Deciding Emptiness of min-automata

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joint work with

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Plan

1. Introduction to the problem
2. Reduce emptiness of min-automata to the *finite section problem*, via a Ramsey-type theorem
3. Solve the finite section problem using Simon's factorization theorem

Min-automata

deterministic automata with counters
transitions invoke counter operations:

$c := c + 1$

$c := \min(d, e)$

acceptance condition is a boolean combination of:

$\liminf(c) = \infty$
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"c tends to ∞"

Example. $L = \{a^{n_1} b\, a^{n_2} b\, a^{n_3} b \ldots : n_1, n_2 \ldots$ does not converge to $\infty\}$
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	a	a	a	b	a	b	a	a	b...
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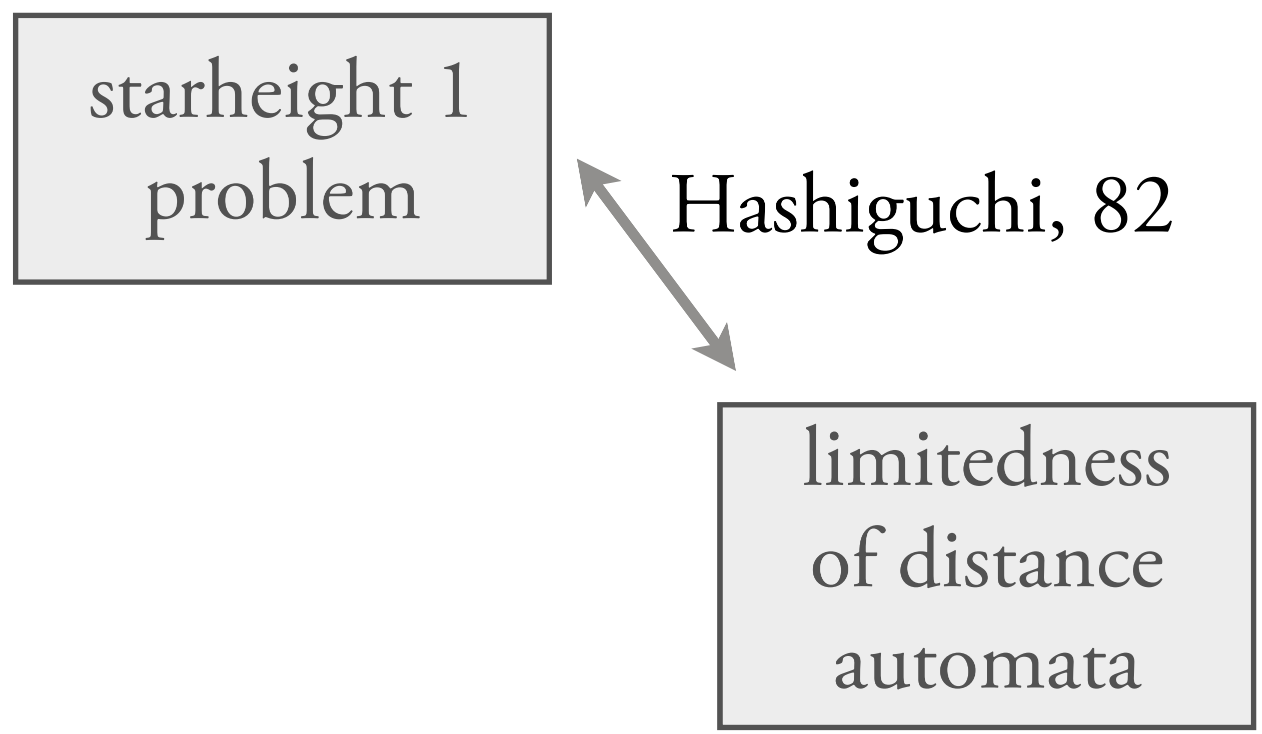

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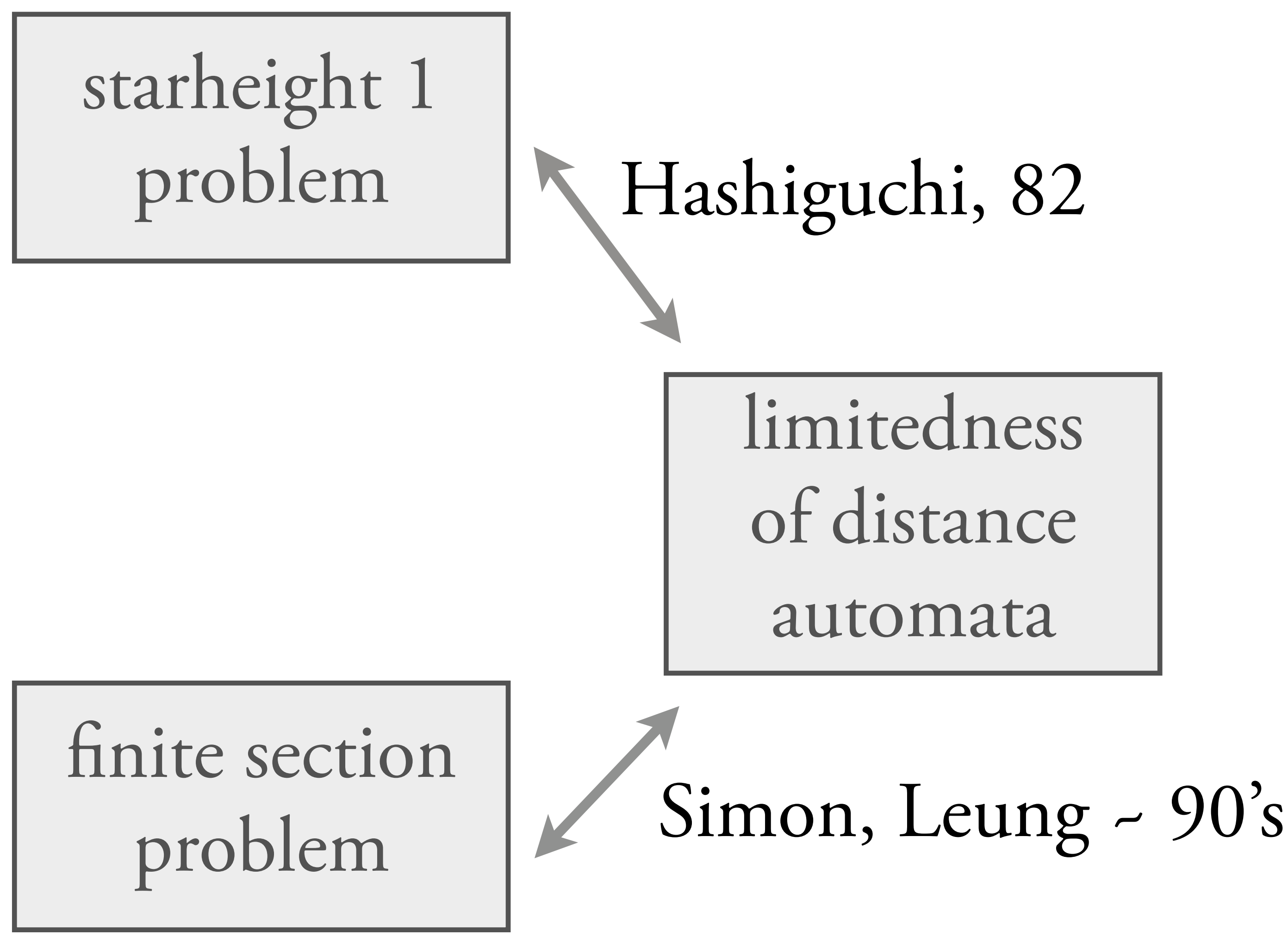

