Cartesian closed categories

CCC

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CCC

typed functional programming
vs.
category theory

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Recall the definitions of these categories and the constructions of products in each of them

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$$\mathbf{K} \xrightarrow{\_ \times B} \mathbf{K}$$

$$C$$

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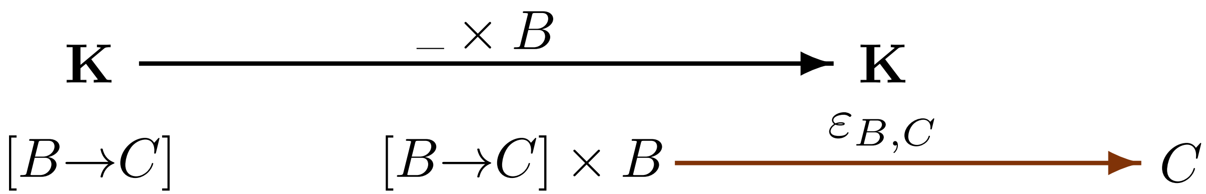

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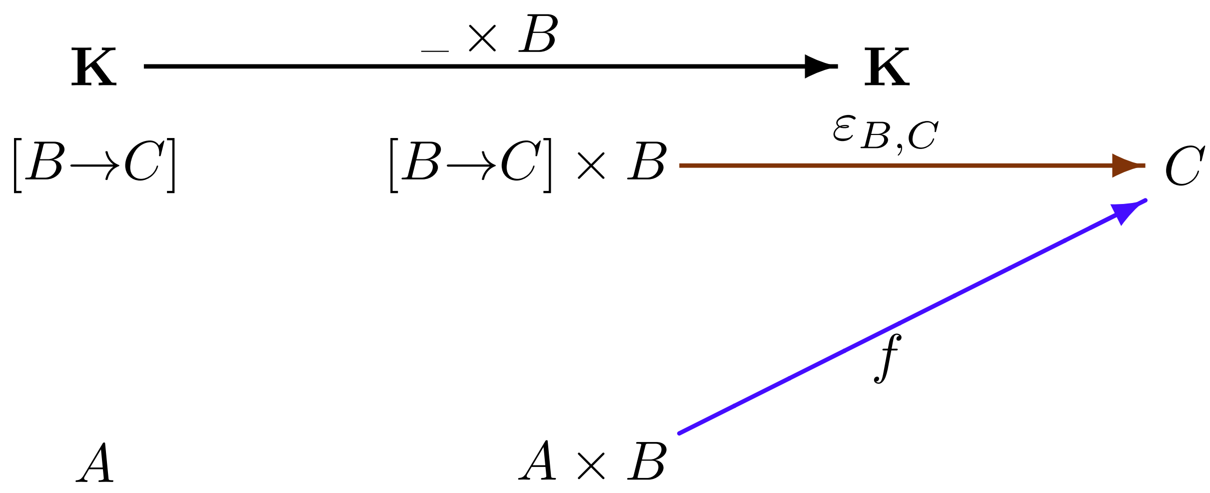

