Szymon Toruńczyk

University of Warsaw

STRUCTURALLY TRACTABLE GRAPH CLASSES

Szymon Toruńczyk

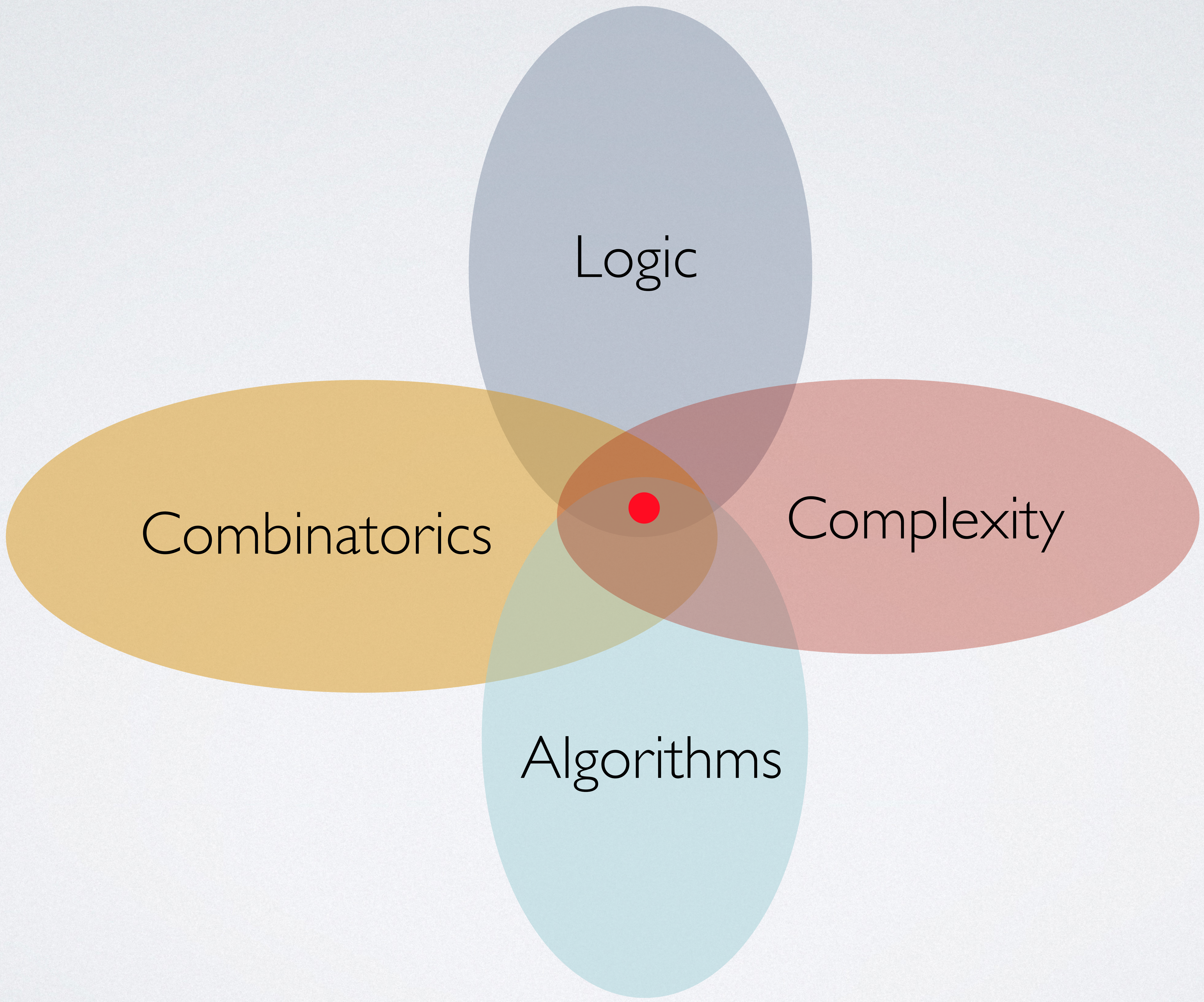
University of Warsaw

MONADICALLY DEPENDENT GRAPH CLASSES

ALGORITHMIC MODEL THEORY

algorithmic
model
theory

ALGORITHMIC MODEL THEORY



OUTLINE

1. The model checking problem
2. Sparsity: monotone case
3. Twin-width: ordered case
4. Monadic dependence
5. Flip-breakability
6. Stability: orderless case

OUTLINE

1. **The model checking problem**
2. Sparsity: monotone case
3. Twin-width: ordered case
4. Monadic dependence
5. Flip-breakability
6. Stability: orderless case

FIRST-ORDER MODEL CHECKING

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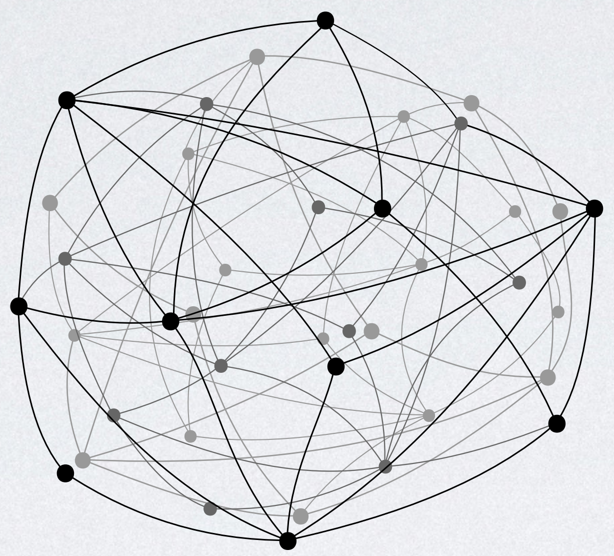
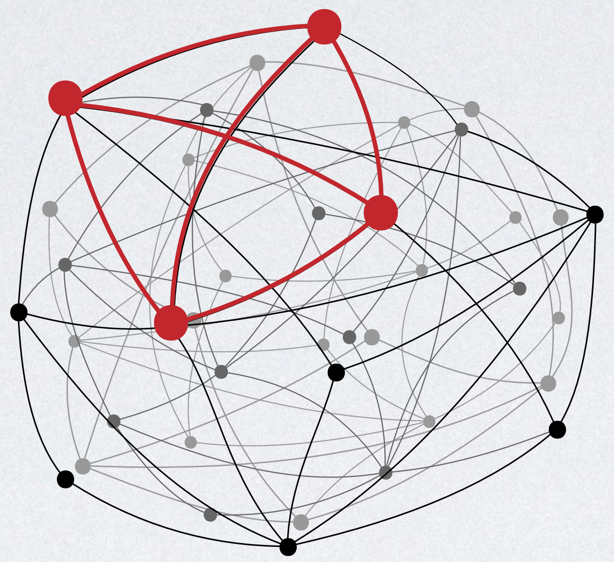


$$\overset{?}{\models} \quad \varphi$$

structure

first-order formula

FIRST-ORDER MODEL CHECKING



$\overset{?}{\models}$ φ

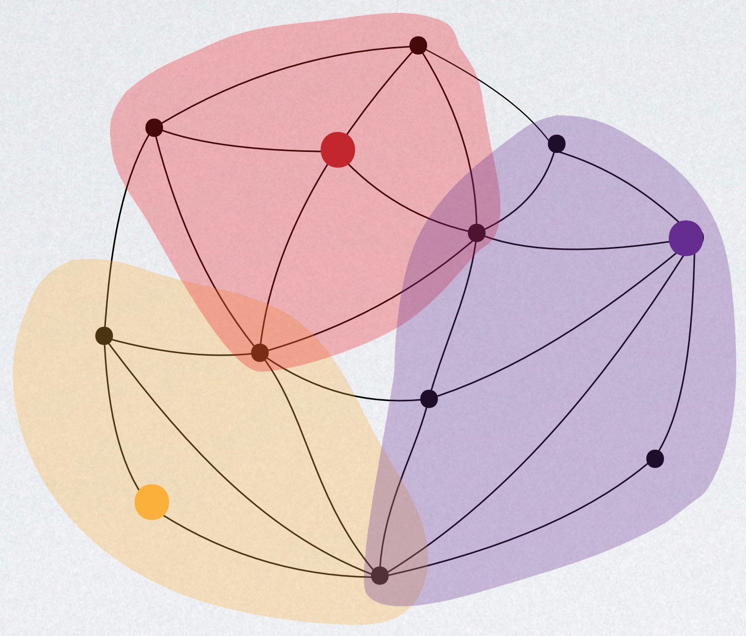
structure

first-order formula

e.g. $\varphi = \exists x.\ \exists y.\ \exists z.\ \exists t.\ (x{\sim}y) \wedge (y{\sim}z) \wedge (z{\sim}t) \wedge (t{\sim}x) \wedge (x{\sim}z) \wedge (y{\sim}t)$

"Is there a clique of size 4?"

FIRST-ORDER MODEL CHECKING



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"Is there a clique of size 4?"

e.g. $\varphi = \exists x.\, \exists y.\, \exists z.\, \forall t.\, (x{\sim}t) \vee (y{\sim}t) \vee (z{\sim}t)$

"Are there 3 nodes whose neighborhoods include all nodes?"

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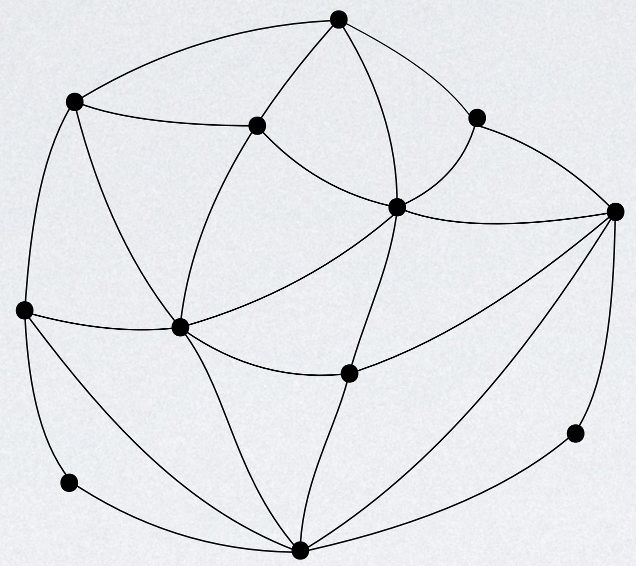

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A fundamental problem in TCS

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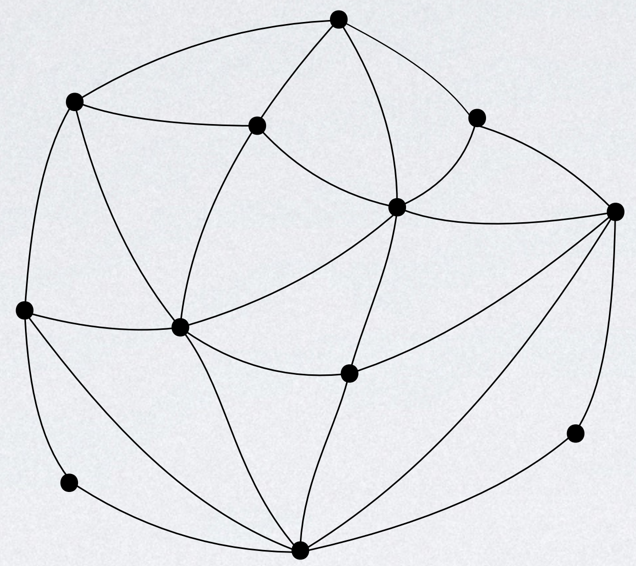

$\overset{?}{\models} \quad \varphi$

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A fundamental problem in TCS

Central in

→ database theory

→ software verification (for other logics)

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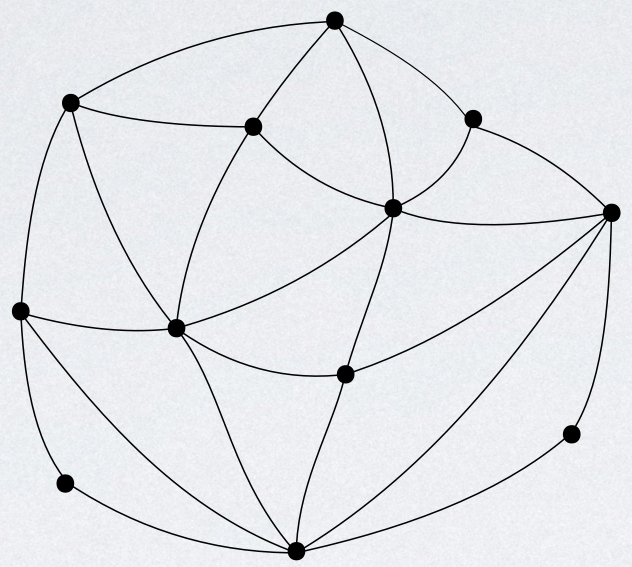
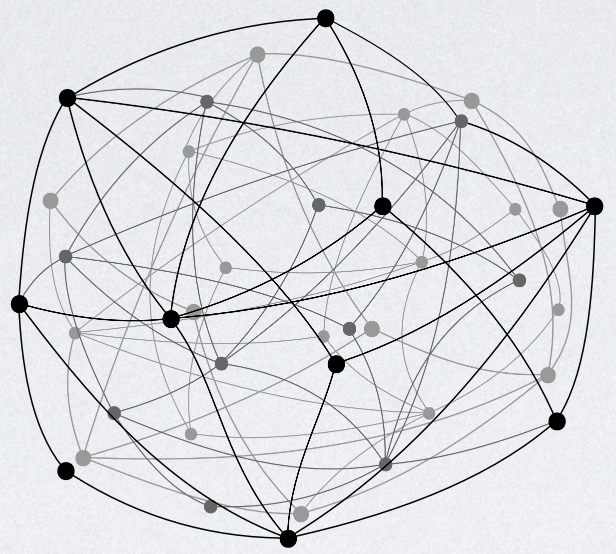


structure

φ

first-order
formula

QUERY EVALUATION



database

Q

SQL query

```
SELECT e1.x AS a, e2.x AS b, e3.x AS c, e4.x AS d
FROM edges e1 JOIN edges e2 ON e2.x = e1.x
JOIN edges e3 ON e3.x = e1.x
JOIN edges e4 ON e4.y = e1.y
JOIN edges e5 ON e5.x = e1.y
JOIN edges e6 ON e6.x = e2.y AND e6.y = e1.y
```

select all 4-cliques

Goal: efficiently evaluate queries in large databases

FIRST-ORDER MODEL CHECKING

PARAMETERISED COMPLEXITY

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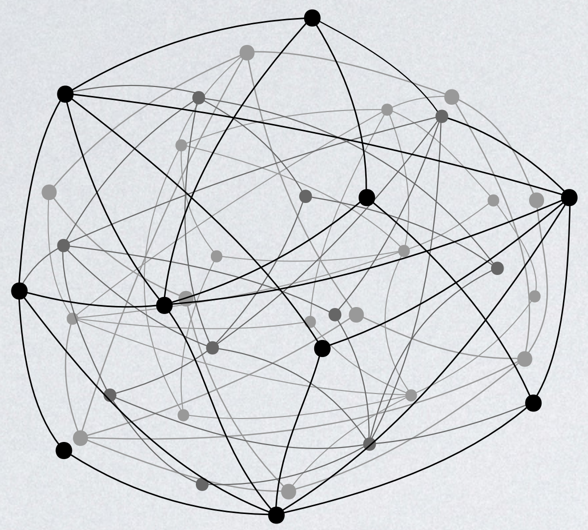
PARAMETERISED COMPLEXITY



General input graphs *highly intractable*: AW[$*$]-hard

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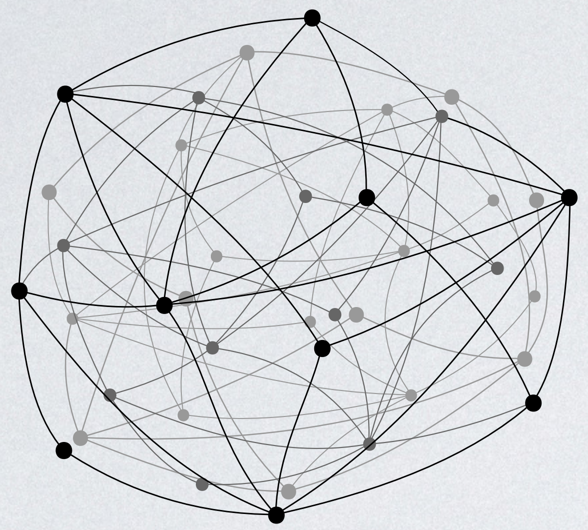

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Does an n-vertex graph contain a clique of size k=4 ?

Naive algorithm: $O(n^k) = O(n^4)$ time

Best known algorithm: $O(n^{0.79k}) = O(n^{3.16})$ time

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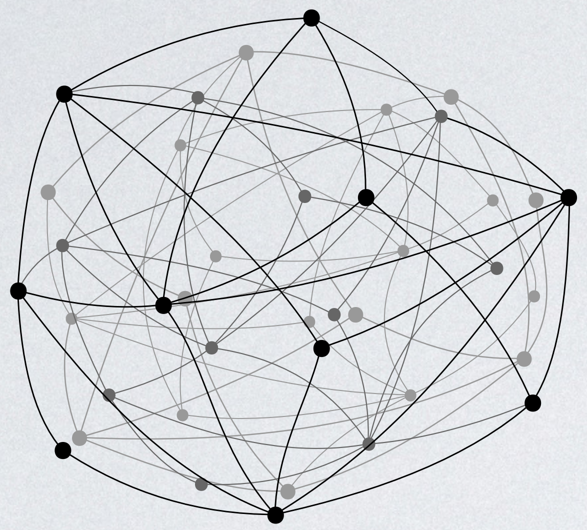

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Impractical for n=100,000

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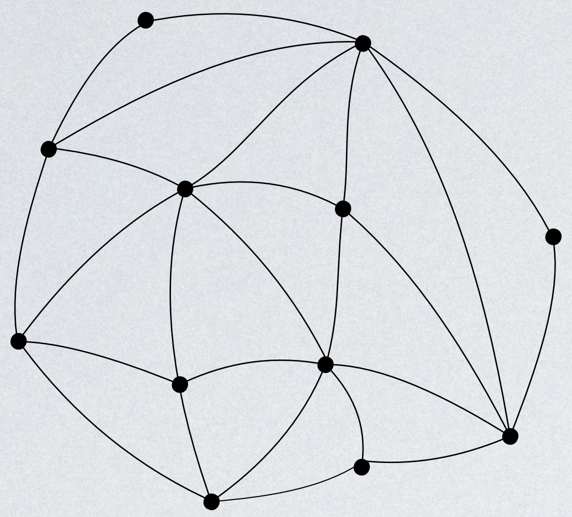

Theorem [Frick, Grohe, 2001]

Every first-order property can be tested in **linear time** on planar graphs.

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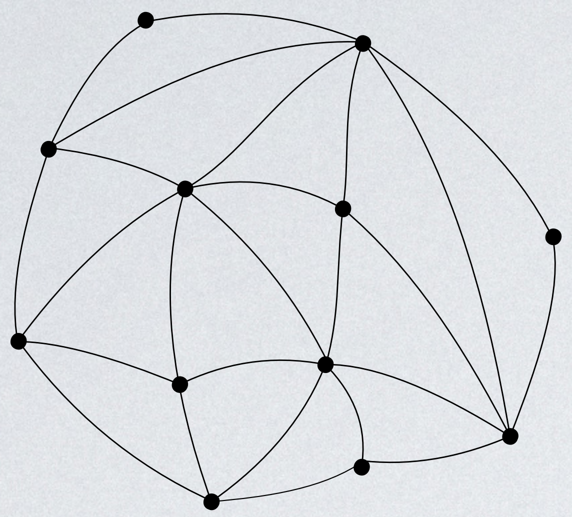

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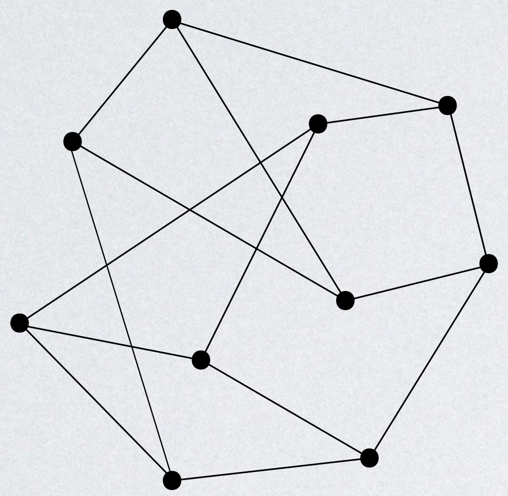

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Same for graphs of maximum degree ≤ Δ, for any fixed Δ

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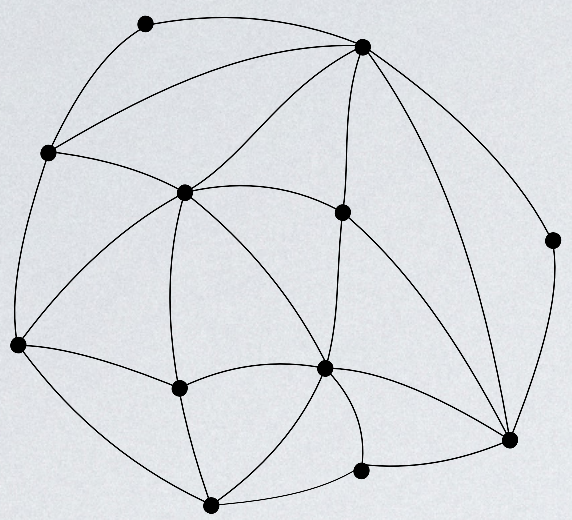

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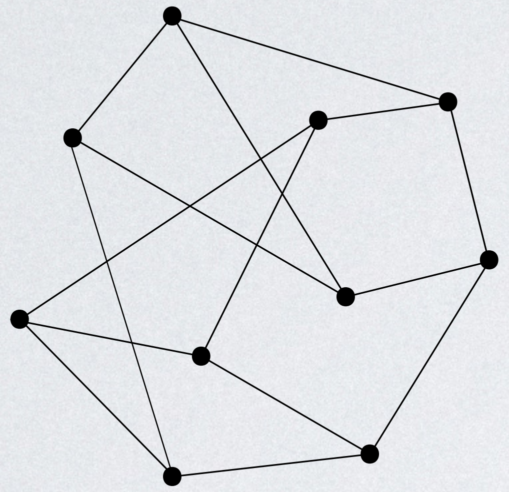

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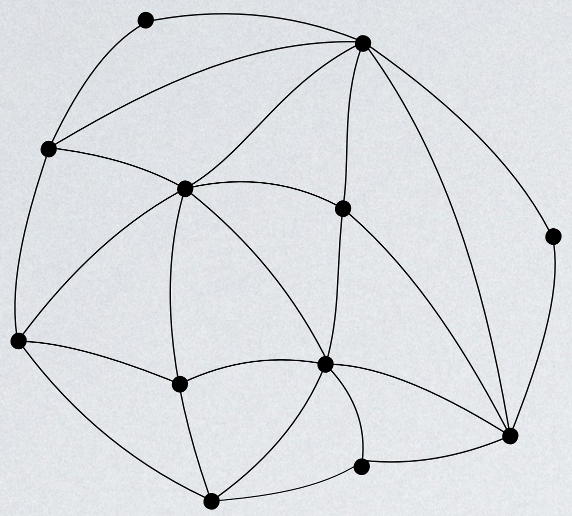

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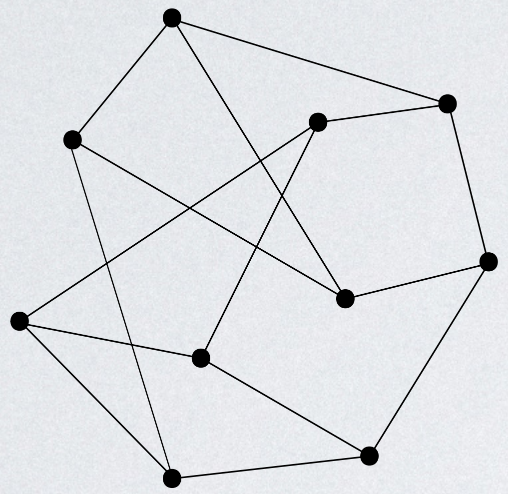

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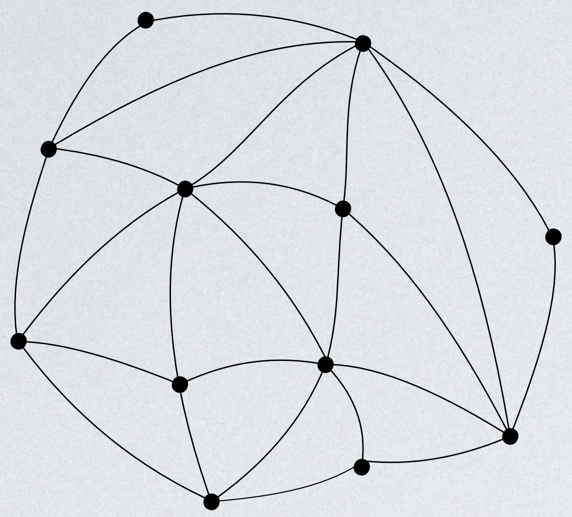

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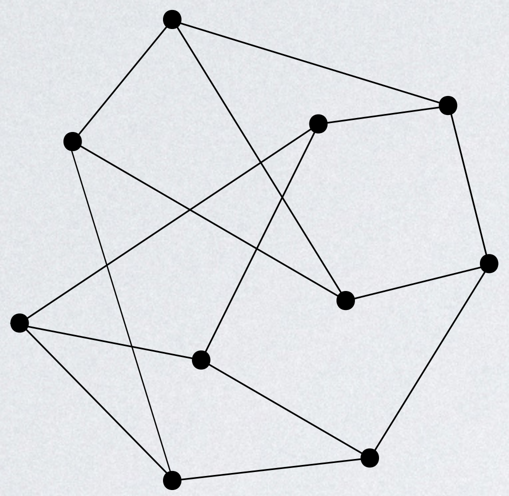

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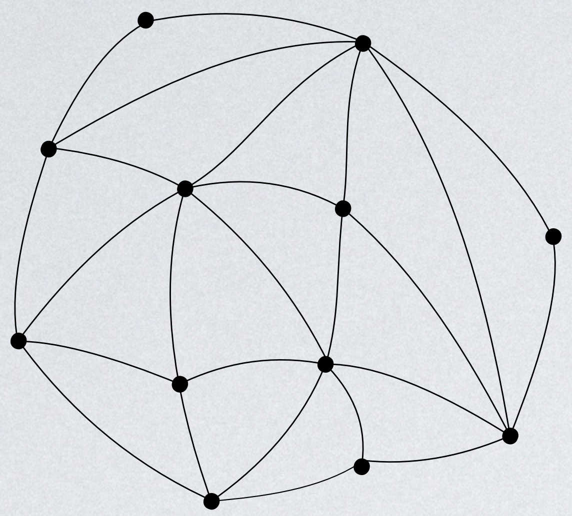

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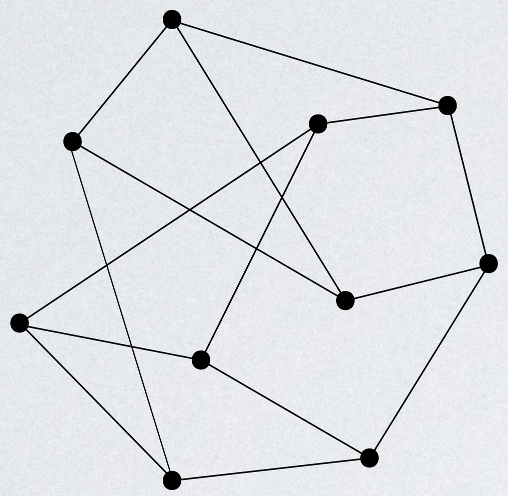

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$$O_{\phi,C}(\ |G|^{d}\ )$$

constant d independent of ϕ.

MSO MODEL CHECKING

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Courcelle's theorem (1990)

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Ingredients:

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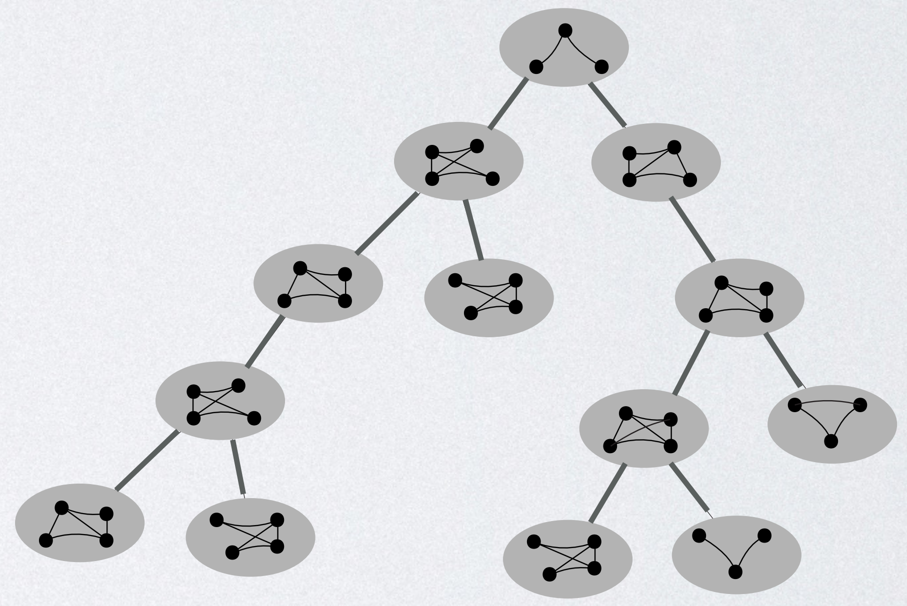

1. existence of tree decompositions (by definition)
2. efficient computation of decomposition (Bodlaender)

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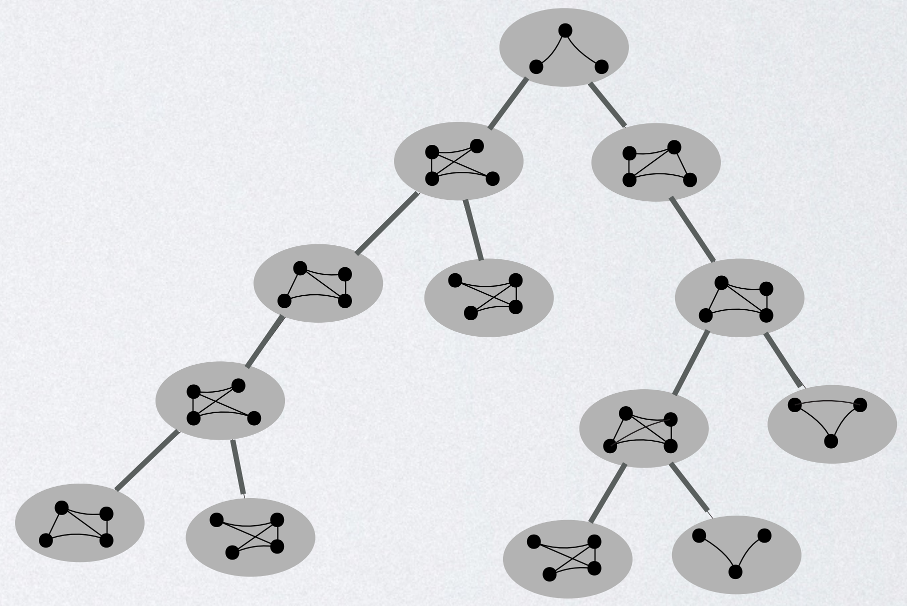

Ingredients:

1. existence of tree decompositions (by definition)
2. efficient computation of decomposition (Bodlaender)
3. dynamic algorithm computing partial solutions to formulas (Courcelle)

FIRST-ORDER MODEL CHECKING

PARAMETERISED COMPLEXITY

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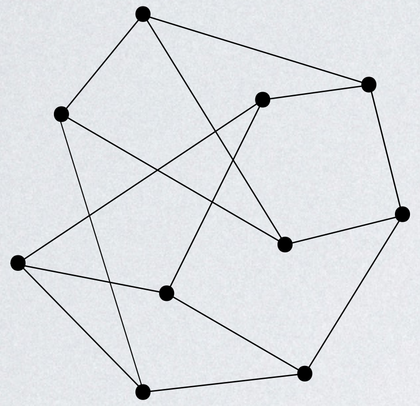
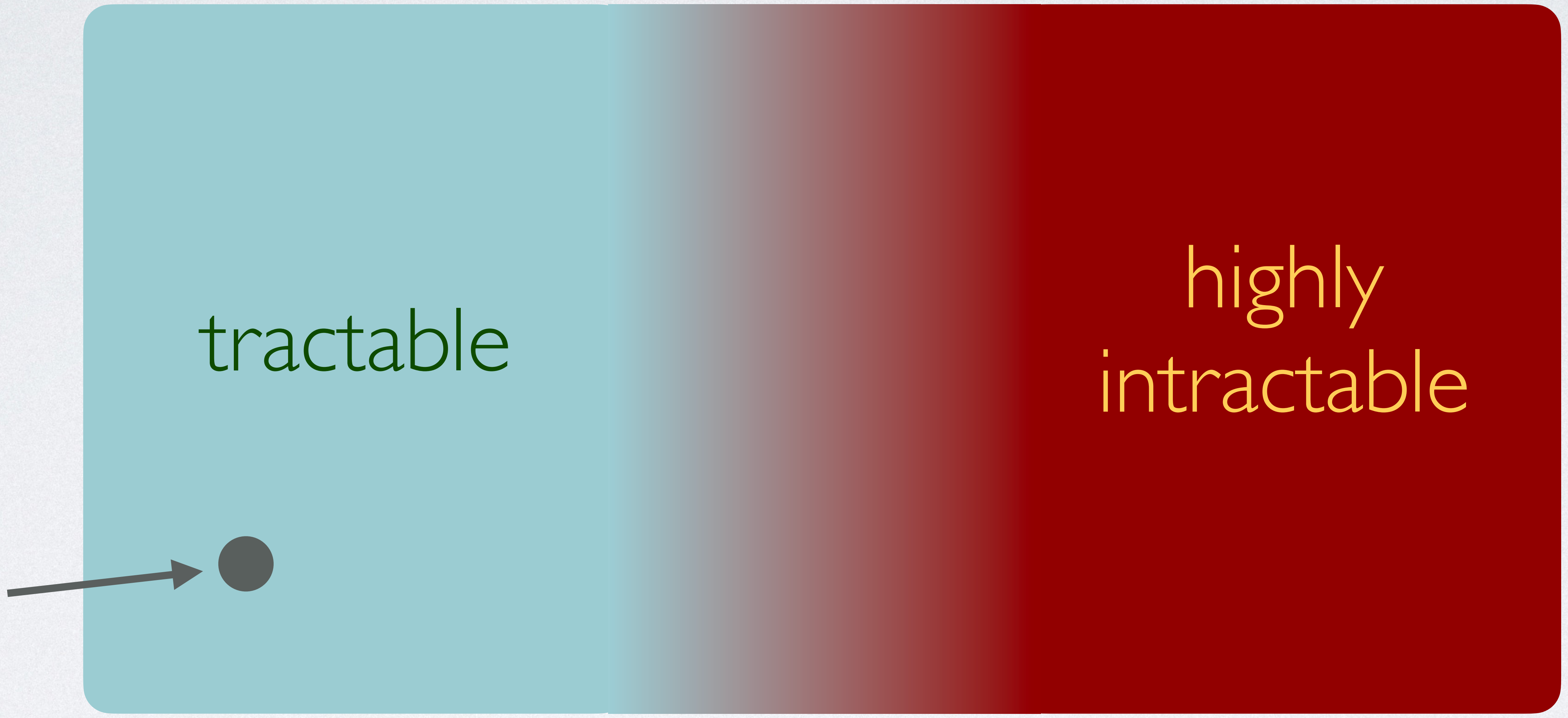


class of
max. degree $\leq\Delta$



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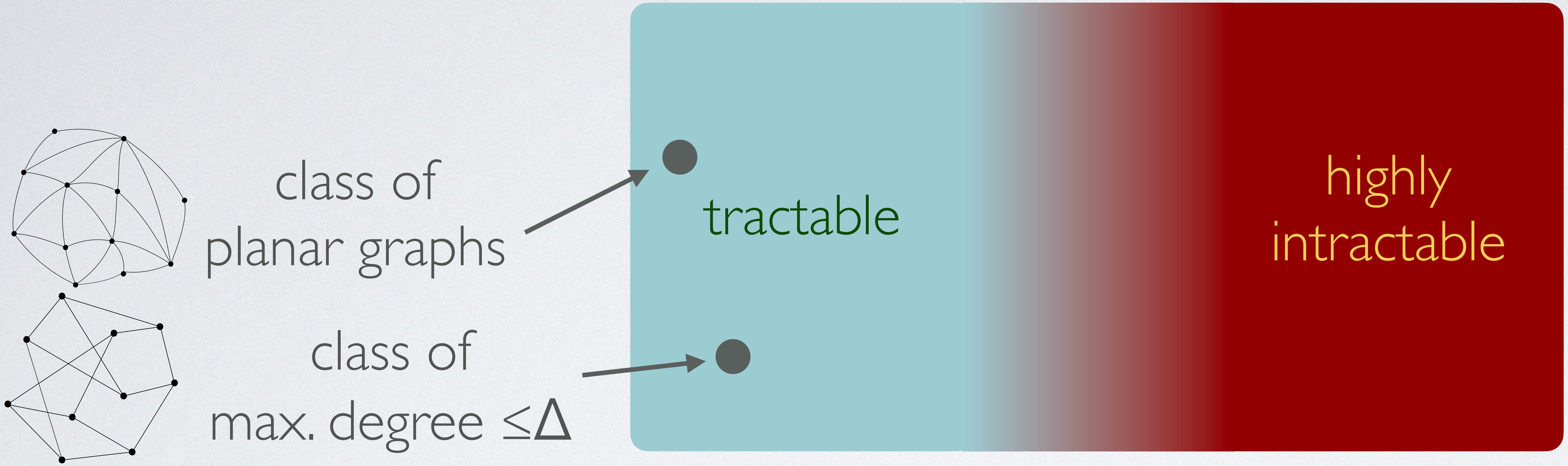

