Category theory

for computer science

- *generality* • *abstraction* • *convenience* • *constructiveness* •

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Overall idea

look at all objects exclusively through relationships between them

capture relationships between objects as appropriate morphisms between them

(Cartesian) product

- *Cartesian product* of two sets A and B, is the set $A \times B = \{\langle a, b \rangle \mid a \in A, b \in B\}$ with projections $\pi_1 \colon A \times B \to A$ and $\pi_2 \colon A \times B \to B$ given by $\pi_1(\langle a, b \rangle) = a$ and $\pi_2(\langle a, b \rangle) = b$.

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$$A \xleftarrow{\pi_1} P \xrightarrow{\pi_2} B$$

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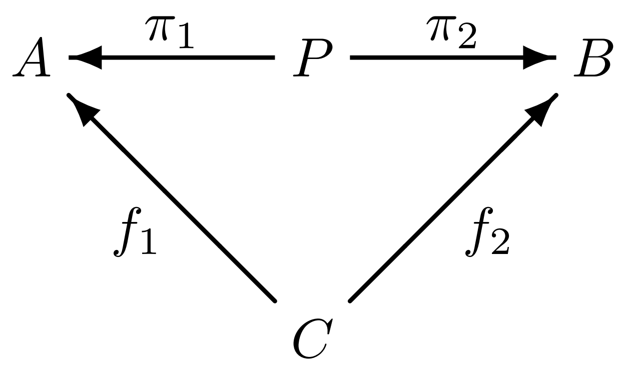

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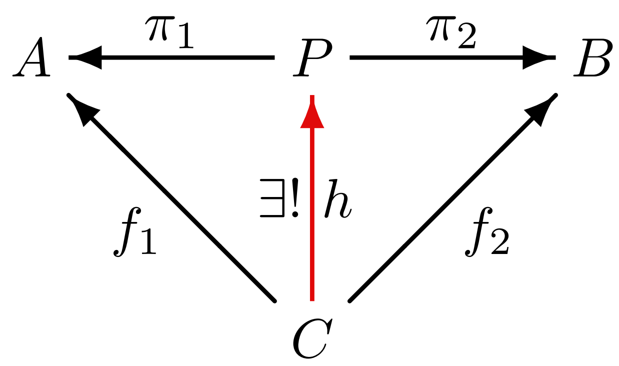

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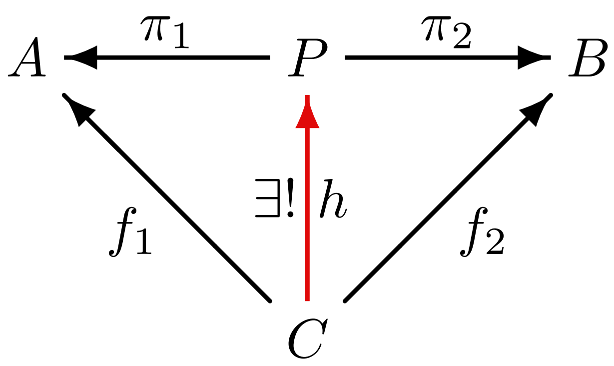
Theorem: *Cartesian product (of sets A and B) is a product (of A and B).*



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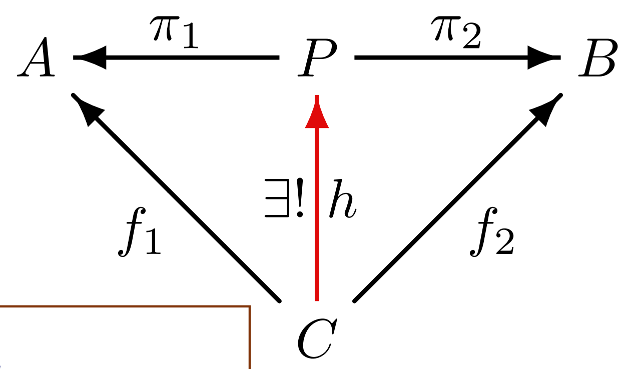

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Recall the definition of (Cartesian) product of Σ-algebras.
Define product of Σ-algebras as above. *What have you changed?*

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$\mathbf{K}$ is *locally small* if for all $A, B \in |\mathbf{K}|$, $\mathbf{K}(A, B)$ is a set.
$\mathbf{K}$ is *small* if in addition $|K|$ is a set.