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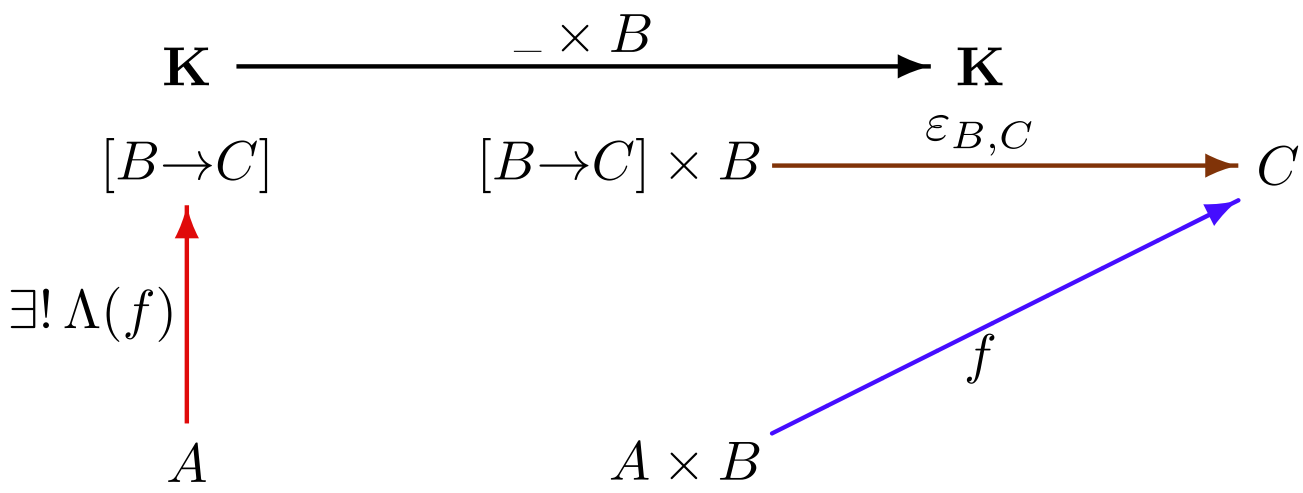

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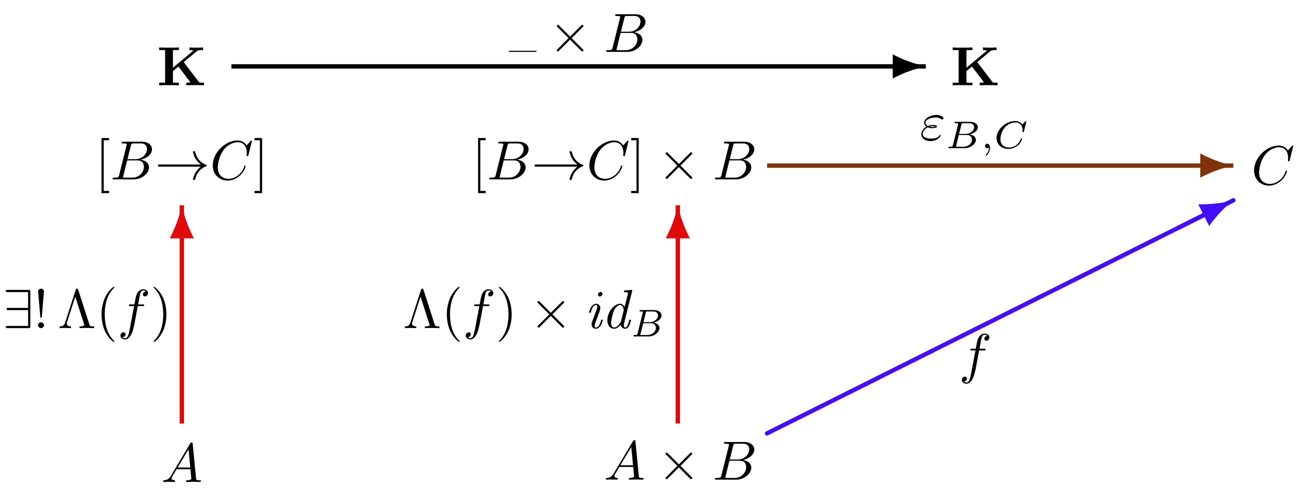