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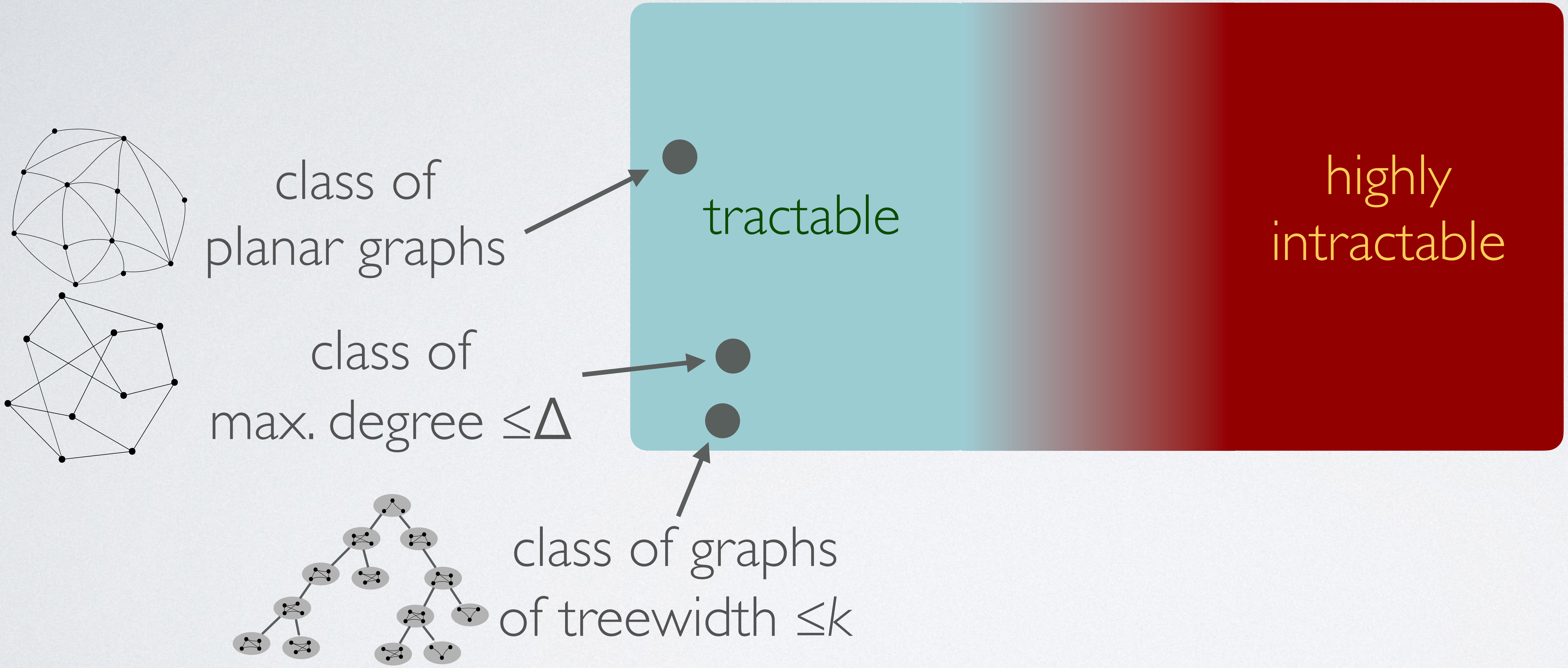

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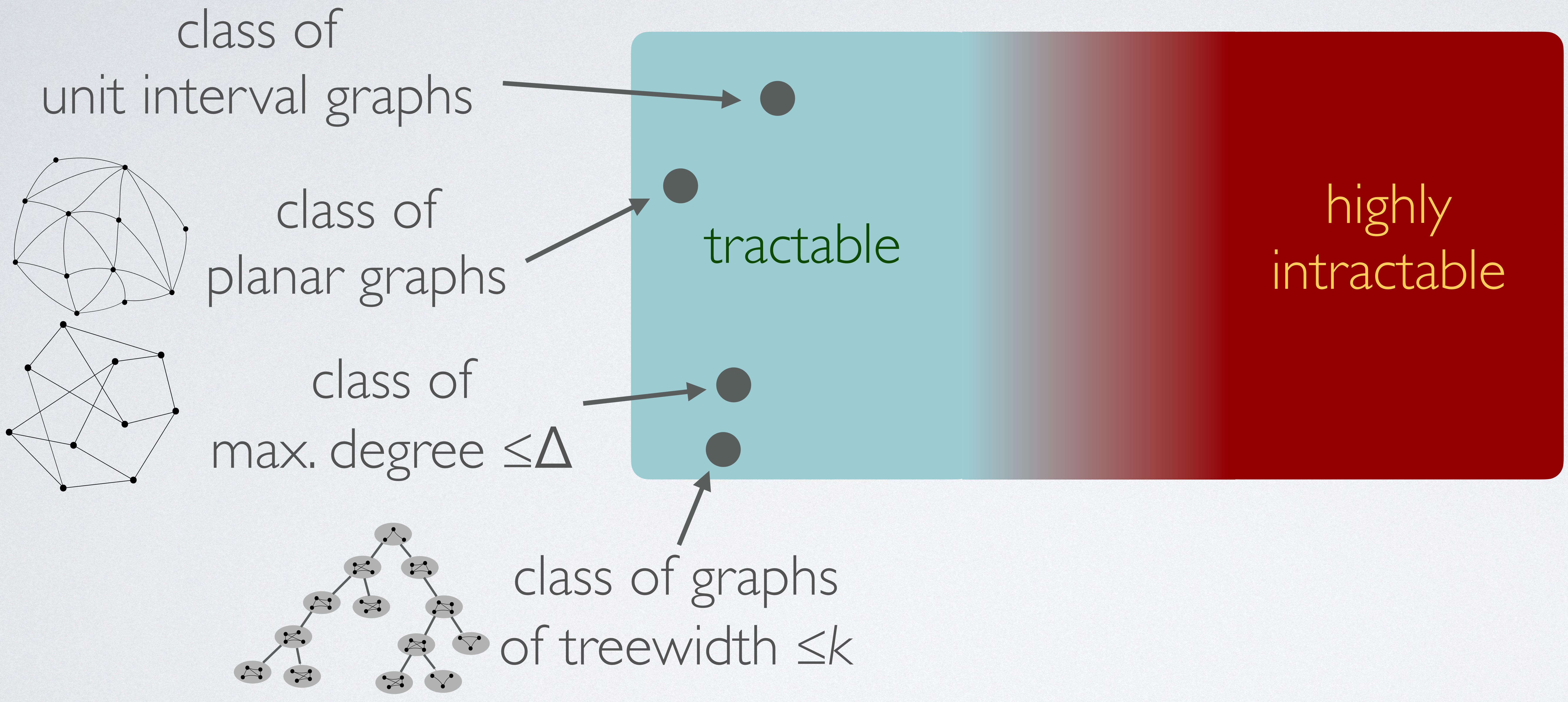

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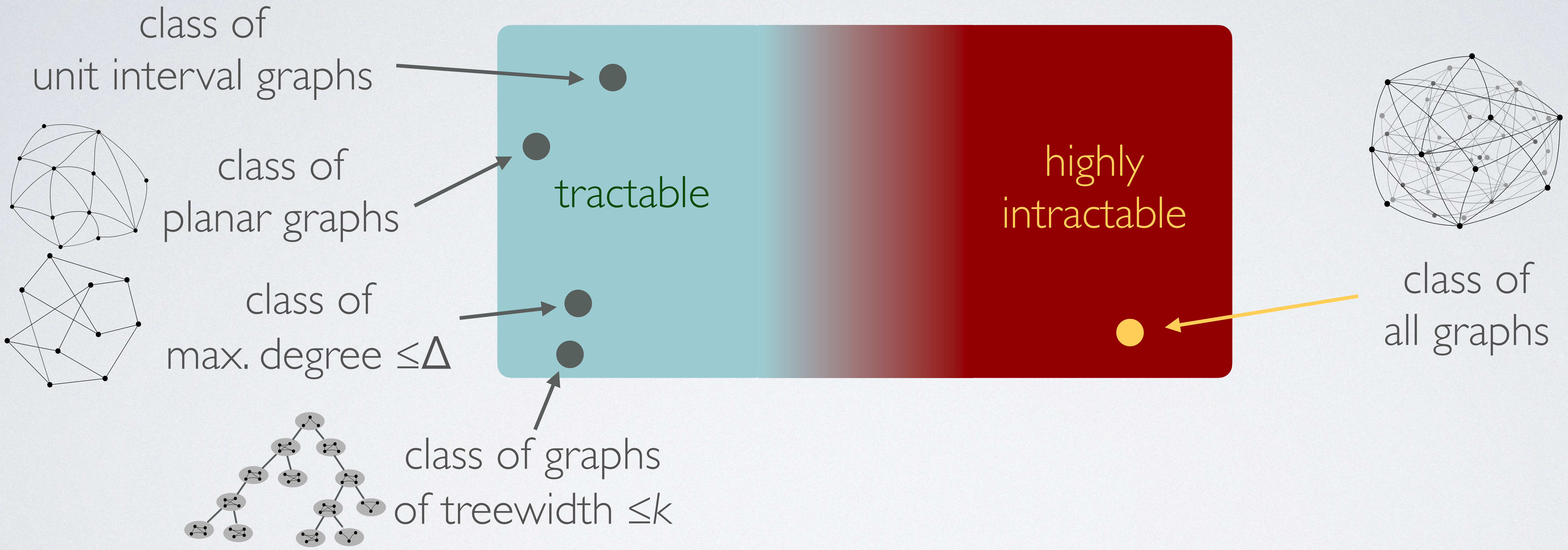

FIRST-ORDER MODEL CHECKING

PARAMETERISED COMPLEXITY

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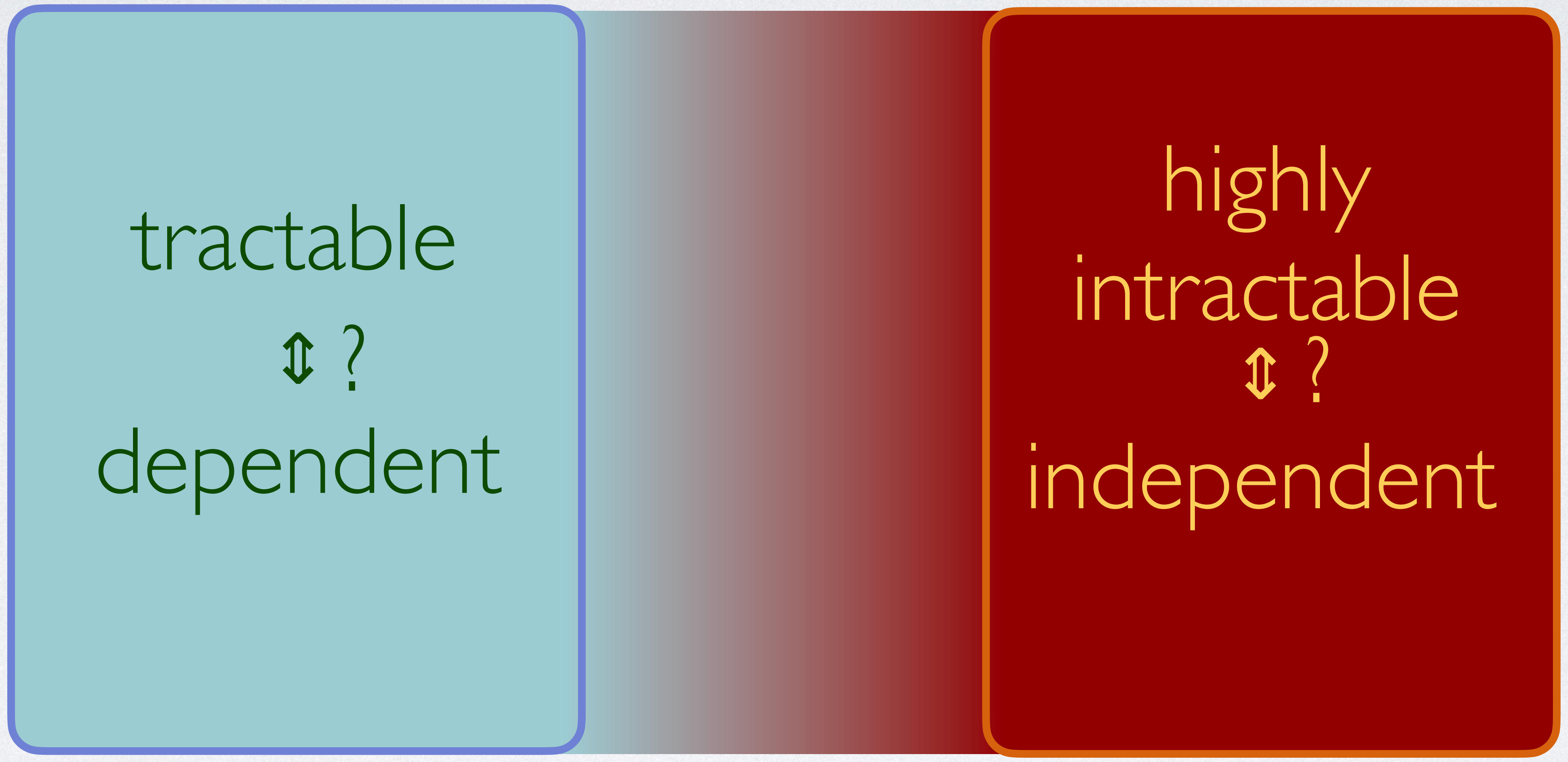
Quest: Characterise all hereditary graph classes with tractable FO model checking

FIRST-ORDER MODEL CHECKING

PARAMETERISED COMPLEXITY



Quest: Characterise all hereditary graph classes with tractable FO model checking

Conjecture: Those are exactly the *dependent* graph classes

FIRST-ORDER MODEL CHECKING

PARAMETERISED COMPLEXITY

tractable
$\Updownarrow$?
dependent

highly
intractable
$\Updownarrow$?
independent

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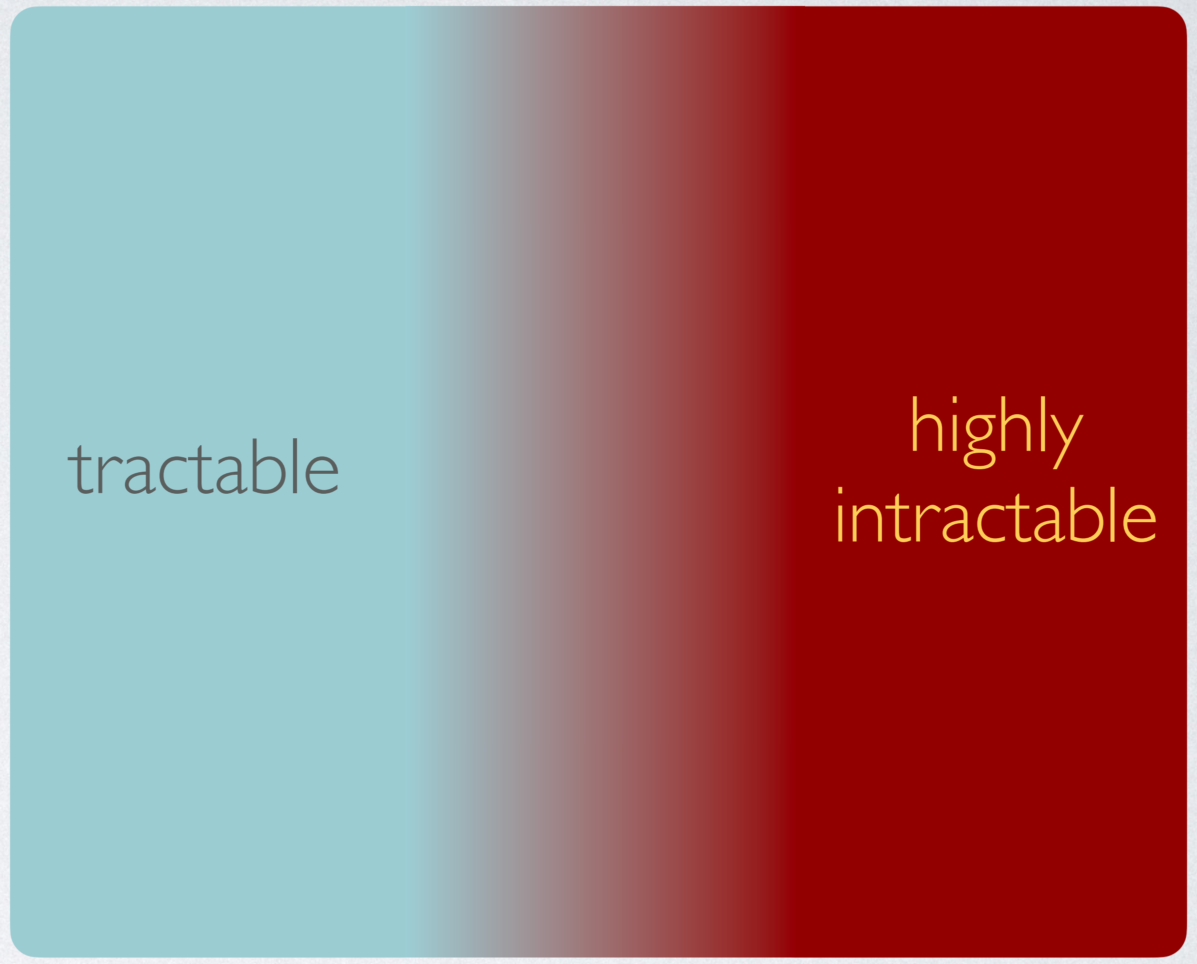

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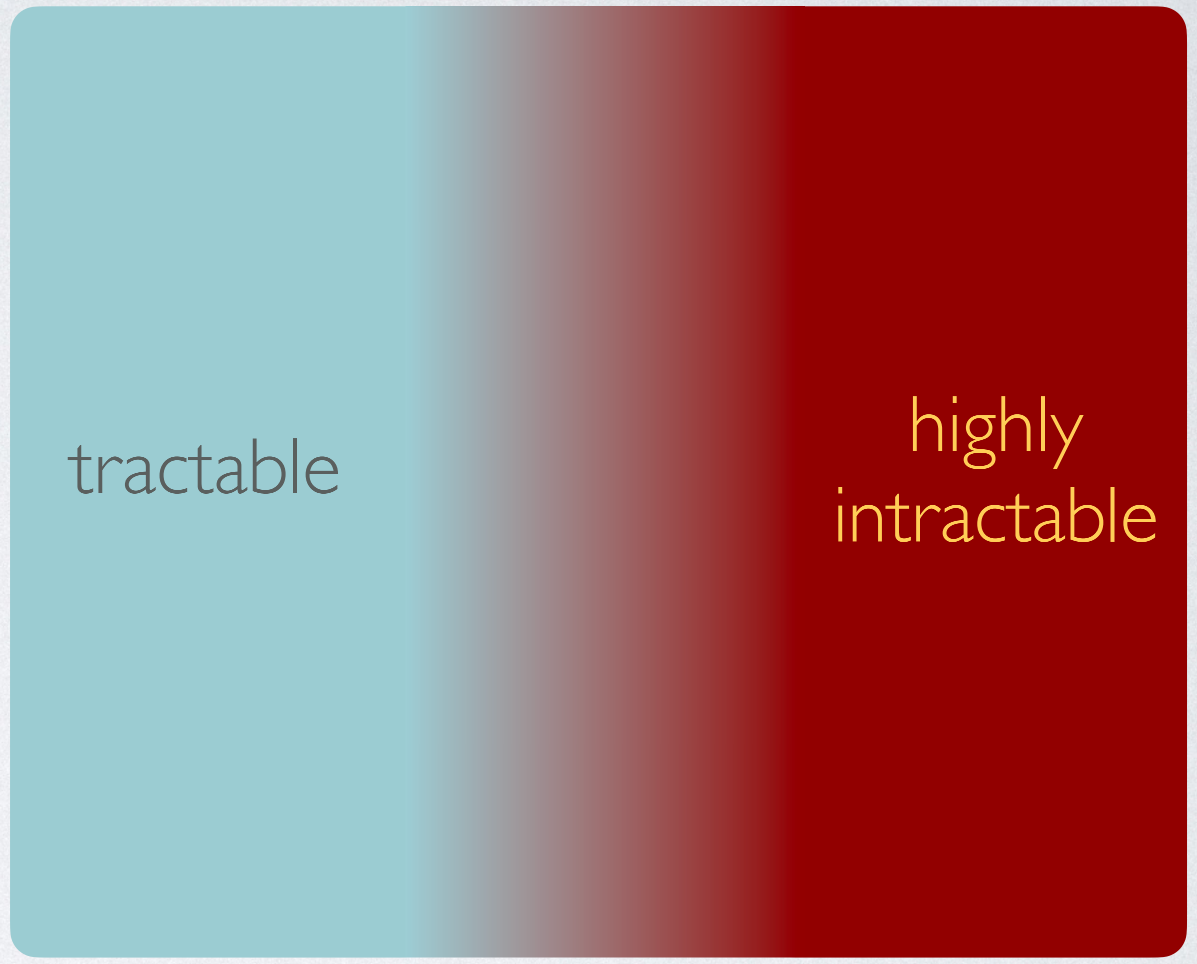

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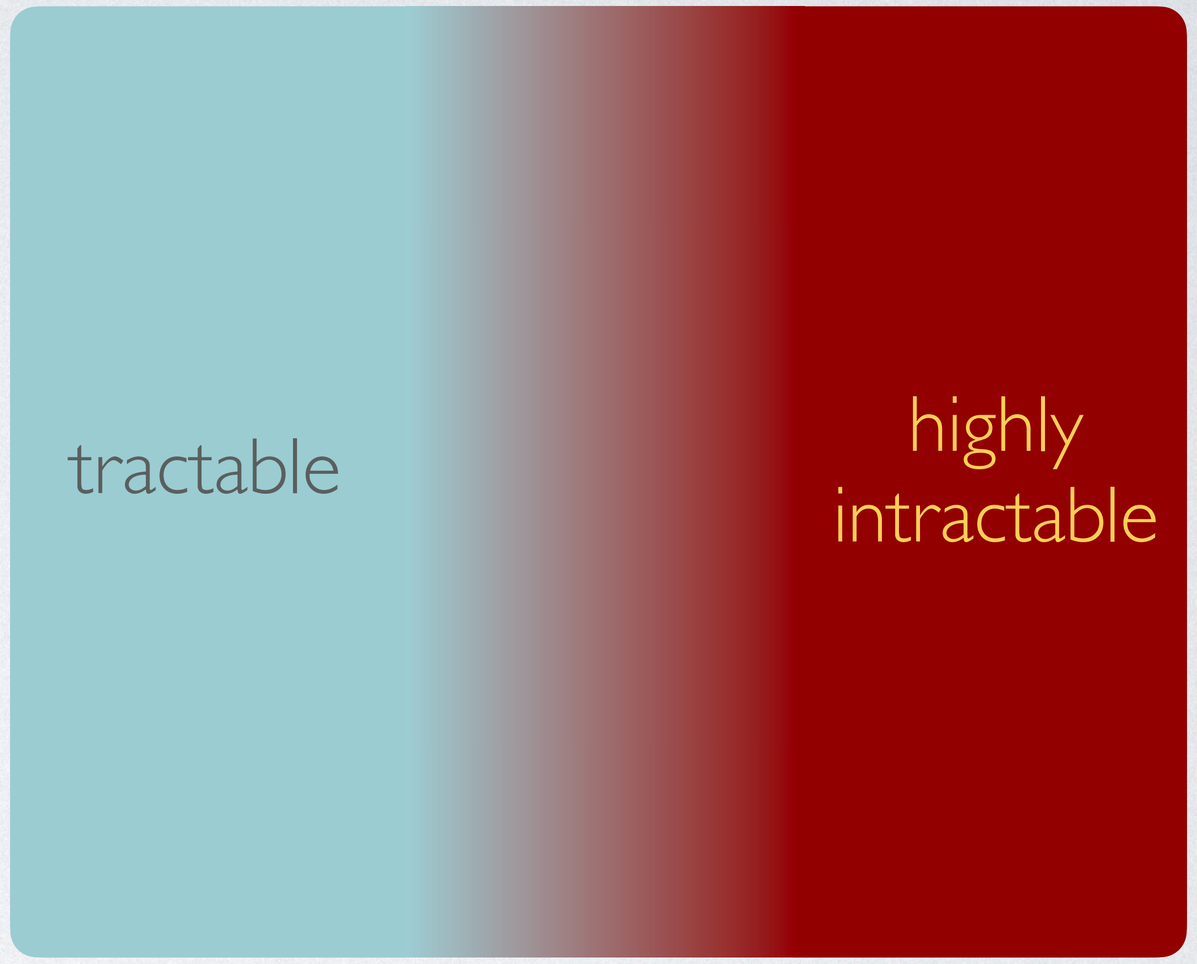

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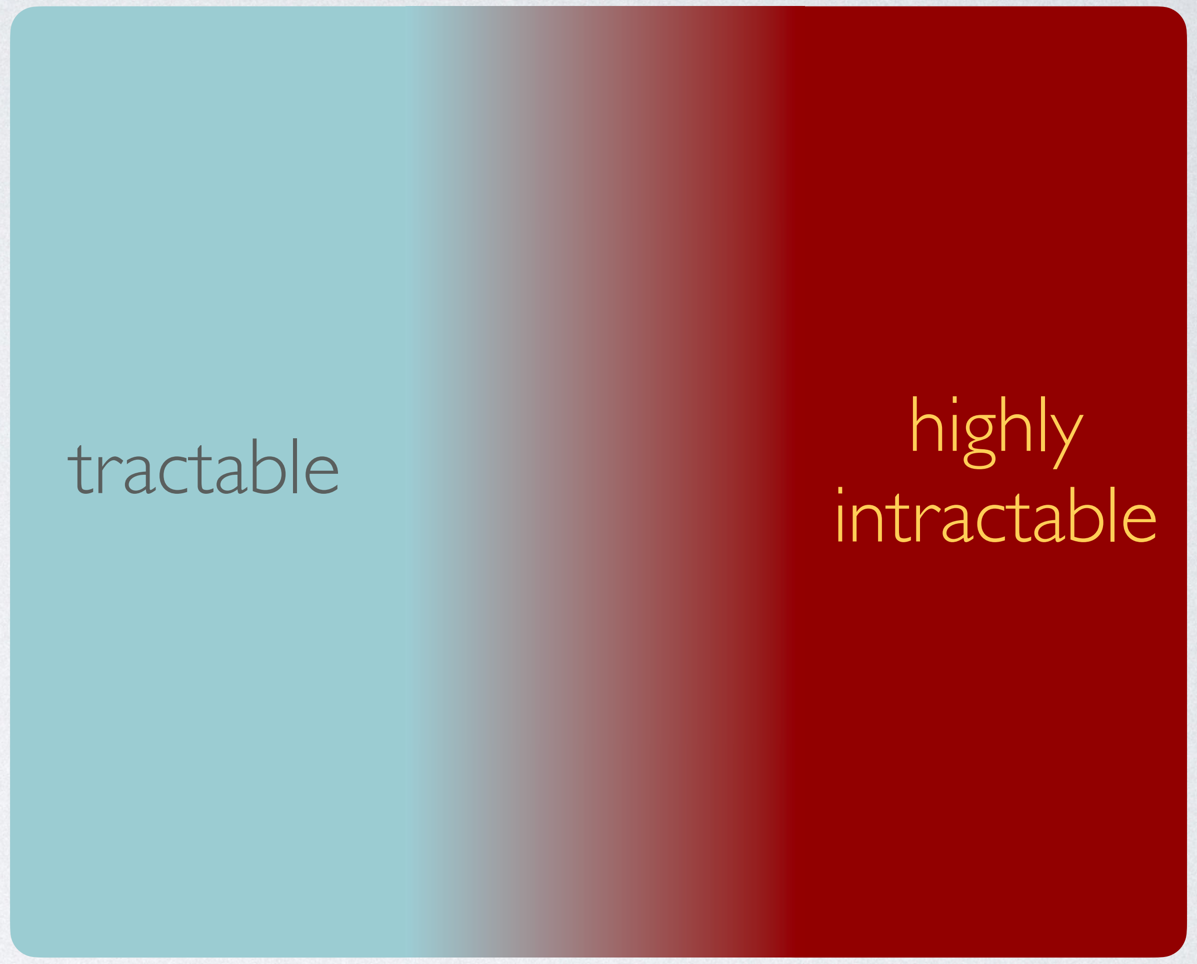

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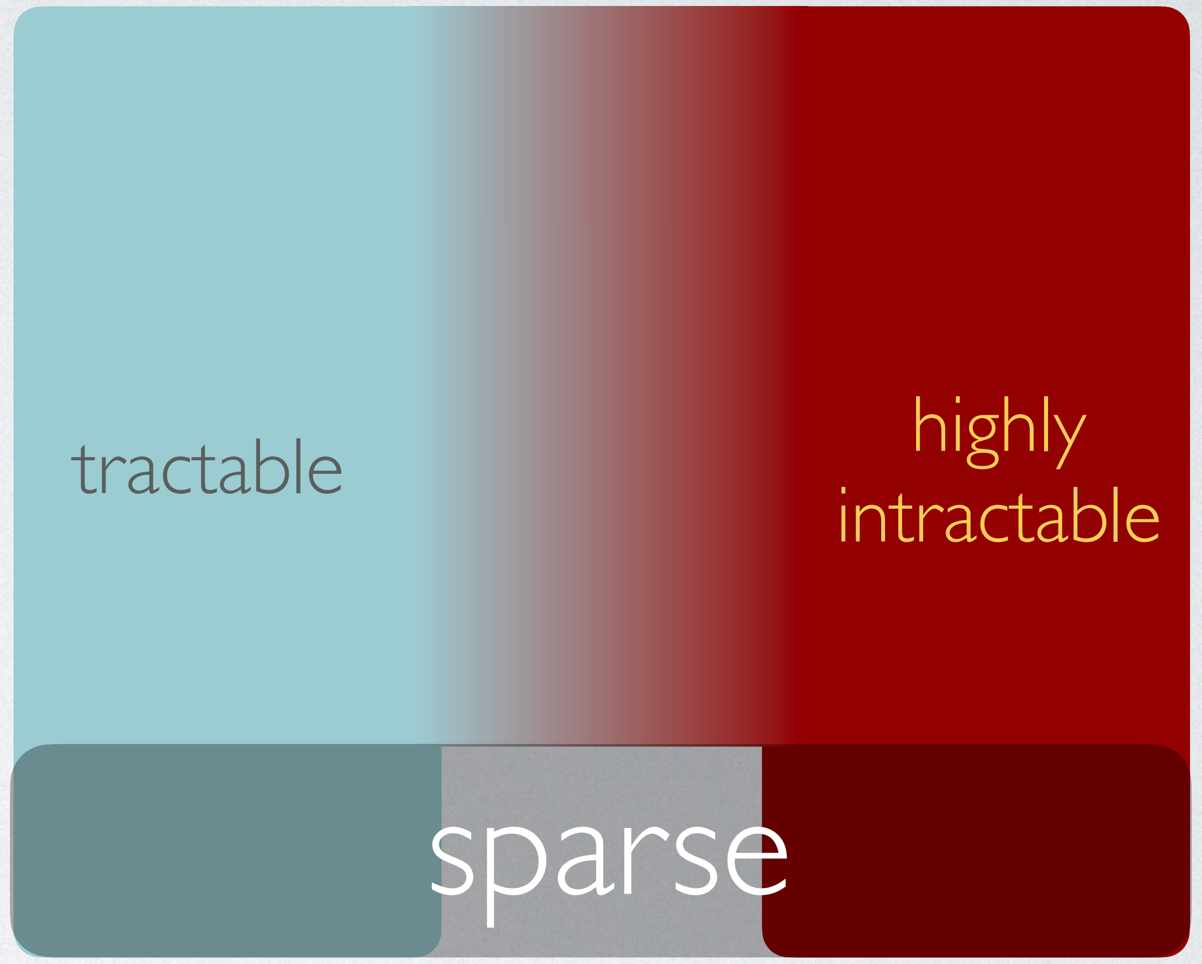