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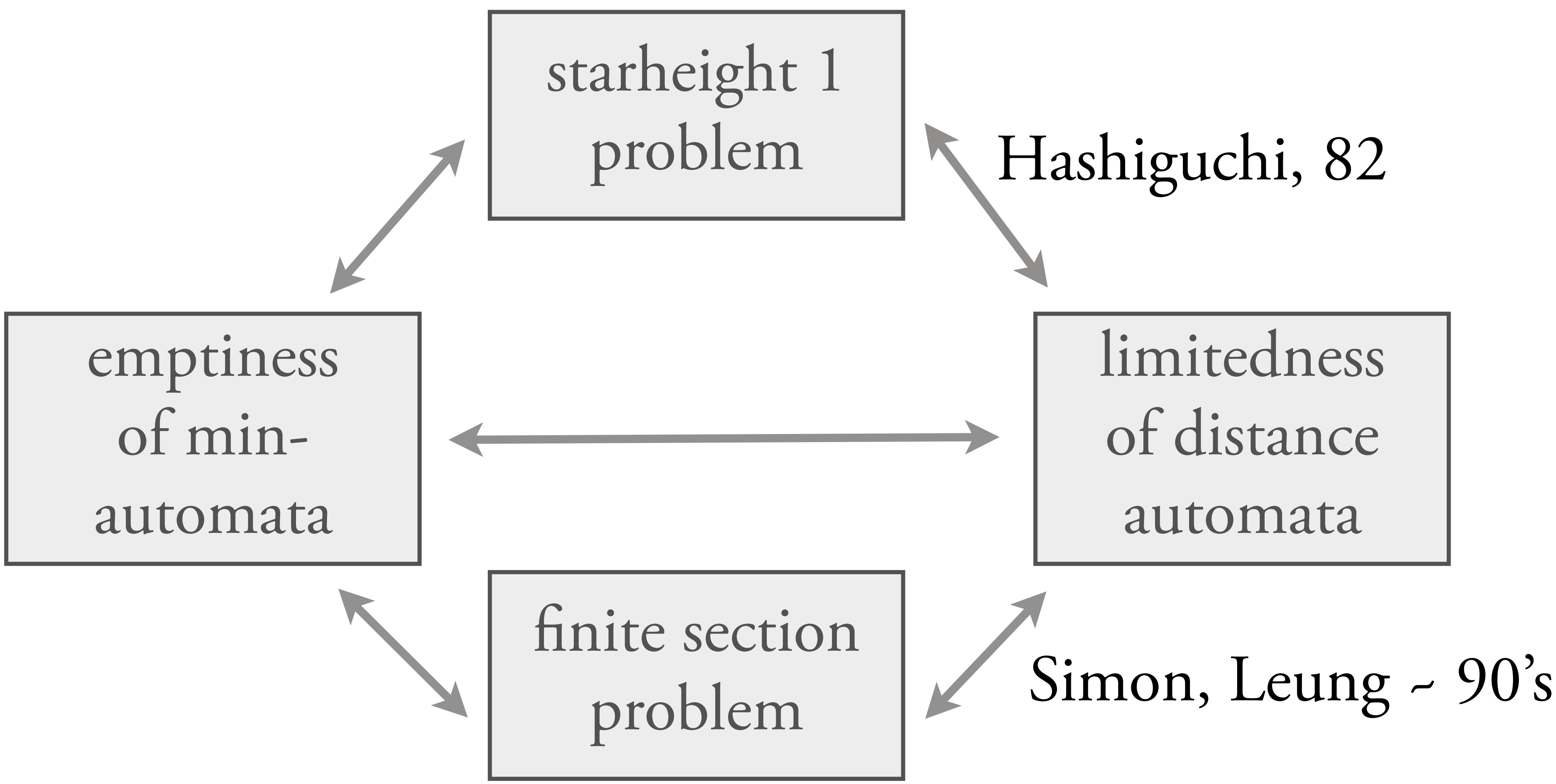

