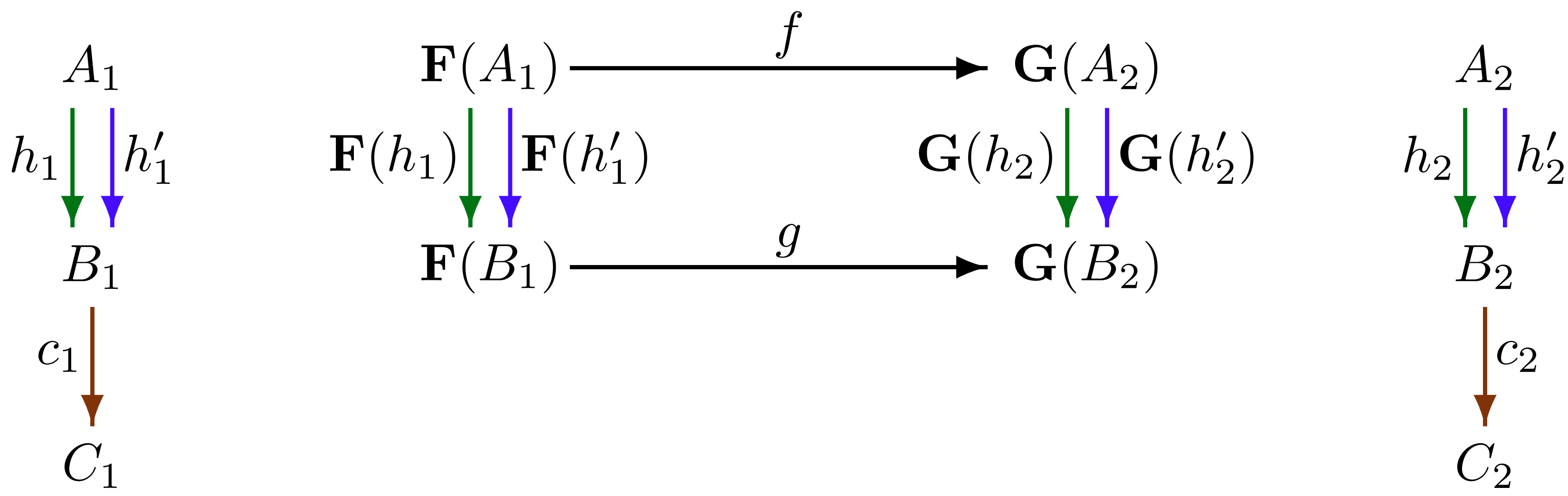
Coequalisers:



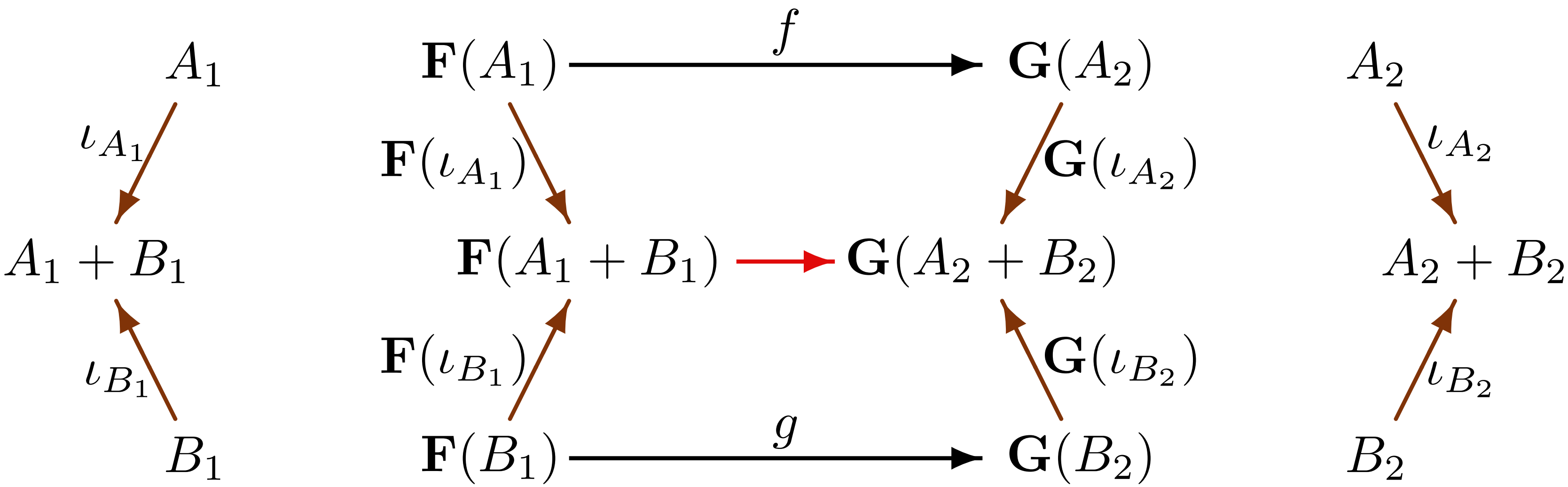
Coproducts:



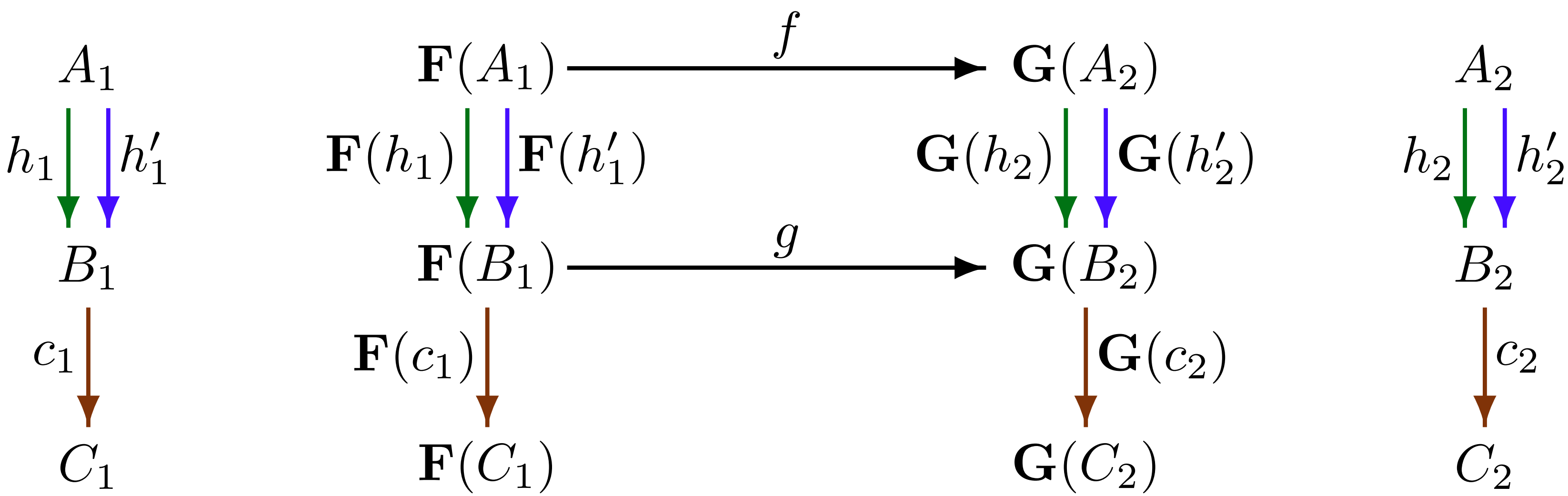
Coequalisers:



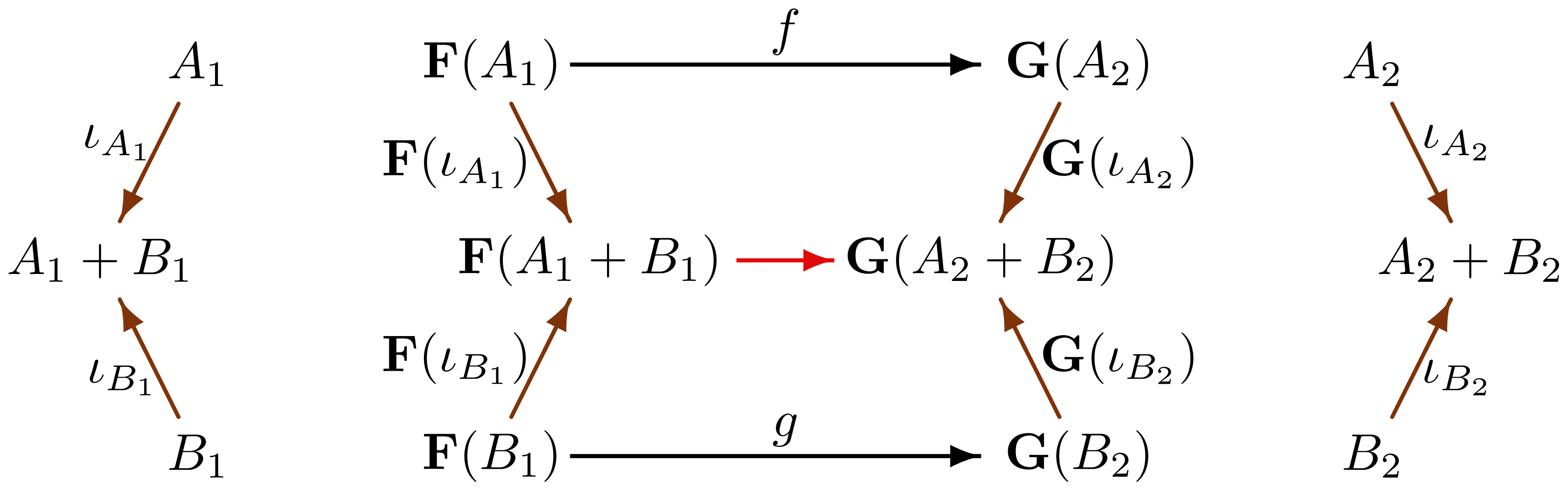
Coproducts:



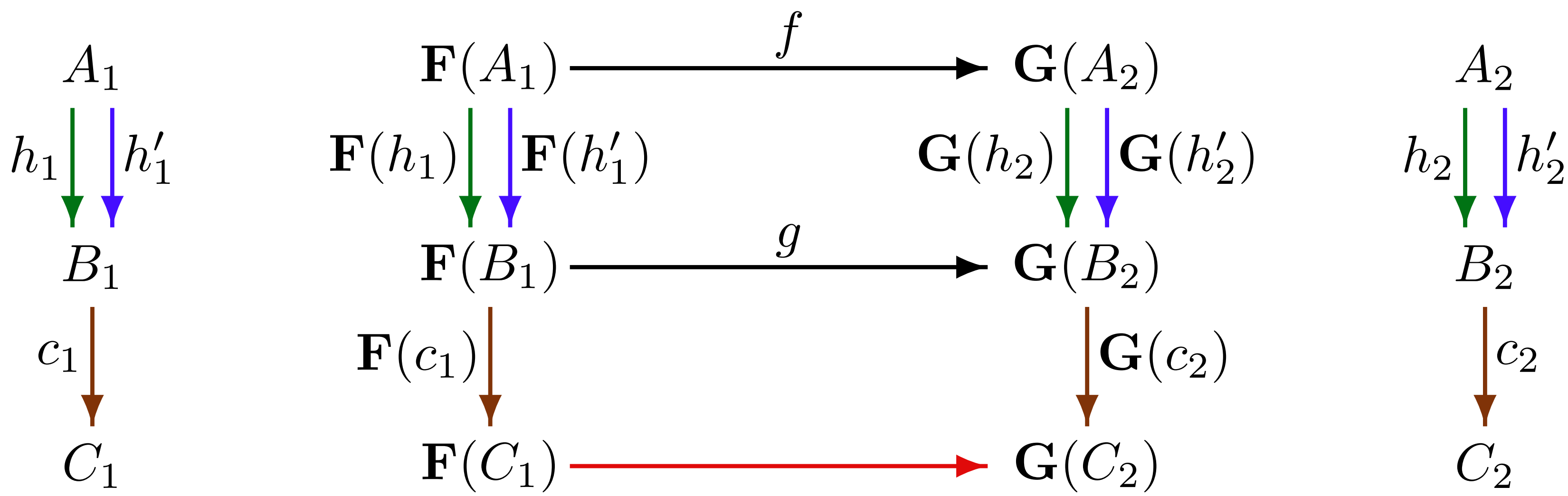
Coequalisers:



Coproducts:



Coequalisers:



Indexed categories

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Standard example: $\mathbf{Alg}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$

Indexed categories

An *indexed category* is a functor $\mathcal{C}\colon \mathbf{Ind}^{op} \to \mathbf{Cat}$.