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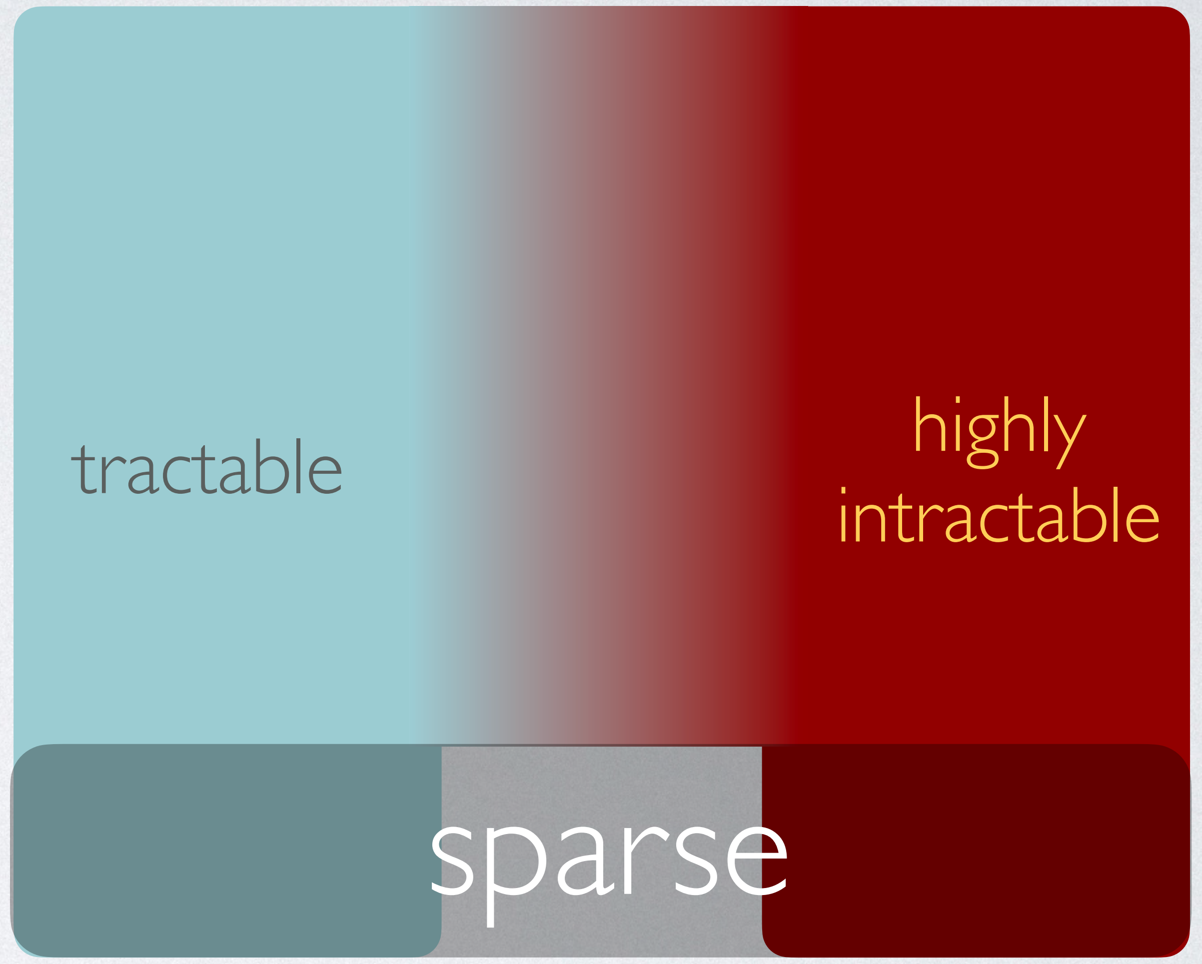

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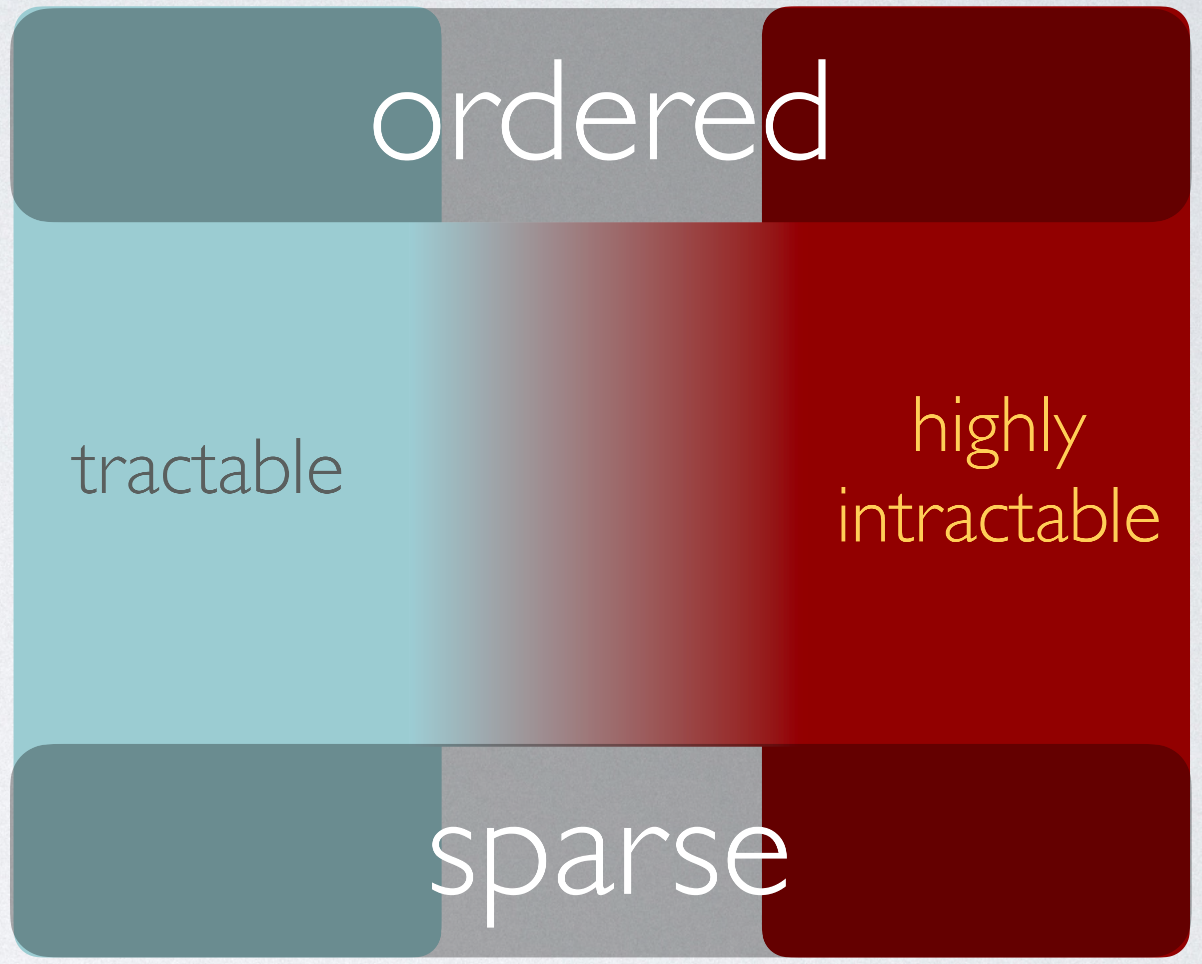

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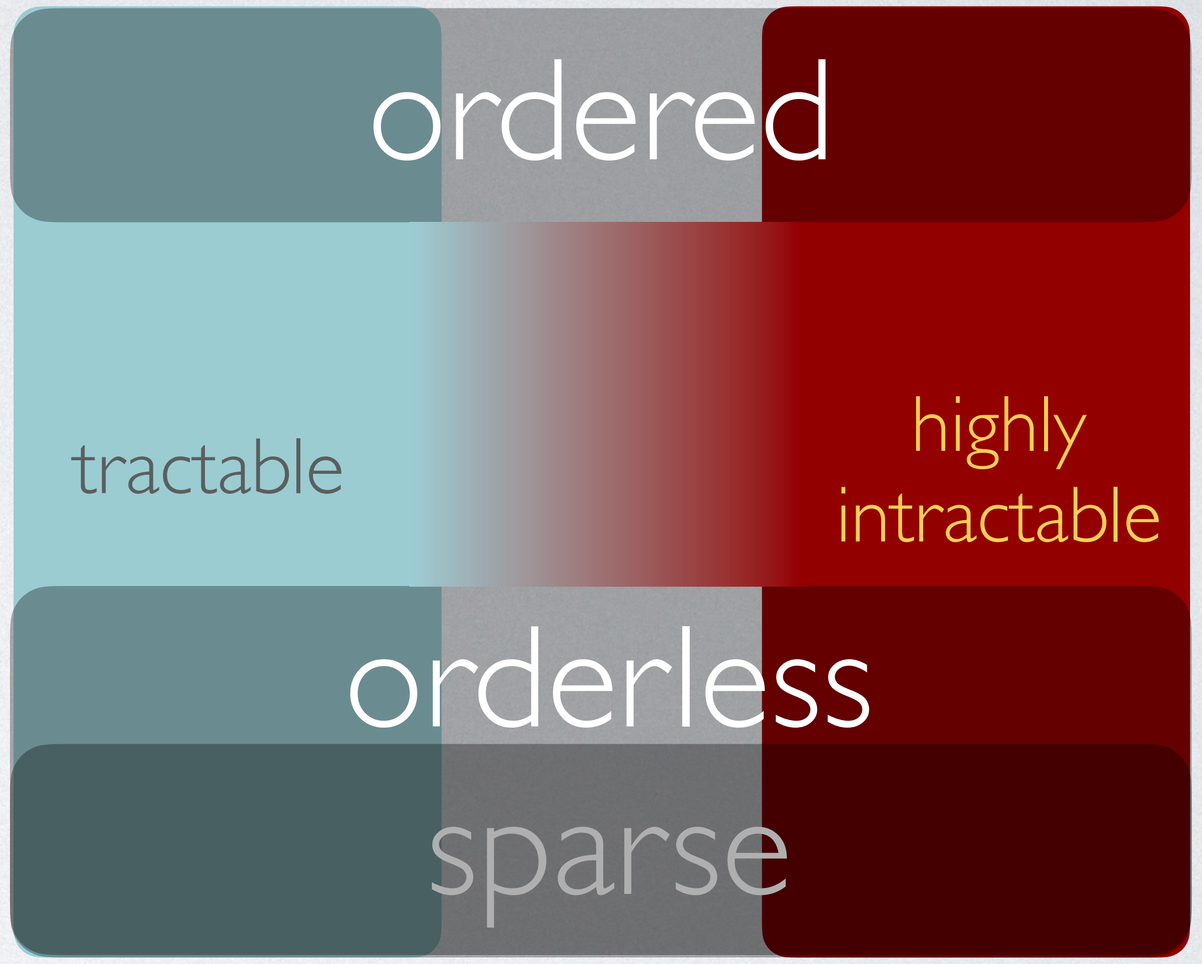

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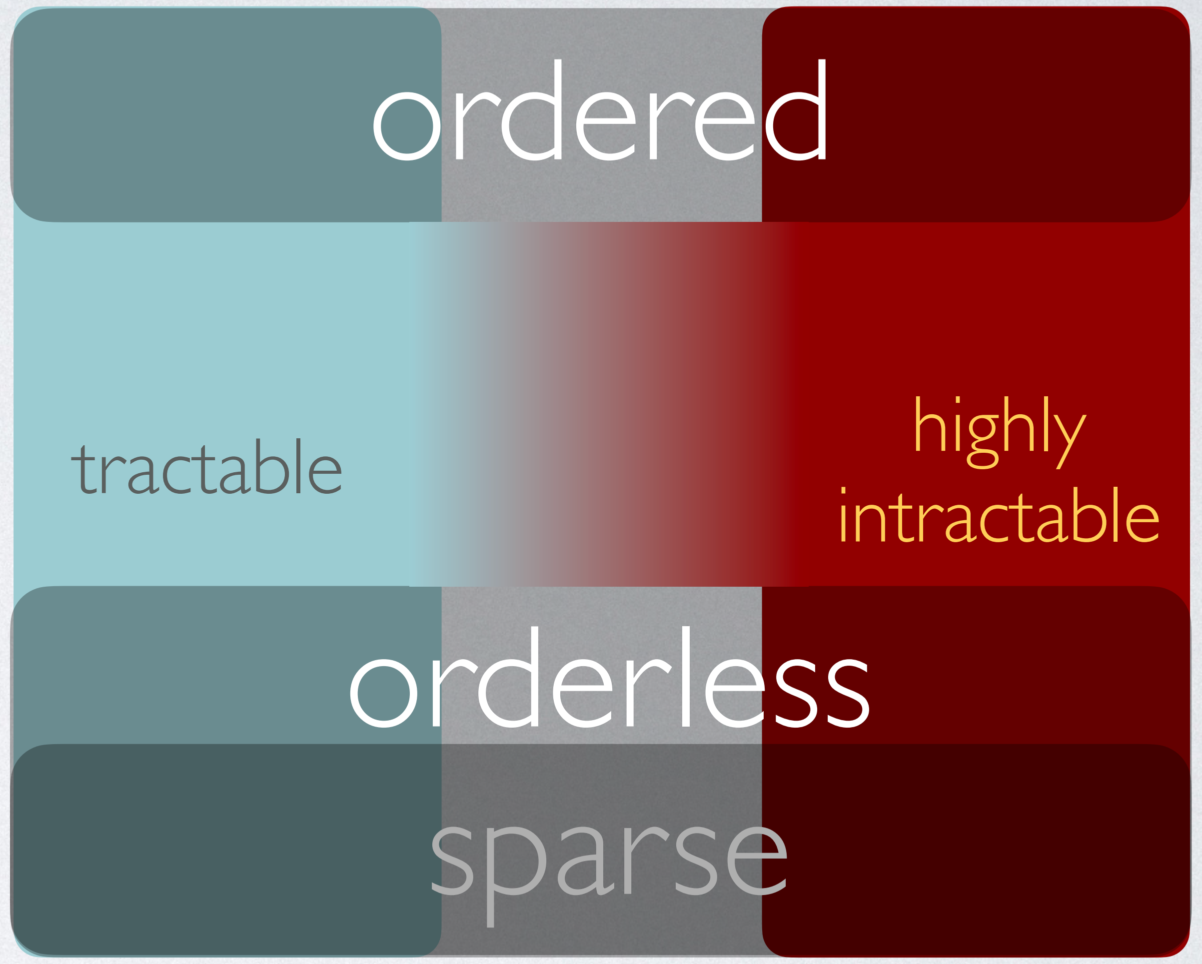
STOC 2024: Dreier, Mählmann, **T**. initial progress in general case



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SPARSITY

Nešetřil and Ossona de Mendez

SPARSITY

Nešetřil and Ossona de Mendez

Definition. *C* is *nowhere dense* if

SPARSITY

Nešetřil and Ossona de Mendez

Definition. C is *nowhere dense* if

$\forall r \geq 0\ \exists k_r.$ C avoids the r-subdivided k_r-clique as a subgraph

SPARSITY

Nešetřil and Ossona de Mendez

Definition. *C* is *nowhere dense* if

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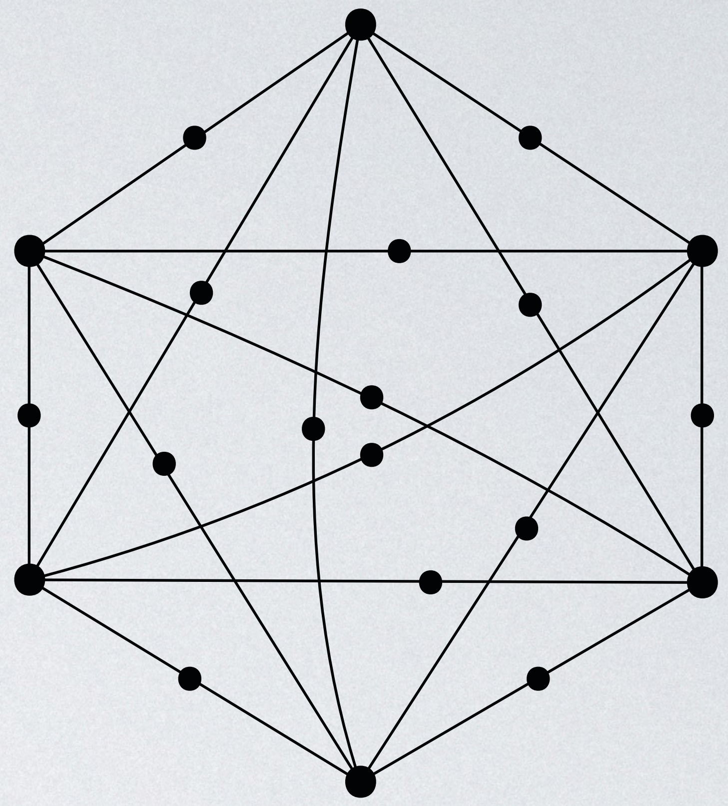

1-subdivided
6-clique

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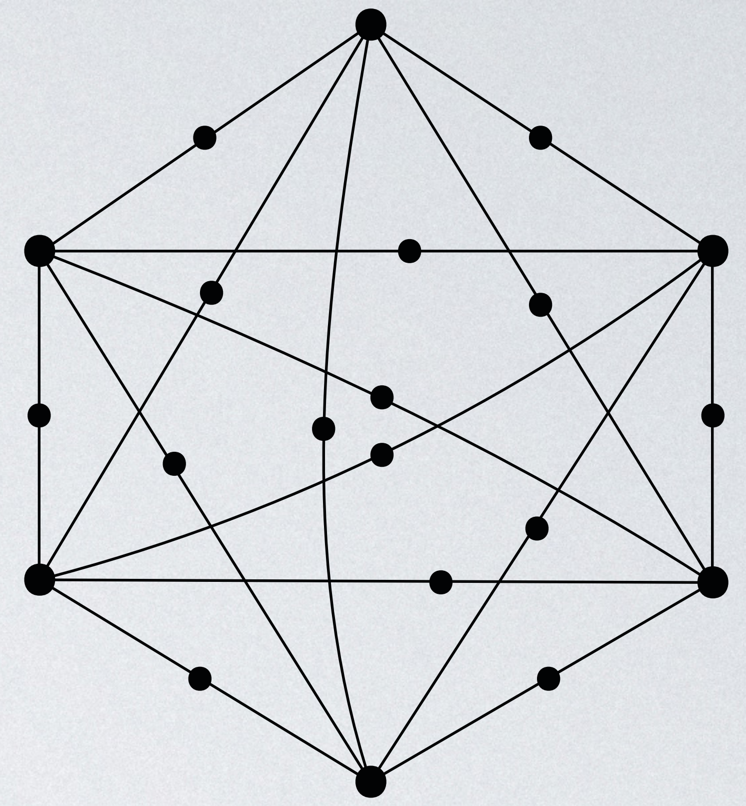

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Example. Class of planar graphs: $\forall r \geq 0, k_r := 5.$

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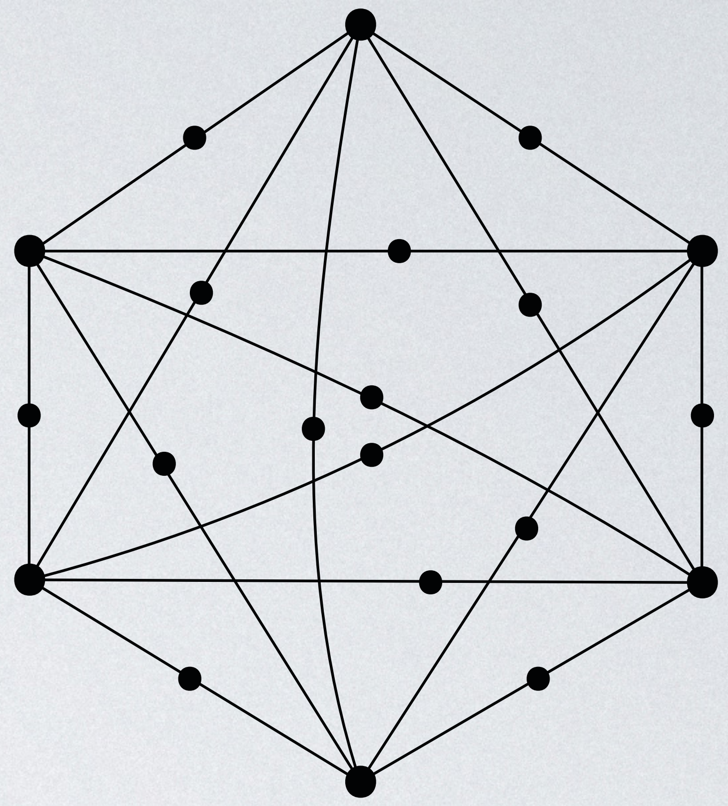

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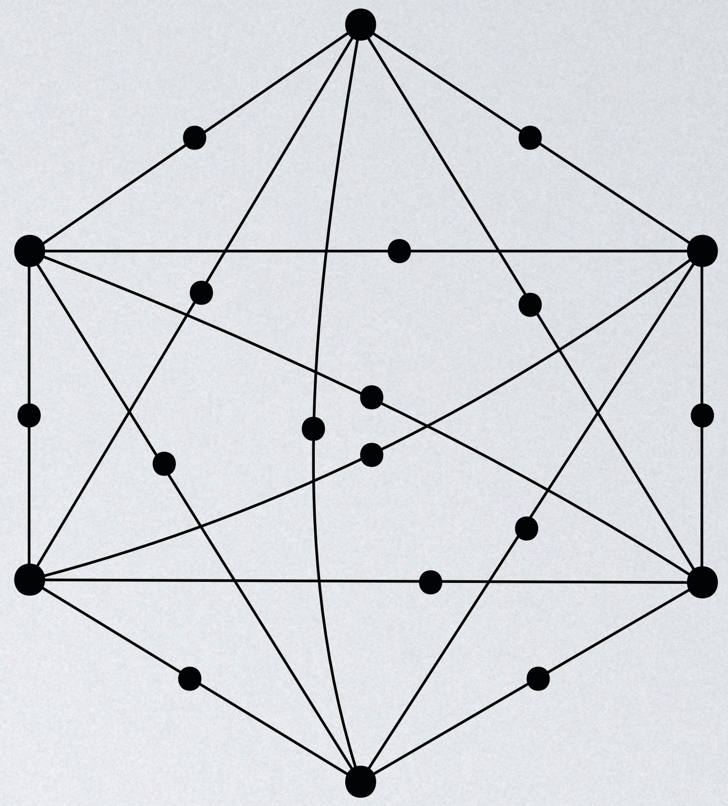

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Fact. If C is nowhere dense, $\varepsilon > 0$, then every $G \in C$ has $O_{\varepsilon,C}(\ |V(G)|^{1+\varepsilon})$ edges.

SPARSITY

bounded treewidth

↓

excluding a minor

↓

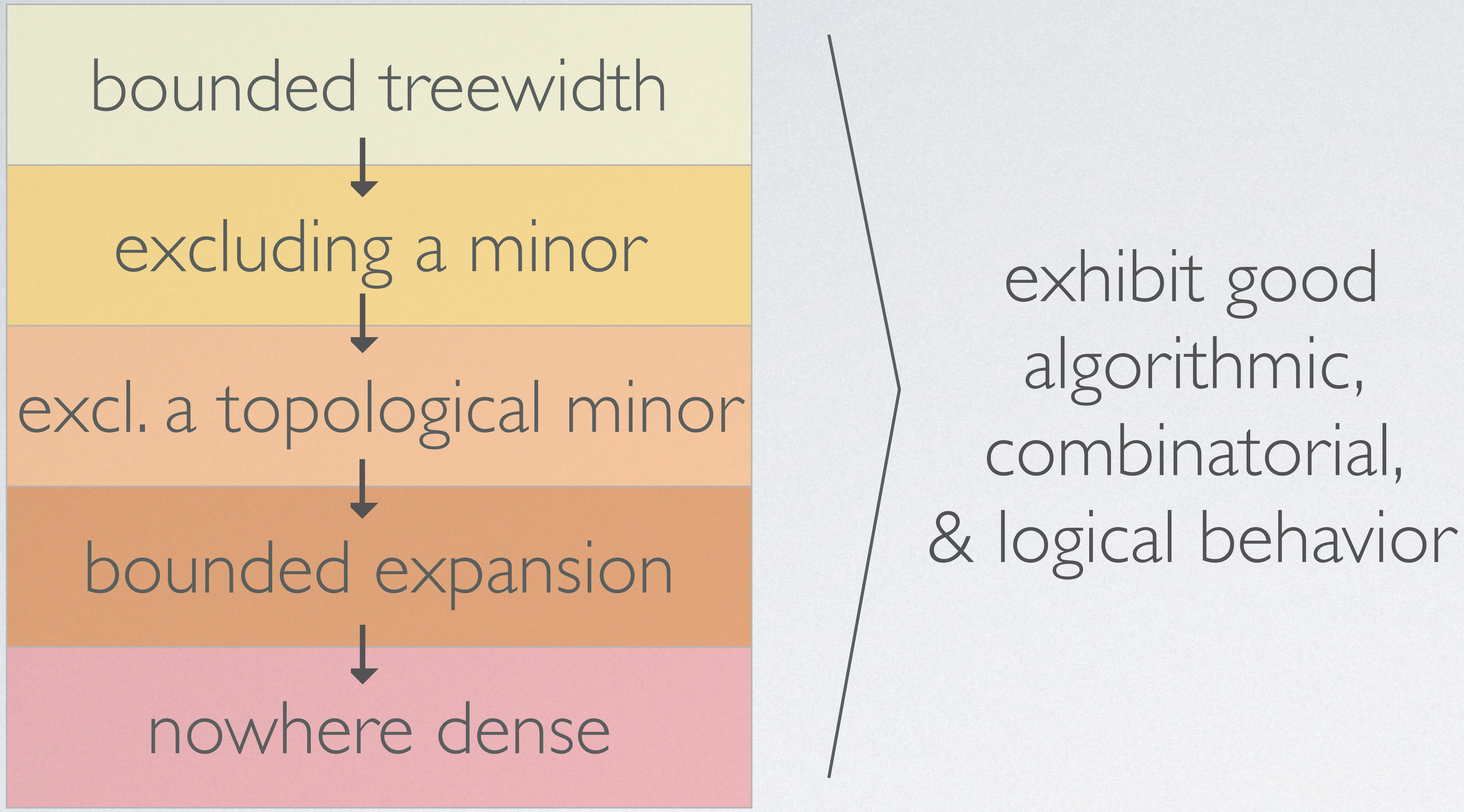
excl. a topological minor

↓

bounded expansion

↓

nowhere dense


bounded treewidth
excluding a minor
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bounded expansion
nowhere dense
exhibit good
algorithmic,
combinatorial,
& logical behavior

bounded treewidth

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Theorem (Courcelle 1990)
Model checking MSO logic is fpt
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MODEL CHECKING ON SPARSE CLASSES

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MONOTONE CLASSES

=subgraph closed classes

Corollary. Let C be a monotone graph class. Then:

C is nowhere dense $^*\Leftrightarrow$ FO-model checking is fpt on C.

*assuming AW[$*$]≠FPT

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$\exists r\geq 0 \forall G$ C contains the r-subdivision of $G \Rightarrow$ *hardness*

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BEYOND SPARSITY

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Prototype: treewidth $\mapsto$ cliquewith/rankwidth

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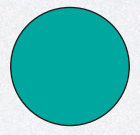

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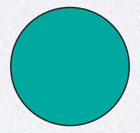
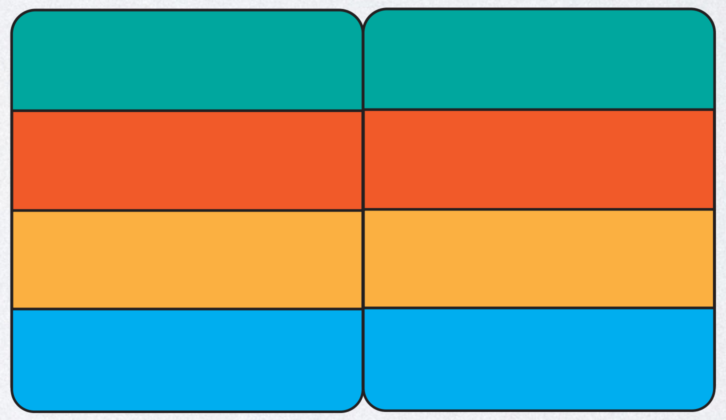


- Take disjoint union of two colored graphs



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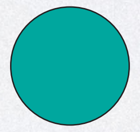

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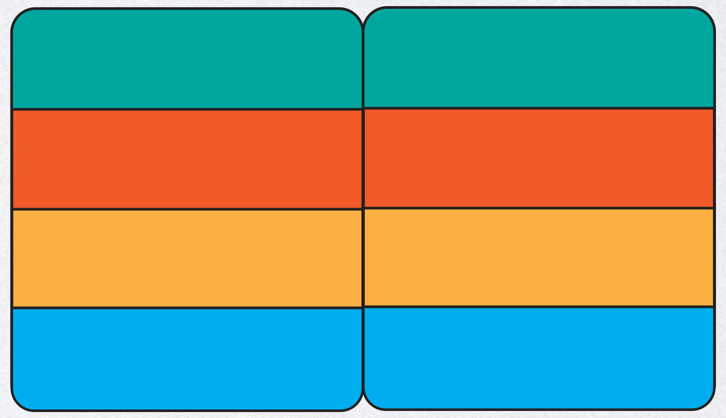

- Join by an edge every vertex colored i to every vertex colored j

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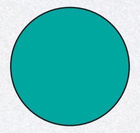

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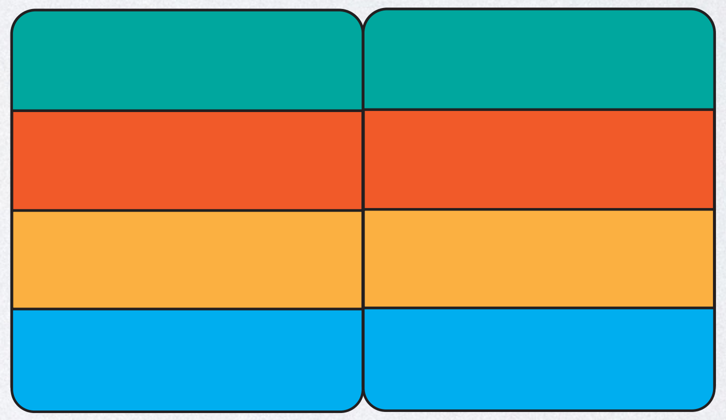

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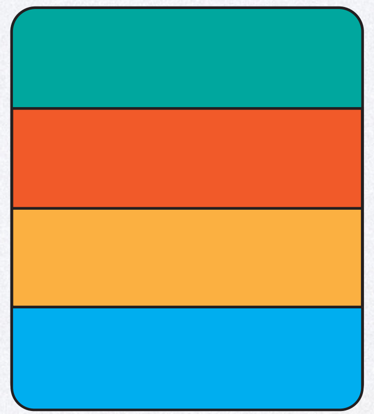

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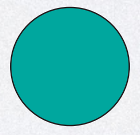
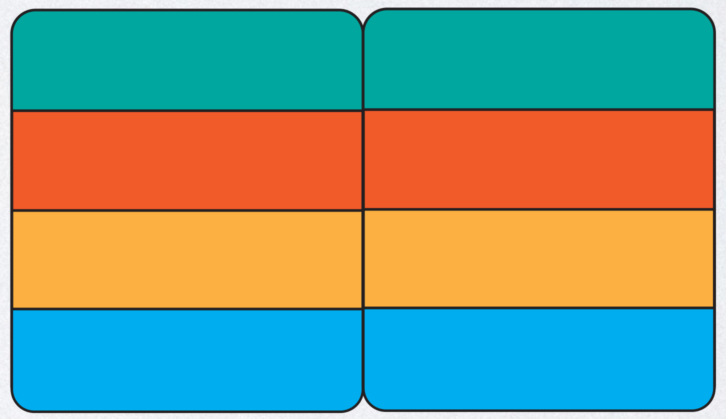
- Create new vertex with color $i \in [k]$ 
- Take disjoint union of two colored graphs 
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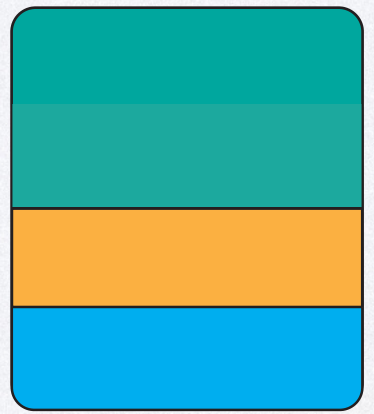
- Recolor i to color j 

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BEYOND SPARSITY

Project: extend from monotone classes to hereditary classes

monotone		*hereditary*
bounded treewidth	→	bounded cliquewidth
↓		↓
excluding a minor	→	?
↓		↓
excl. a top. minor	→	?
↓		↓
bounded expansion	→	?
↓		↓
nowhere dense	→	?