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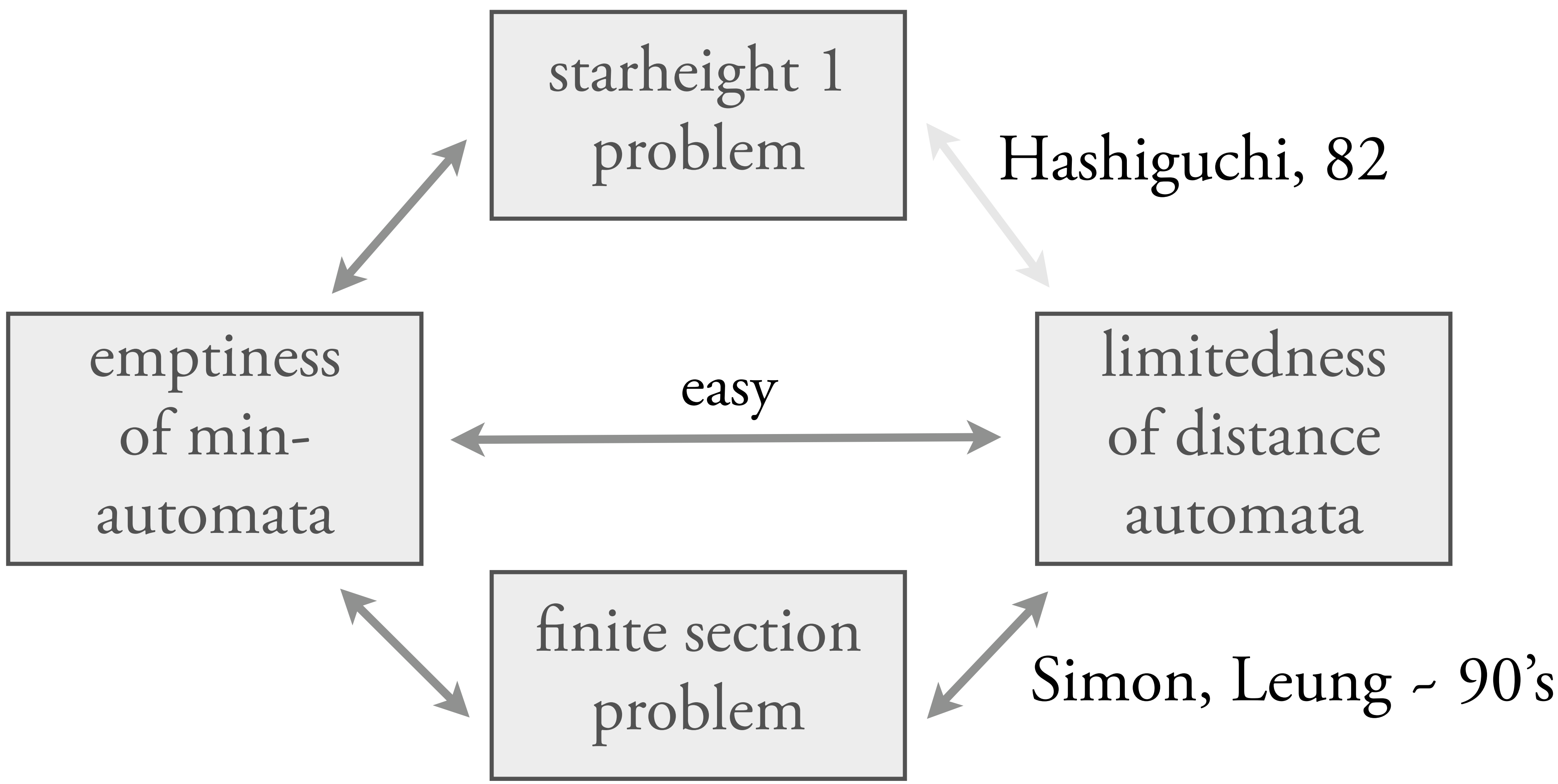

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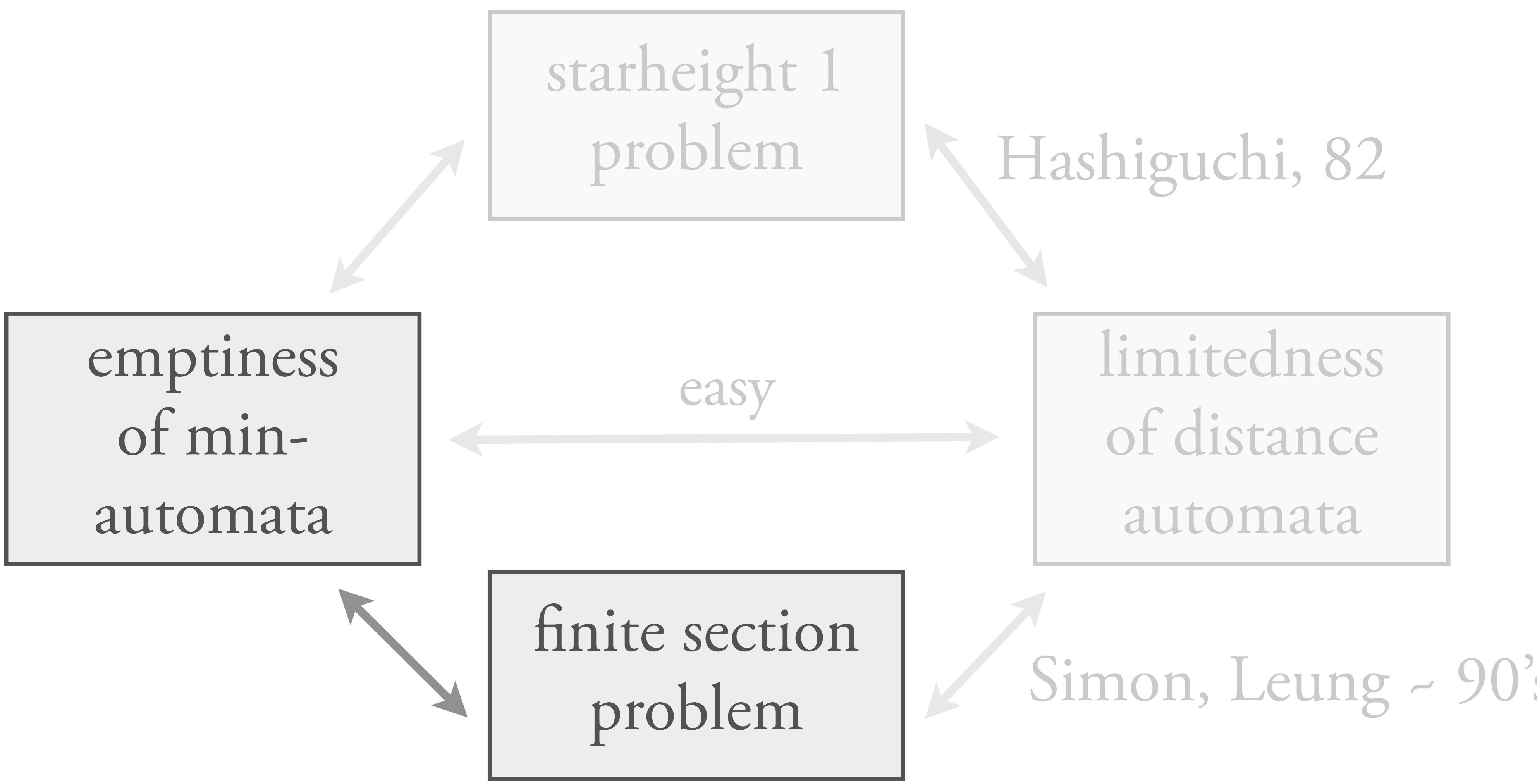
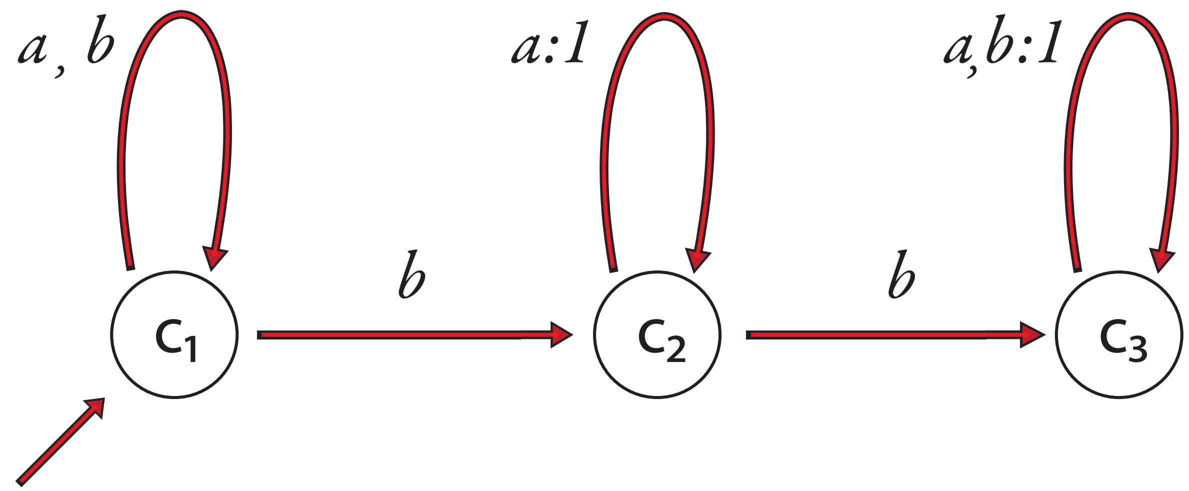
1st proof. min-automata are a special case of ωBS-automata (Bojańczyk, Colcombet [06]), so emptiness is decidable. This gives bad complexity, however.