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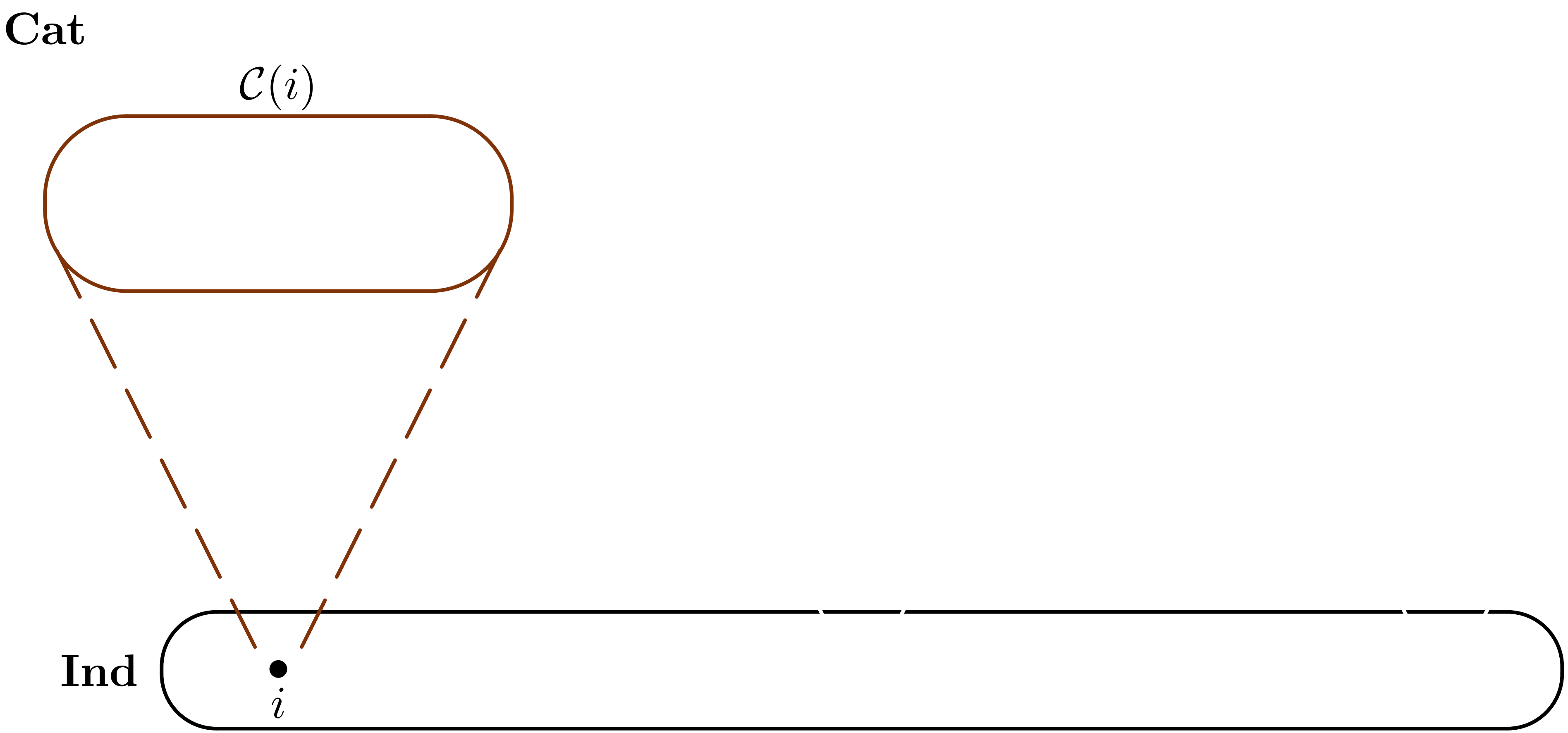
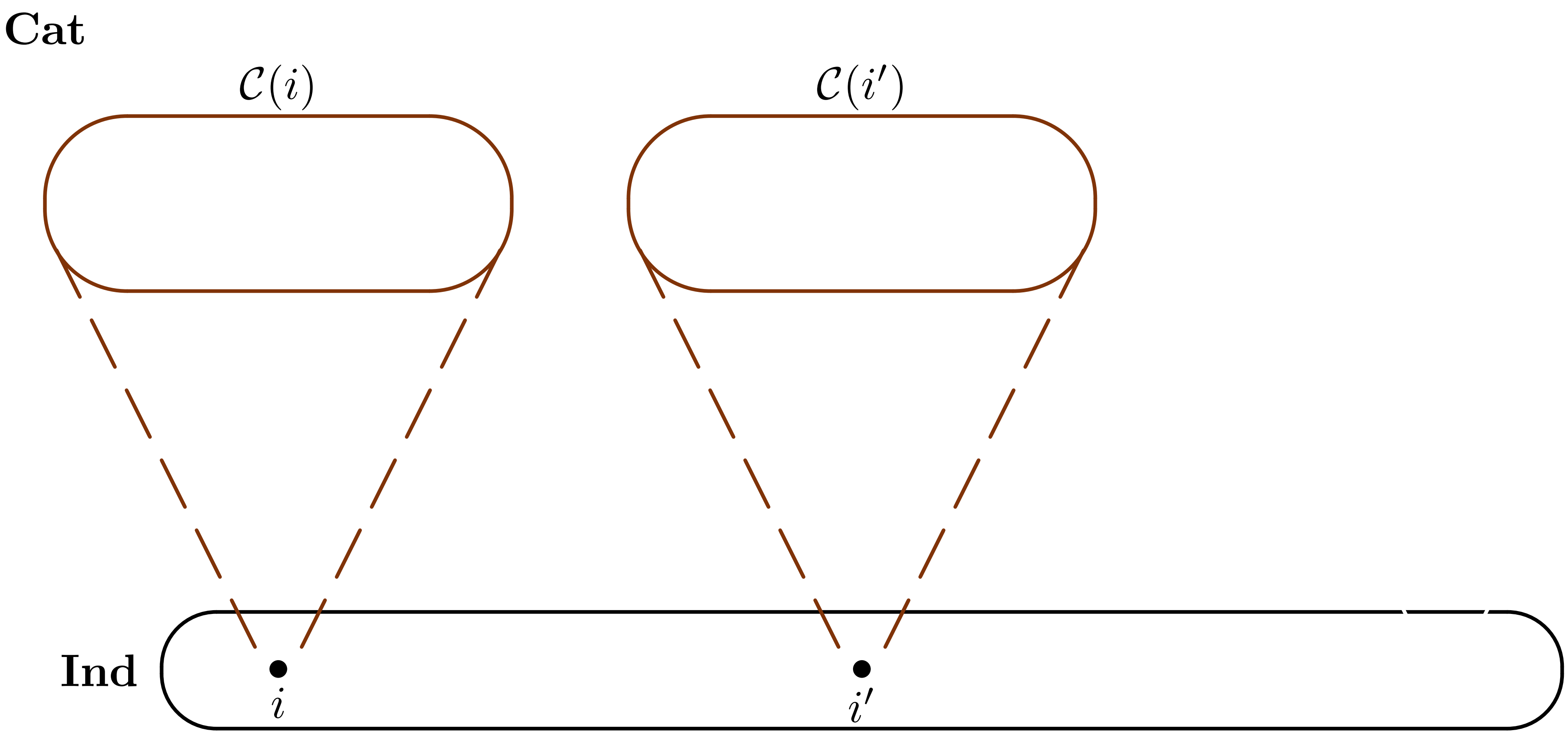
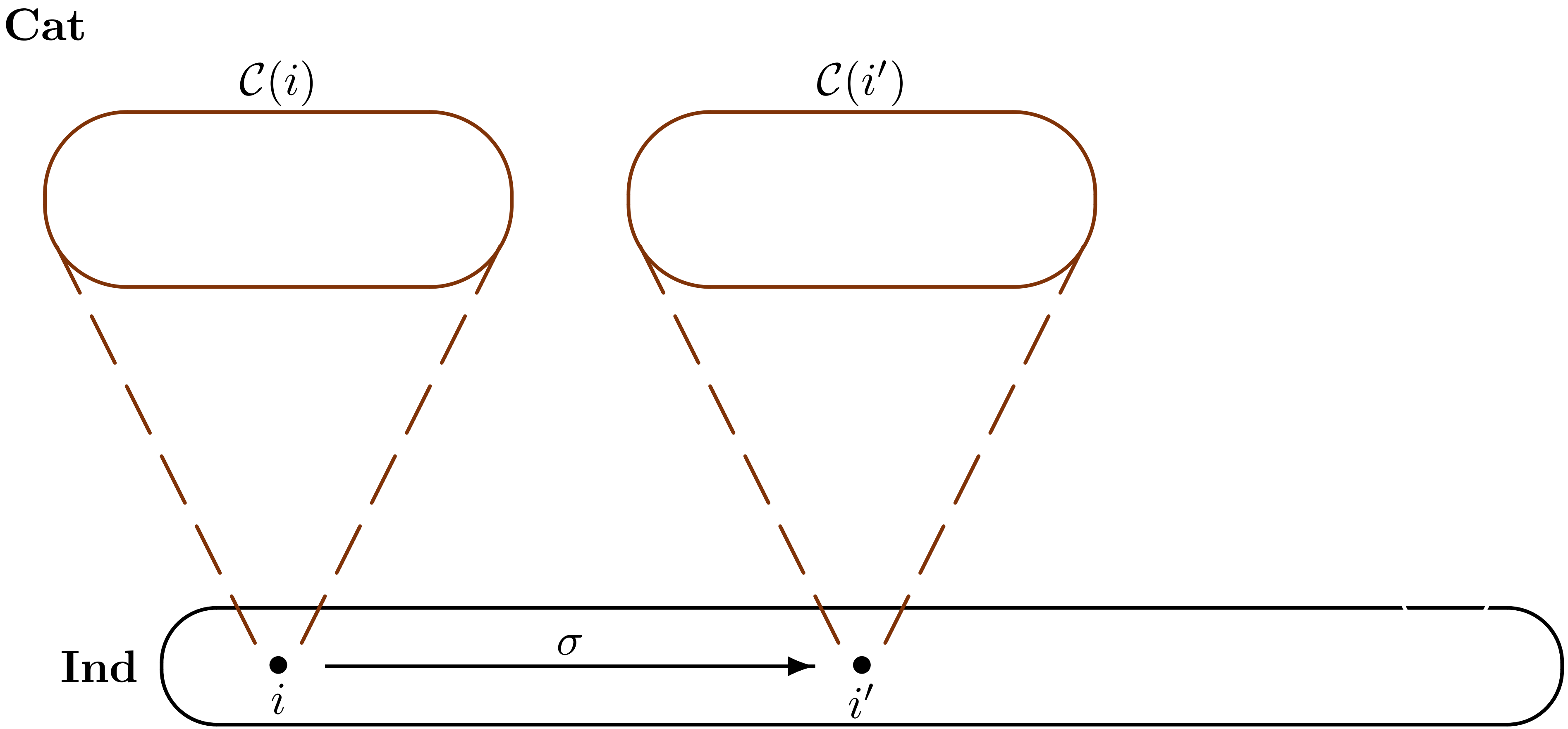
Ind
i