OUTLINE

1. The model checking problem
2. Sparsity: monotone case
3. **Twin-width: ordered case**
4. Monadic dependence
5. Flip-breakability
6. Stability: orderless case

TWIN-WIDTH

Bonnet, Thomassé, and coauthors

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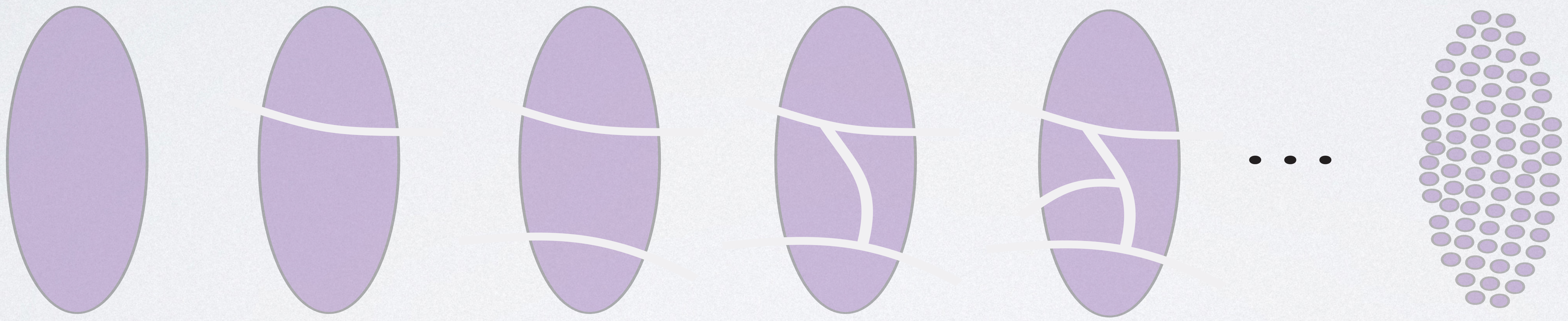
Definition. G has *twin-width* $\leq d$ if there is a refining sequence $P_1,\ldots,P_n$ of partitions of $V(G)$ – a *merge sequence* –



P_1: $\{V(G)\}$ P_2 P_3 P_4 P_5 … P_n: singletons

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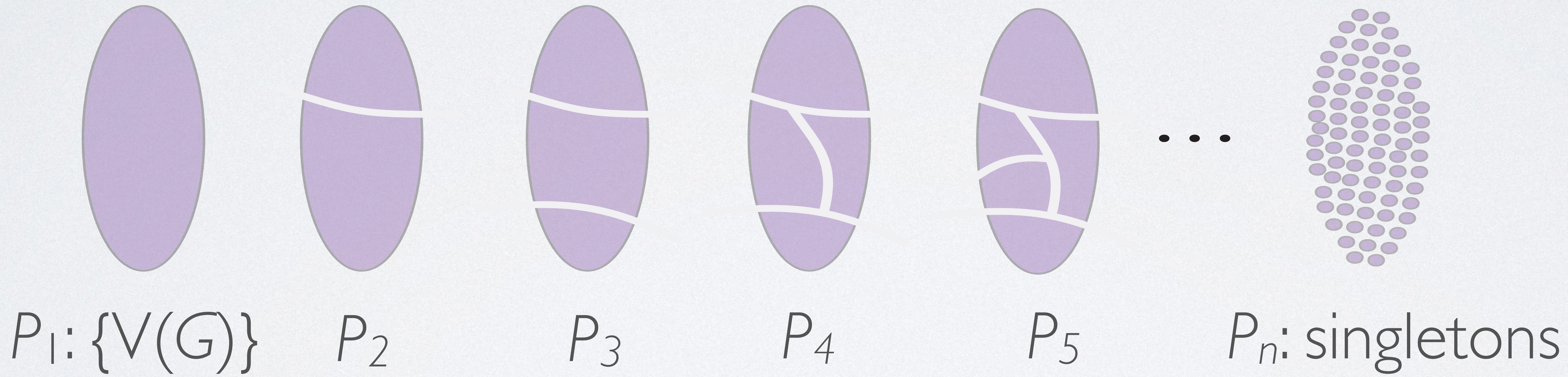

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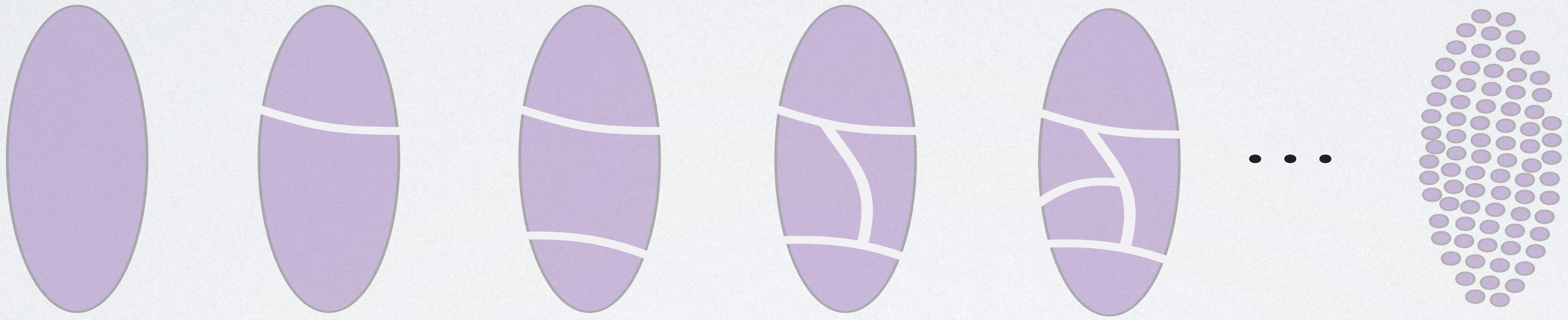

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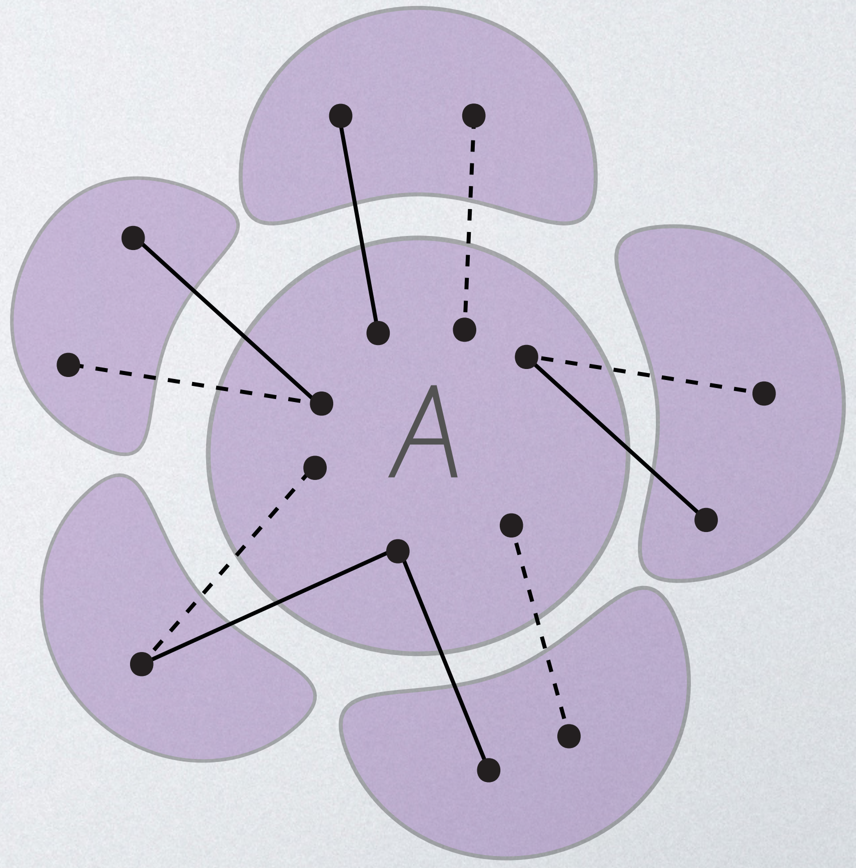
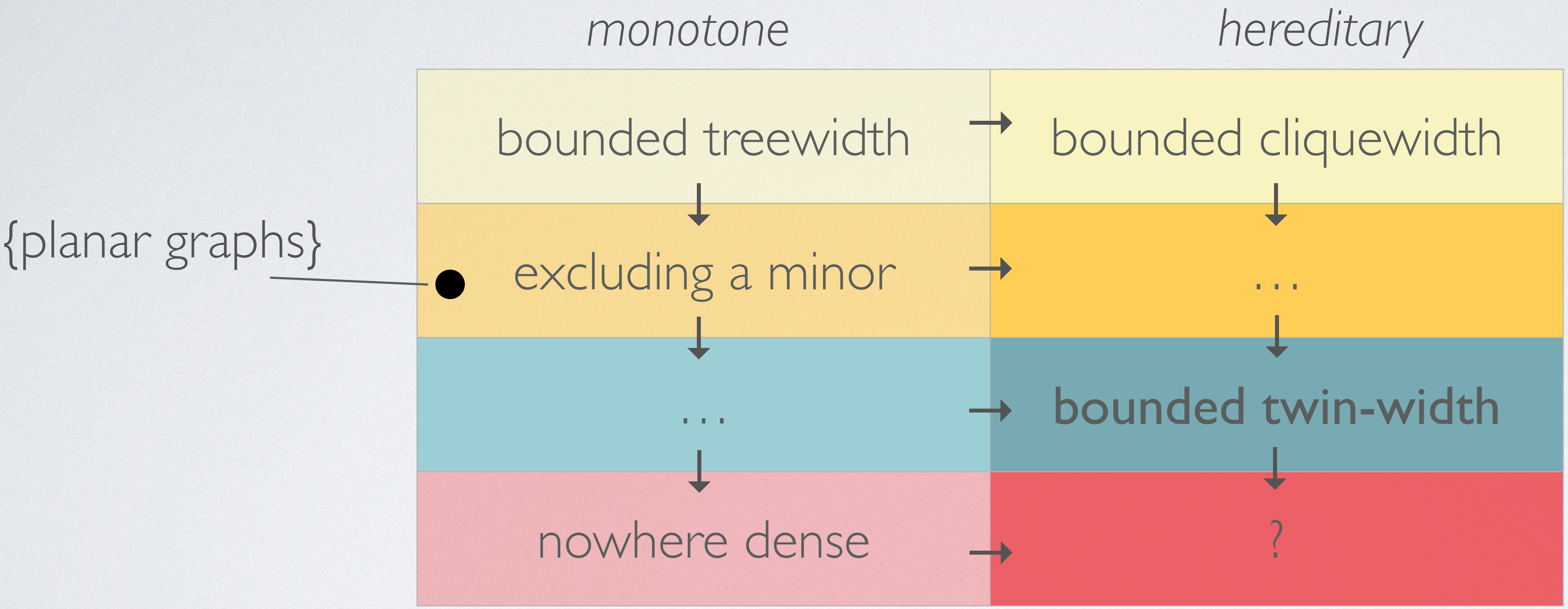


TWIN-WIDTH

monotone	*hereditary*
bounded treewidth	bounded cliquewidth
excluding a minor	…
…	**bounded twin-width**
nowhere dense	?

TWIN-WIDTH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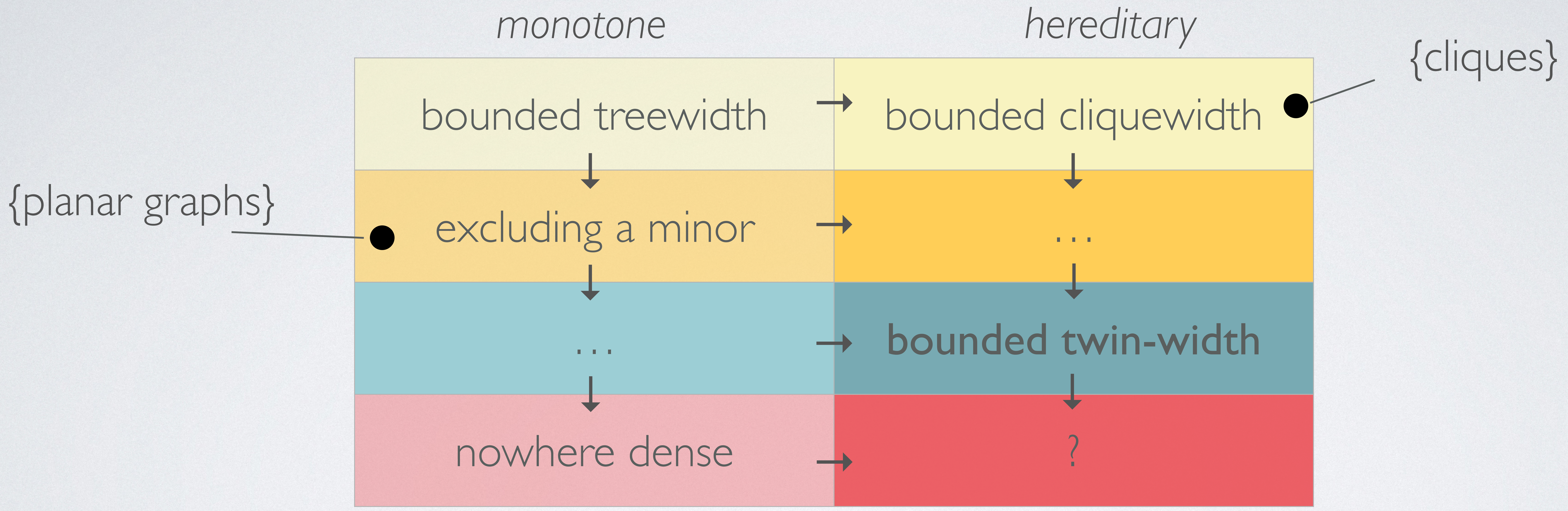
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{graphs with max. degree 3}

MODEL CHECKING ON CLASSES OF BOUNDED TWIN-WIDTH

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Given $\phi \in FO$, a graph G of twin-width $\leq d$ **with its merge sequence,**

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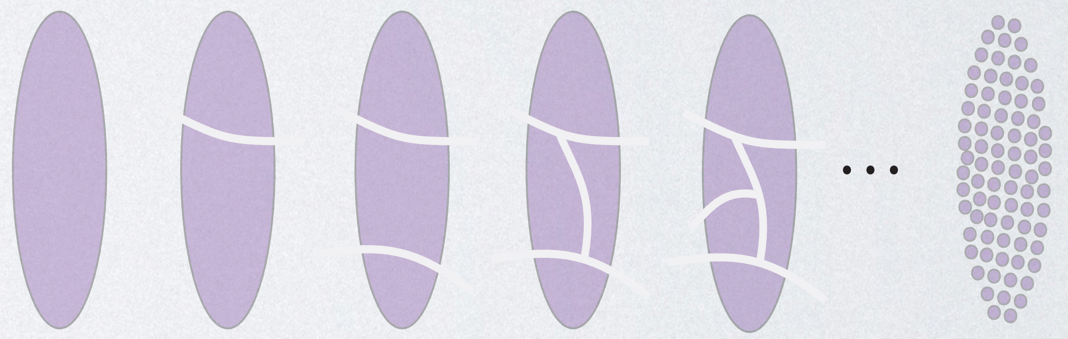

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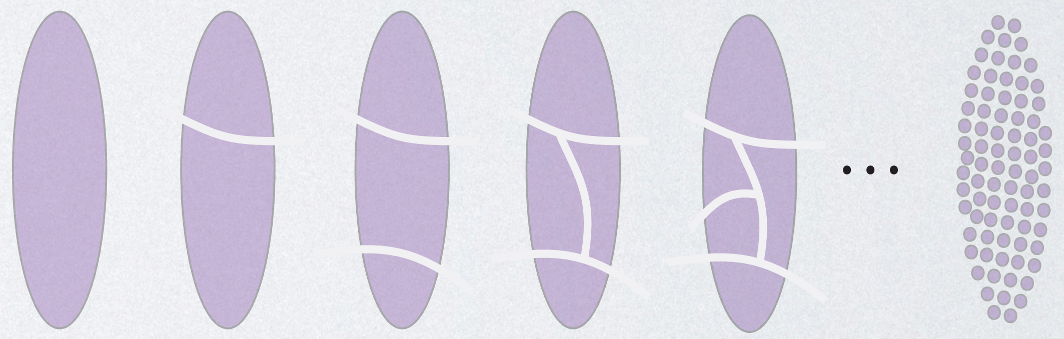

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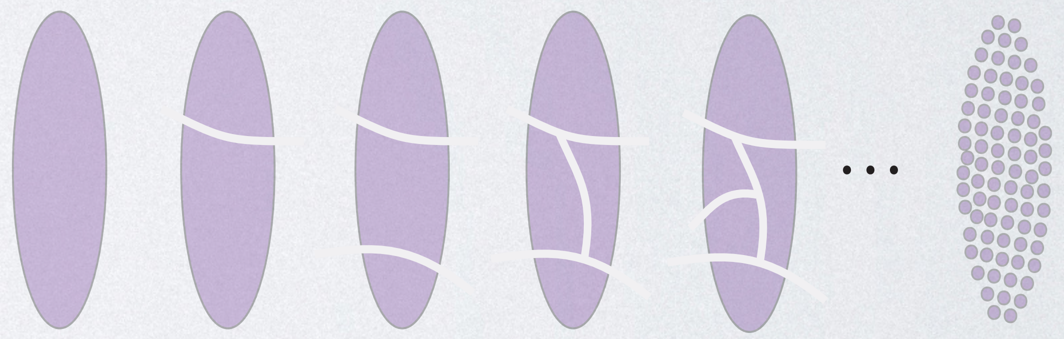

ORDERED GRAPHS

$G = (V,E,<)$

$<$ contributes to the logic, and to the twin-width.

Theorem (Bonnet, Ossona de Mendez, Thomassé, Simon, **T.**, 2022)
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Ingredients:

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3. dynamic algorithm [Twin-width I]

CLASSES OF ORDERED GRAPHS

Theorem (Bonnet, O. de Mendez, Thomassé, Simon, **T.**, 2022).
Let C be a hereditary class of *ordered* graphs. Then:

C has bounded twin-width $^{*}\Leftrightarrow$ FO-model checking is fpt on C

*assuming AW[$*$]$\neq$FPT

GRAND UNIFICATION

Common generalization of Sparsity and Twin-width

monotone	*hereditary*
bounded treewidth	bounded cliquewidth
excluding a minor	…
…	bounded twin-width
nowhere dense	?

OUTLINE

1. The model checking problem
2. Sparsity: monotone case
3. Twin-width: ordered case
4. **Monadic dependence**
5. Flip-breakability
6. Stability: orderless case

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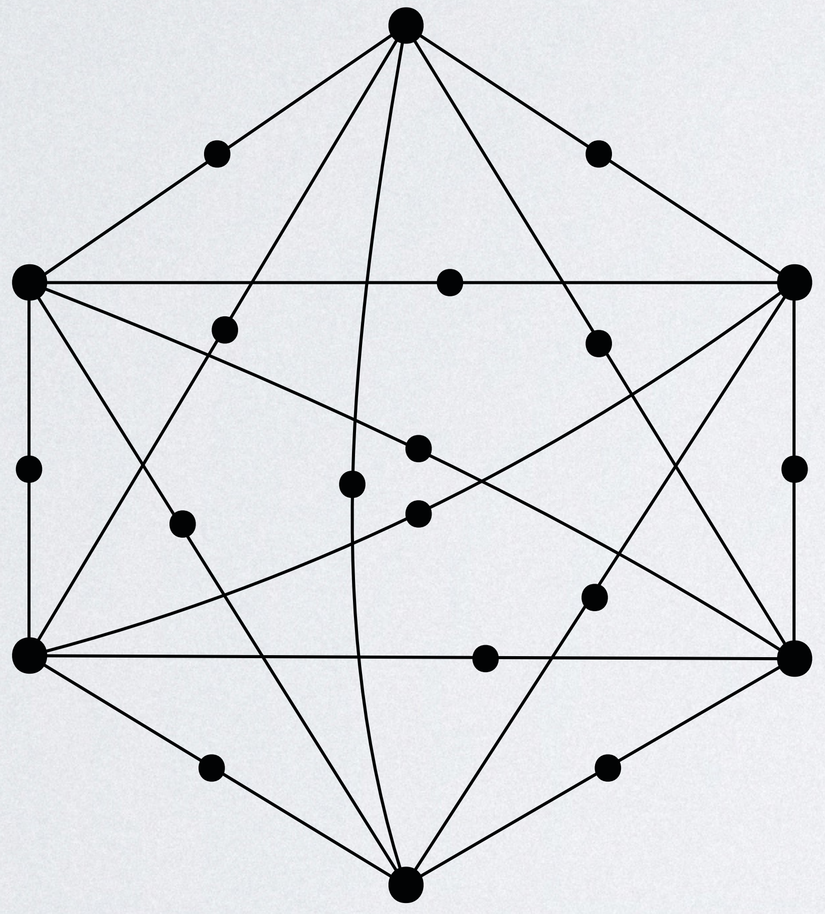

1-subdivided 6-clique

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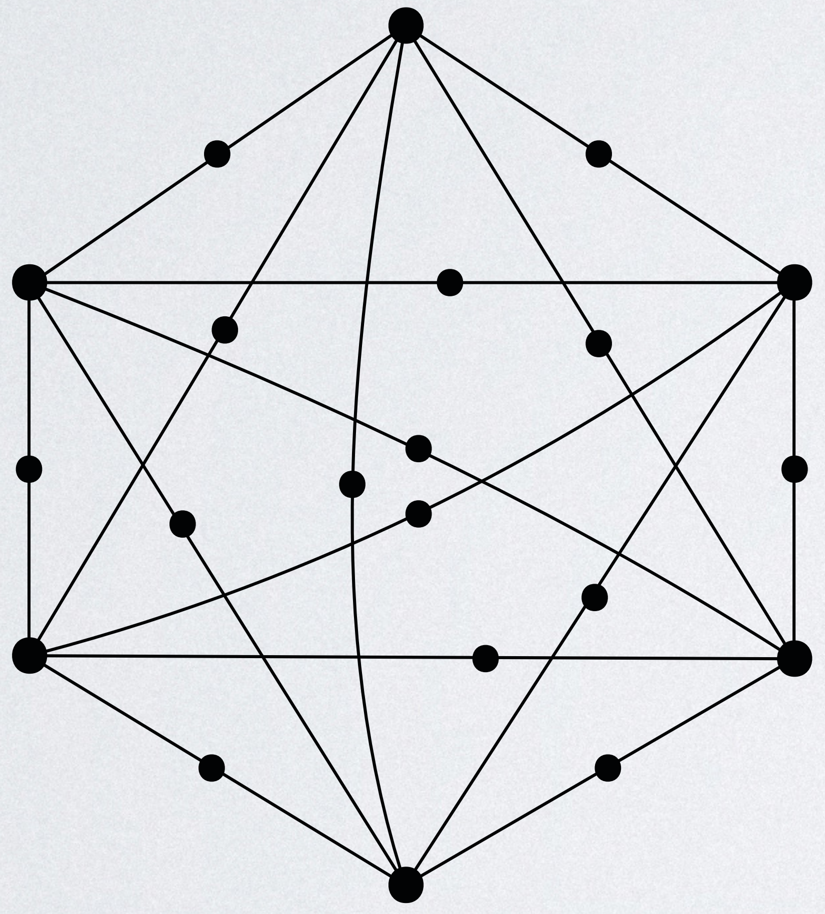

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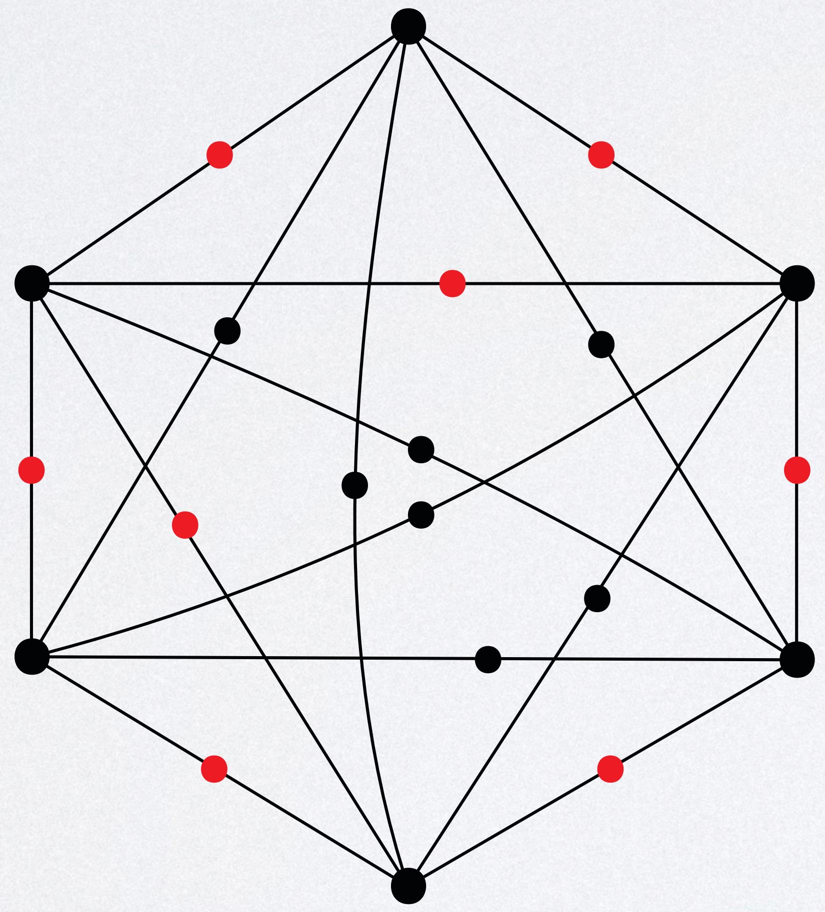

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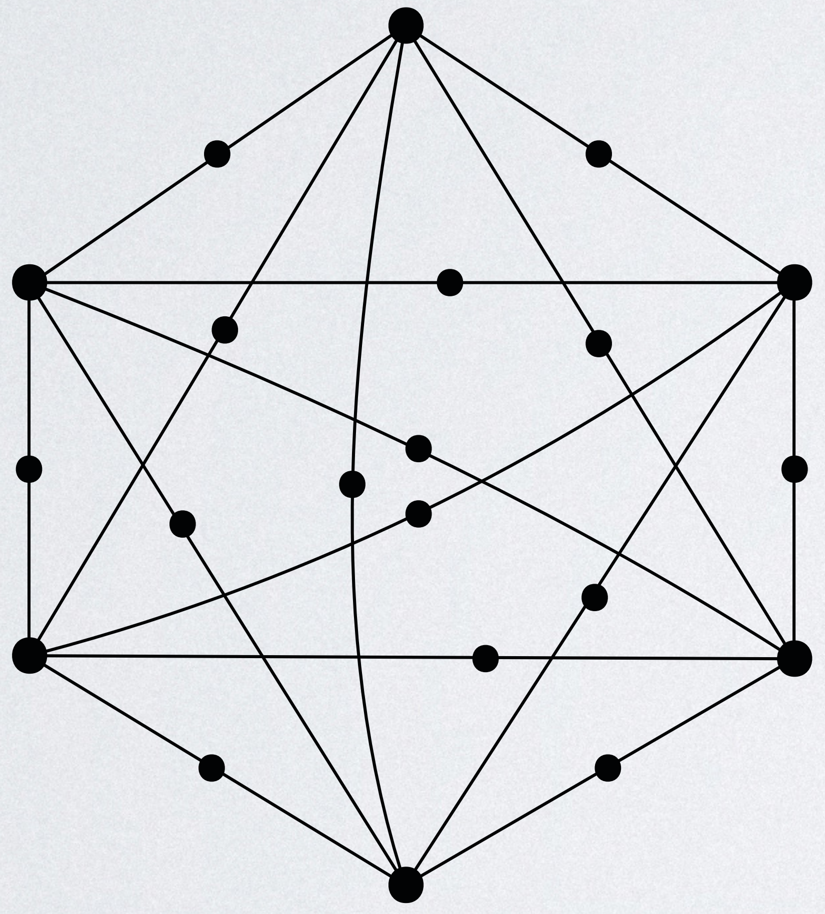

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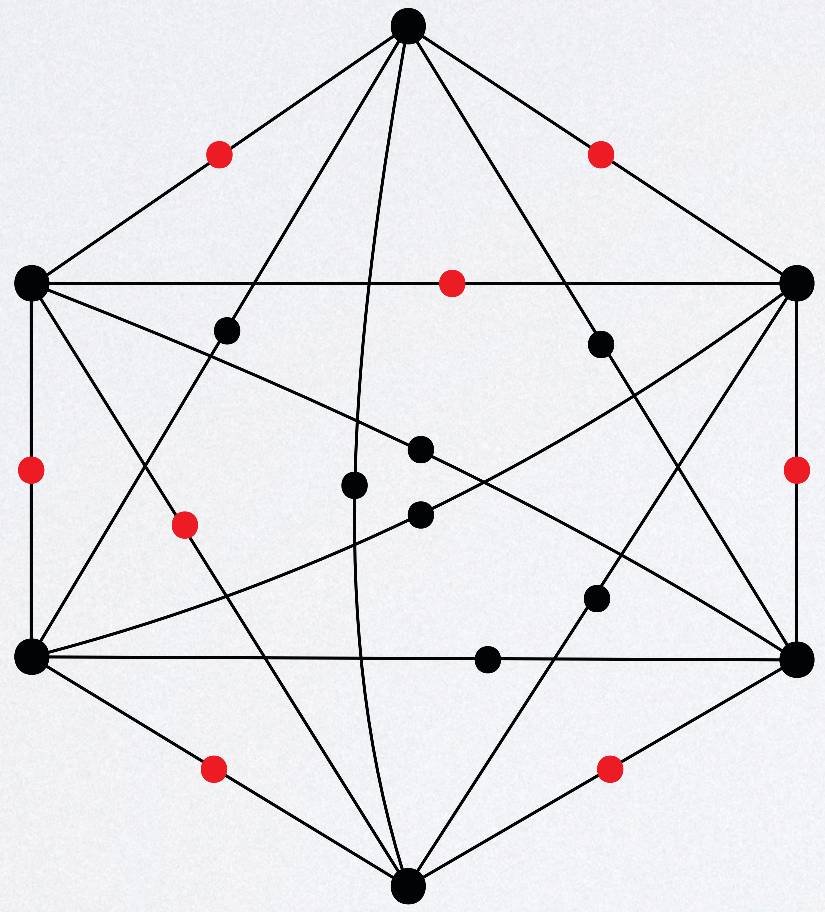

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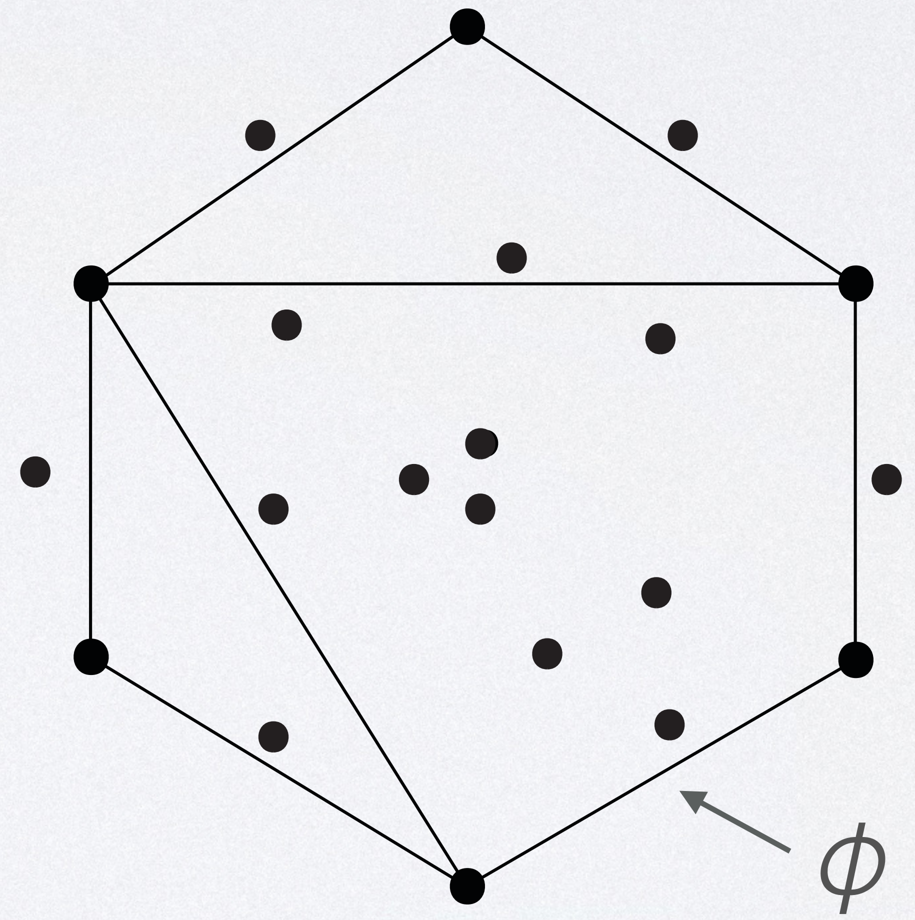

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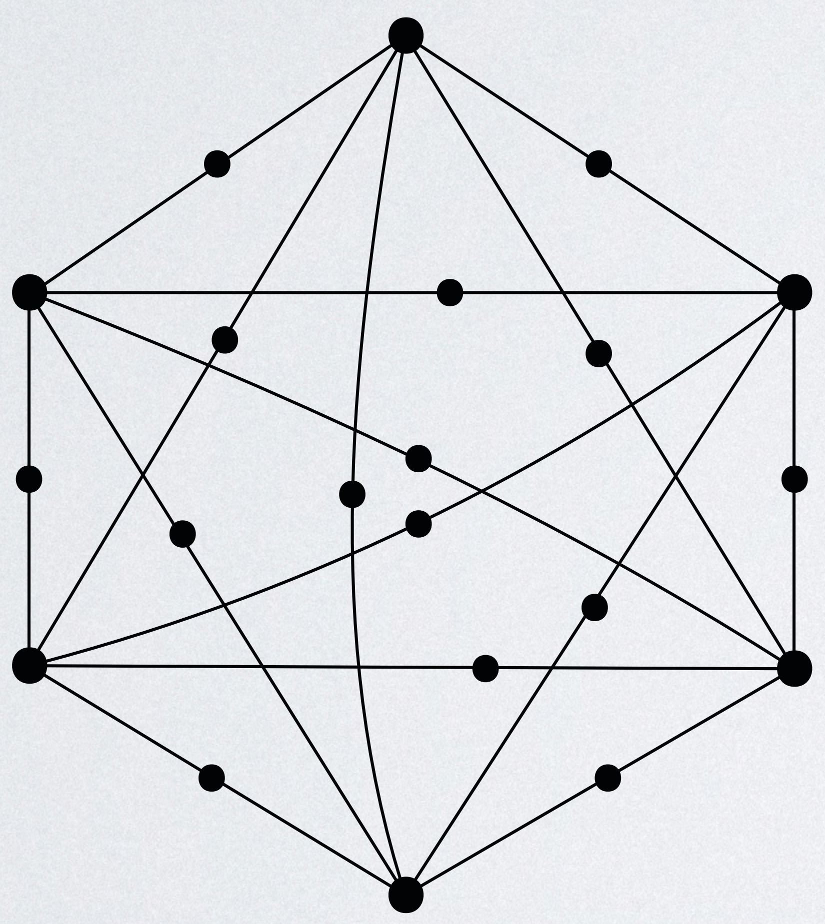

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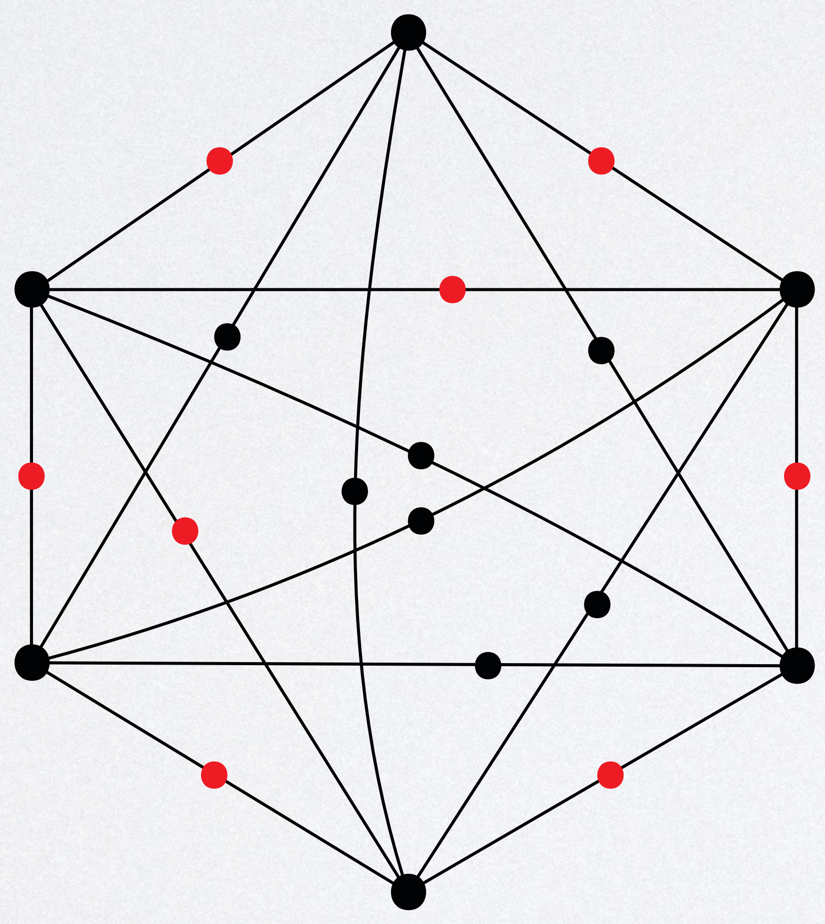