2nd proof. Reduction to the *finite section problem* over the *tropical semiring*. Gives PSPACE algorithm, which is optimal.



An example



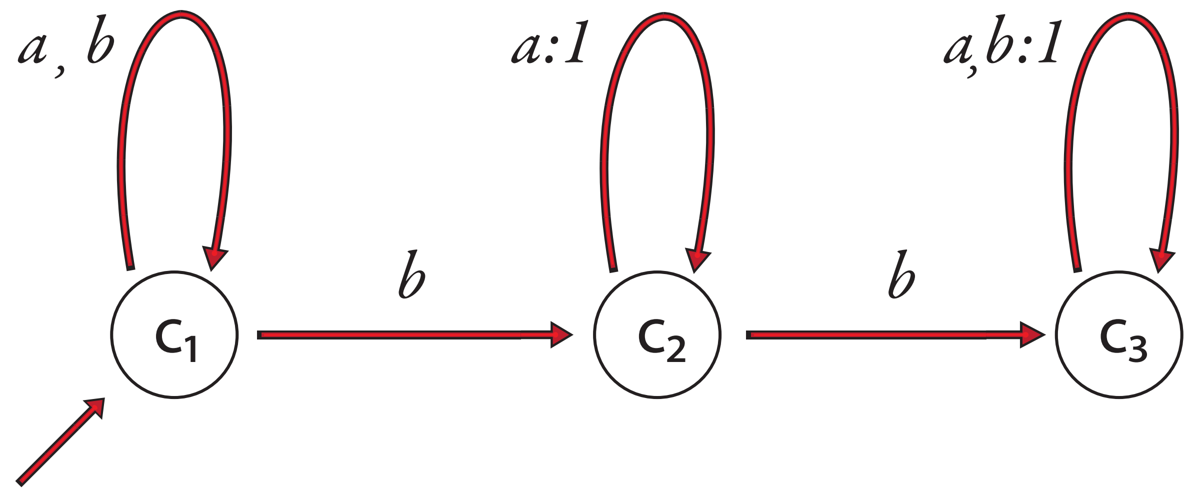
initial counter

$$a: \begin{pmatrix} 0 & \top & \top \\ \top & 1 & \top \\ \top & \top & 1 \end{pmatrix} \qquad b: \begin{pmatrix} 0 & 0 & \top \\ \top & \top & 0 \\ \top & \top & 1 \end{pmatrix}$$

Initial valuation: $\begin{pmatrix} 0 & \top & \top \end{pmatrix}$

Acceptance condition: $\neg \operatorname{liminf}(c_3) = \infty$

An example



initial counter

a:

0	$\top$	$\top$
$\top$	1	$\top$
$\top$	$\top$	1

b:

0	0	$\top$
$\top$	$\top$	0
$\top$	$\top$	1

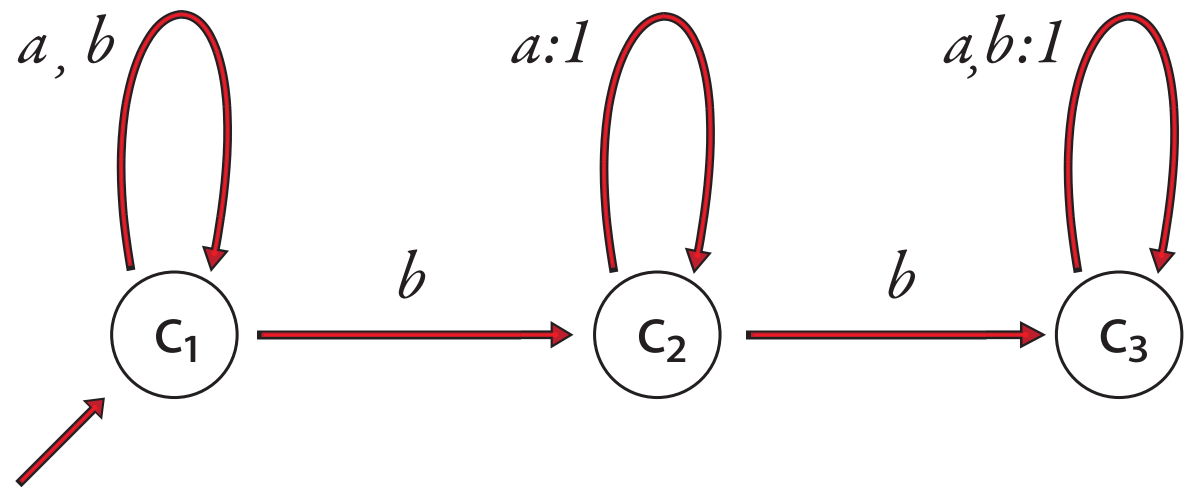
Initial valuation:

0	$\top$	$\top$

Acceptance condition: $\neg\ \liminf(c_3) = \infty$

What are the values of the counters after reading *baab*?