Presenting finite categories

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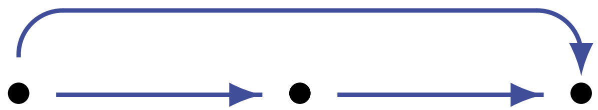
3: 

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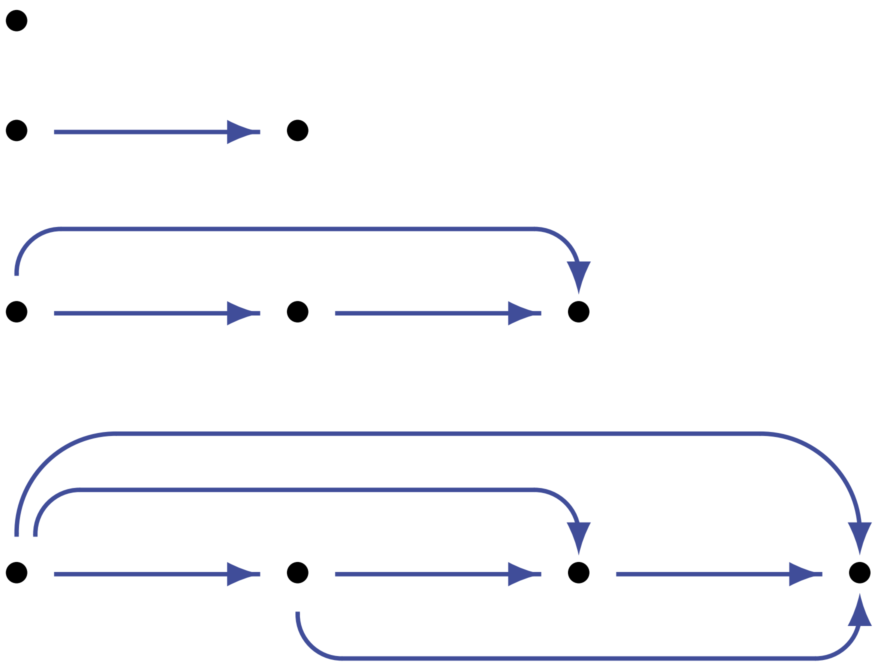

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- reflexivity: $x \leq x$
- transitivity: if $x \leq y$ and $y \leq z$ then $x \leq z$

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- For any signature Σ, partial Σ-algebras (as objects) and their weak homomorphisms (as morphisms) form the category $\mathbf{PAlg}(\Sigma)$.

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- Algebraic signatures (as objects) and their morphisms (as morphisms) with the composition defined in the obvious way form the category $\mathbf{AlgSig}$.

Substitutions

For any signature $\Sigma = (S, \Omega)$, the category of Σ-substitutions $\mathbf{Subst}_\Sigma$ is defined as follows:

- objects of $\mathbf{Subst}_\Sigma$ are S-sorted sets (of variables);
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Subcategories

Given a category $\mathbf{K}$, a *subcategory* of $\mathbf{K}$ is any category $\mathbf{K}'$ such that

- $|\mathbf{K}'| \subseteq |\mathbf{K}|$,
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- The category $\mathbf{FinSet}$ of finite sets is a full subcategory of $\mathbf{Set}$.
- The discrete category of sets is a subcategory of the category of sets with inclusions as morphisms, which is a subcategory of the category of sets with injective functions as morphisms, which is a subcategory of $\mathbf{Set}$.
- The category of single-sorted signatures is a full subcategory of $\mathbf{AlgSig}$.

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Theorem: *If a property W holds for all categories then co-W holds for all categories as well.*

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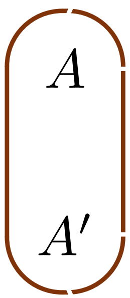

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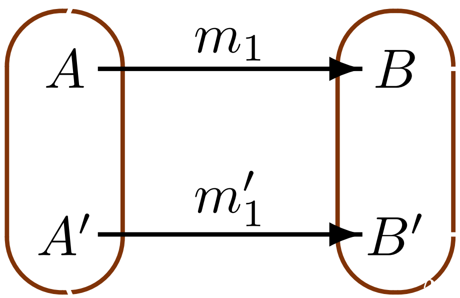

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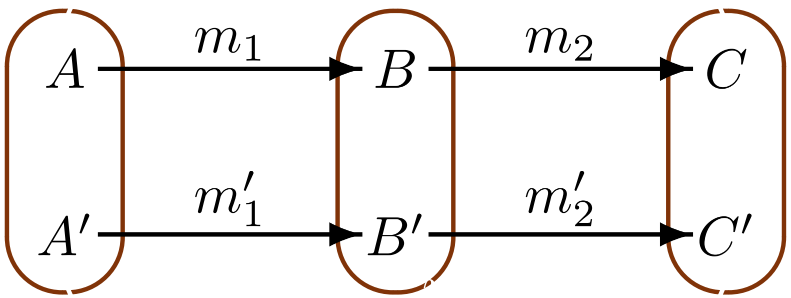