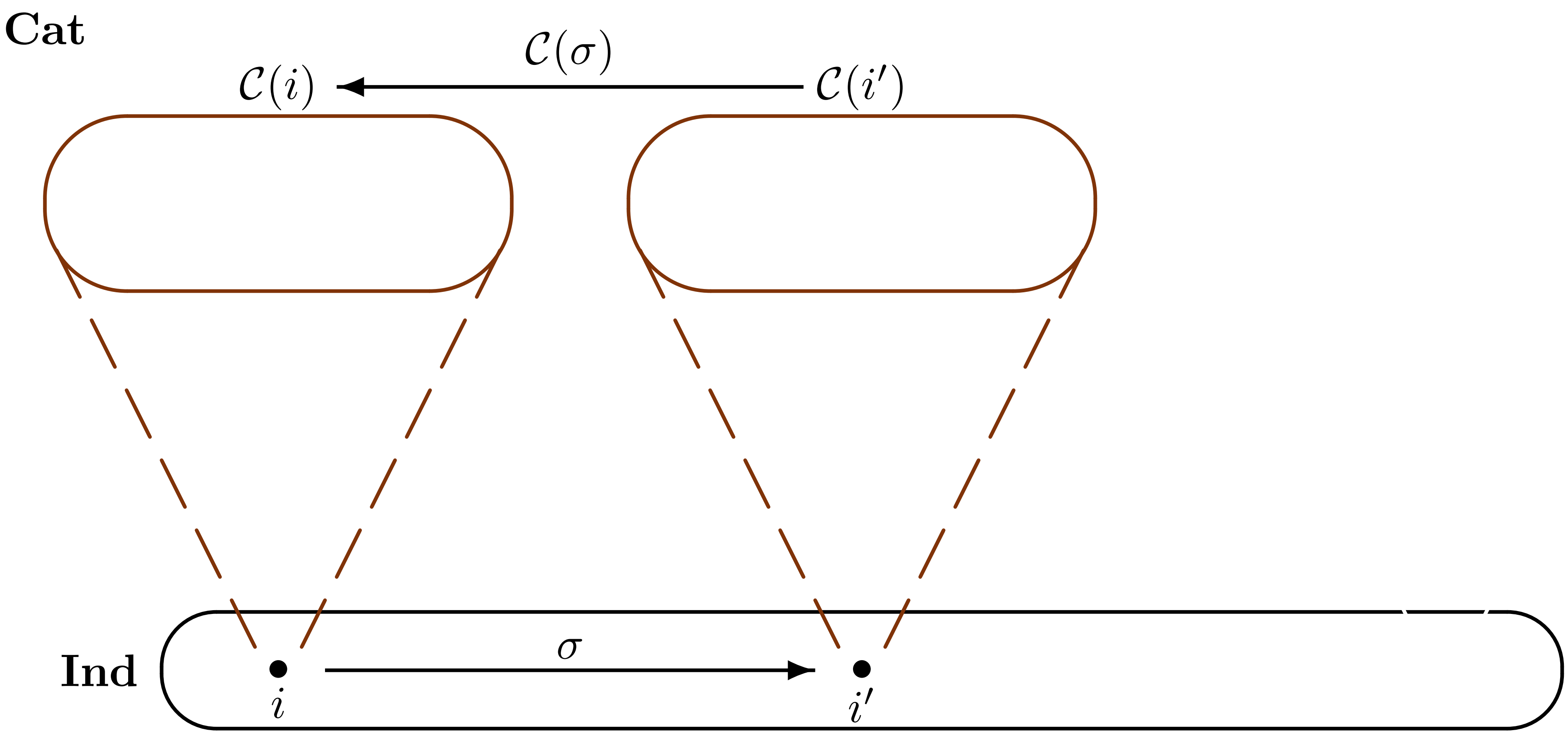
Cat
$\mathcal{C}(i)$
Ind
i



Cat
$\mathcal{C}(i)$
$\mathcal{C}(i')$
Ind
i
i'



Cat
$\mathcal{C}(i)$
$\mathcal{C}(i')$
Ind
σ
i
i'