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg \liminf(c_3) = \infty$

What are the values of the counters after reading *baab?*

0	⊤	⊤

0	0	⊤
⊤	⊤	0
⊤	⊤	1

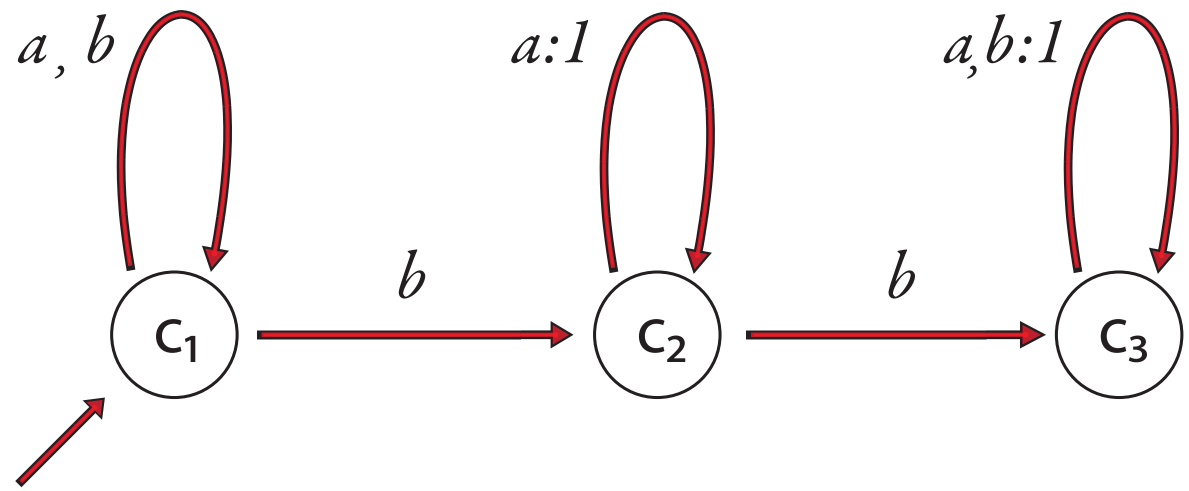
0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

0	0	⊤
⊤	⊤	0
⊤	⊤	1

=

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg \liminf(c_3) = \infty$

What are the values of the counters after reading *baab?*

0	⊤	⊤

0	0	⊤
⊤	⊤	0
⊤	⊤	1

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

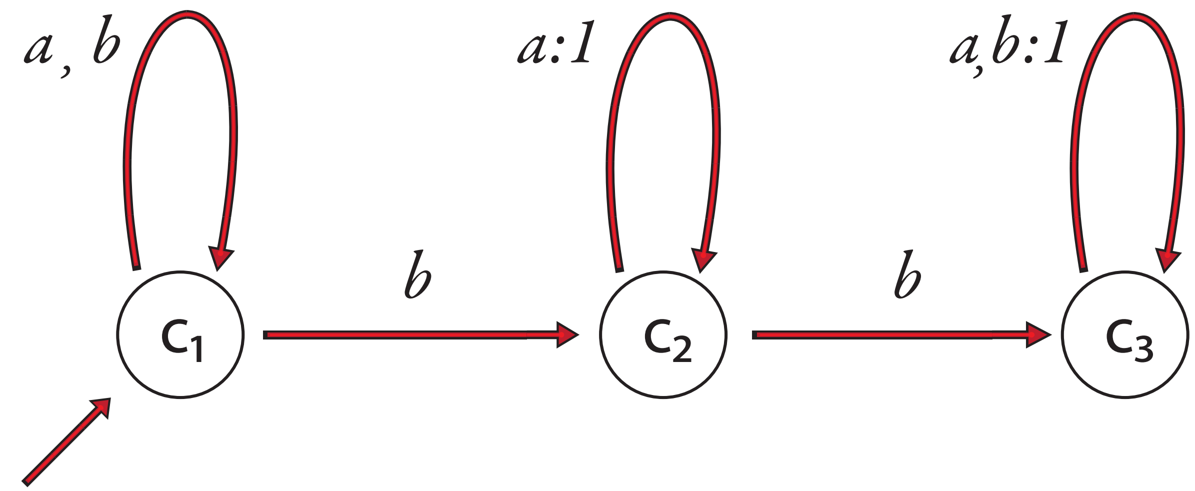
0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