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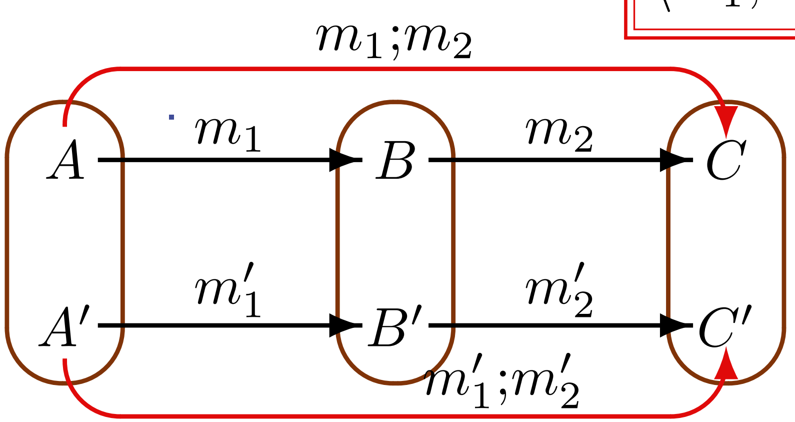

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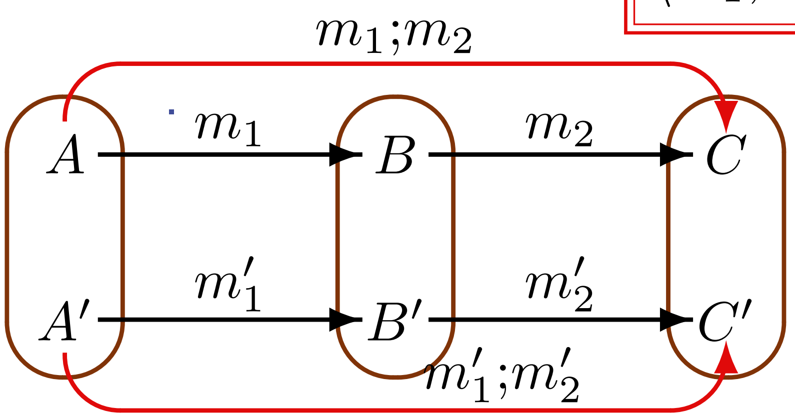

Define $\mathbf{K}^n$, where $\mathbf{K}$ is a category and $n \geq 1$.
Extend this definition to $n = 0$.

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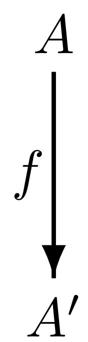

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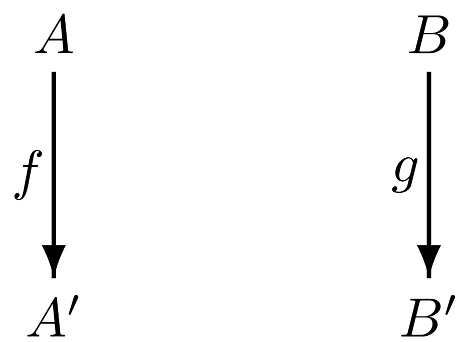

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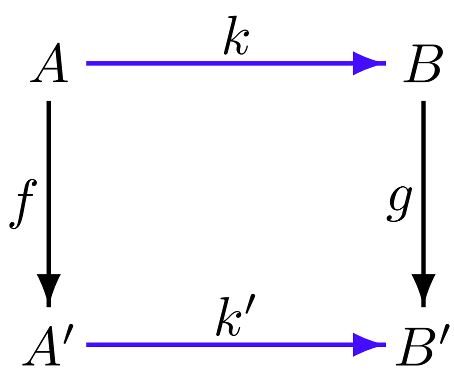

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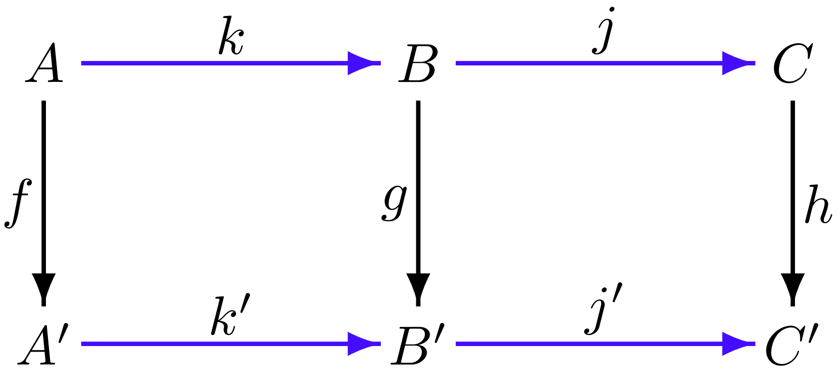