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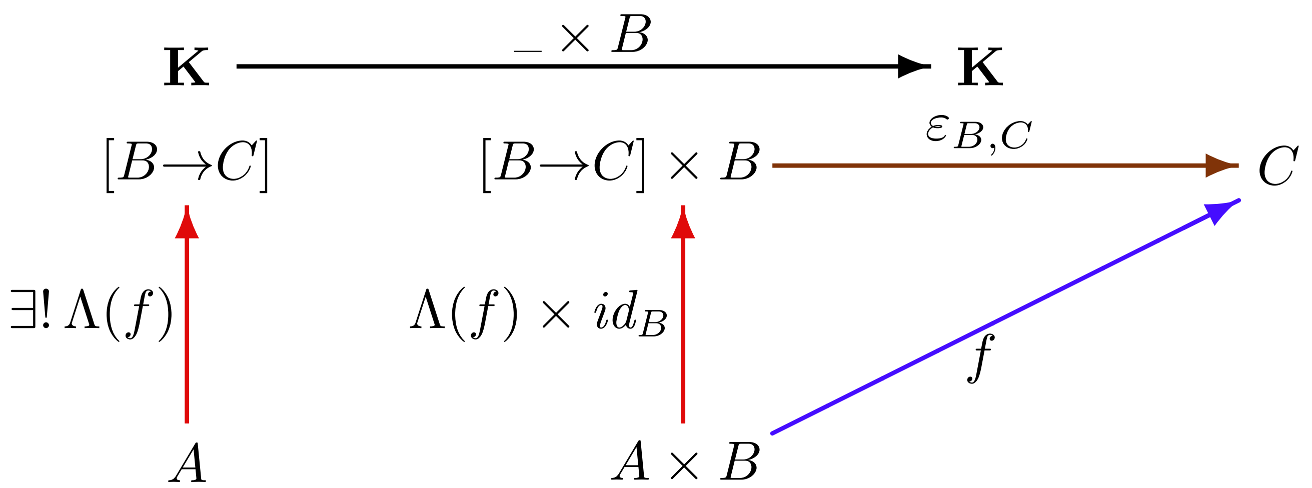

BTW: for $g : A \to A'$ and $h : B \to B'$

$$g \times h = \langle \pi_{A,B};g, \pi'_{A,B};h \rangle : A \times B \to A' \times B'$$

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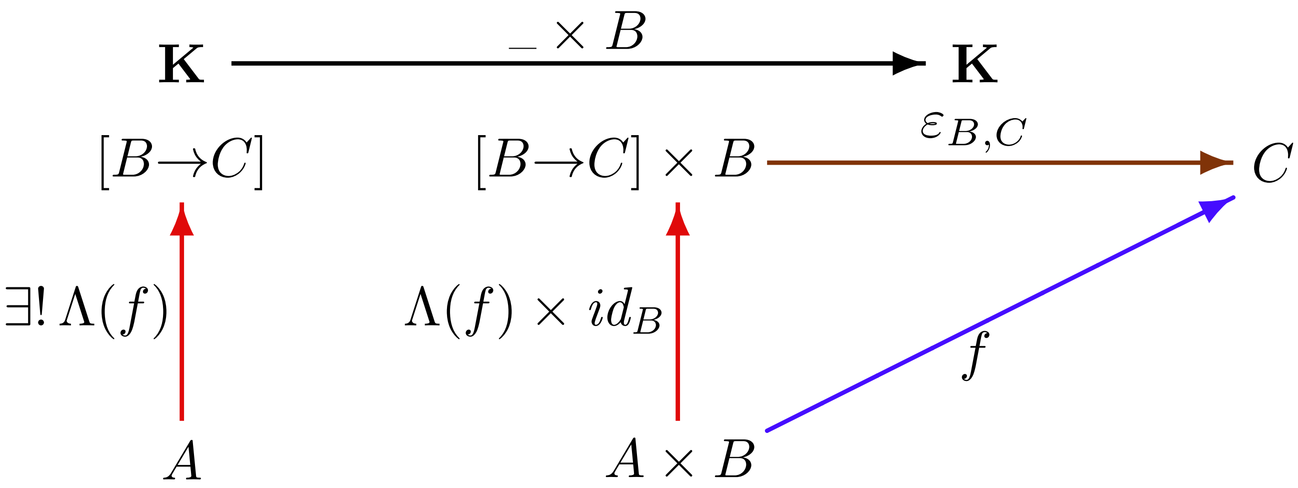