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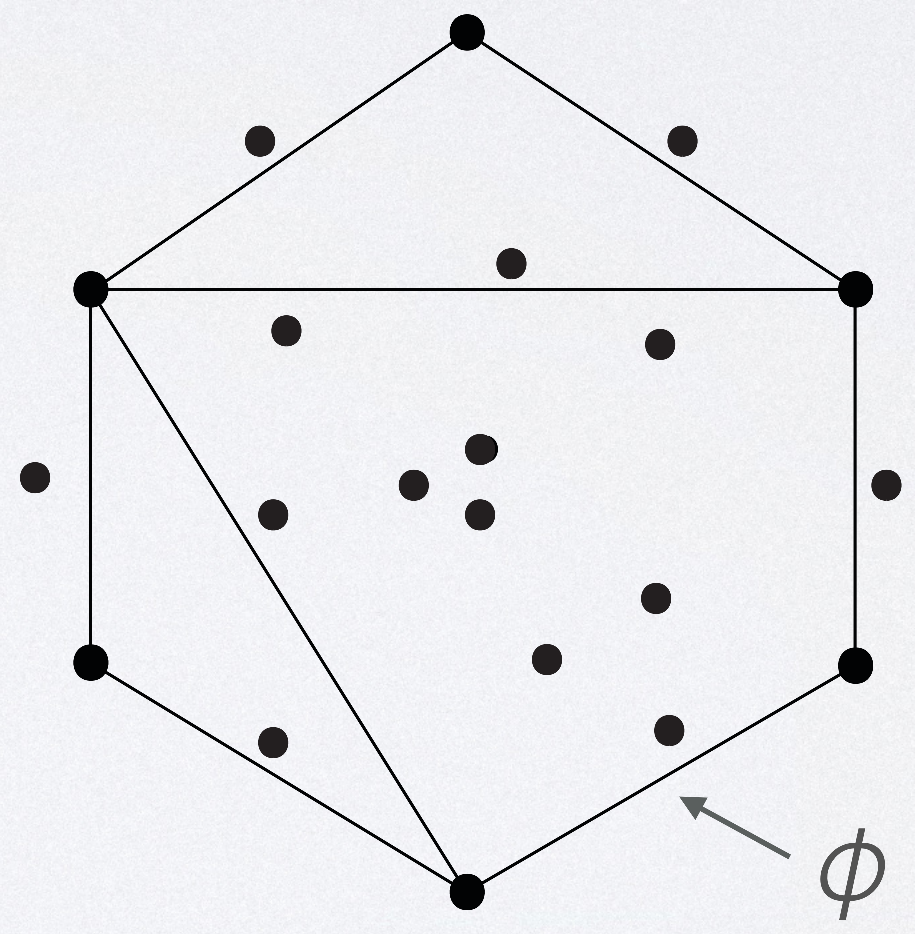

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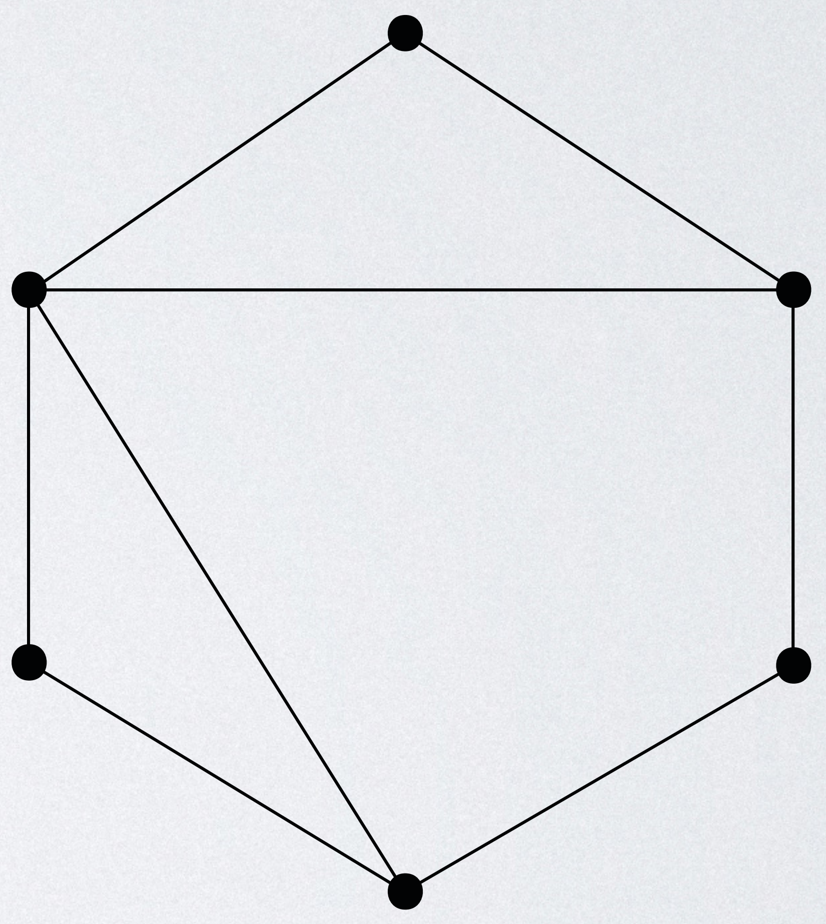

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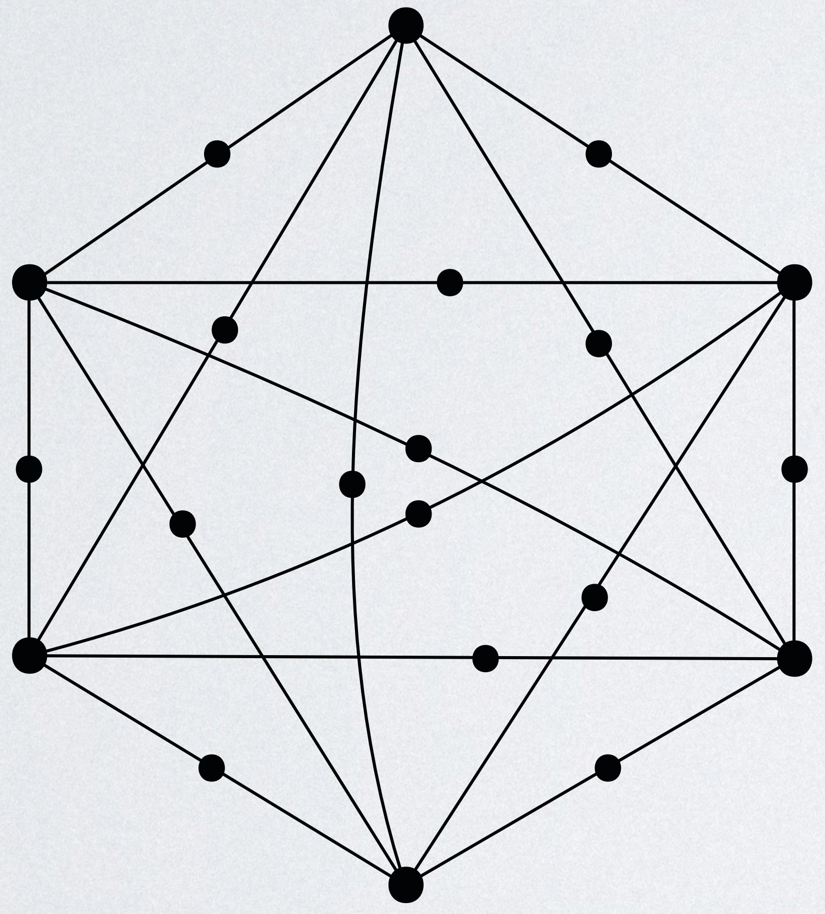

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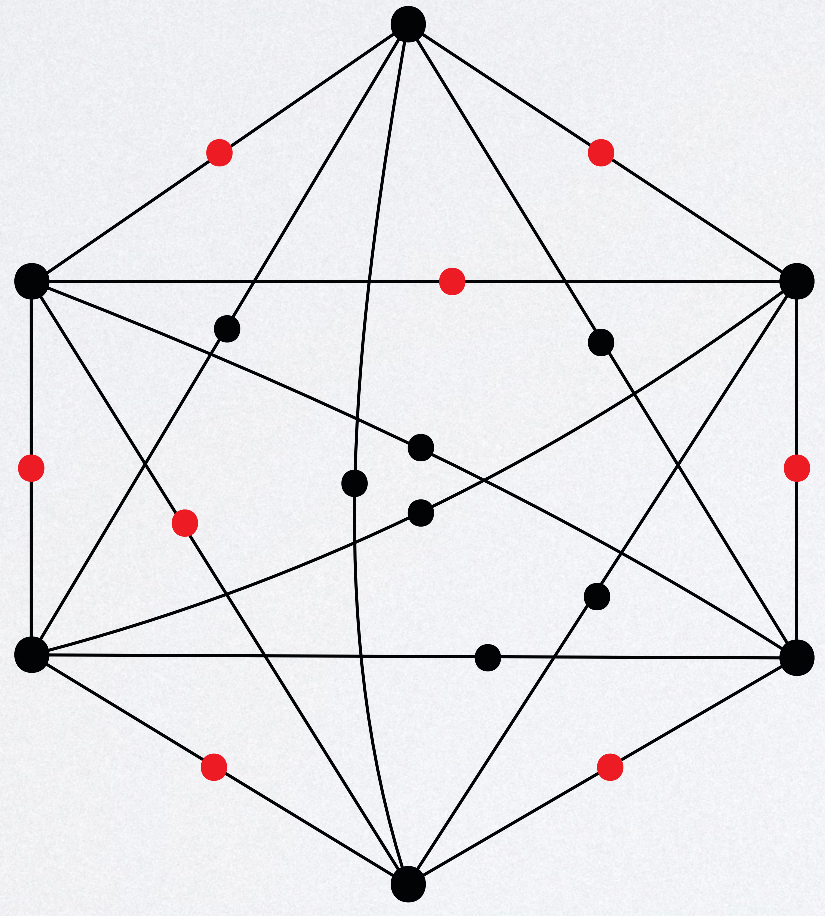

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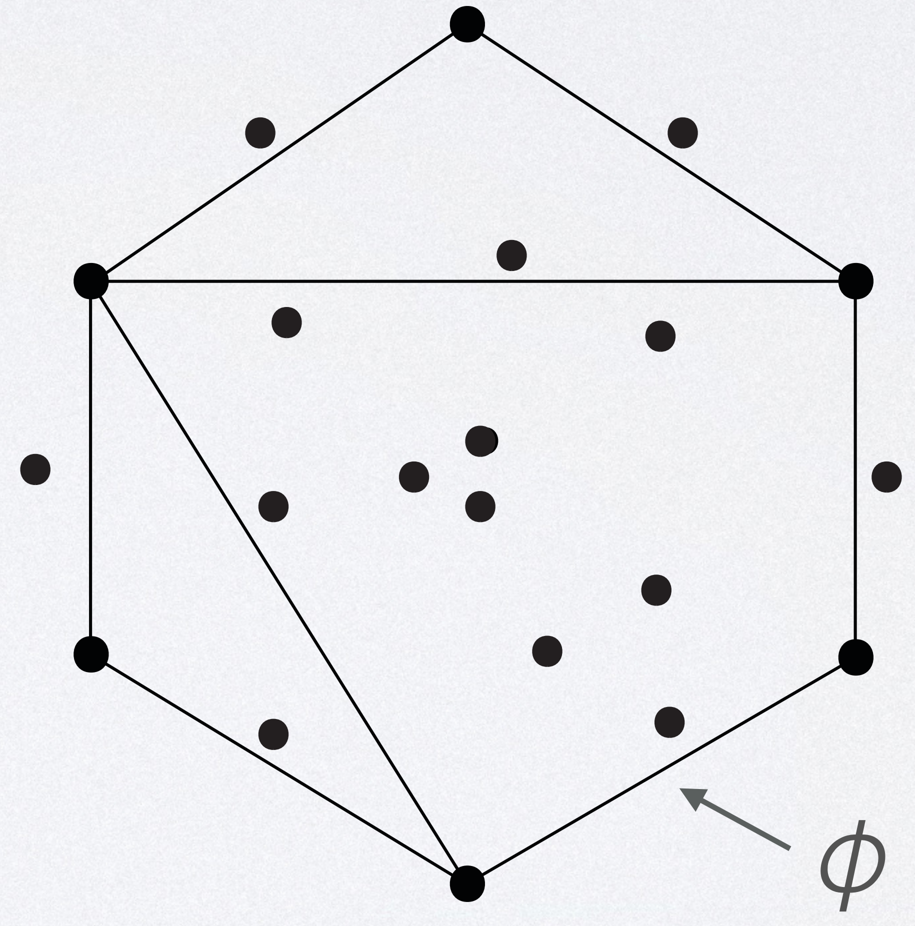

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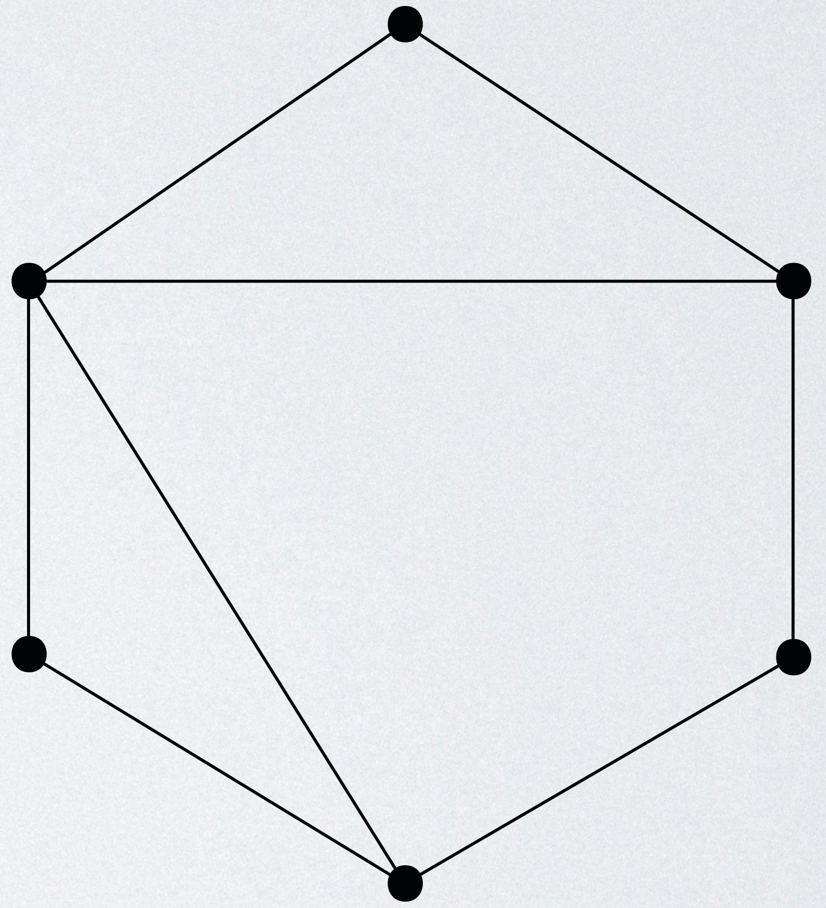

Write: $G \rightsquigarrow_{\phi} H$

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graph classes

C transduces D if for some FO formula $\phi(x,y)$

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{1-subdivided cliques} $\geq_{FO}$ {all graphs} $\geq_{FO}$ any class

MONADIC DEPENDENCE

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Definition (Shelah, 1986) A graph class C is *monadically dependent* if C does not transduce the class of all graphs:

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$$C \not<_{FO} \{\text{all graphs}\}$$

monotone	*hereditary*
bounded treewidth	bounded cliquewidth
excluding a minor	. . .
. . .	bounded twin-width
nowhere dense	**monadically dependent**

Theorem [Podewski-Ziegler '78, Adler-Adler'10, Grohe-Kreutzer-Siebertz '14]
For ***monotone*** graph classes:

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Conjecture [Pilipczuk, Siebertz, **T.** '16]
For all hereditary graph classes:

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OUTLINE

1. The model checking problem
2. Sparsity: monotone case
3. Twin-width: ordered case
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	monotone: deletion	*hereditary:* flip
radius ∞ MSO	bounded treewidth $\Leftrightarrow$ ∞-**deletion**-breakable	bounded cliquewidth $\Leftrightarrow$ ∞-**flip**-breakable
finite radii FO	nowhere dense $\Leftrightarrow$ **deletion**-breakable	monadically dependent $\Leftrightarrow$ **flip**-breakable

∞-DELETION-BREAKABILITY

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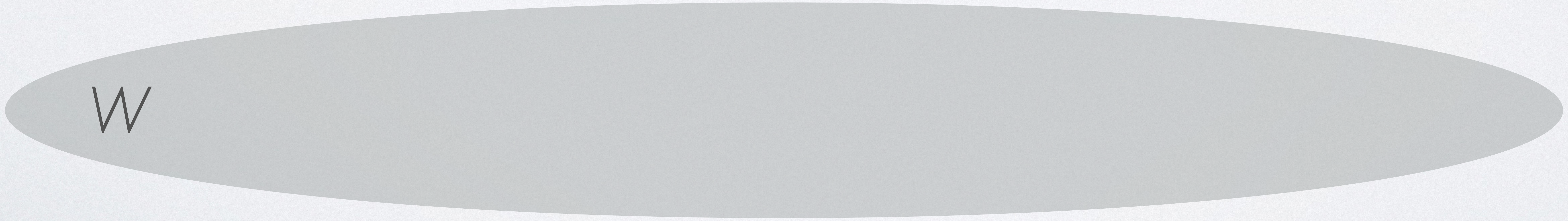

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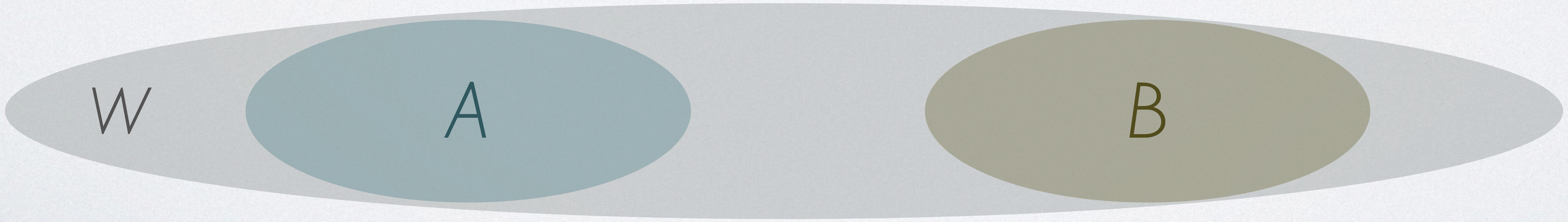

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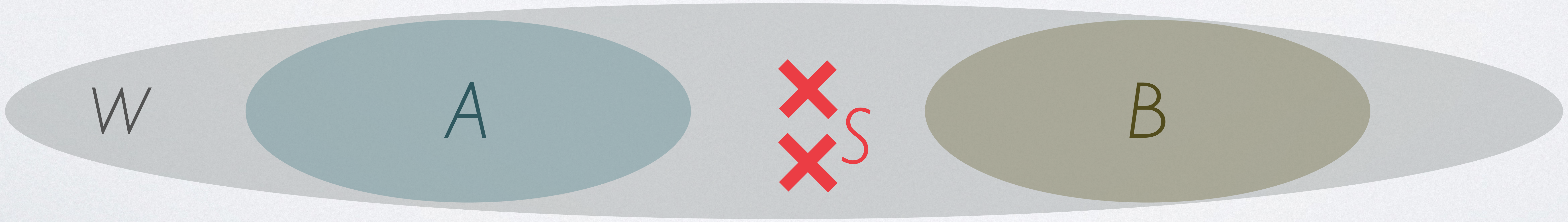

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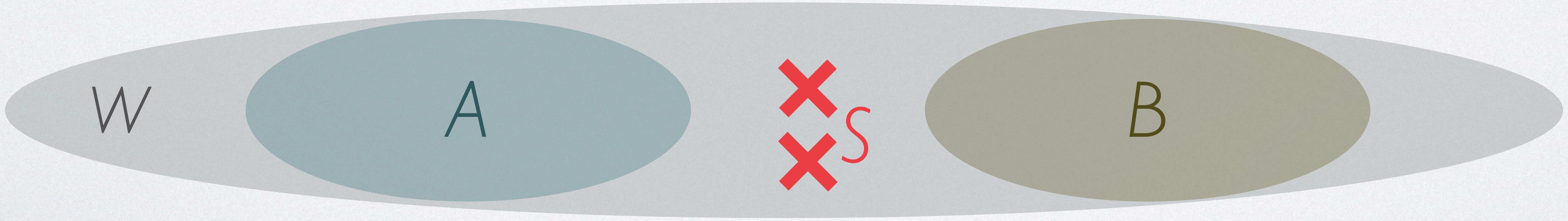

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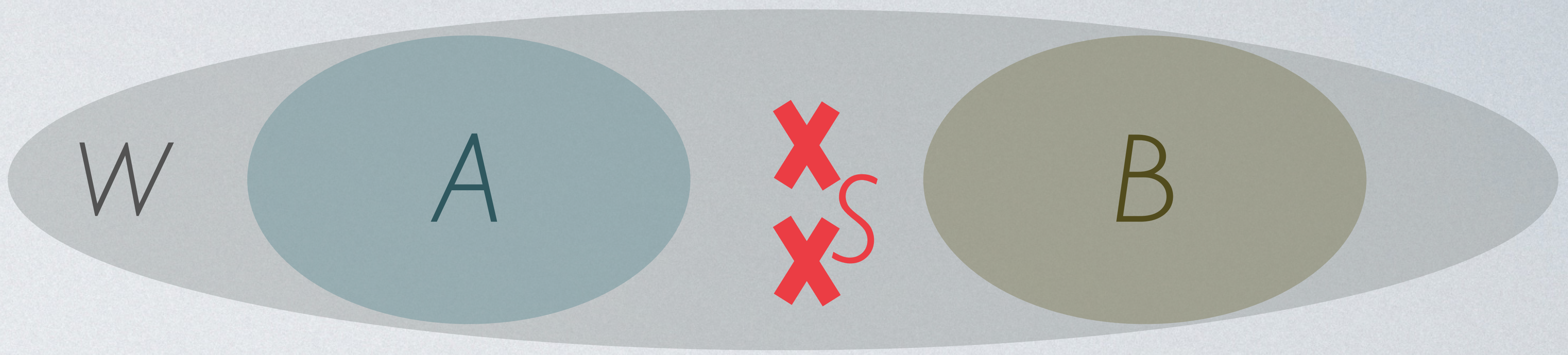

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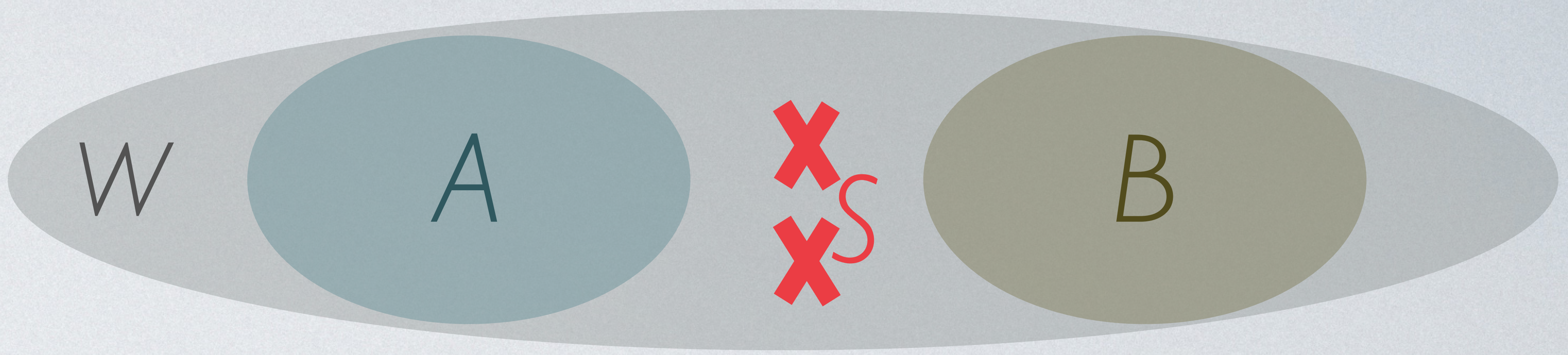
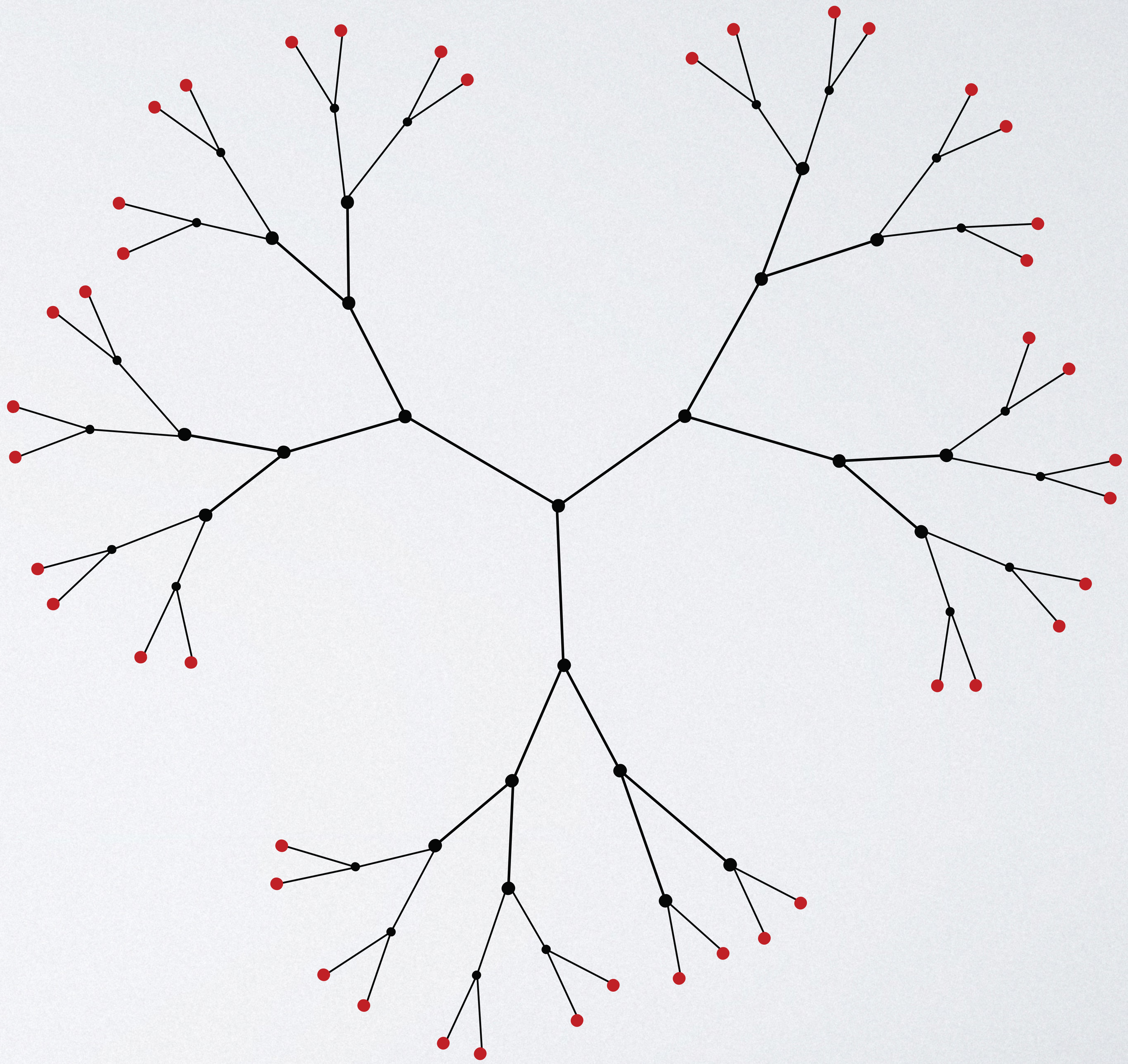


Example. Class of trees: $k := 1,\ \mathrm{U}(n) := n/3$



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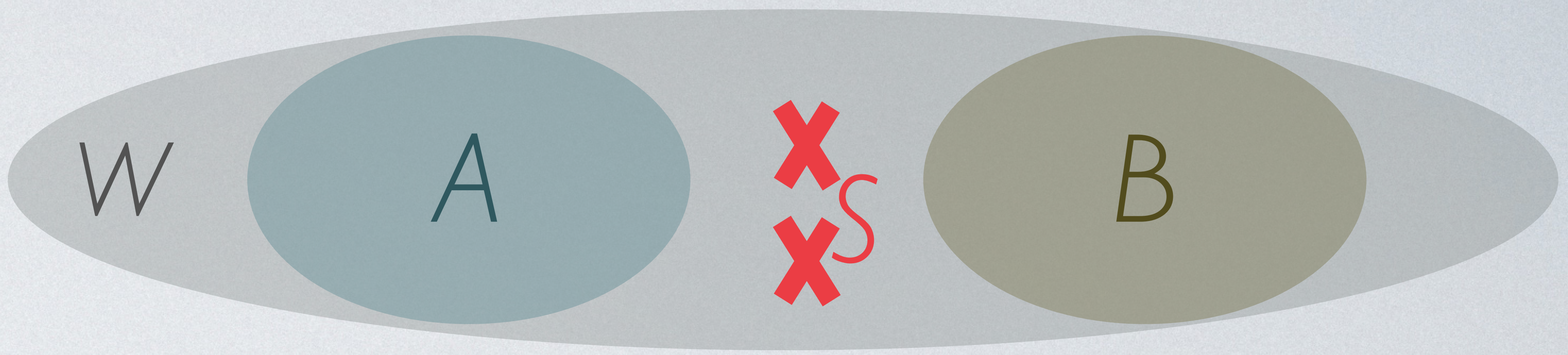

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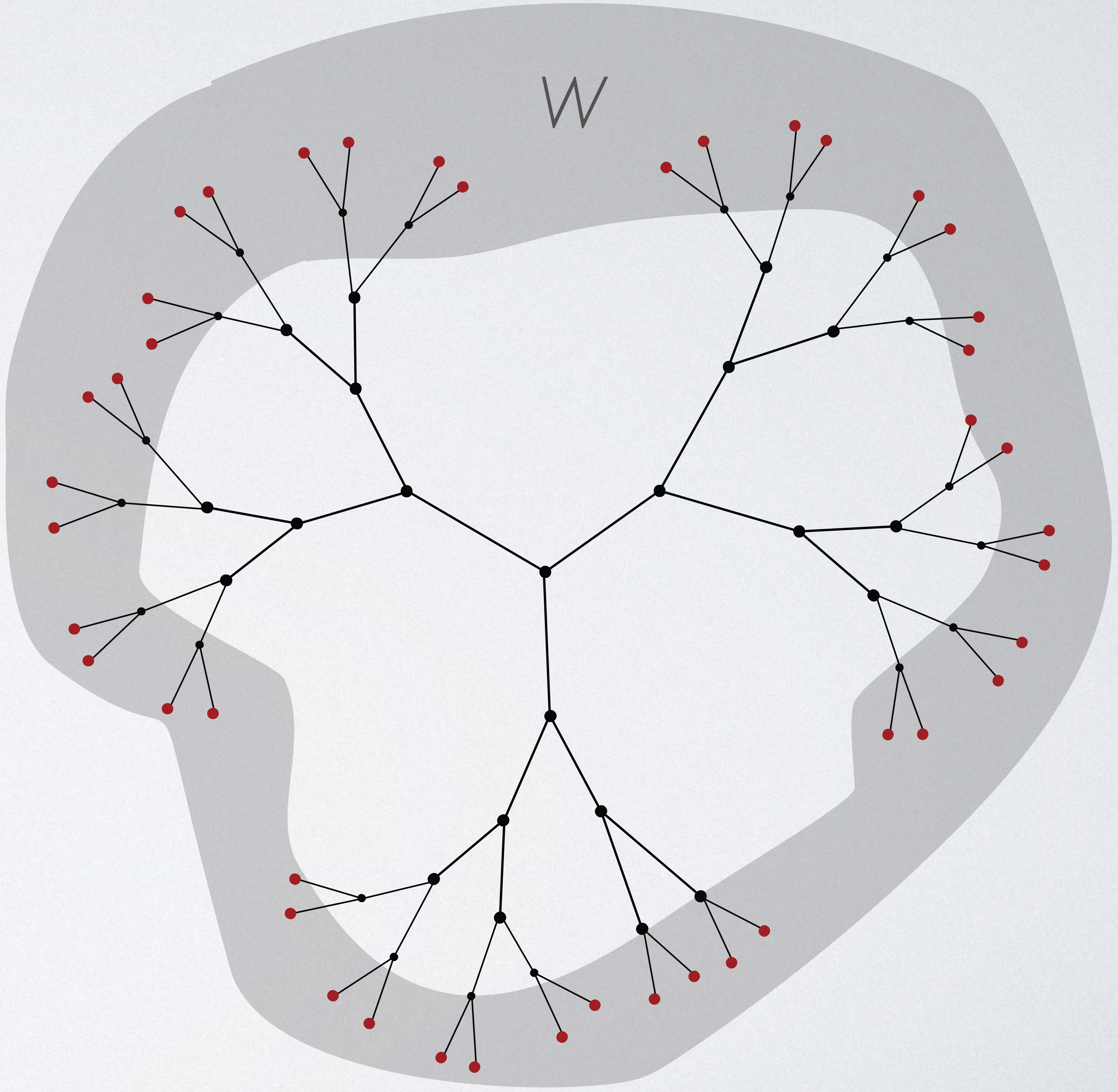

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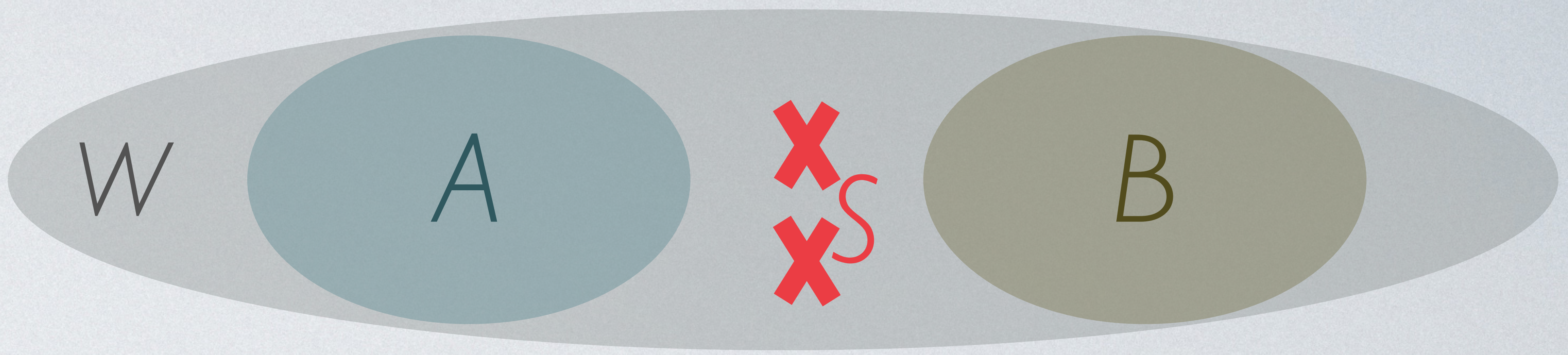