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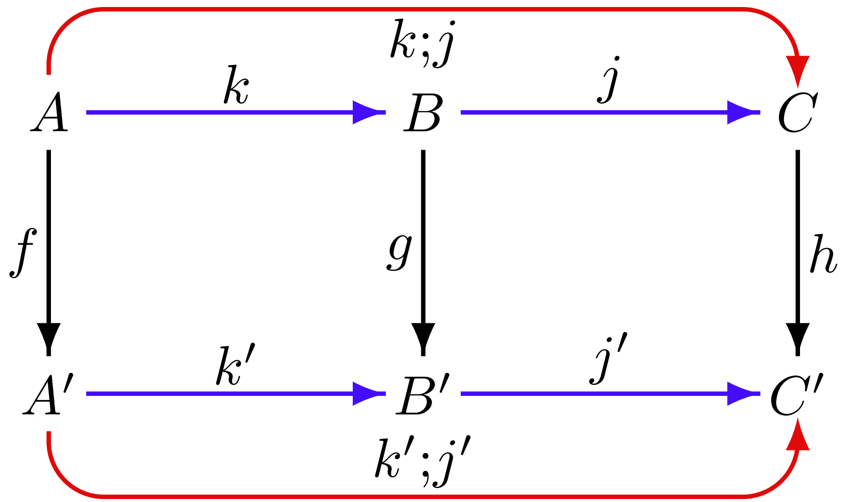

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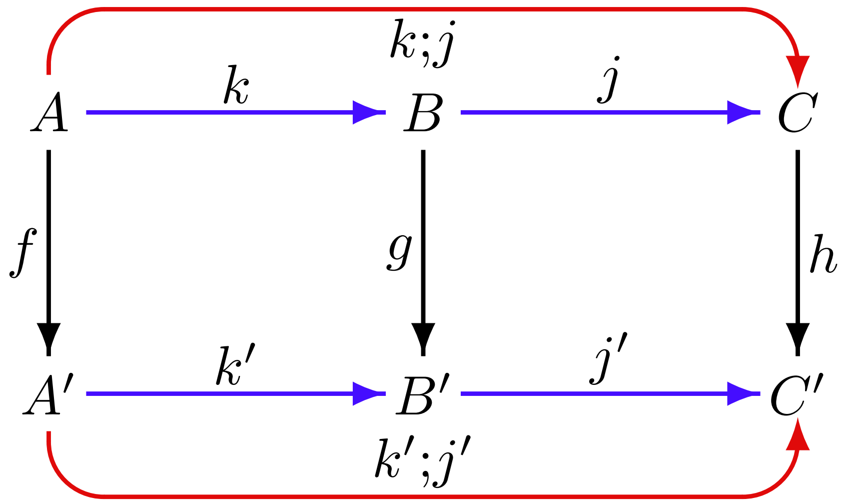
Check that the composition is well-defined.



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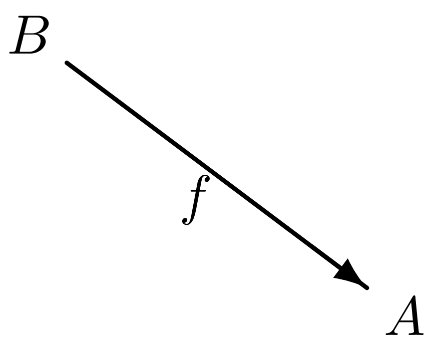

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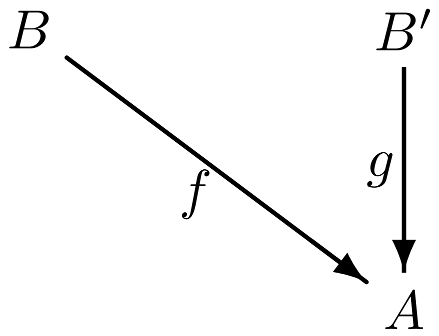

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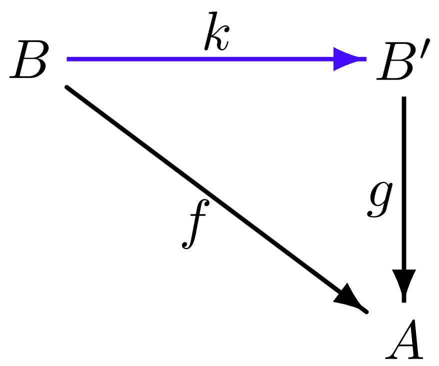