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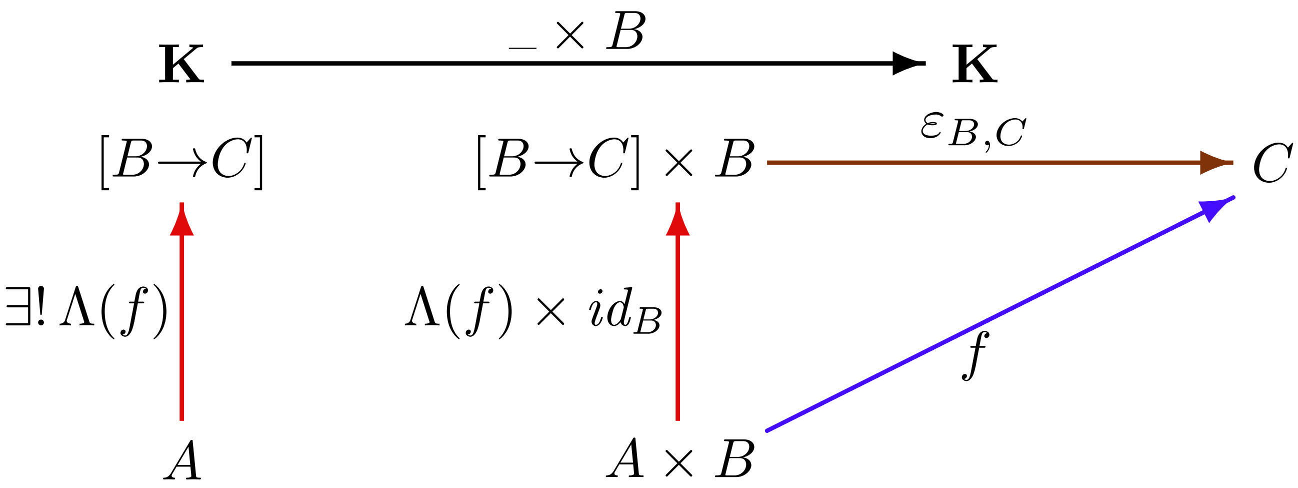

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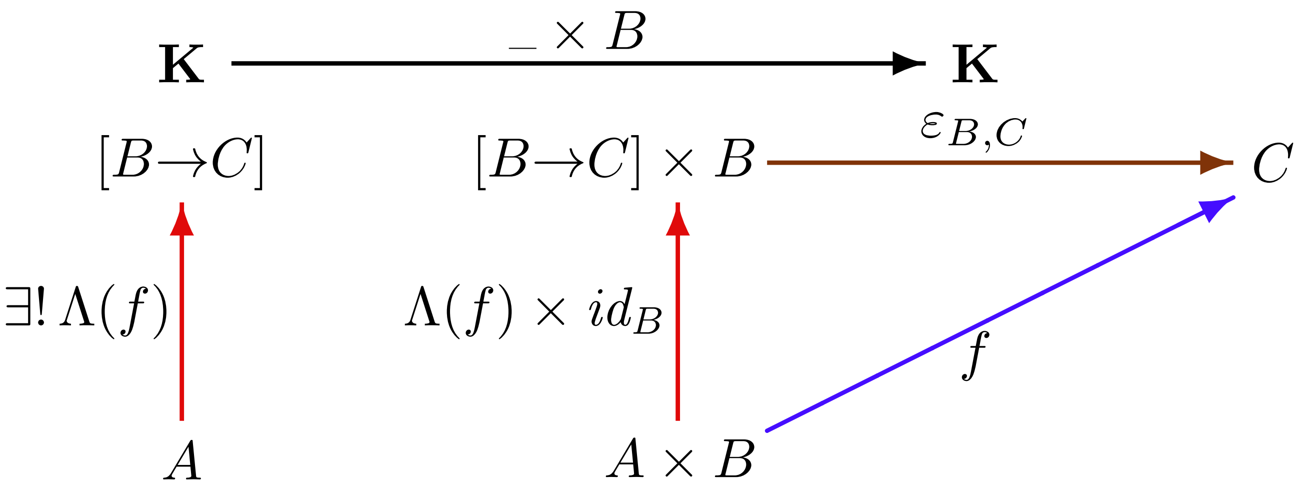

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Non-examples: $\mathbf{Pfn}$, $T^{op}_{\Sigma,\Phi}$.