0	0	⊤
⊤	⊤	0
⊤	⊤	1

=

0	0	2

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

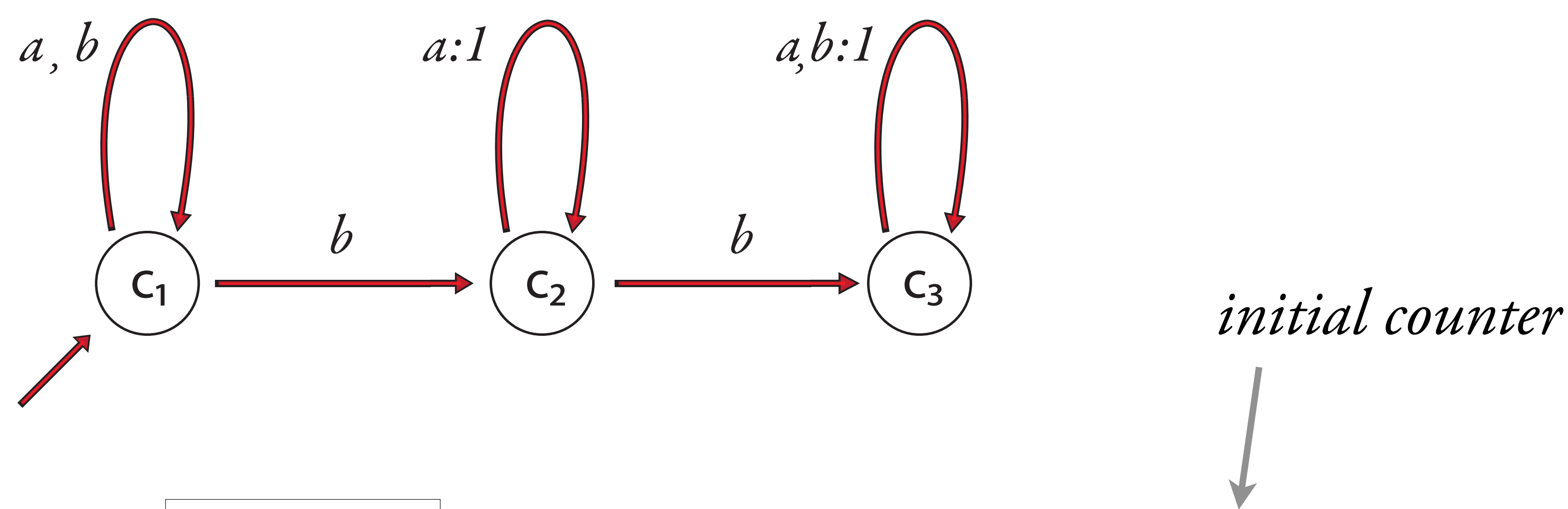
b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation: | 0 | ⊤ | ⊤ |

Acceptance condition: $\neg\, \liminf(c_3) = \infty$

An example



a:	0	⊤	⊤
	⊤	1	⊤
	⊤	⊤	1

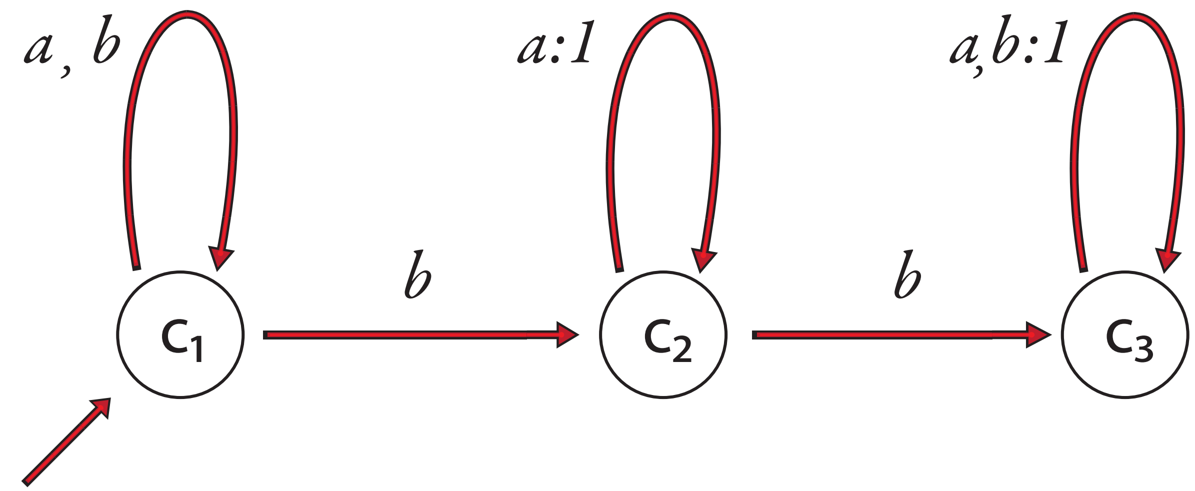
b:	0	0	⊤
	⊤	⊤	0
	⊤	⊤	1

Initial valuation: | 0 | ⊤ | ⊤ |

Acceptance condition: $\neg \liminf(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

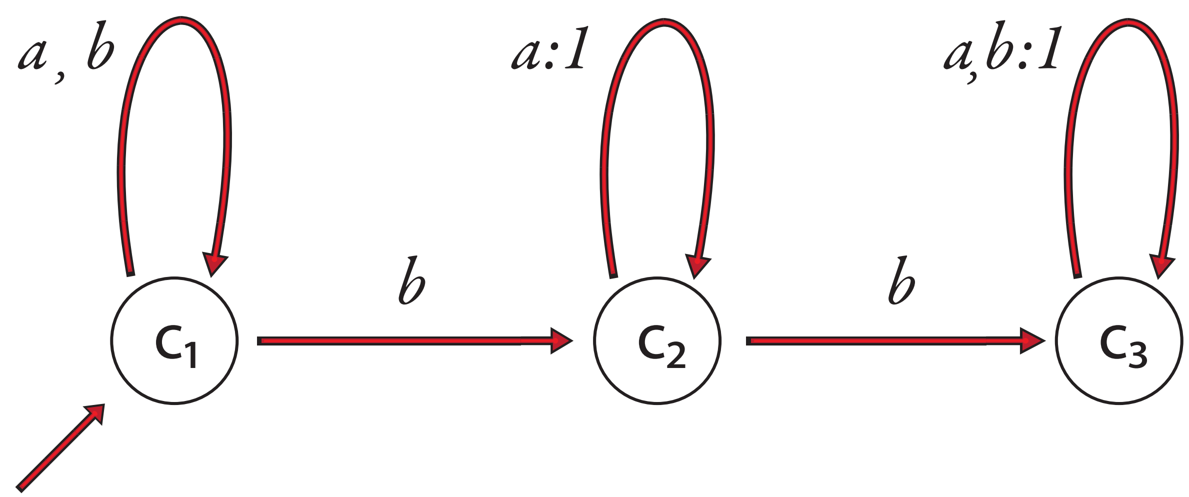
0	⊤	⊤

Acceptance condition: $\neg \operatorname{liminf}(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg \operatorname{liminf}(c_3) = \infty$ holds