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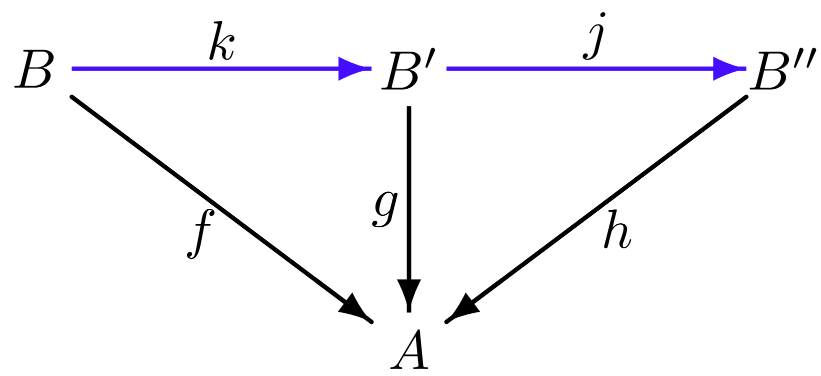

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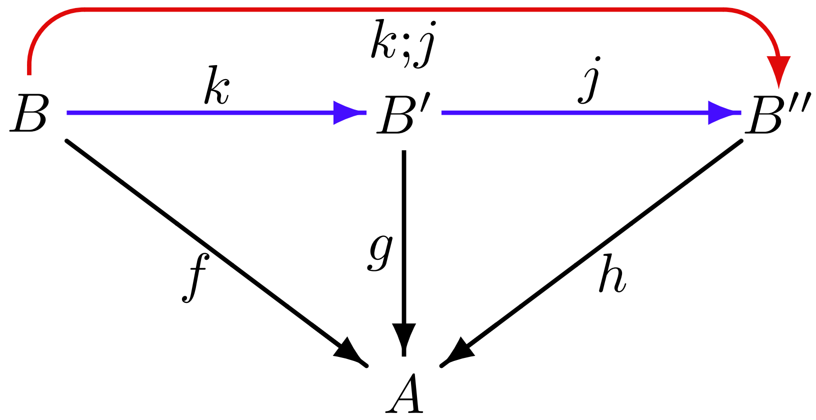

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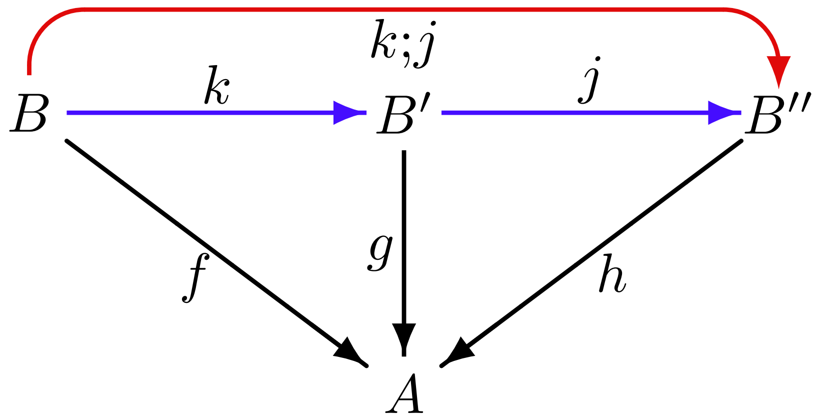

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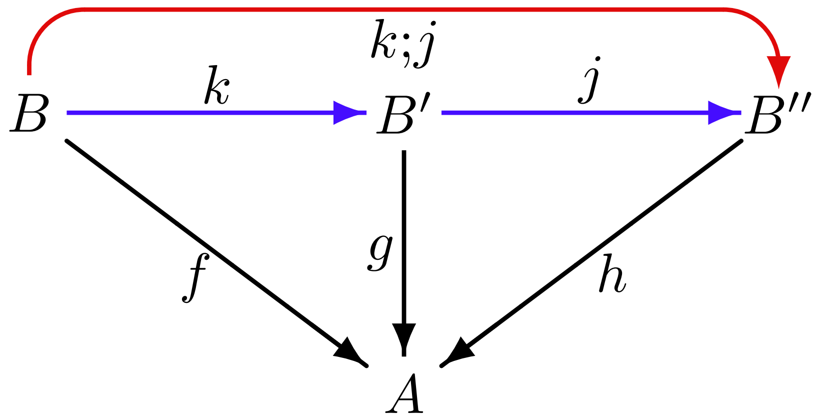