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- for each $B \in |\mathbf{K}|$, $\_ \times B\colon \mathbf{K} \to \mathbf{K}$ has right adjoint $[B {\to} \_]\colon \mathbf{K} \to \mathbf{K}$, with counit given by $\varepsilon_{B,C}\colon [B{\to}C] \times B \to C$, for $C \in |\mathbf{K}|$.

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Contexts Γ are of the form:

- $x_1{:}\tau_1, \ldots, x_n{:}\tau_n$,
  where $n \geq 0$, $x_1, \ldots, x_n$ are distinct variables, and $\tau_1, \ldots, \tau_n \in \mathcal{T}$

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Weakening – context extension:

If $\Gamma \vdash M : \tau$ then $\Gamma, \Gamma' \vdash M : \tau$

– $[\![M]\!]^{\Gamma} \colon [\![\Gamma]\!] \to [\![\tau]\!]$,

– $[\![M]\!]^{\Gamma,\Gamma'} \colon [\![\Gamma, \Gamma']\!] \to [\![\tau]\!]$, and

$$[\![M]\!]^{\Gamma,\Gamma'} = \rho ; \pi_{[\![\Gamma]\!],[\![\Gamma']\!]} ; [\![M]\!]^{\Gamma}$$

where $\rho \colon [\![\Gamma, \Gamma']\!] \to [\![\Gamma]\!] \times [\![\Gamma']\!]$ is the obvious isomorphism.

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Easy: $\rho;[\![M]\!] \colon [\![\Gamma']\!] \times [\![\tau_i]\!] \to [\![\tau]\!]$,

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Easy: $\rho;[\![M]\!] \colon [\![\Gamma']\!] \times [\![\tau_i]\!] \to [\![\tau]\!]$, hence $\Lambda(\rho;[\![M]\!]) \colon [\![\Gamma']\!] \to [[\![\tau_i]\!]{\to}[\![\tau]\!]]$.

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– $\boxed{[\![\langle\rangle]\!] = \langle\rangle_{[\![\Gamma]\!]} \colon [\![\Gamma]\!] \to 1}$

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$$\frac{\Gamma \vdash M : \tau \qquad \Gamma \vdash N : \tau'}{\Gamma \vdash \langle M, N \rangle : \tau{\times}\tau'}$$

Given $[\![M]\!] \colon [\![\Gamma]\!] \to [\![\tau]\!]$ and $[\![N]\!] \colon [\![\Gamma]\!] \to [\![\tau']\!]$,

Semantics of λ-terms

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$$\frac{}{\Gamma \vdash \pi_{\tau,\tau'} : \tau{\times}\tau'{\to}\tau}$$

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Equational β, η-calculus

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Judgements:

$$\boxed{\Gamma \vdash M = N : \tau}$$

for $\tau \in \mathcal{T}$, $\Gamma \vdash M : \tau$, $\Gamma \vdash N : \tau$

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Axioms:

Equational β, η-calculus

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Axioms:

(β) $\Gamma \vdash (\lambda x{:}\tau.M)N = M[N/x] : \tau'$, for $\Gamma \vdash \lambda x{:}\tau.M : \tau \to \tau'$, $\Gamma \vdash N : \tau$

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Rules: reflexivity, symmetry, transitivity, congruence.

Soundness

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Given $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$

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Warning: The "real work" is in the proof of soundess for (β), where induction on the structure of terms is needed.

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SUMMING UP:

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