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

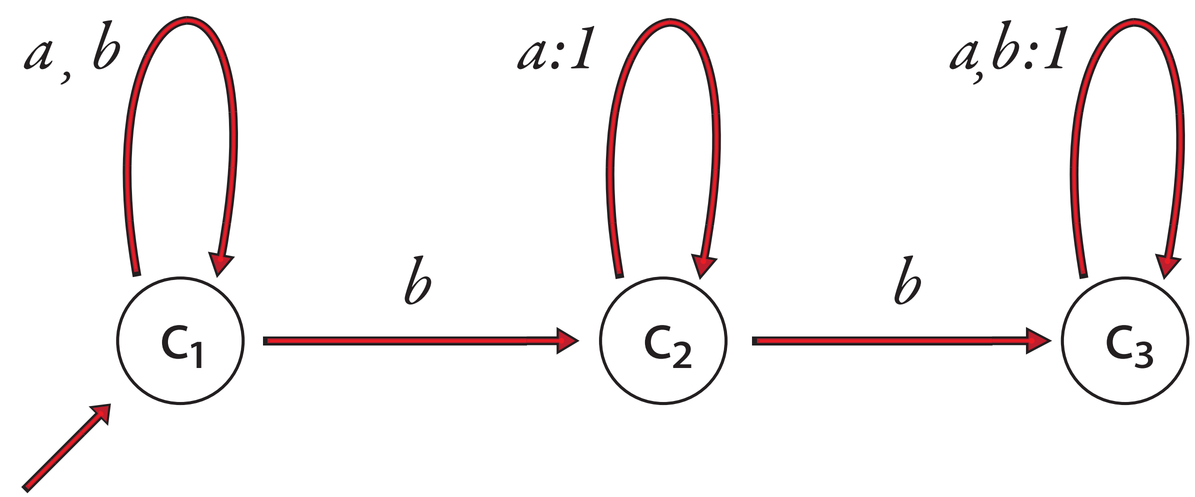
Acceptance condition: $\neg\ \mathrm{liminf}(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg\ \mathrm{liminf}(c_3) = \infty$ holds

iff

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg\ \text{liminf}(c_3) = \infty$

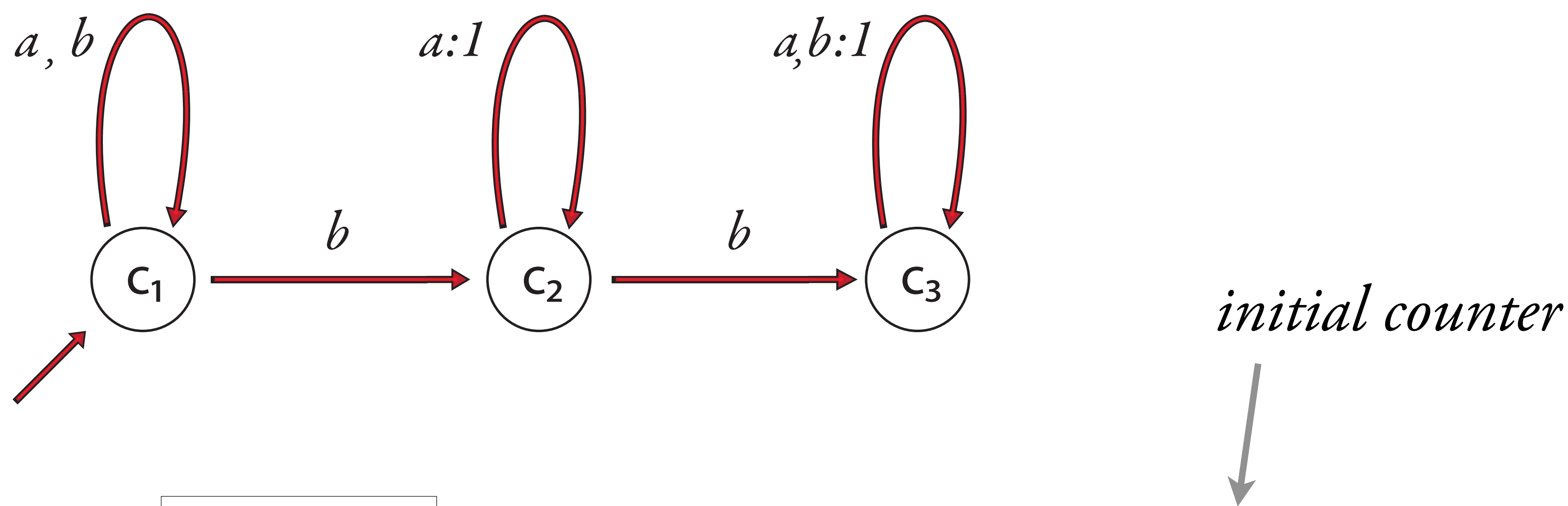
Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg\ \text{liminf}(c_3) = \infty$ holds

iff

val(c_3) *has a bounded subsequence*

An example



a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg\ \mathrm{liminf}(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

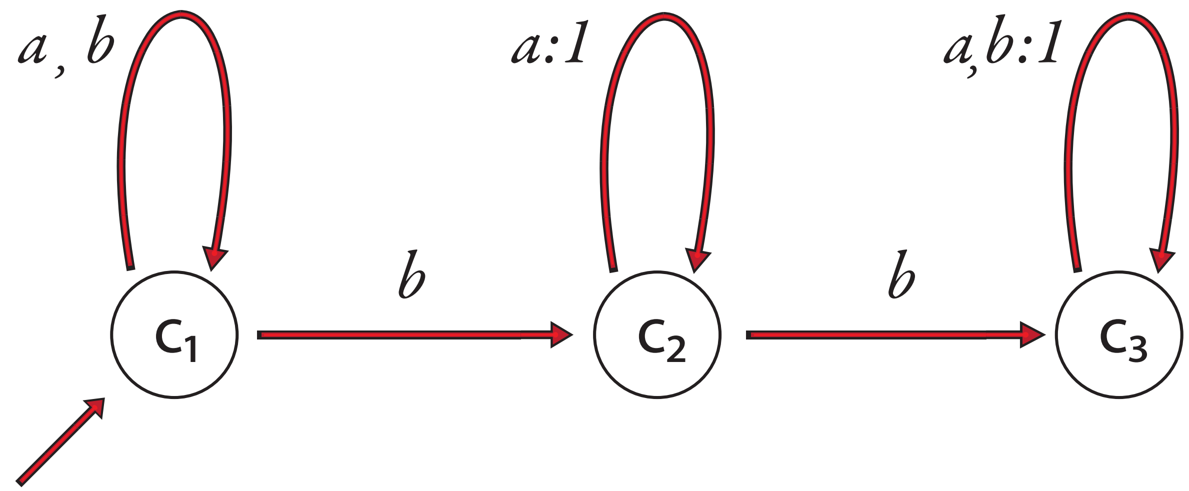
$\neg\ \mathrm{liminf}(c_3) = \infty$ holds

iff

val(c_3) *has a bounded subsequence*

iff

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg \operatorname{liminf}(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg \operatorname{liminf}(c_3) = \infty$ holds