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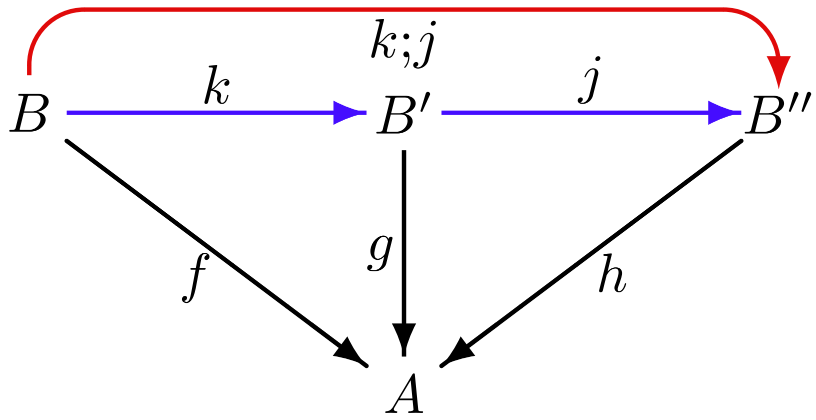

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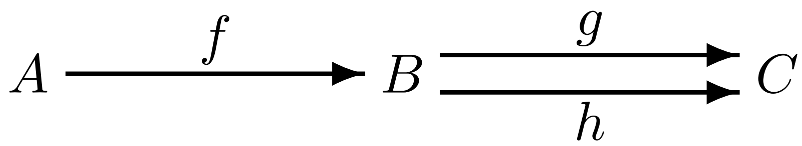

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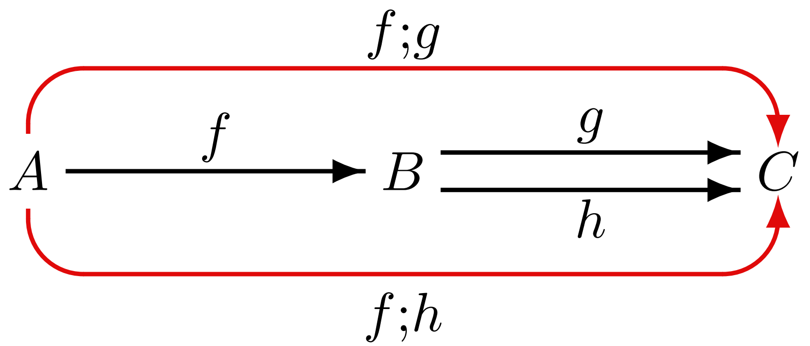

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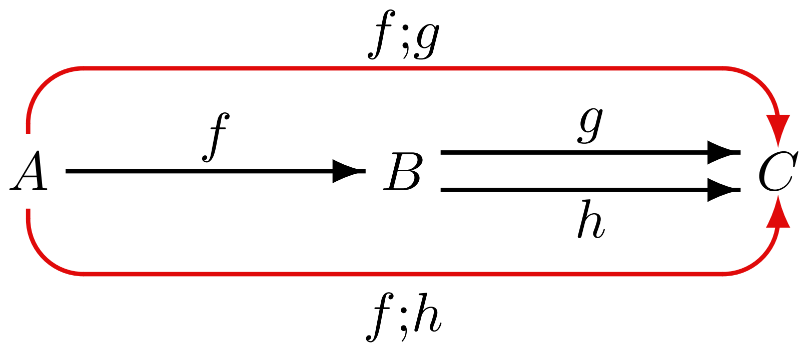

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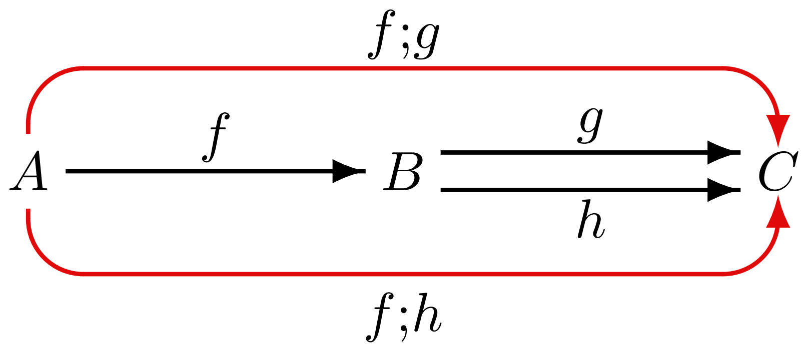