Cat
C(σ)
C(i)
C(i′)
Ind
σ
i
i′

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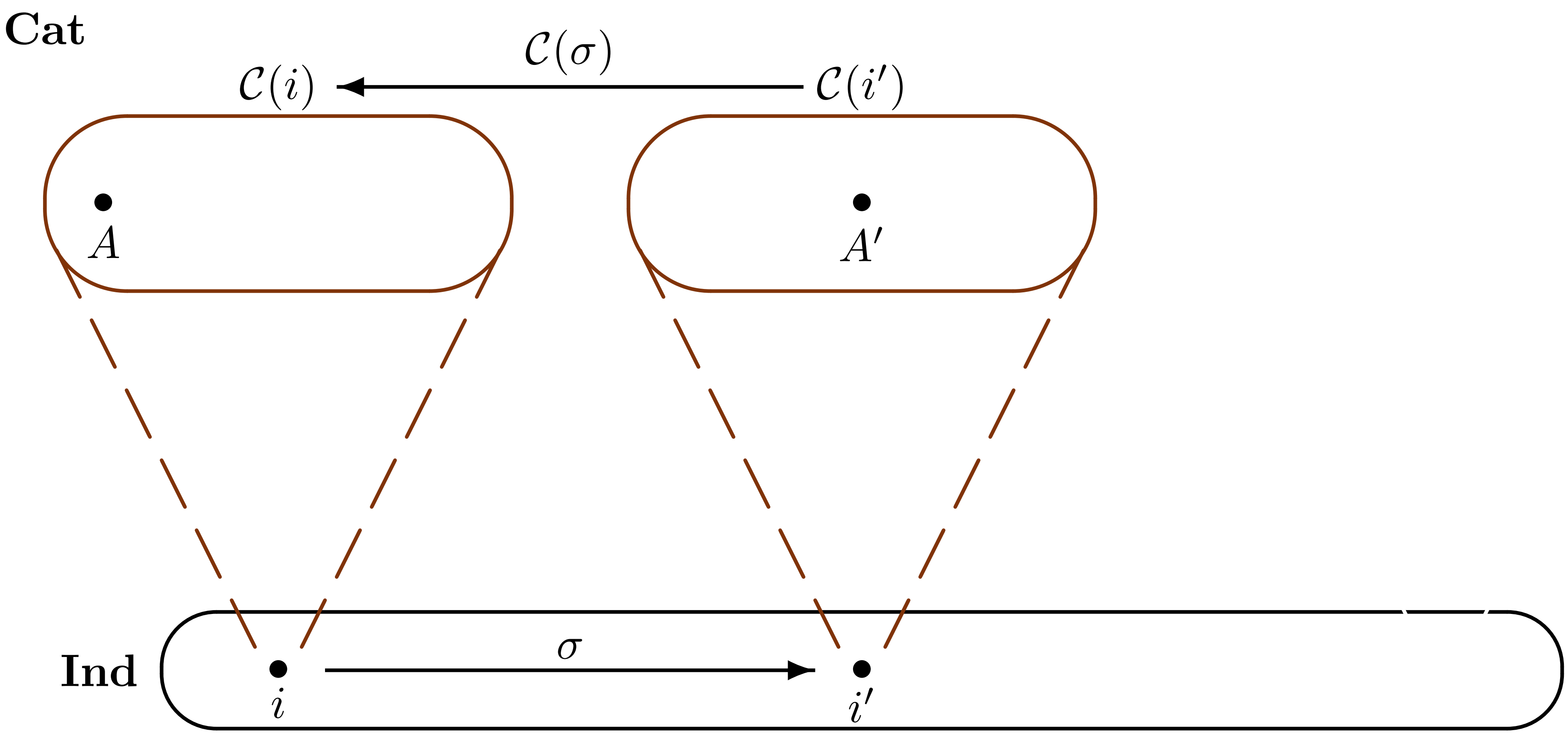
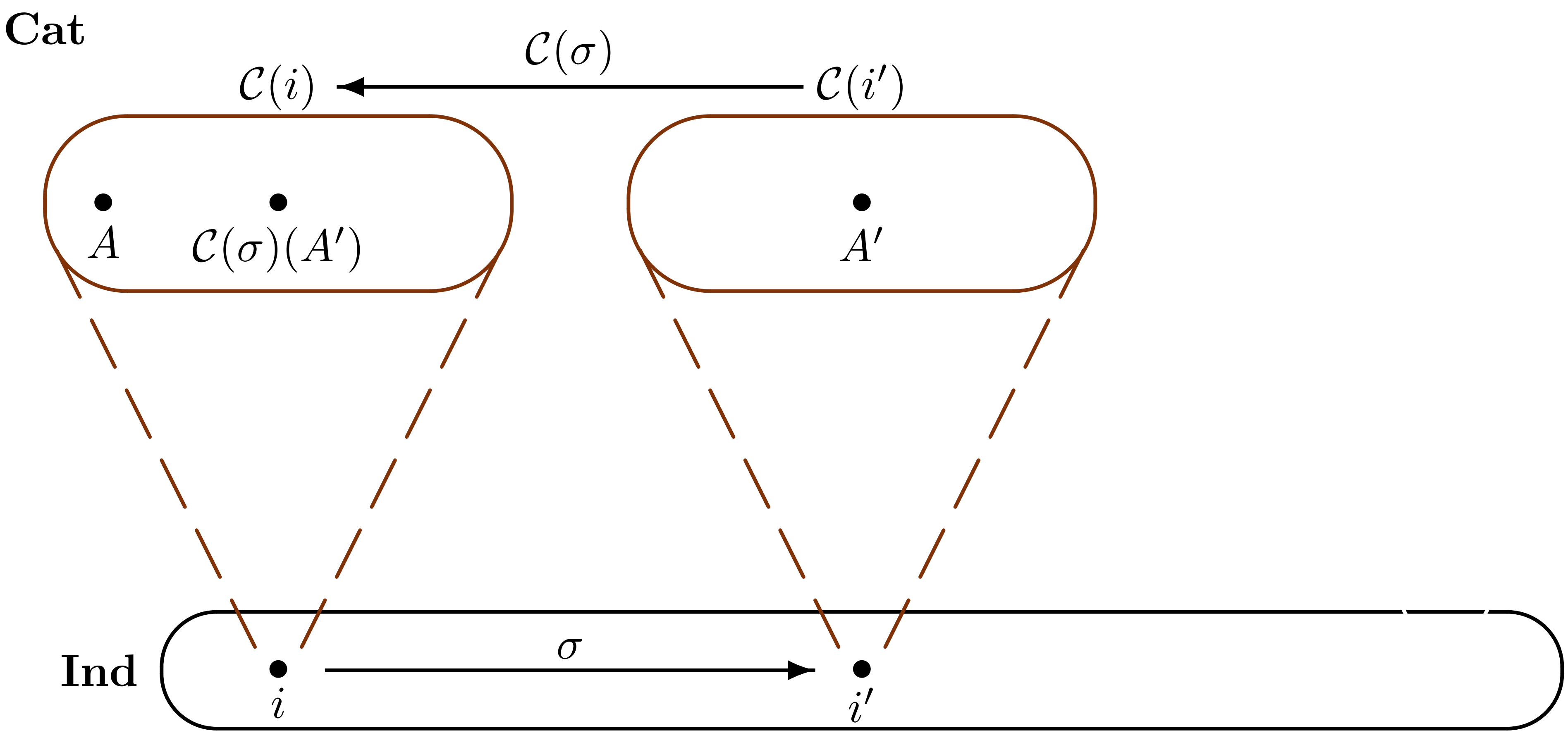
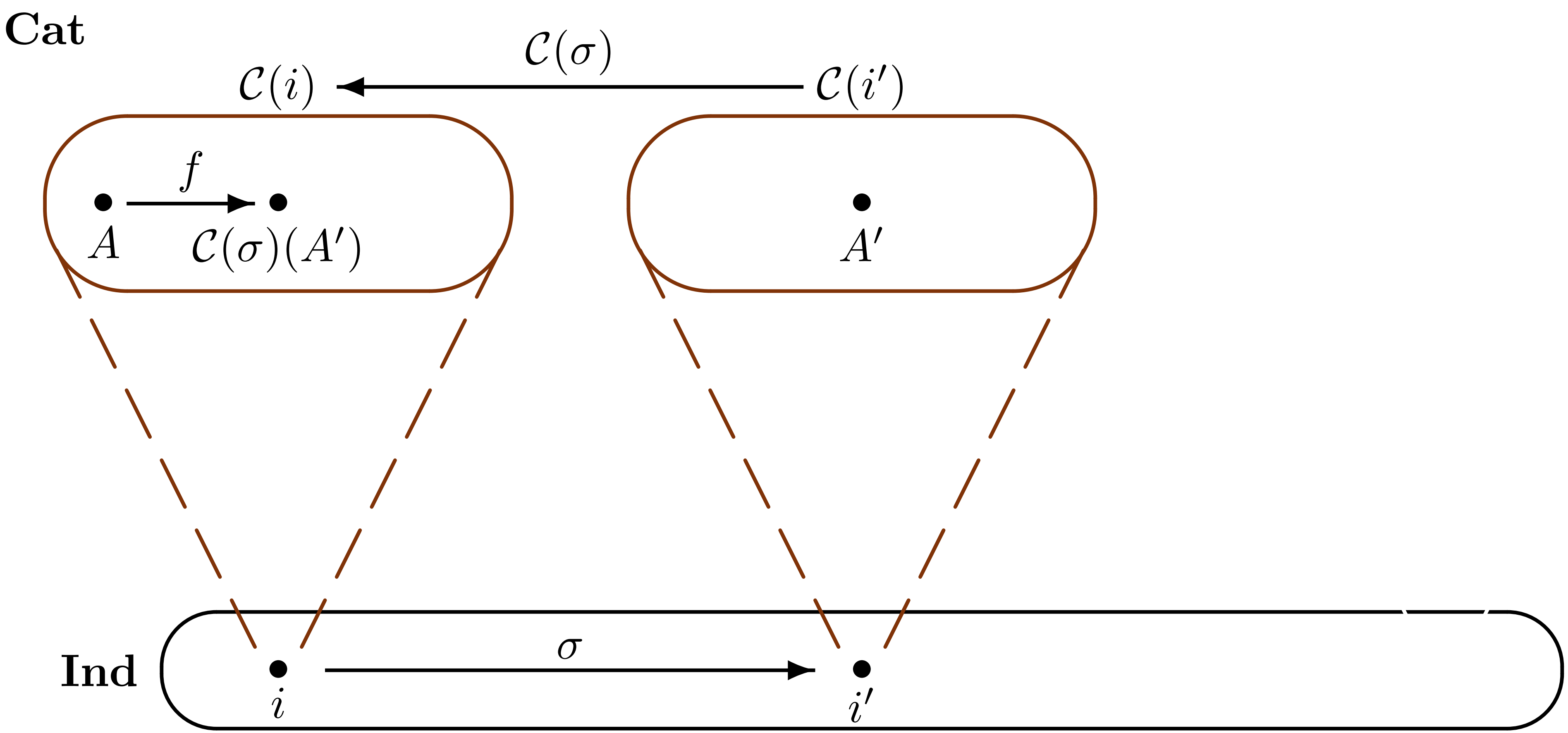