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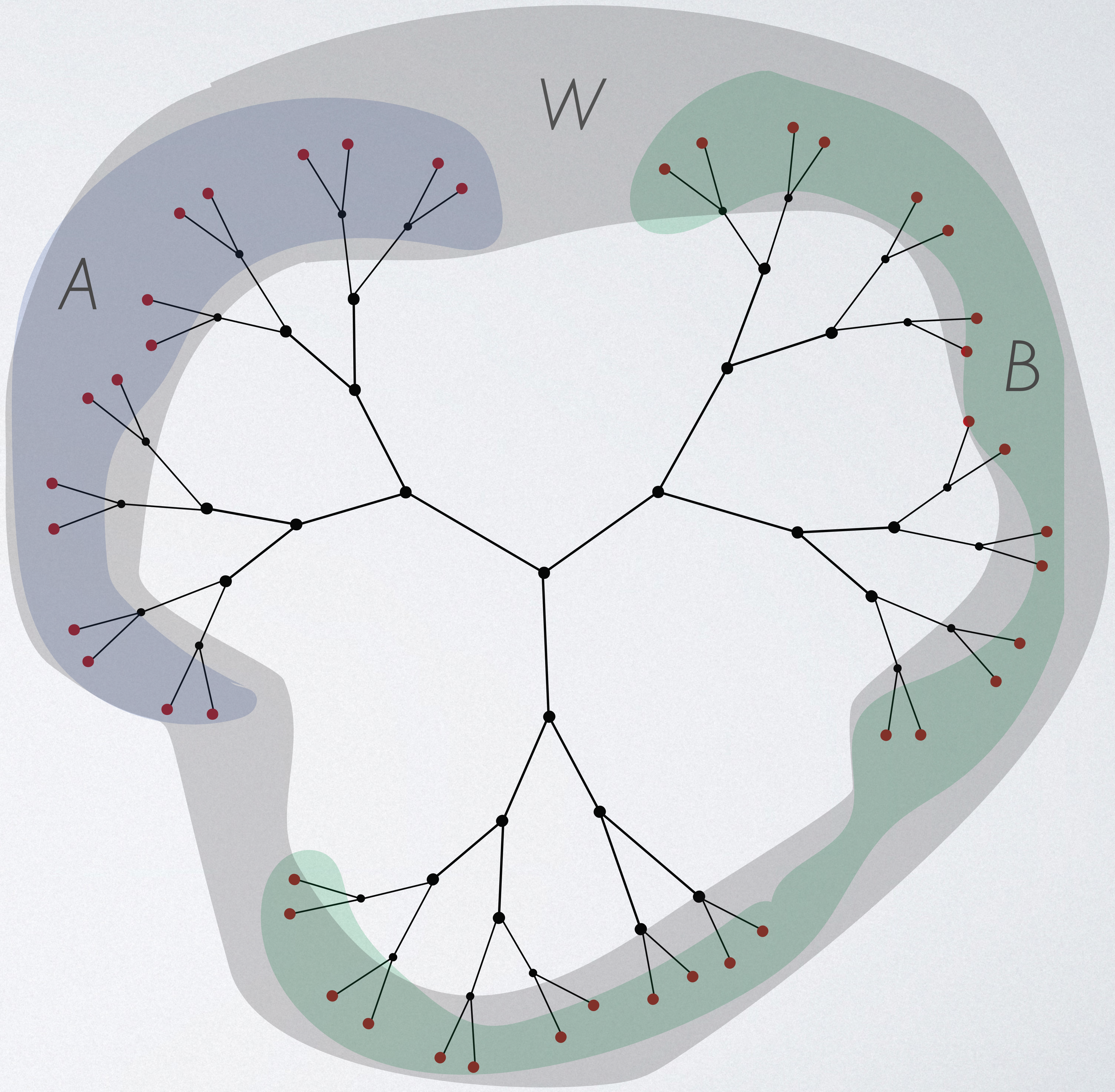

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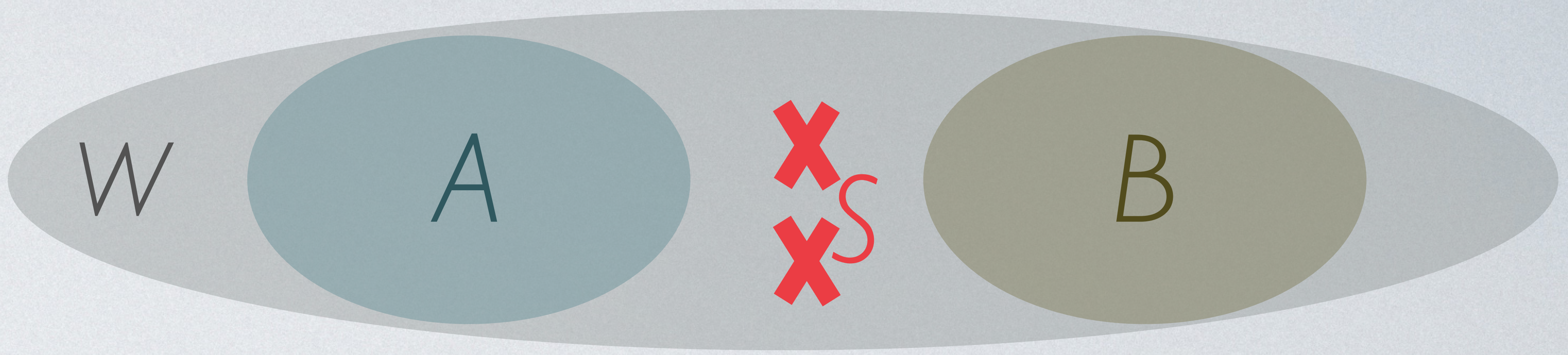

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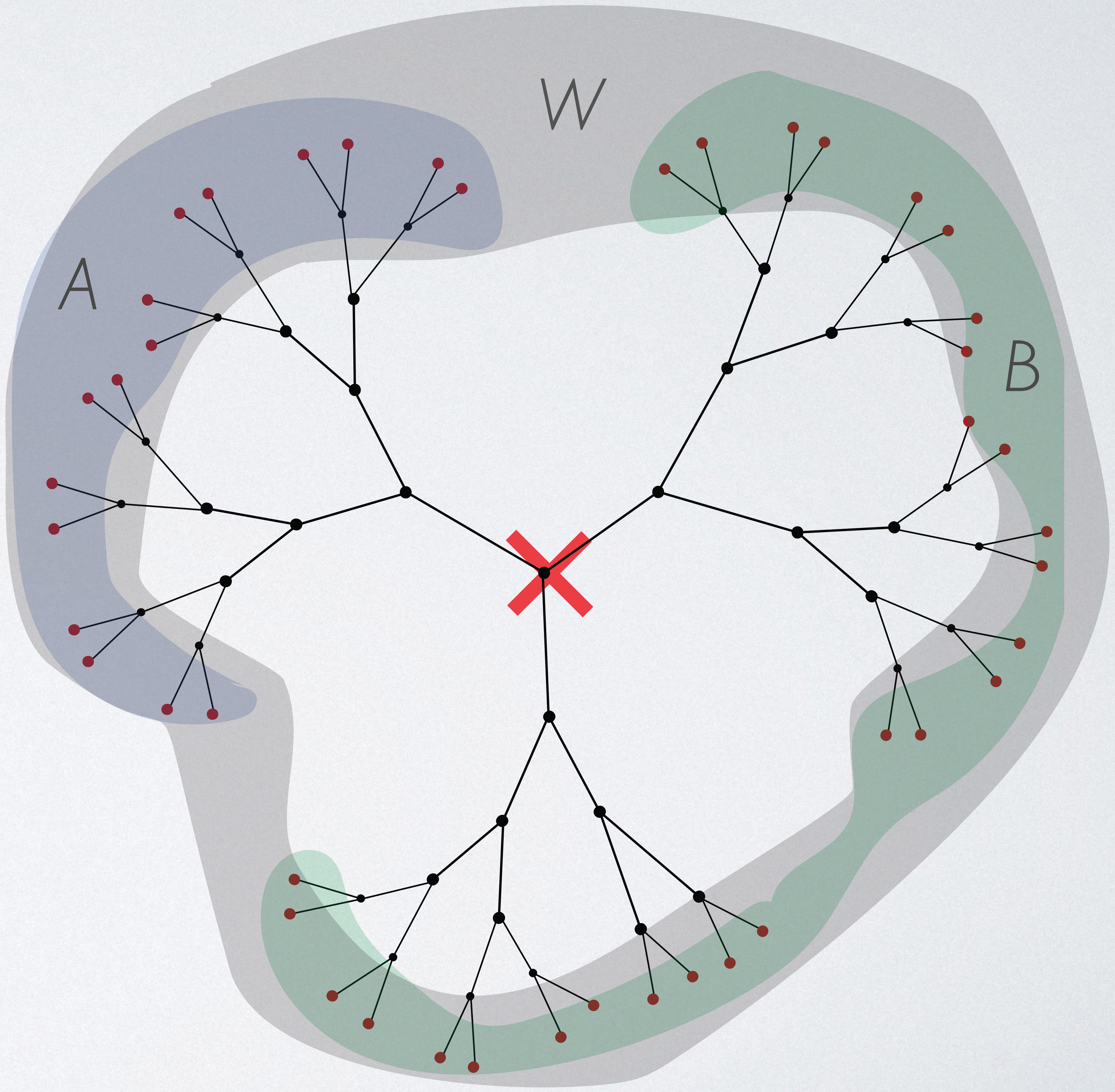

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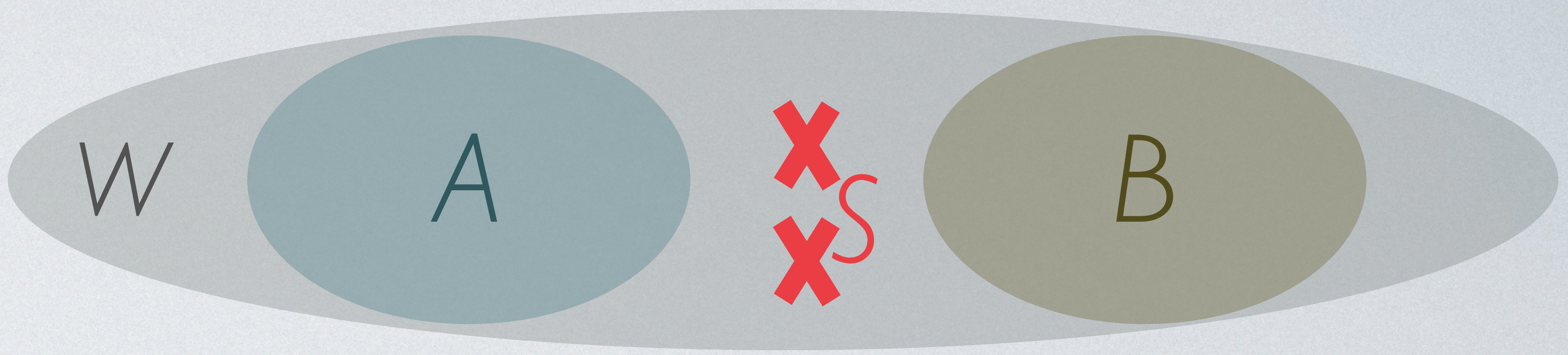

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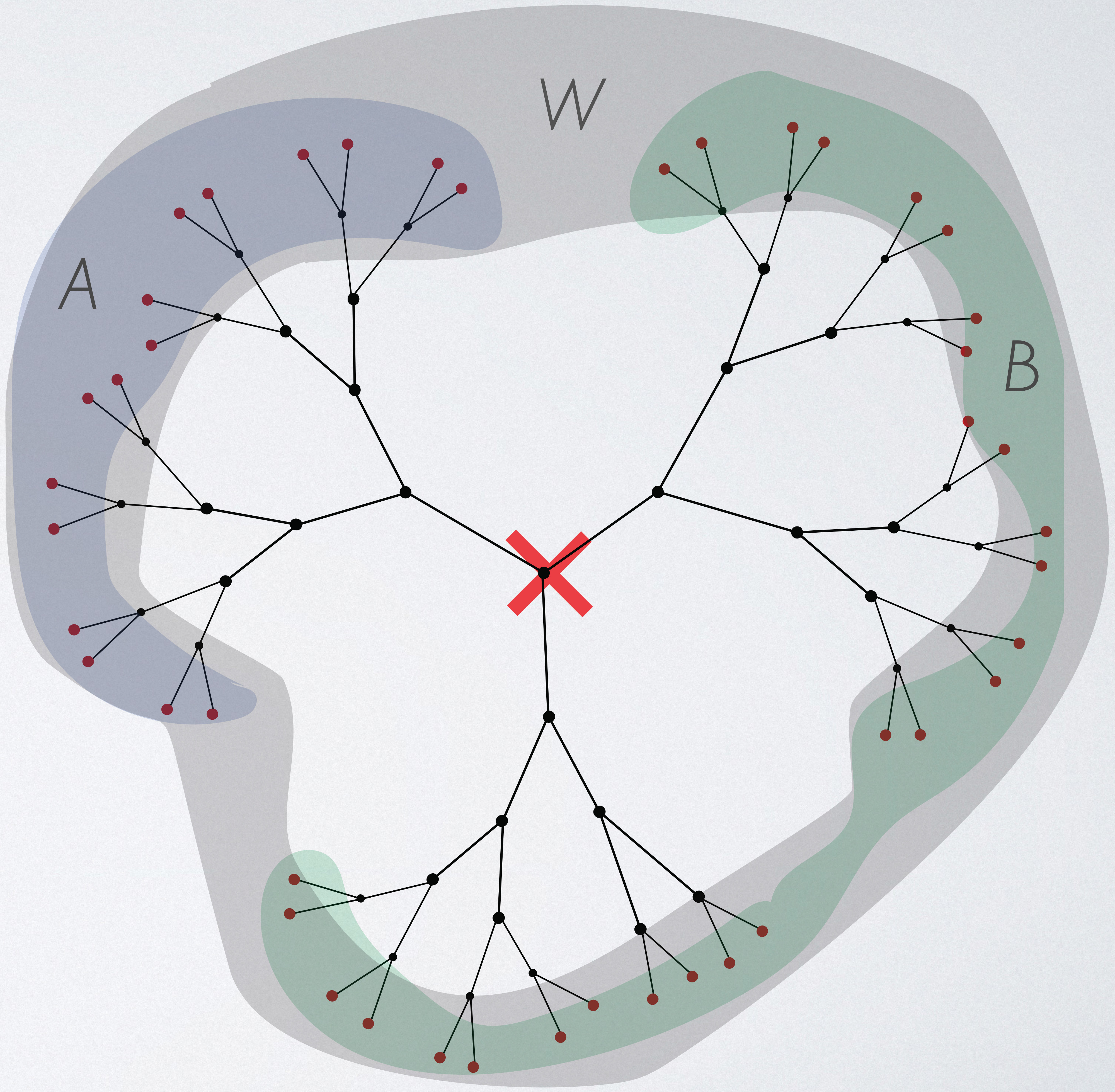
Example. Class of graphs of treewidth $\leq t$:
$k := t+1$, $U(n) := n/3$.



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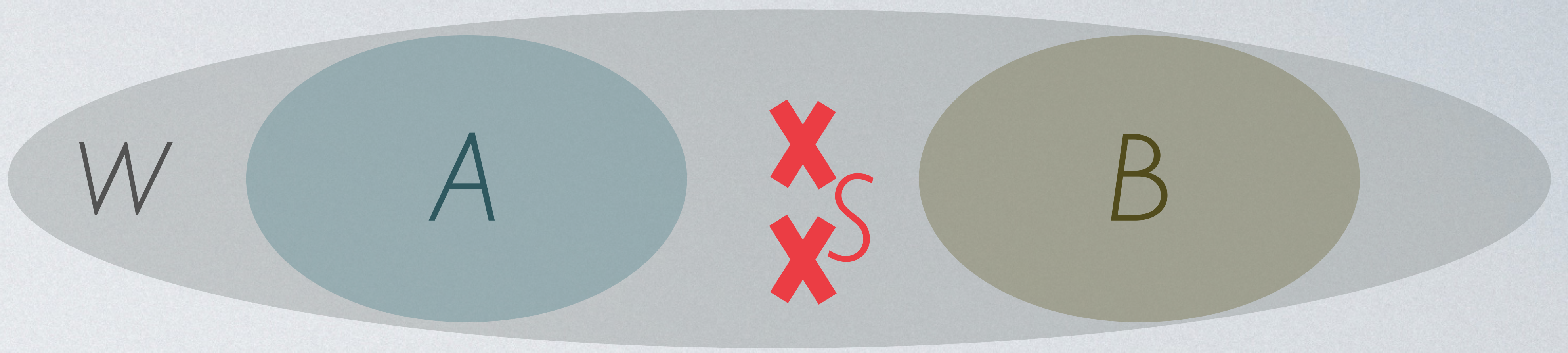

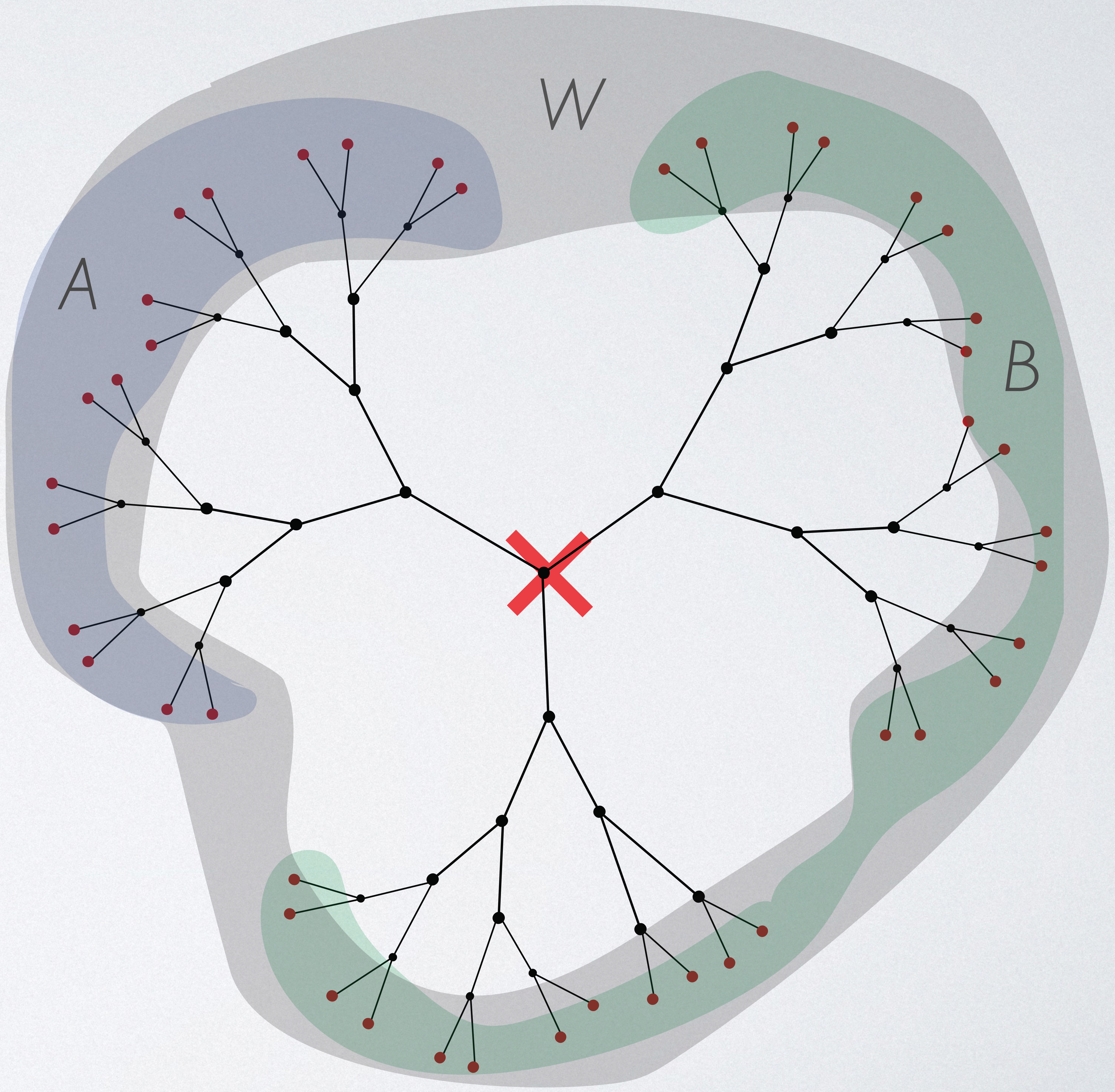

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Theorem. C is ∞-deletion-breakable $\Leftrightarrow$

C has bounded tree-width.

DELETION-BREAKABILITY

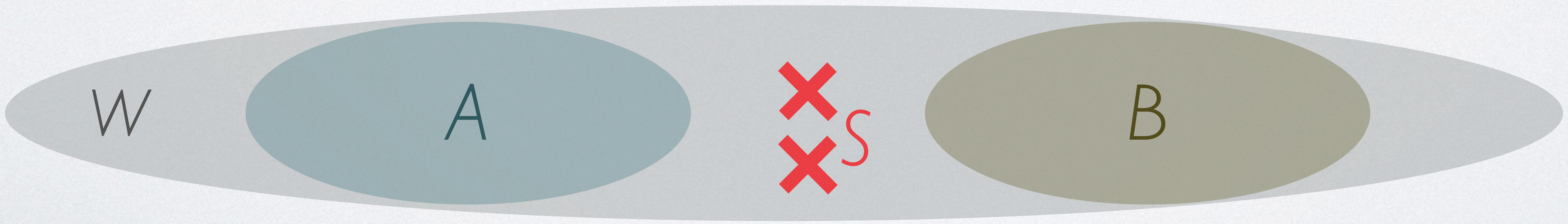
[Dreier, Mählmann, **T.**, STOC '24]

Definition. A class C of graphs is *deletion-breakable* if

$\forall r \geq 1.\ \exists k_r \geq 1.\ \exists U_r : \mathbf{N} \to \mathbf{N}$ – unbounded function s.t.

$\forall G \in C \quad \forall W \subseteq V(G) \quad \exists A,B \subseteq W,\ |A|=|B| \geq U_r(|W|), \quad \exists S \subseteq V(G), |S| \leq k_r$

dist(A,B)≥*r* in $G - S$



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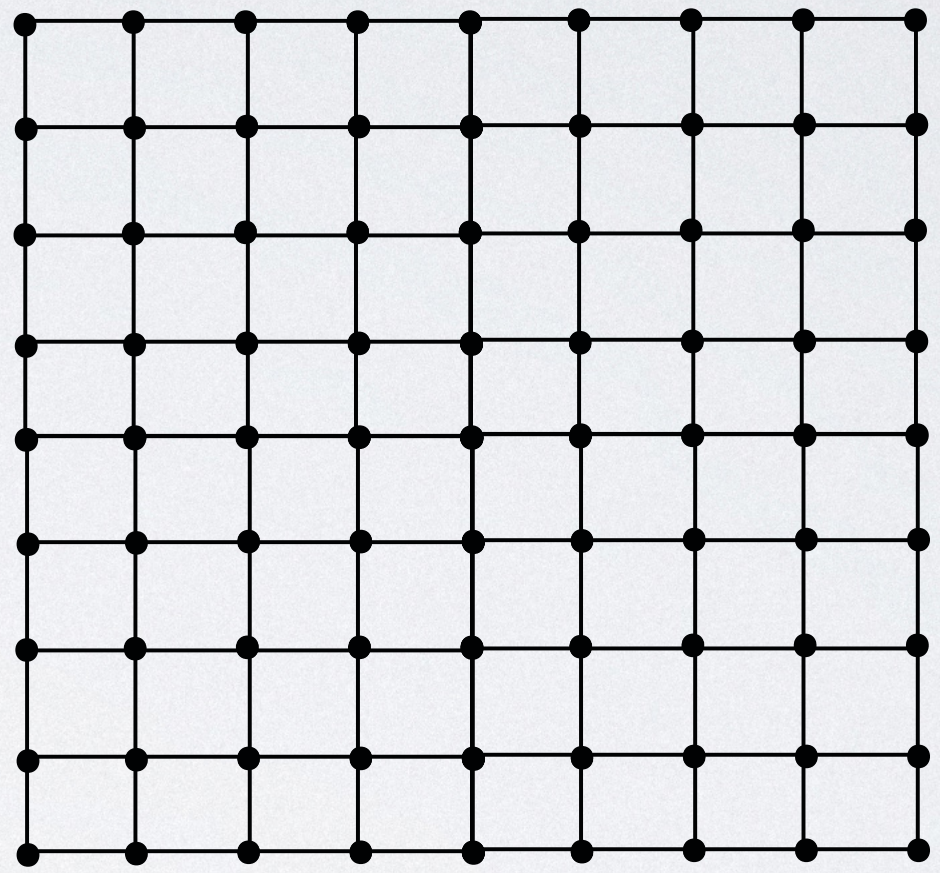
Example. Class of grids:
$k_r := 0, \quad U_r(n) := \Omega(n/r^2)$



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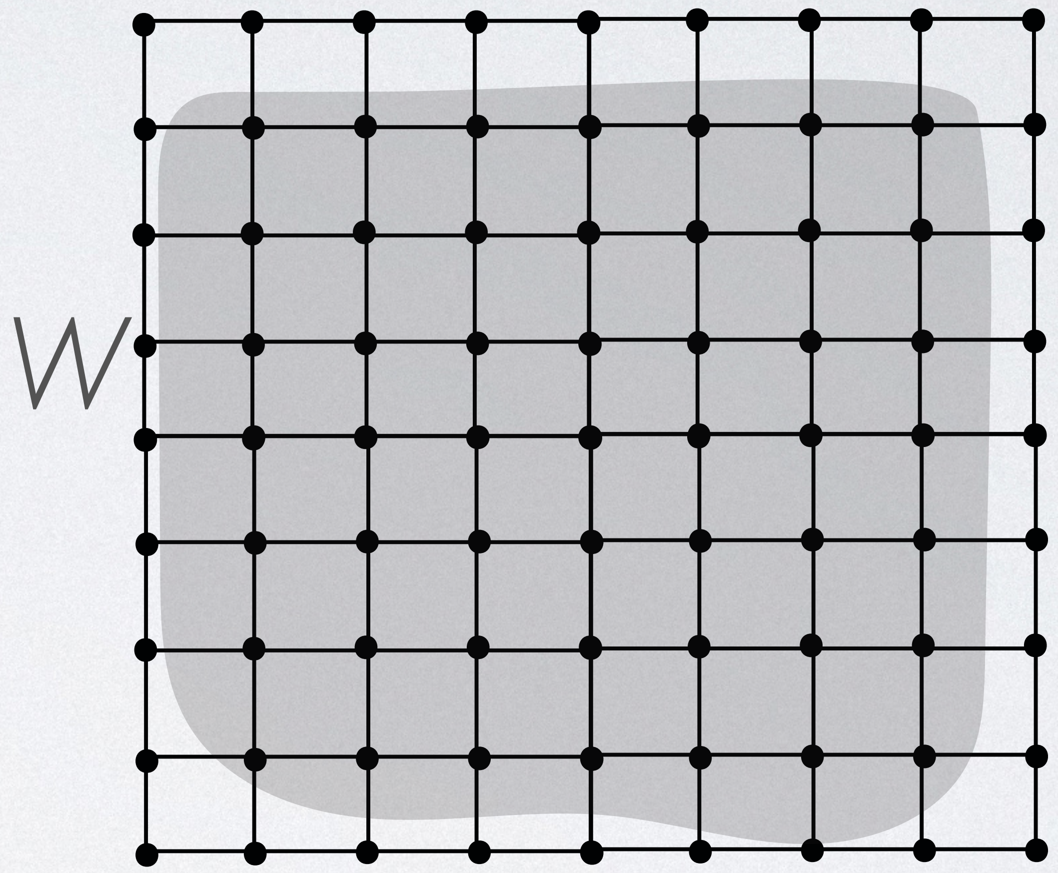
– every set $W \subseteq V(G)$ contains



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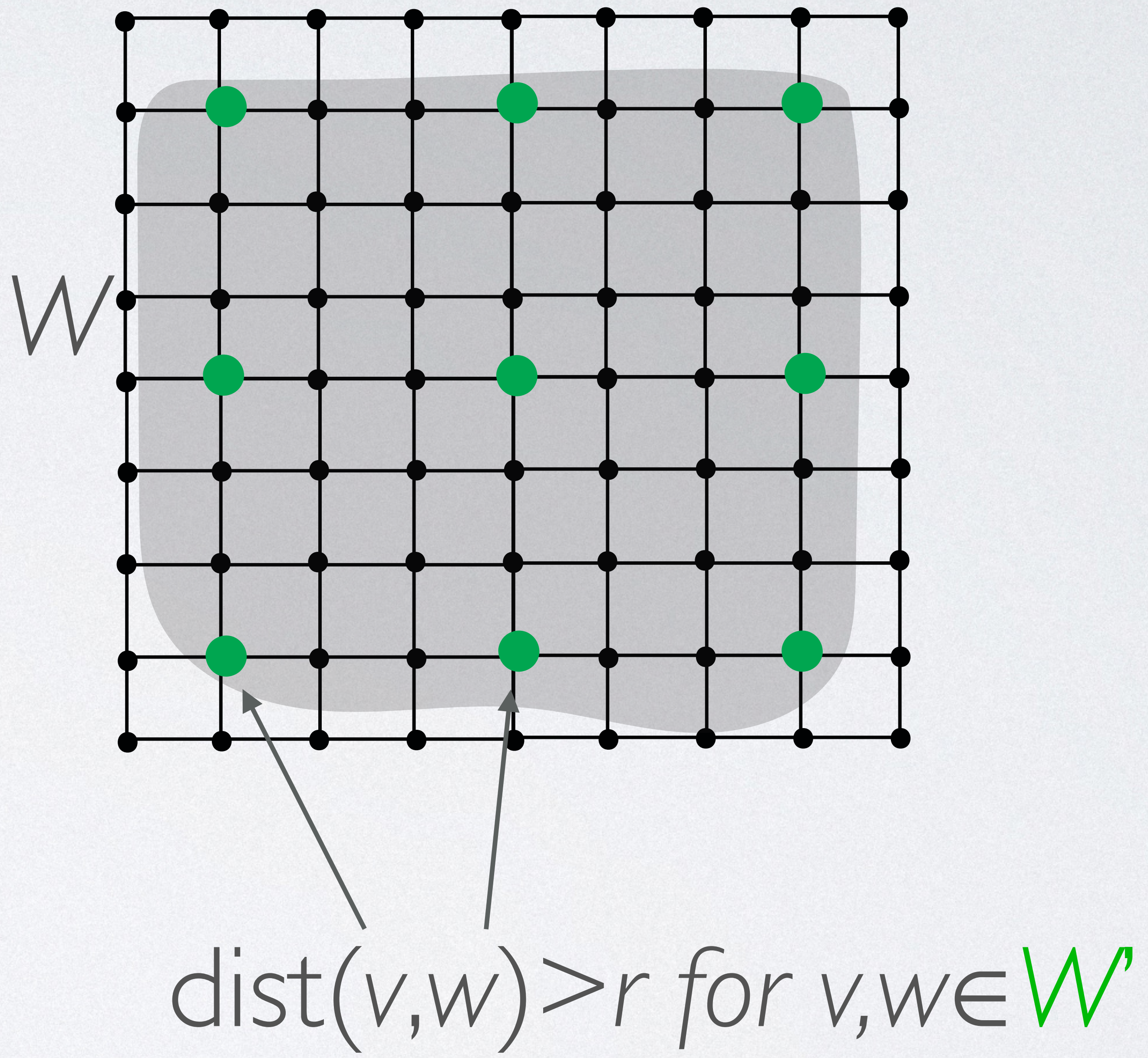

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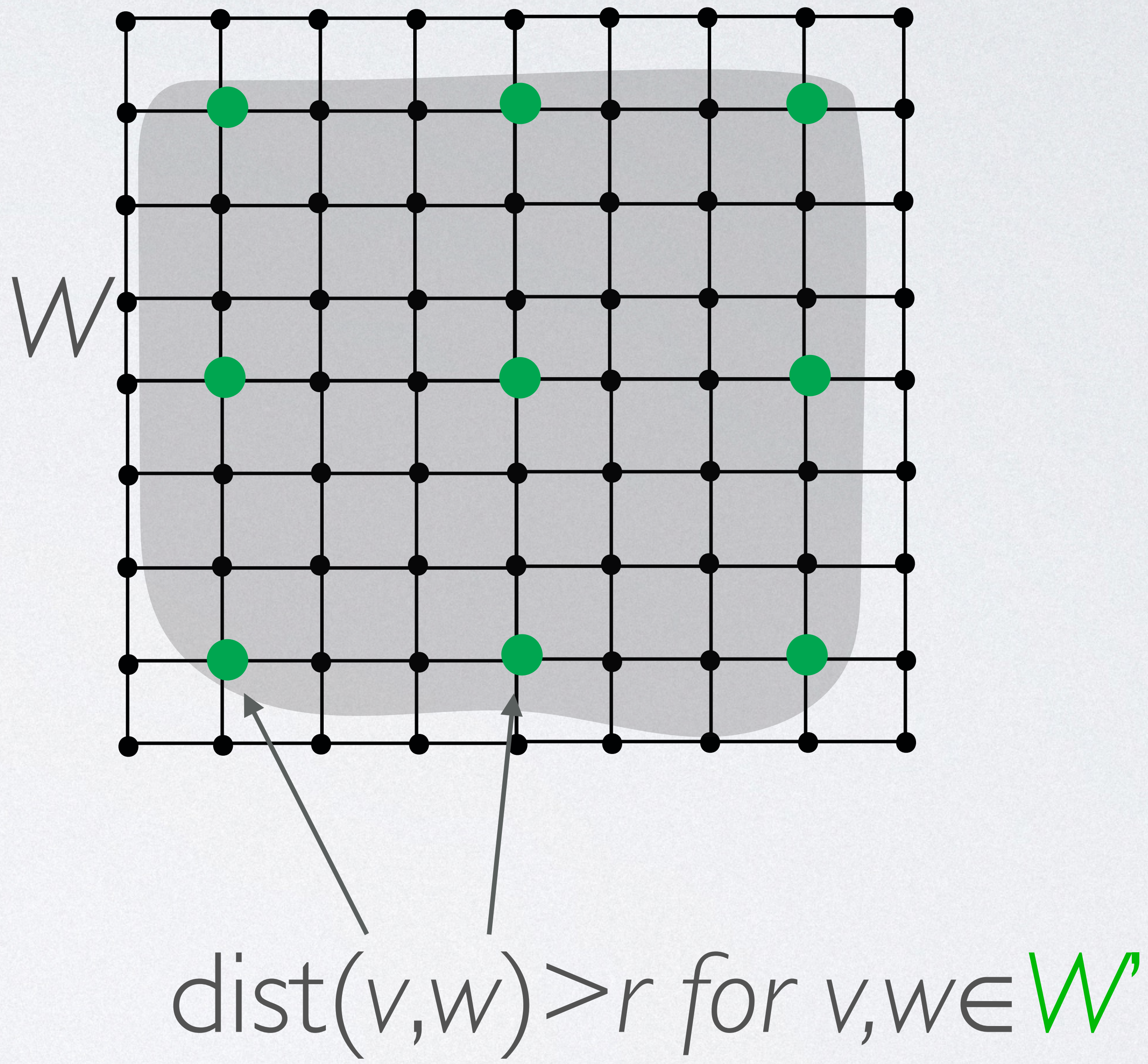