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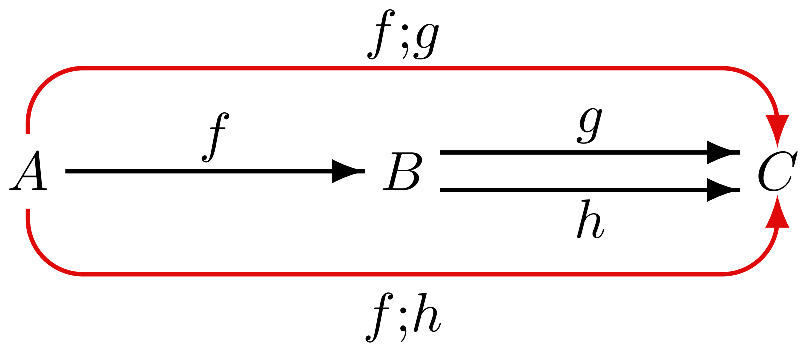

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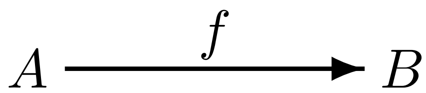

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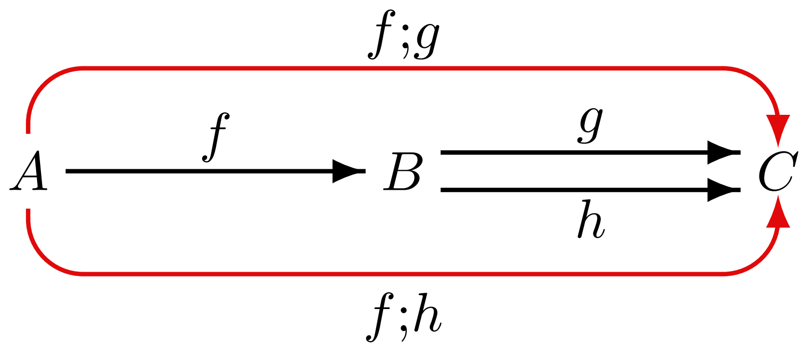

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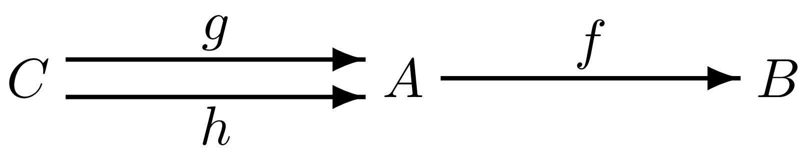

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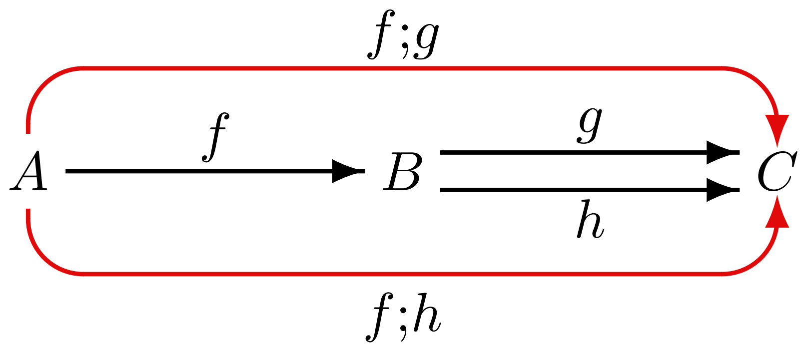

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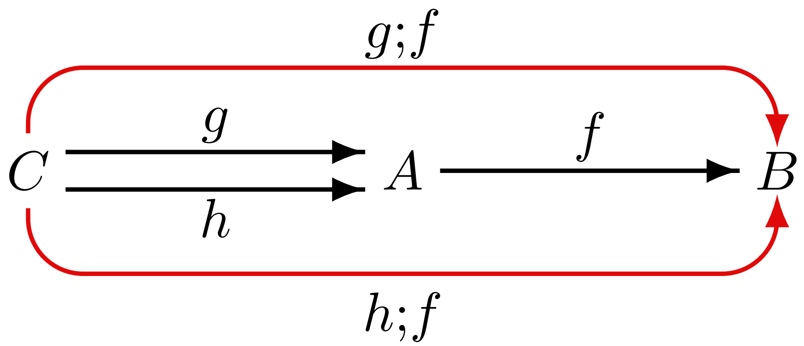