Cat
$\mathcal{C}(\sigma)$
$\mathcal{C}(i)$
$\mathcal{C}(i')$
A
A'
Ind
σ
i
i'



Cat
$\mathcal{C}(\sigma)$
$\mathcal{C}(i)$
$\mathcal{C}(i')$
A
$\mathcal{C}(\sigma)(A')$
A'
Ind
σ
i
i'


Cat
C(σ)
C(i)
C(i′)
f
A
C(σ)(A′)
A′
Ind
σ
i
i′

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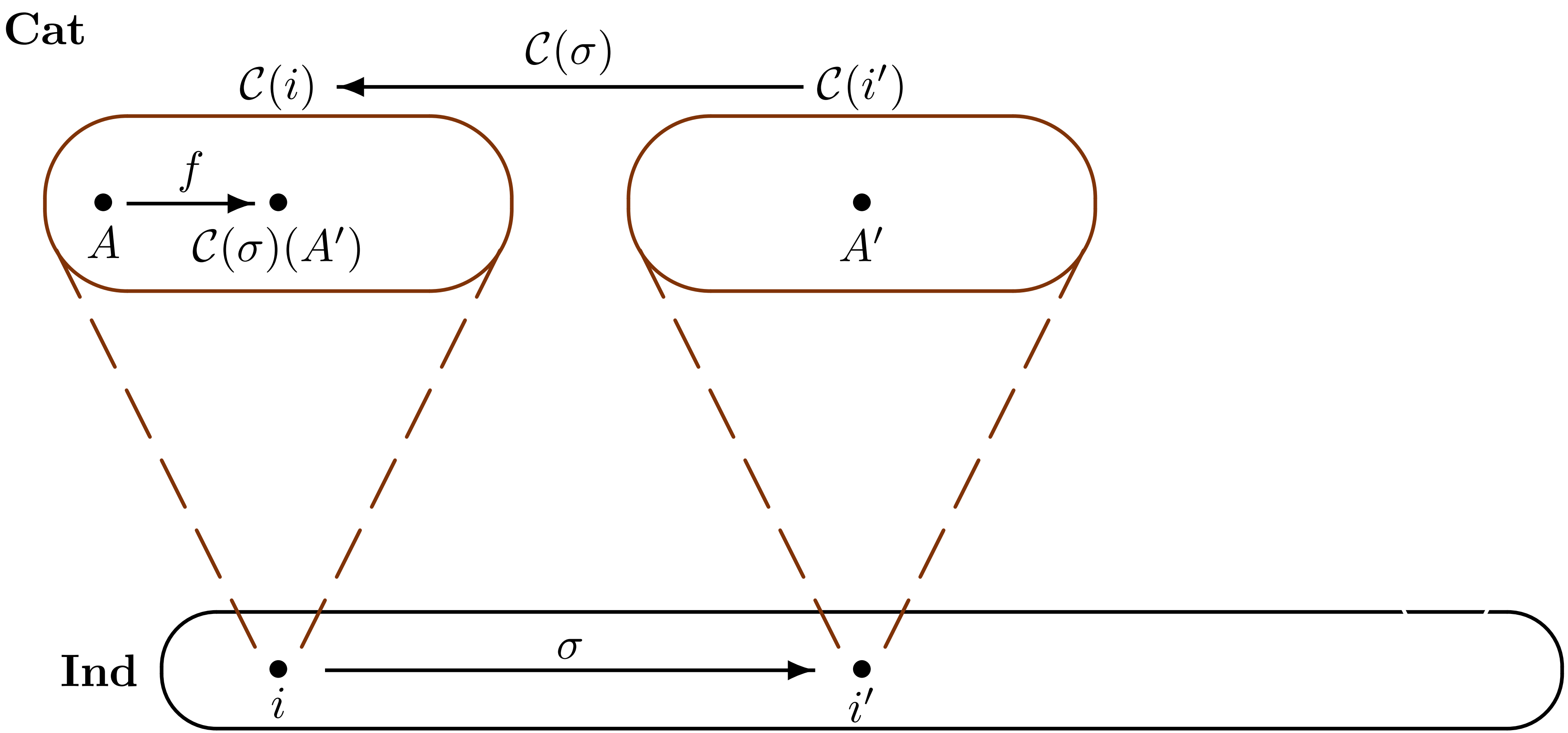
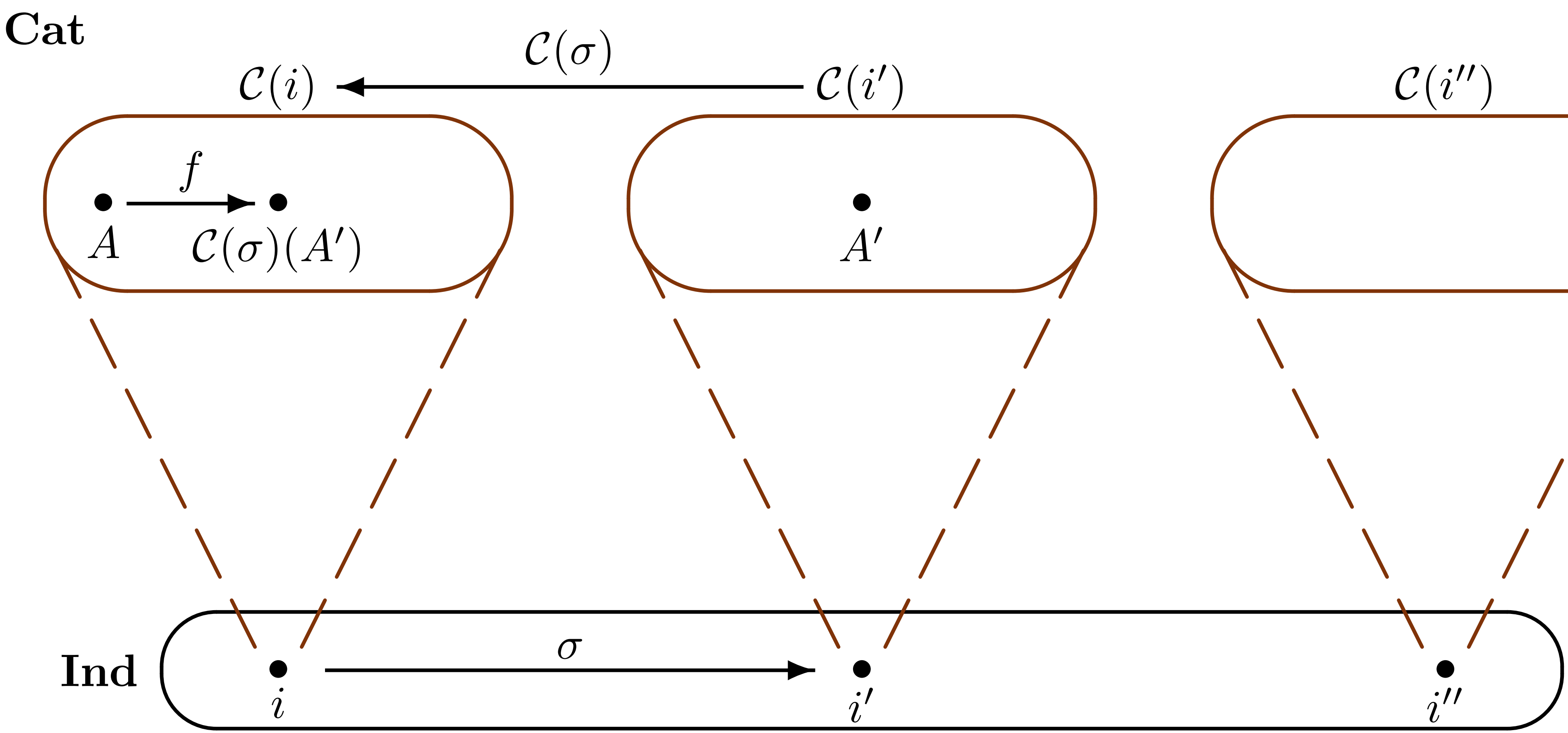
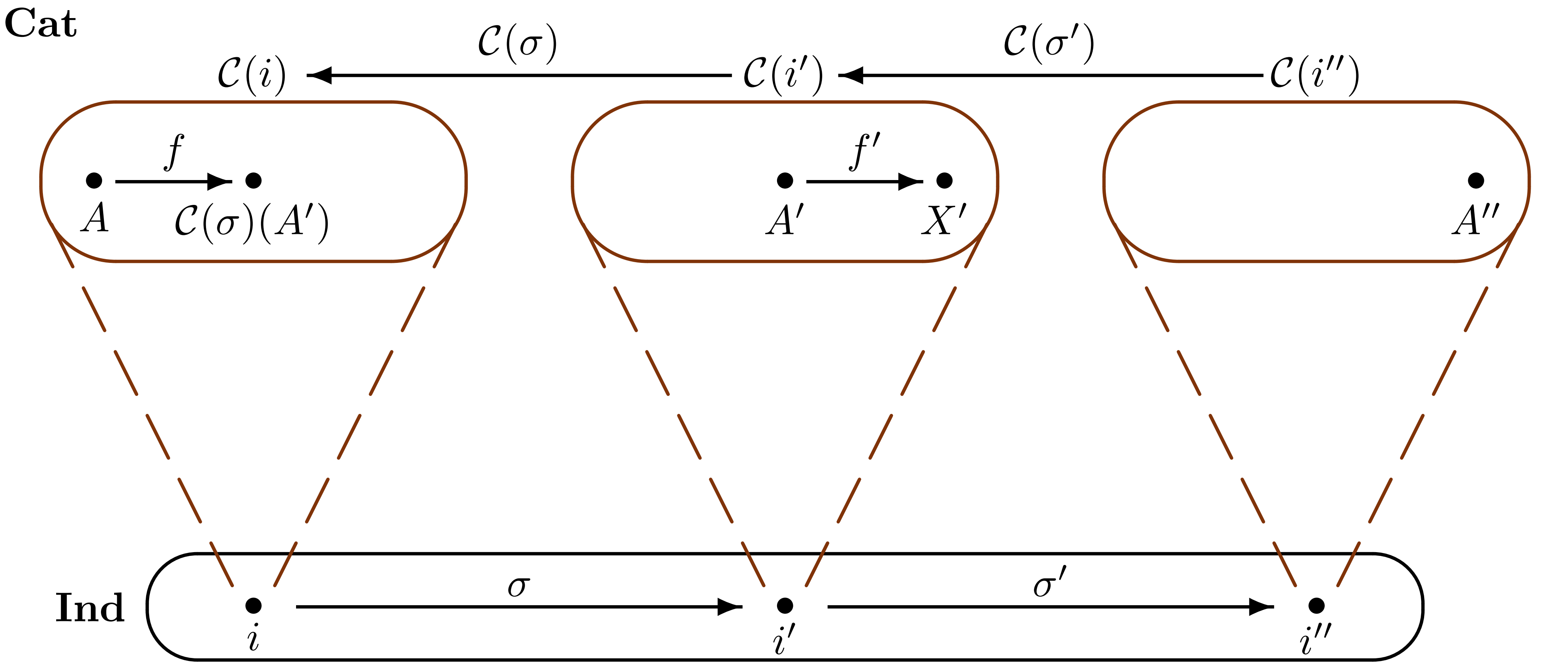
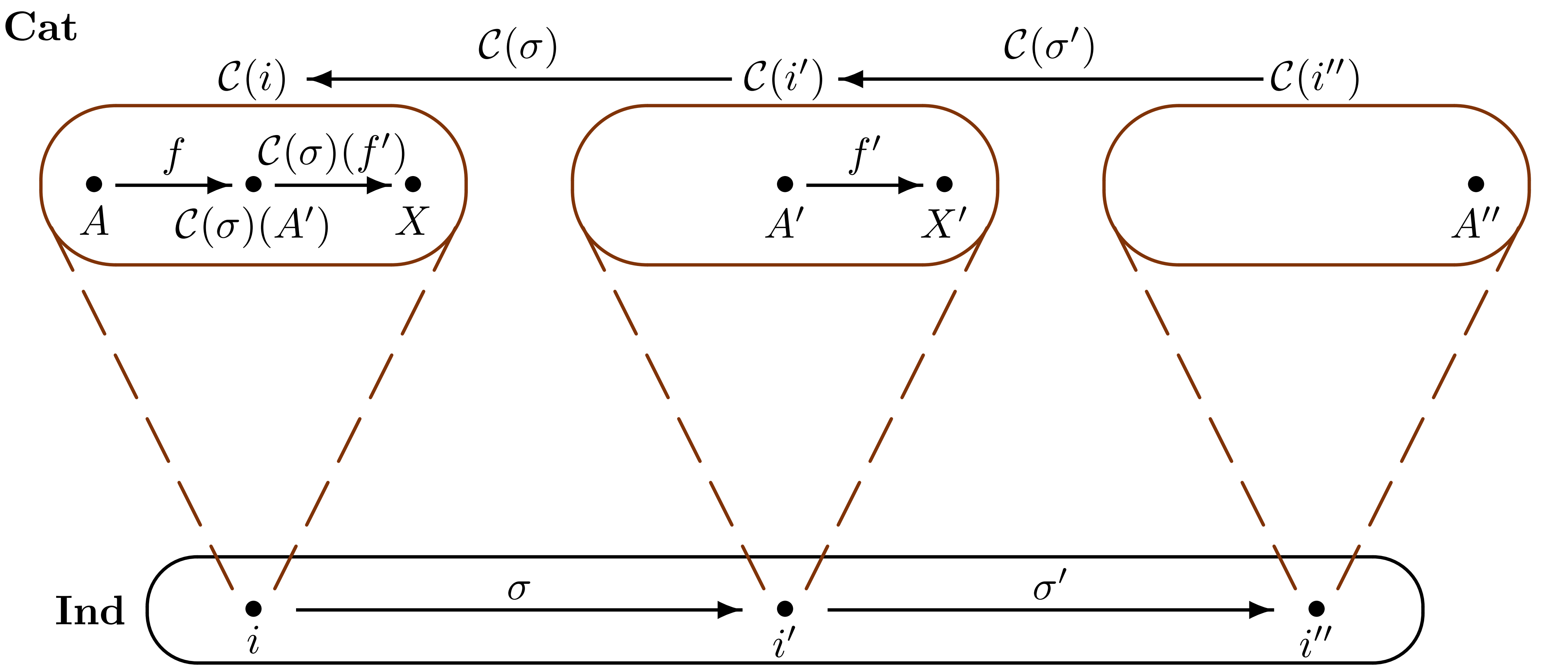