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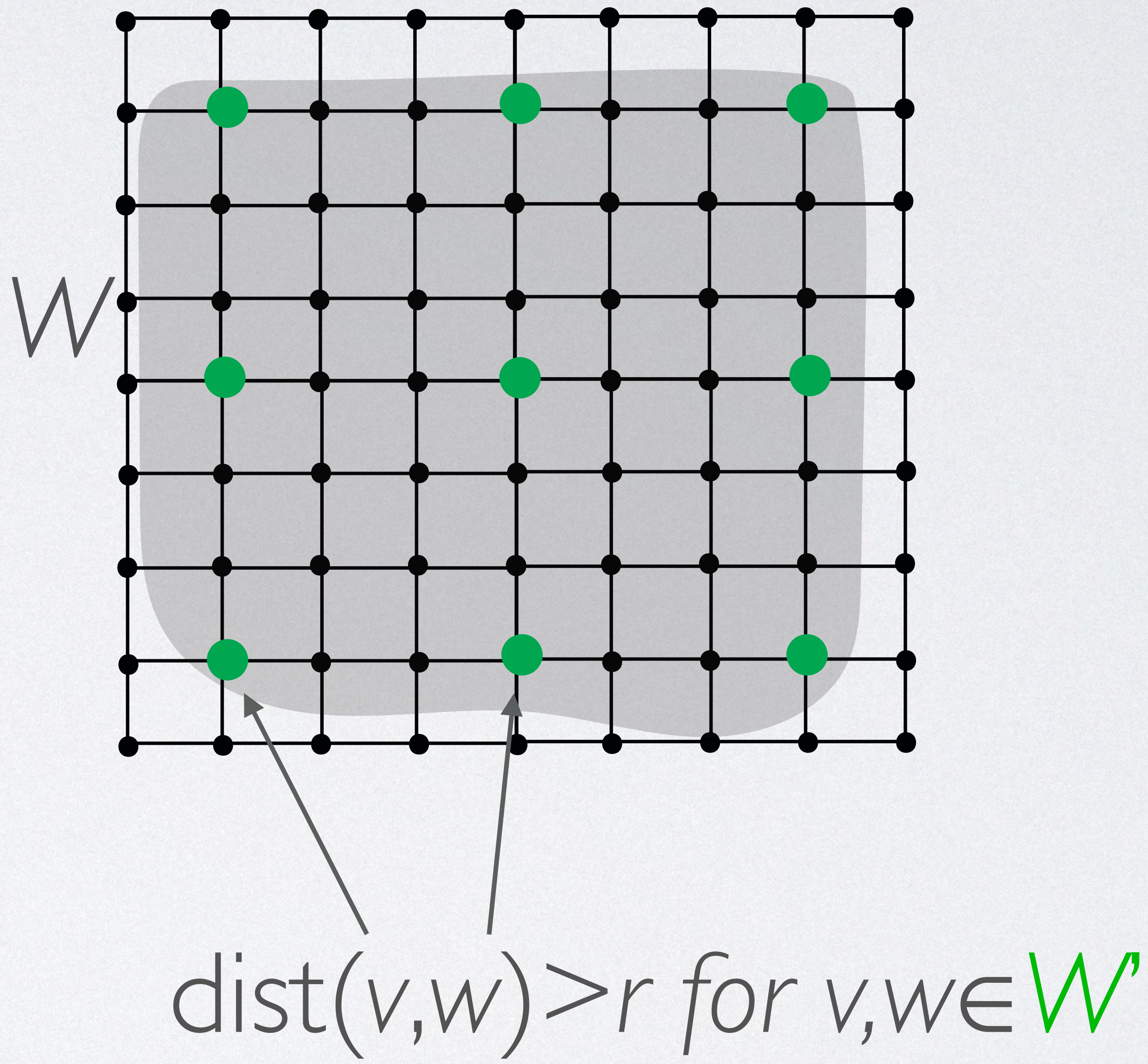
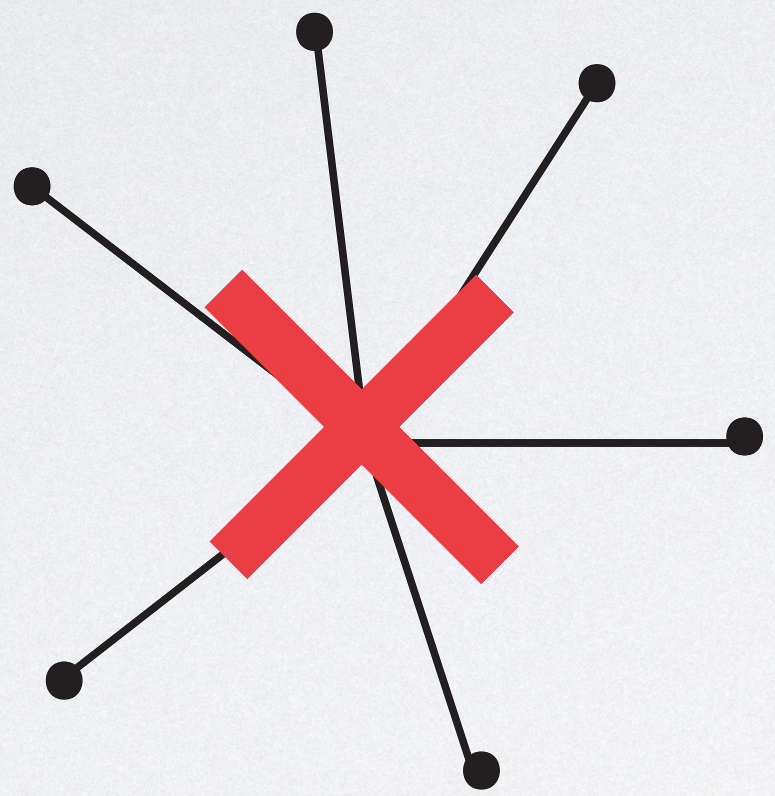


Theorem. C is deletion-breakable $\Leftrightarrow$ C is nowhere dense.

GOING DENSE



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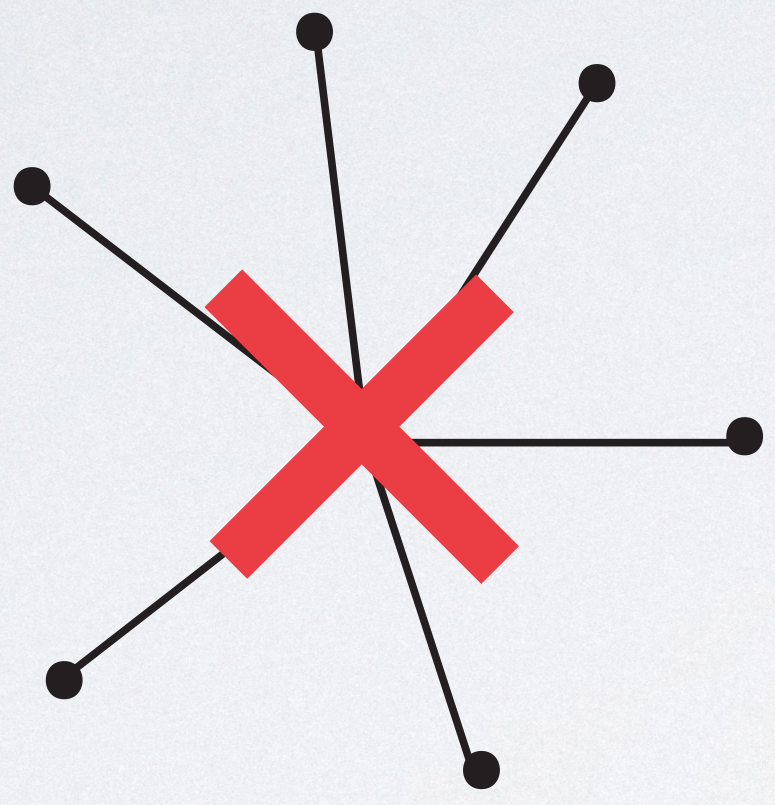

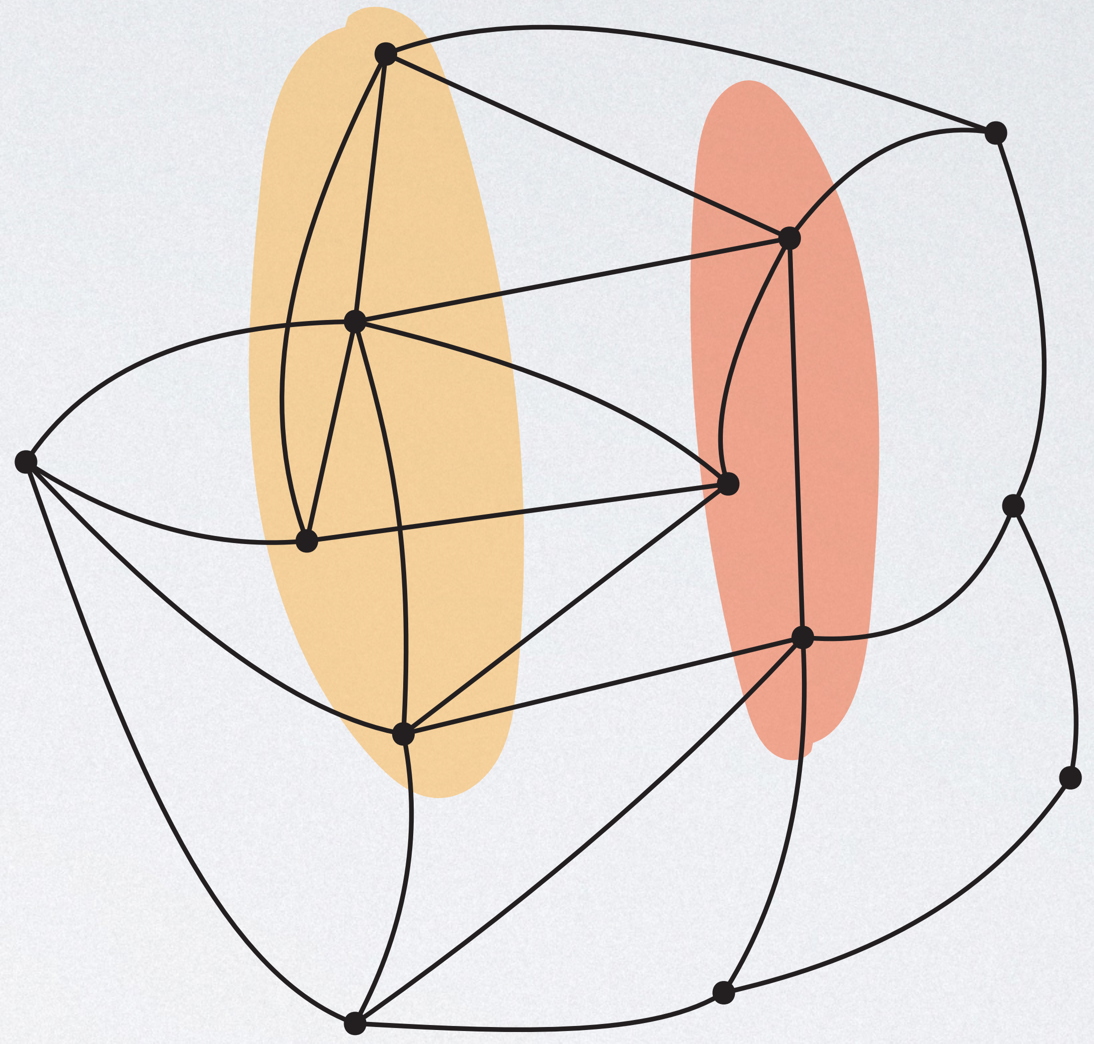

deleting a vertex → **flipping a pair of sets**

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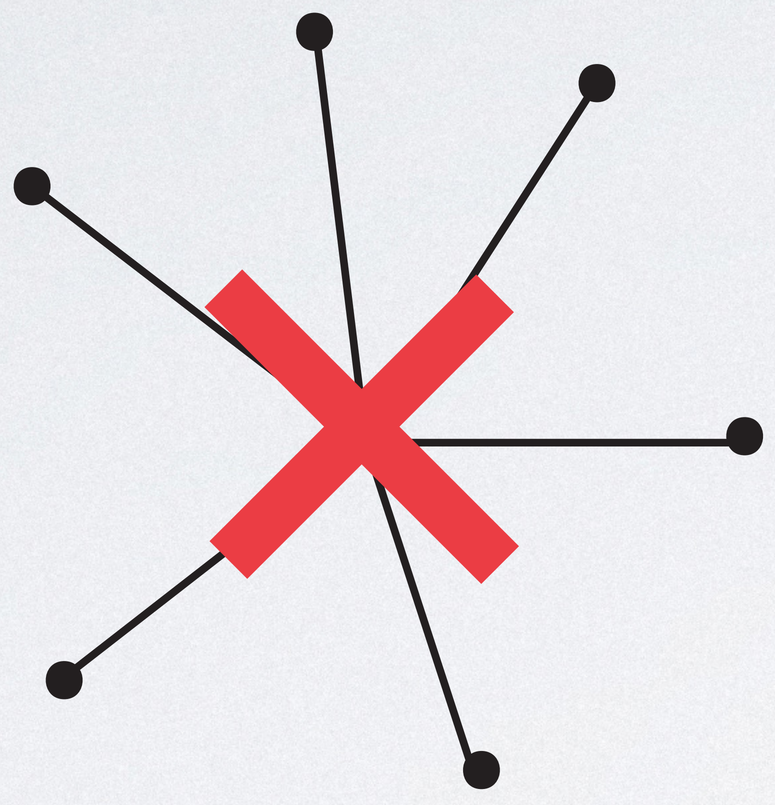

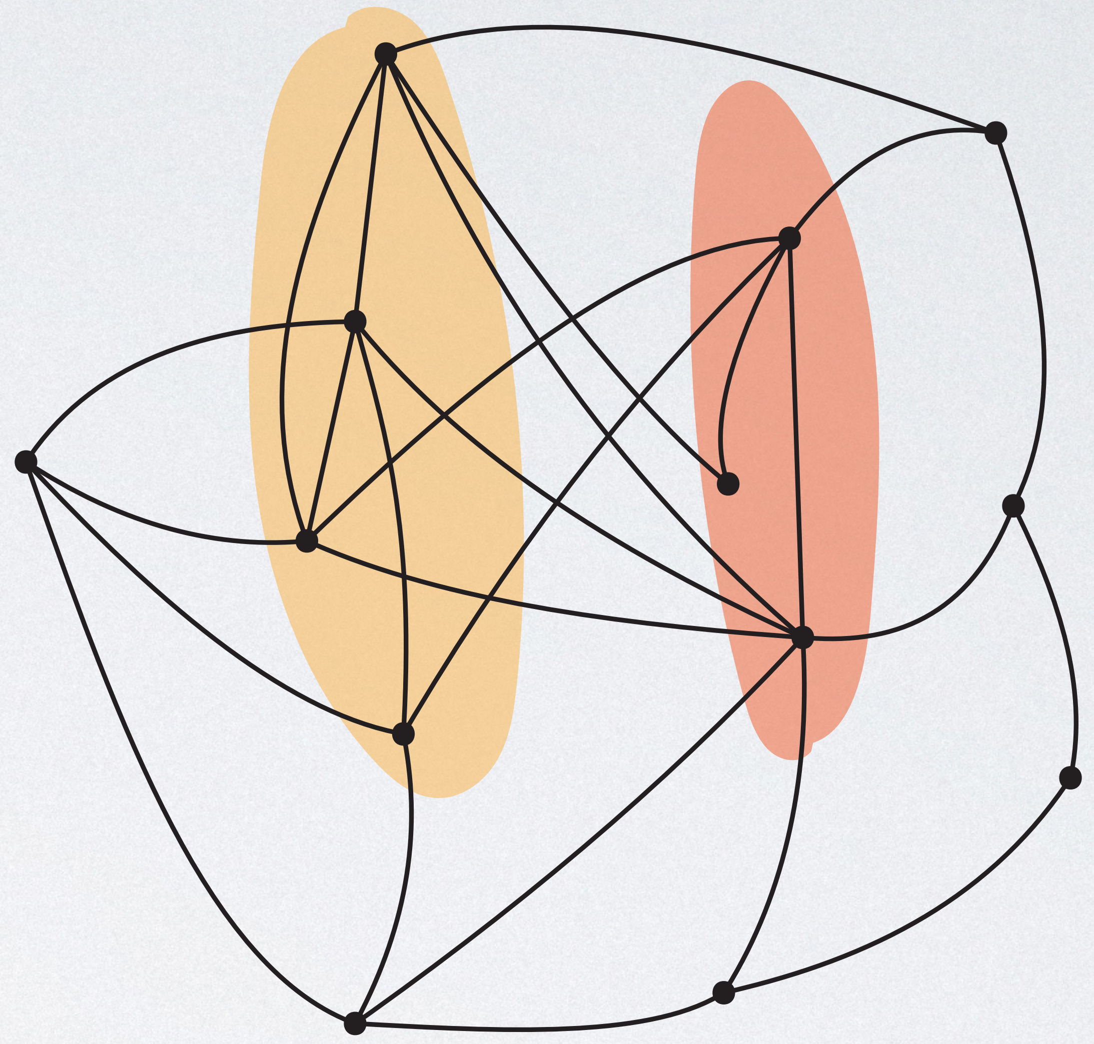

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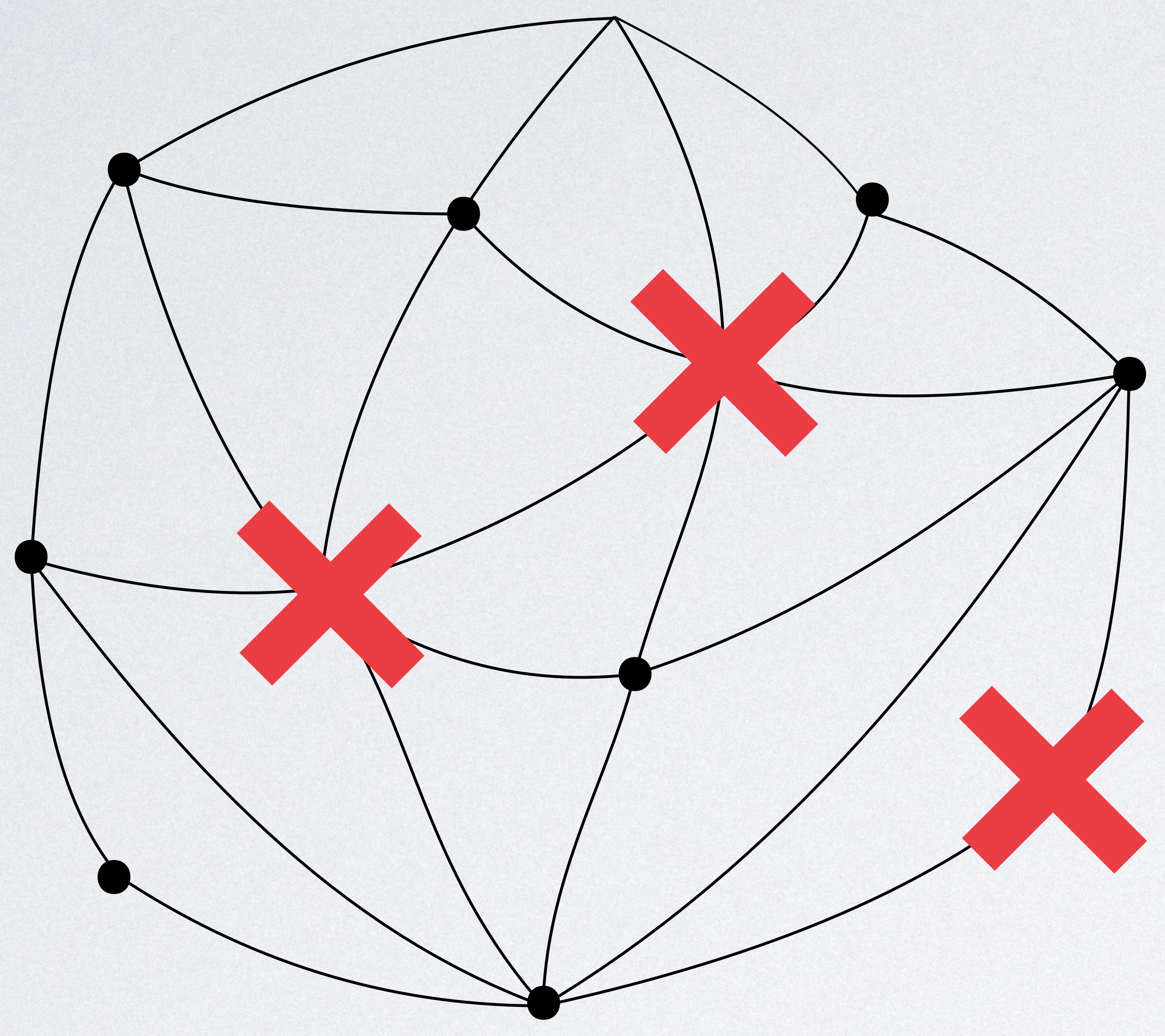

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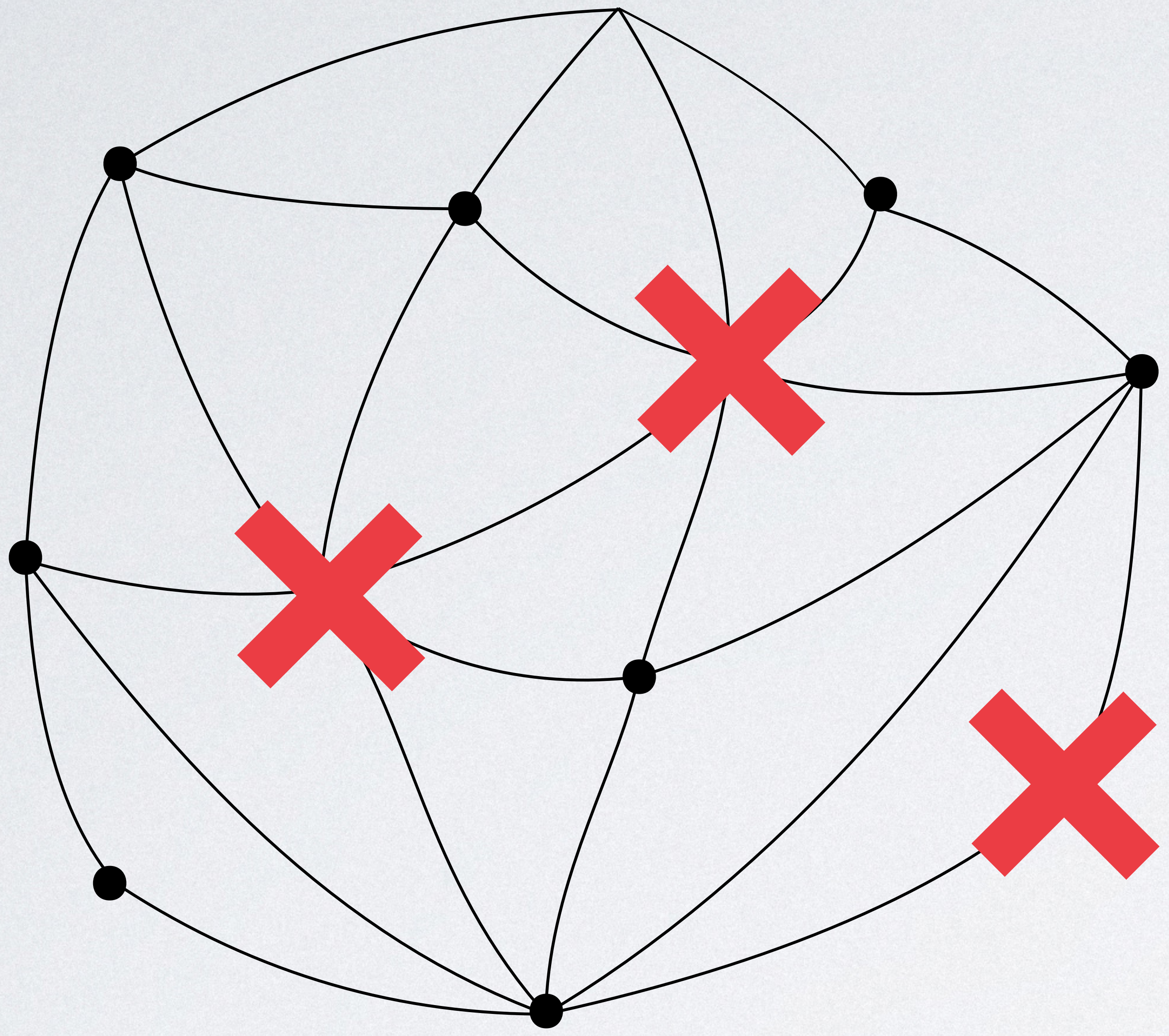

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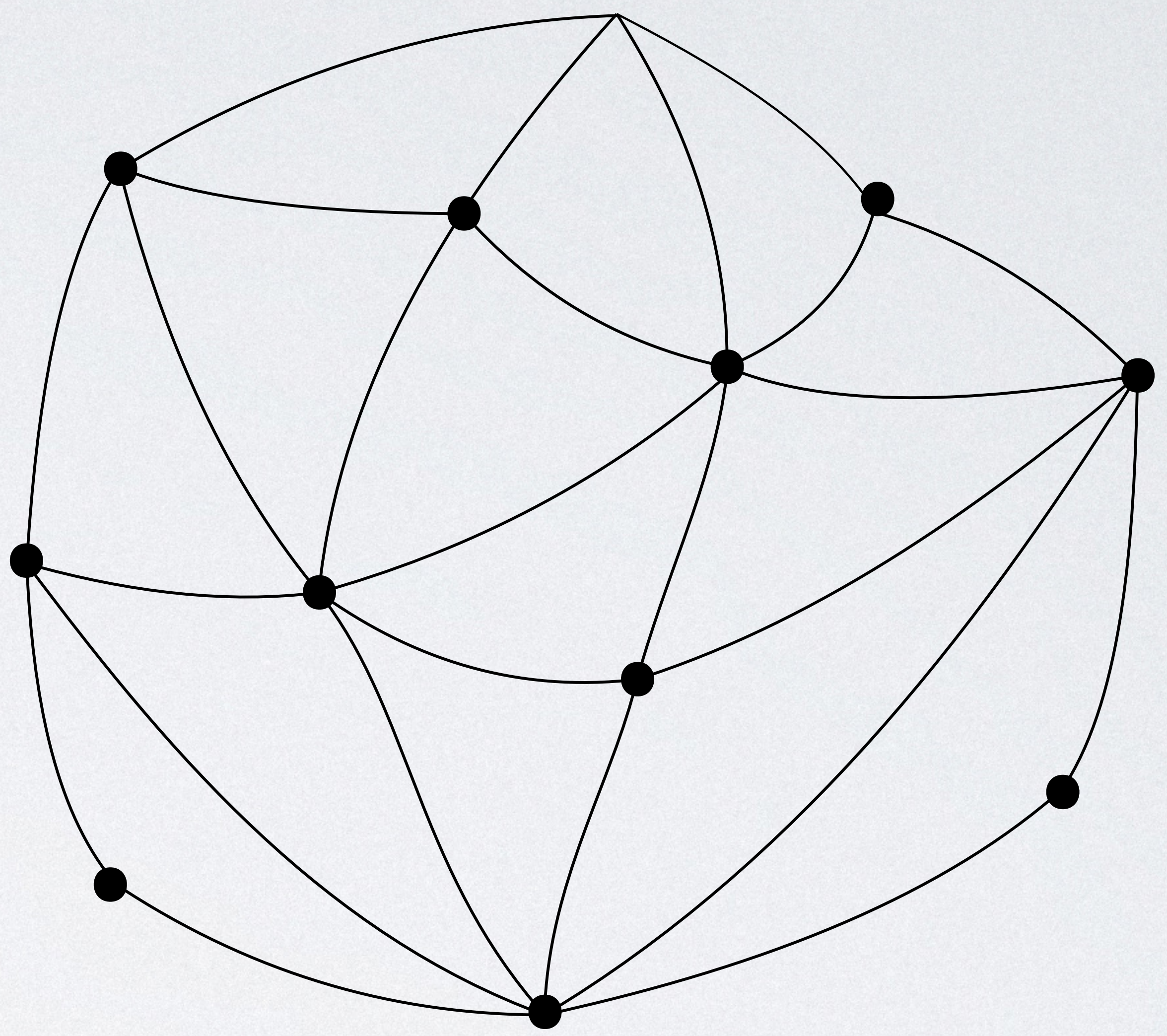

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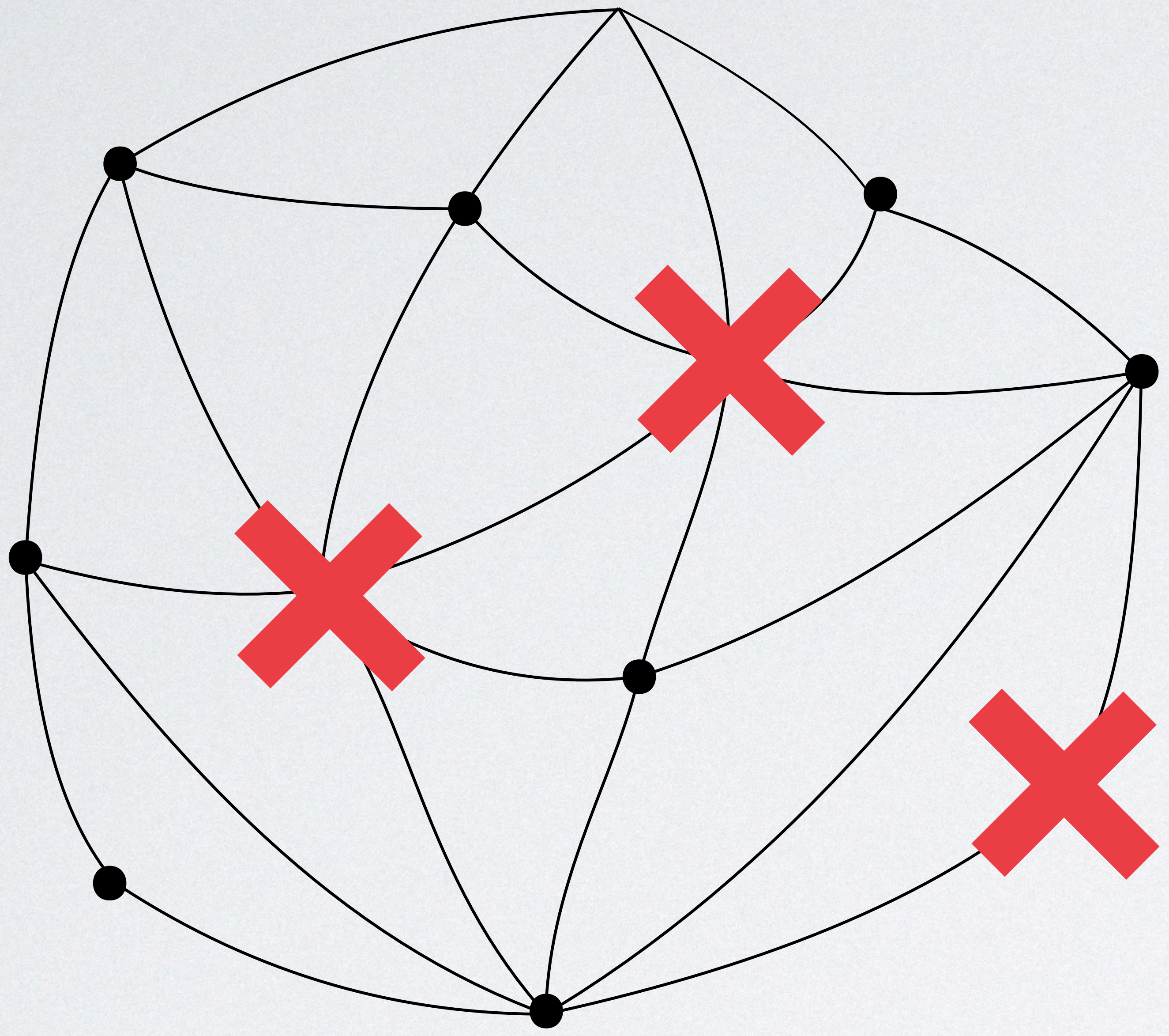

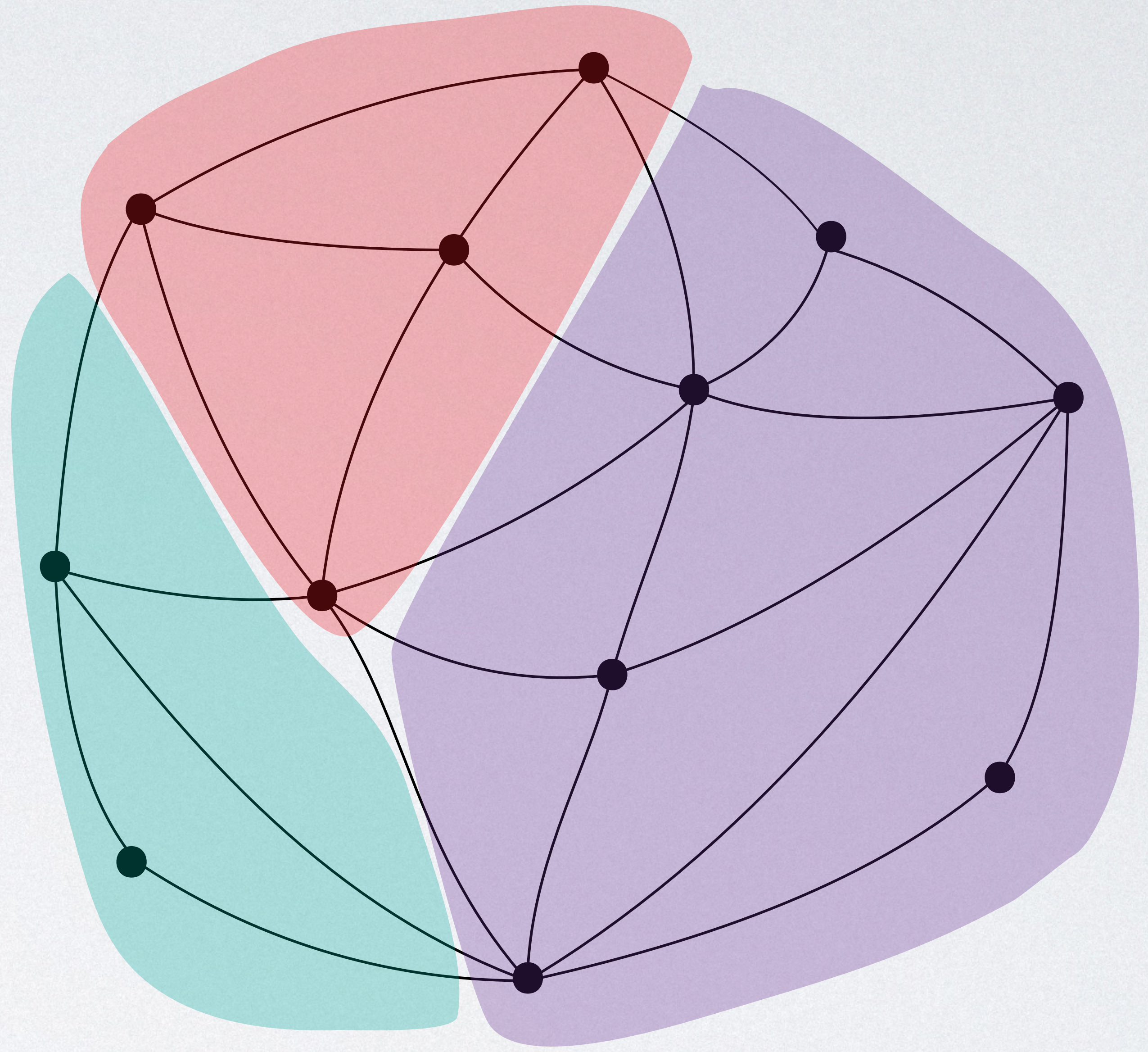

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partition $V(G)= A_1 \cup \ldots \cup A_k$