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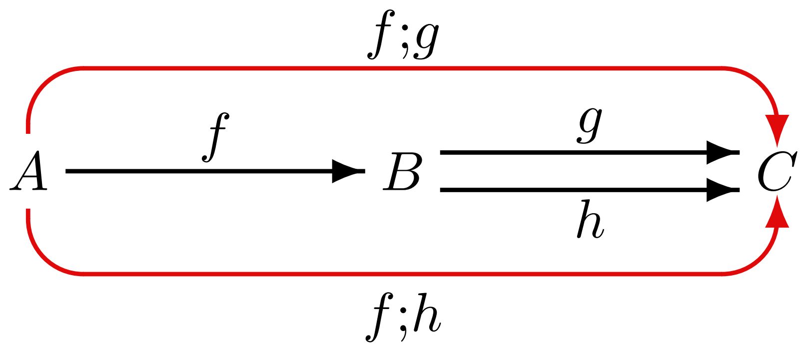

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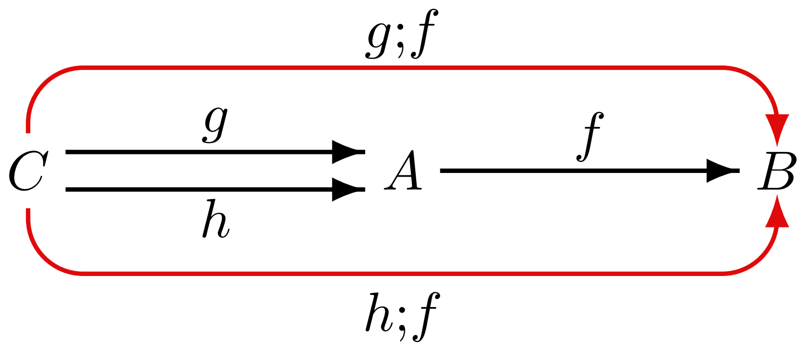

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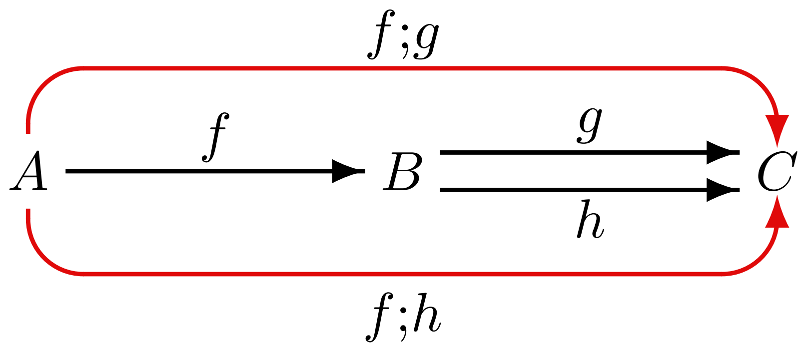

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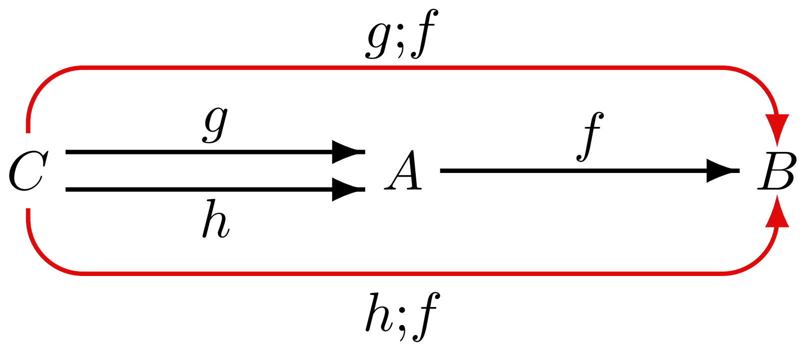

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Proof: Suppose f is epi and $f;g_2 = id_A$. Then $f;id_B = f = (f;g_2);f = f;(g_2;f)$.

Isomorphisms

$f\colon A \to B$ is an *isomorphism* (is *iso*)
if there is $g\colon B \to A$ such that $f;g = id_A$ and $g;f = id_B$.

Then g is the (unique) *inverse of* f, $g = f^{-1}$.

In $\mathbf{Set}$, a function is iso iff it is both epi and mono

Theorem: *If f is iso then it is both epi and mono.*

Give counterexamples to show that the opposite implication fails.

Theorem: *$f\colon A \to B$ is iso iff*

- *f is a retraction, i.e., there is $g_1\colon B \to A$ such that $g_1;f = id_B$, and*
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This yields $g_2;f = id_B$, and so g_2 is the inverse of f.

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Proof: $(i_1;i_2)^{-1} = (i_2)^{-1};(i_1)^{-1}$

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Dualise!