Cat
$\mathcal{C}(\sigma)$
$\mathcal{C}(i)$
$\mathcal{C}(i')$
f
A
$\mathcal{C}(\sigma)(A')$
A'
Ind
σ
i
i'

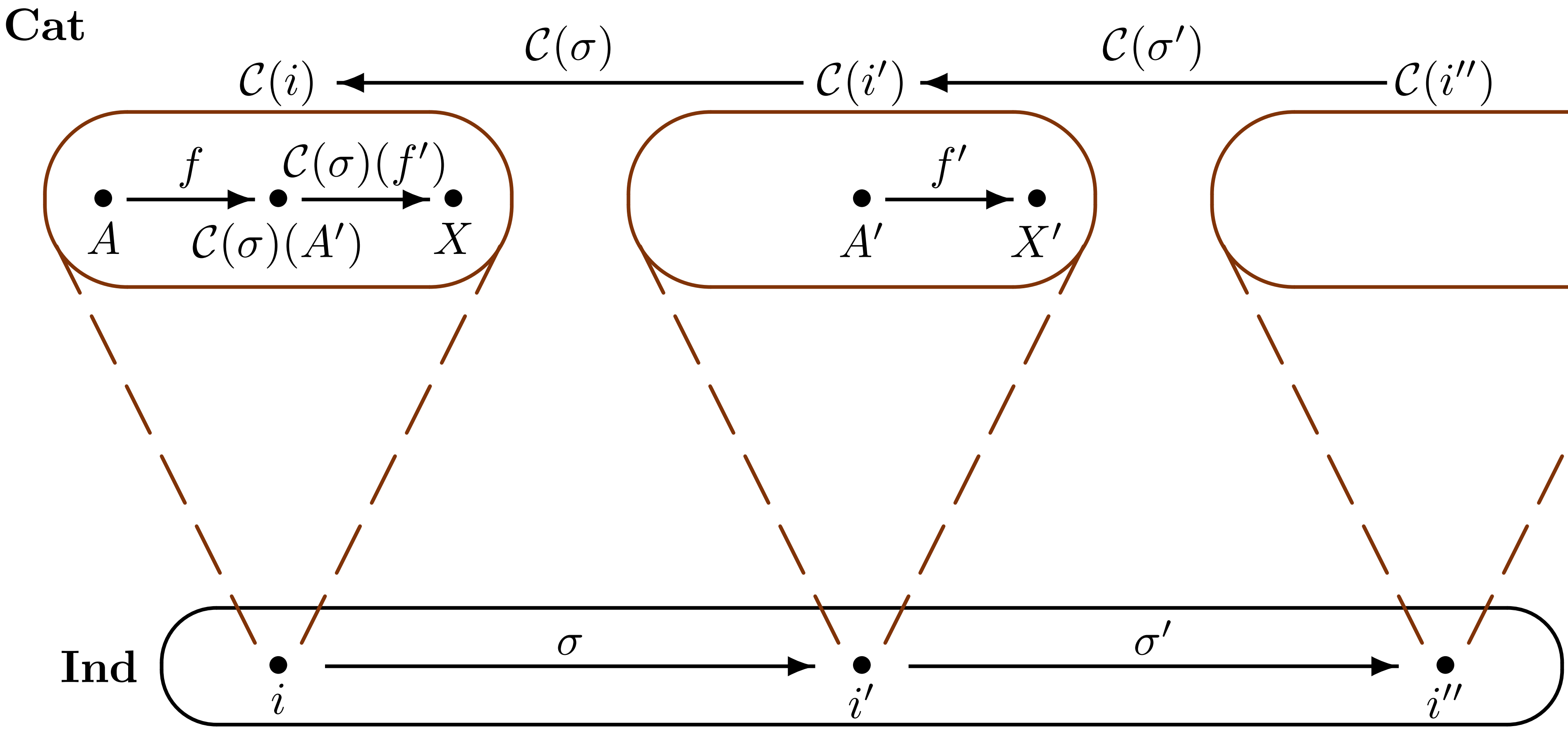

Cat
C(σ)
C(i)
C(i′)
C(i″)
f
A
C(σ)(A′)
A′
A″
Ind
σ
i
i′
i″



where $X' = \mathcal{C}(\sigma')(A'')$



where $X' = \mathcal{C}(\sigma')(A'')$ and $X = \mathcal{C}(\sigma)(X') = \mathcal{C}(\sigma)(\mathcal{C}(\sigma')(A''))$.



where $X' = \mathcal{C}(\sigma')(A'')$ and $X = \mathcal{C}(\sigma)(X') = \mathcal{C}(\sigma)(\mathcal{C}(\sigma')(A''))$.

This works fine, since $\mathcal{C}(\sigma;\sigma') = \mathcal{C}(\sigma');\mathcal{C}(\sigma)$, and so:

$X = \mathcal{C}(\sigma)(\mathcal{C}(\sigma')(A'')) = \mathcal{C}(\sigma;\sigma')(A'')$, and so $f;\mathcal{C}(\sigma)(f') \colon A \to \mathcal{C}(\sigma;\sigma')(A'')$.

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Theorem: *If $\mathbf{Ind}$ is complete, $\mathcal{C}(i)$ are complete for all $i \in |\mathbf{Ind}|$, and $\mathcal{C}(\sigma)$ are continuous for all $\sigma\colon i \to j$ in $\mathbf{Ind}$, then $\mathbf{Flat}(\mathcal{C})$ is complete.*

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Try to formulate and prove a theorem concerning cocompleteness of $\mathbf{Flat}(\mathcal{C})$