For each pair A_iA_j

flip or not

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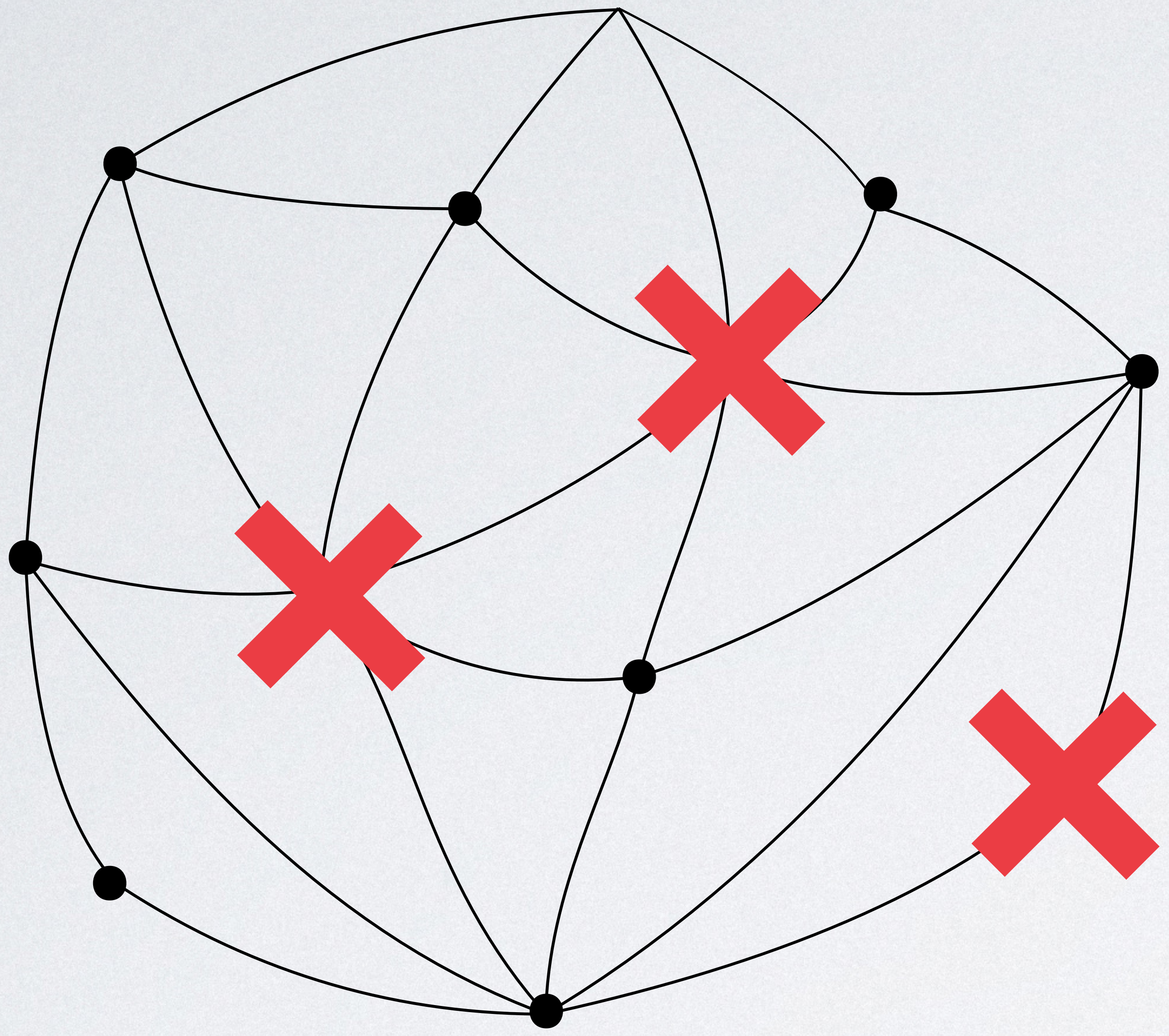

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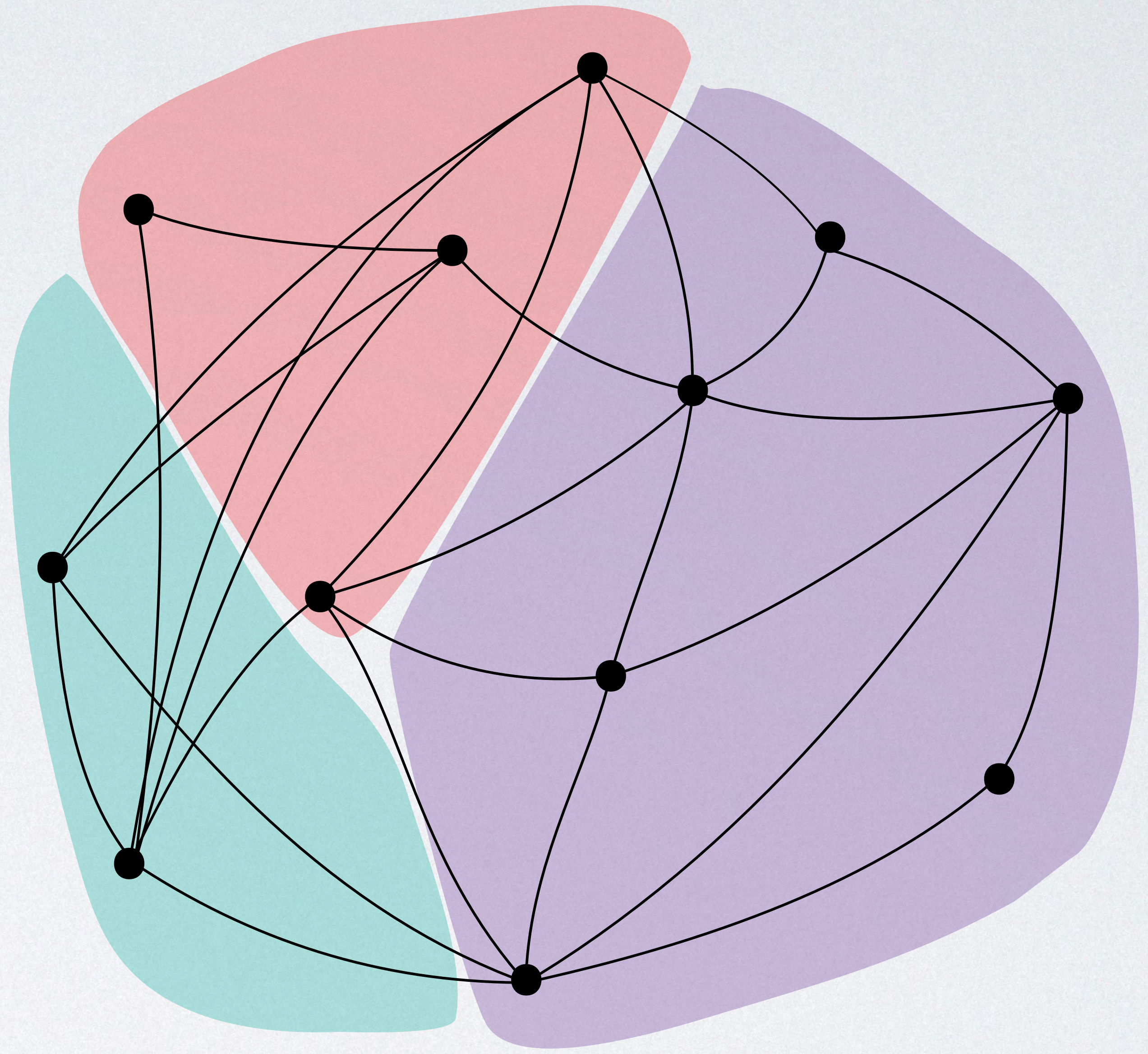

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[Dreier, Mählmann, **T.**, STOC '24]

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Theorem. C is ∞-flip-breakable ⇔ C has bounded cliquewidth.

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Theorem. C is flip-breakable ⇔ C is monadically dependent.

	monotone: delete	*hereditary:* flip
radius ∞ MSO	bounded treewidth $\Leftrightarrow$ ∞-**deletion**-breakable	bounded cliquewidth $\Leftrightarrow$ ∞-**flip**-breakable
finite radii FO	nowhere dense $\Leftrightarrow$ **deletion**-breakable	monadically dependent $\Leftrightarrow$ **flip**-breakable

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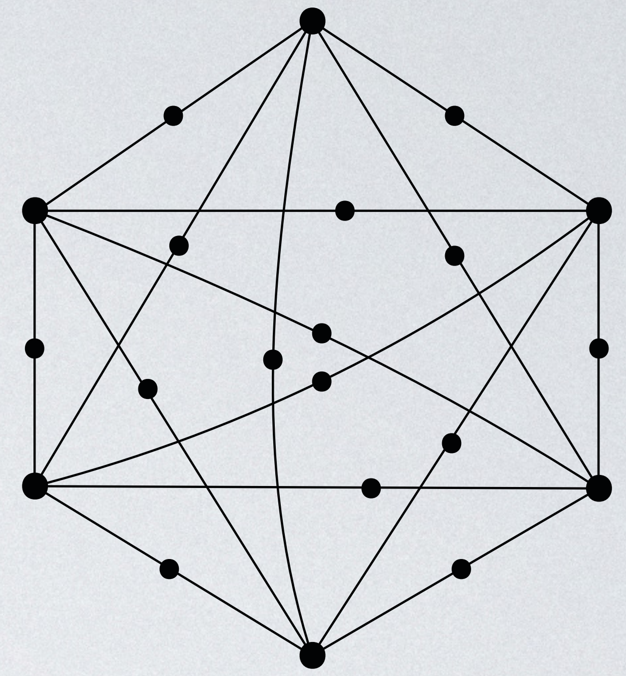

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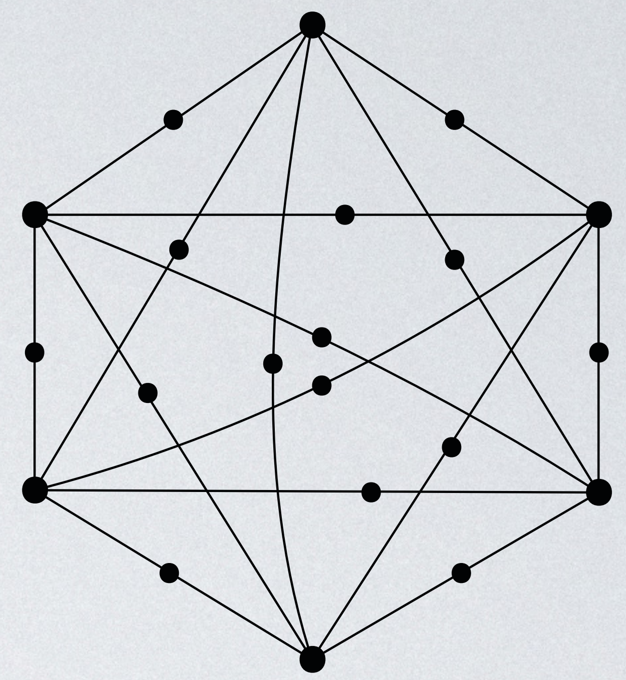

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Corollary. For hereditary graph classes,
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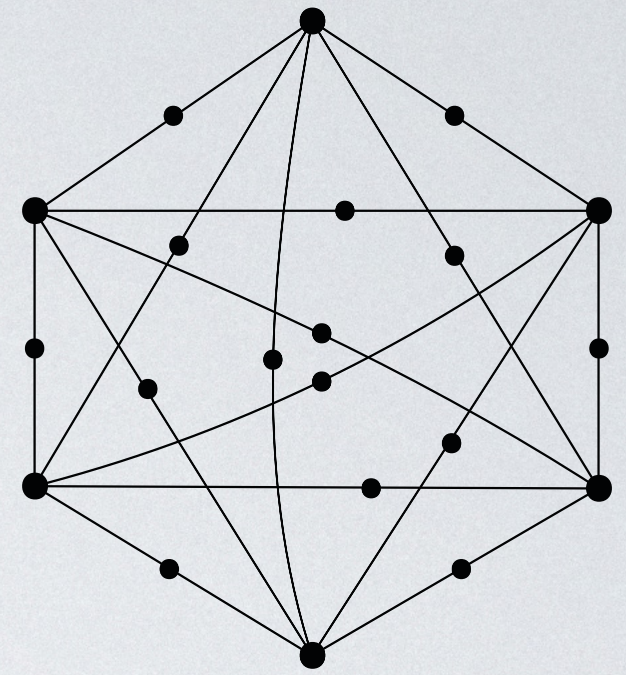

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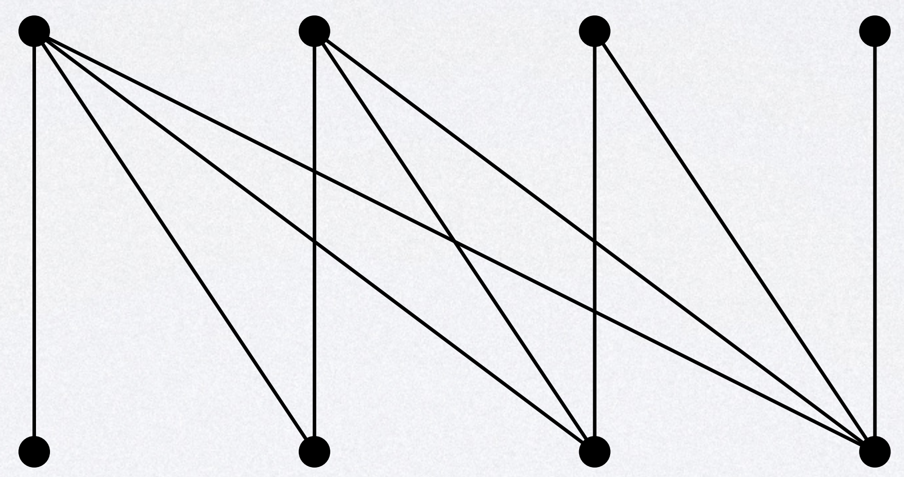
'⇒' implication in *"tractable* ⇔ *monadically dependent"* conjecture.

OUTLINE

1. The model checking problem
2. Sparsity: monotone case
3. Twin-width: ordered case
4. Monadic dependence
5. Flip-breakability
6. **Stability: orderless case**

ORDERLESS CLASSES

Definition
A graph class is *orderless* if it avoids some half-graph as a semi-induced subgraph.



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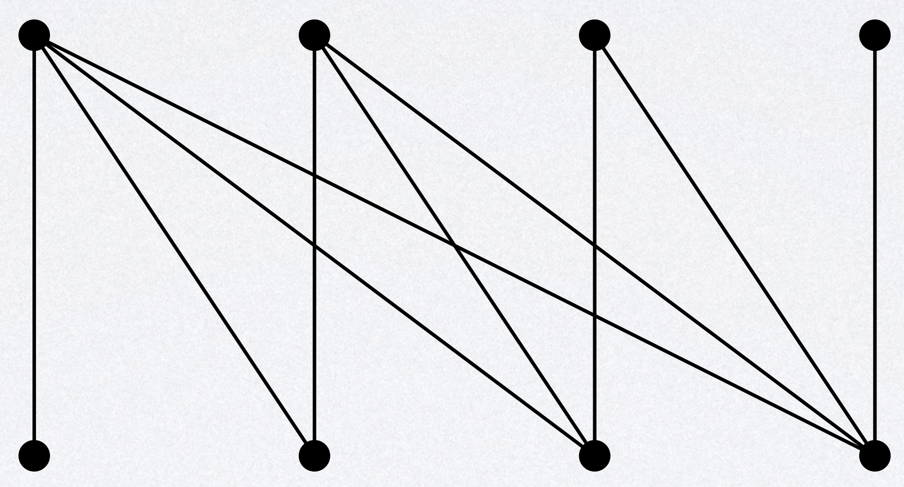

Example:

- all nowhere dense classes C, and
- all classes that can be transduced from C

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Ingredients: stability theory, sparsity theory, VC theory, geometric range queries

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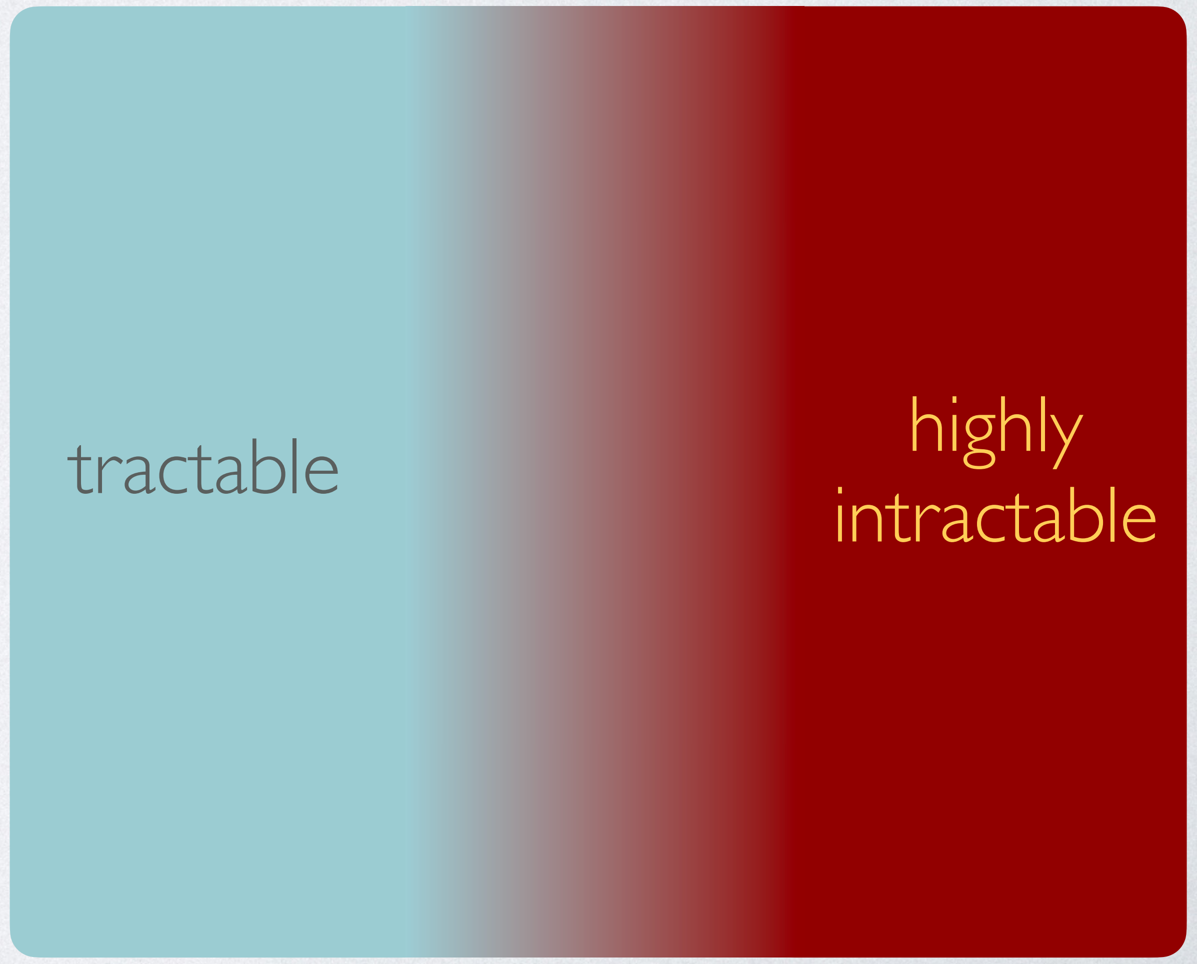

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For **monotone** graph classes:

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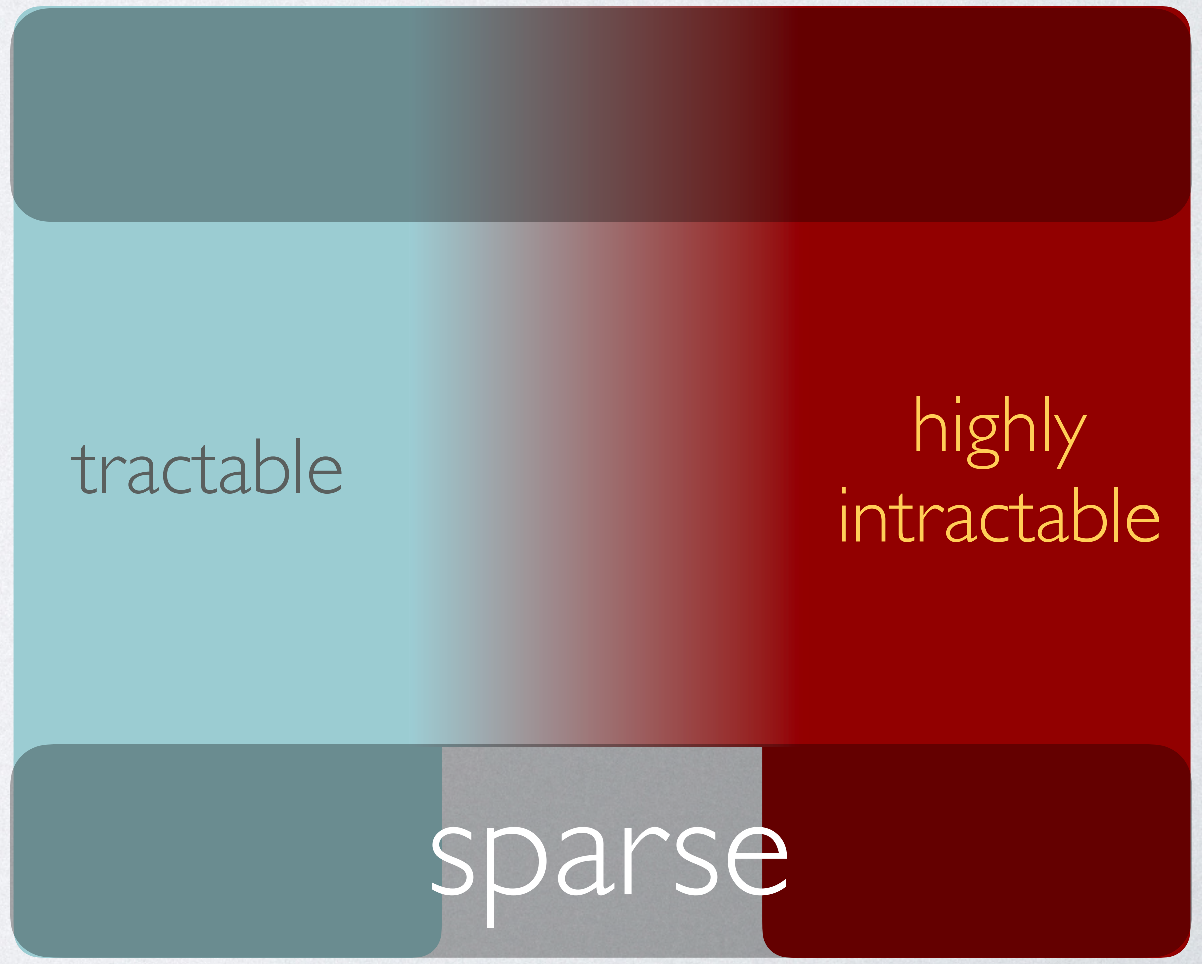

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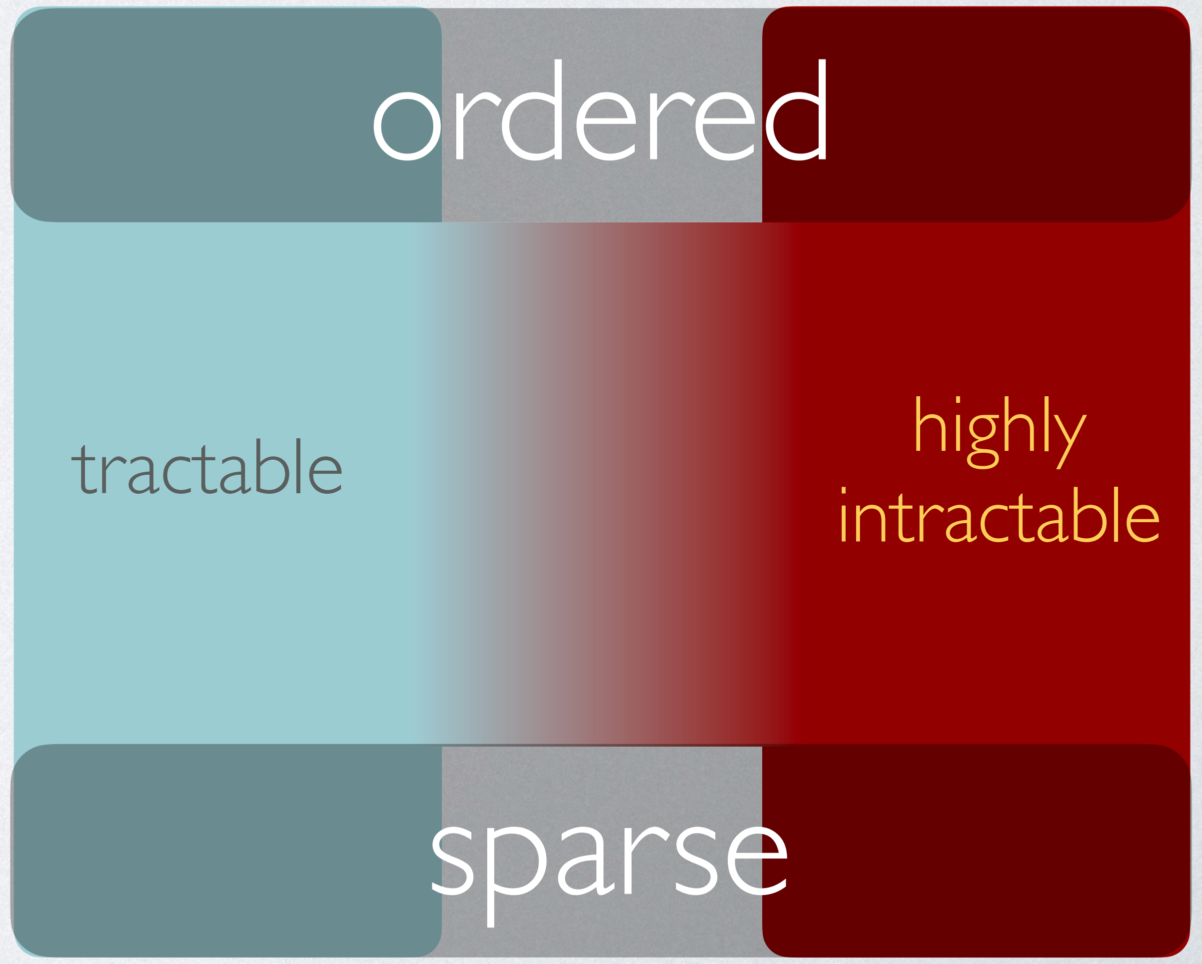

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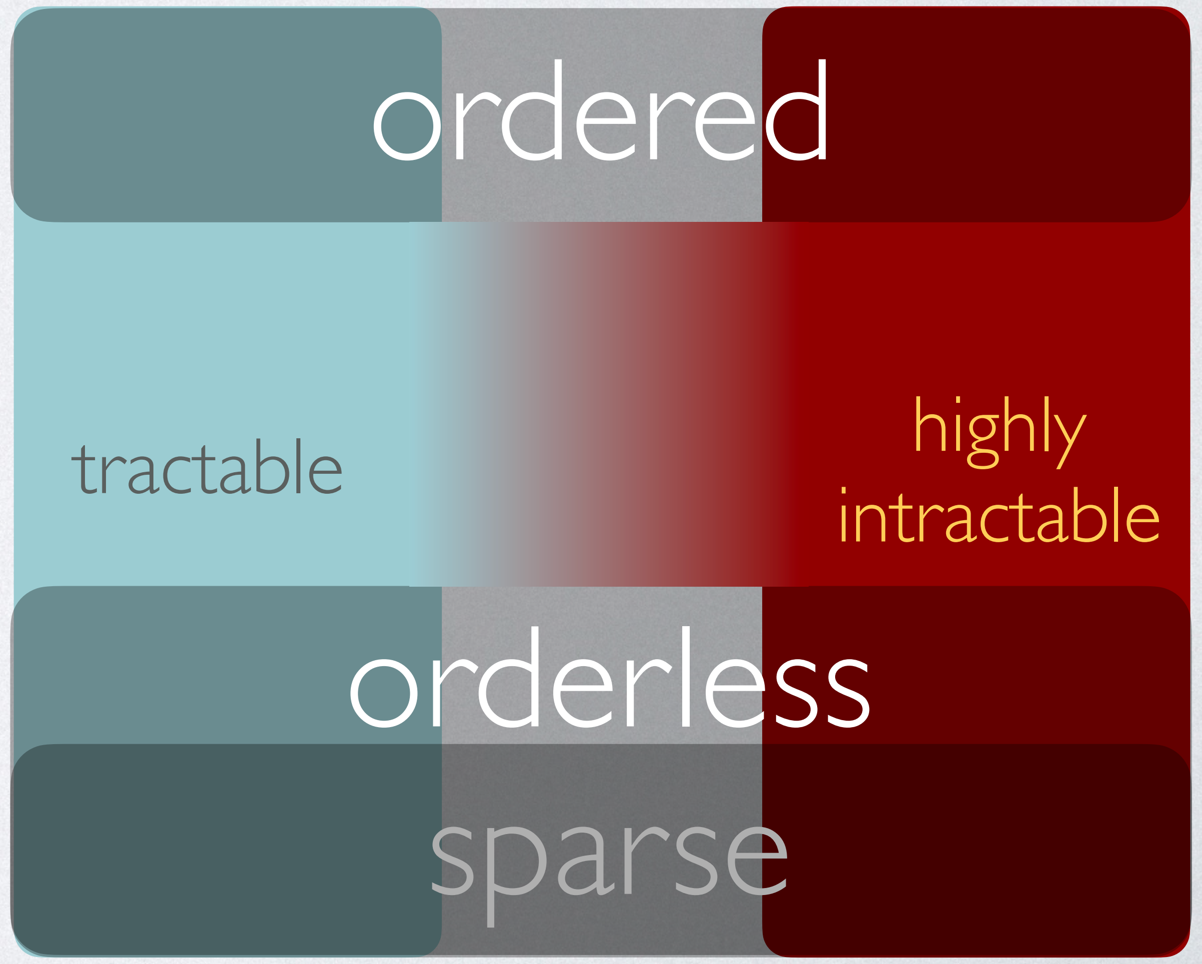

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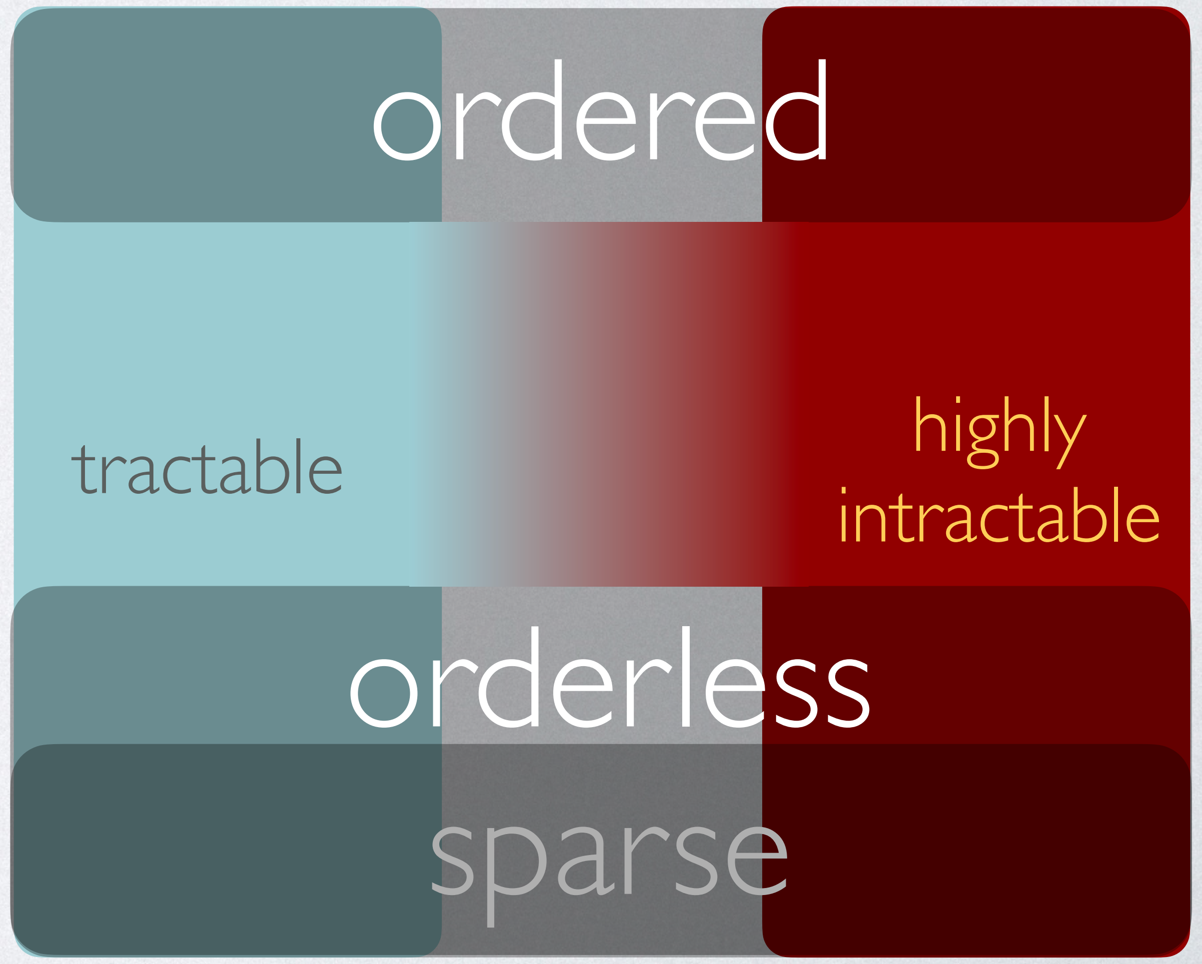
For **all** hereditary classes:

monadically dependent $\Leftrightarrow$ flip-breakable $^*\Leftrightarrow$? tractable



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