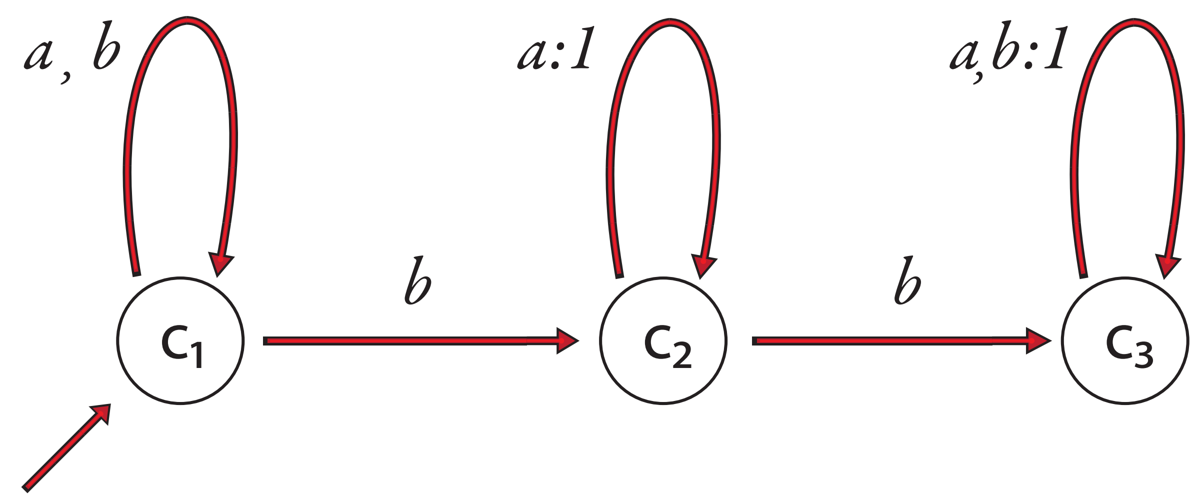
iff

val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of *w*,

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg\ \mathrm{liminf}(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg\ \mathrm{liminf}(c_3) = \infty$ holds

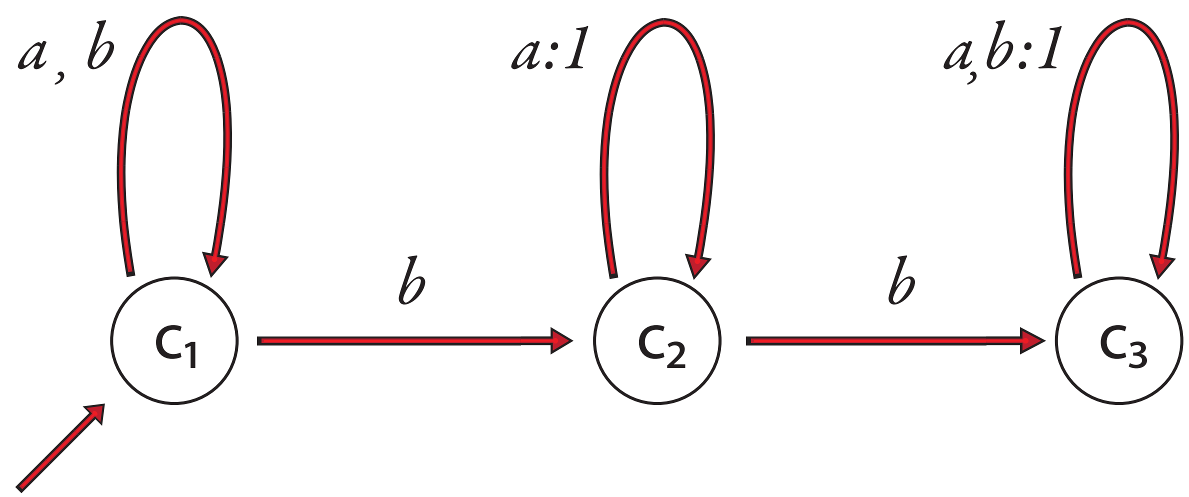
iff

val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg\ \mathrm{liminf}(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg\ \mathrm{liminf}(c_3) = \infty$ holds

iff

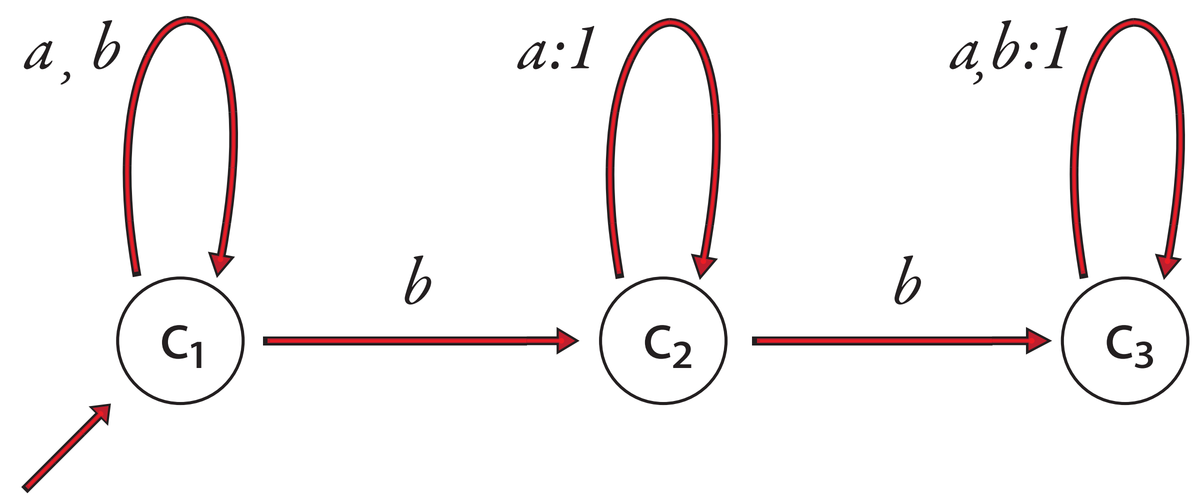
val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of *w*, starting in c_1, ending in c_3 with a *bounded* number of 1's

$a\, ba^2 ba^3 ba^4 ba^5 \ldots$

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg\, \liminf(c_3) = \infty$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg\, \liminf(c_3) = \infty$ holds

iff

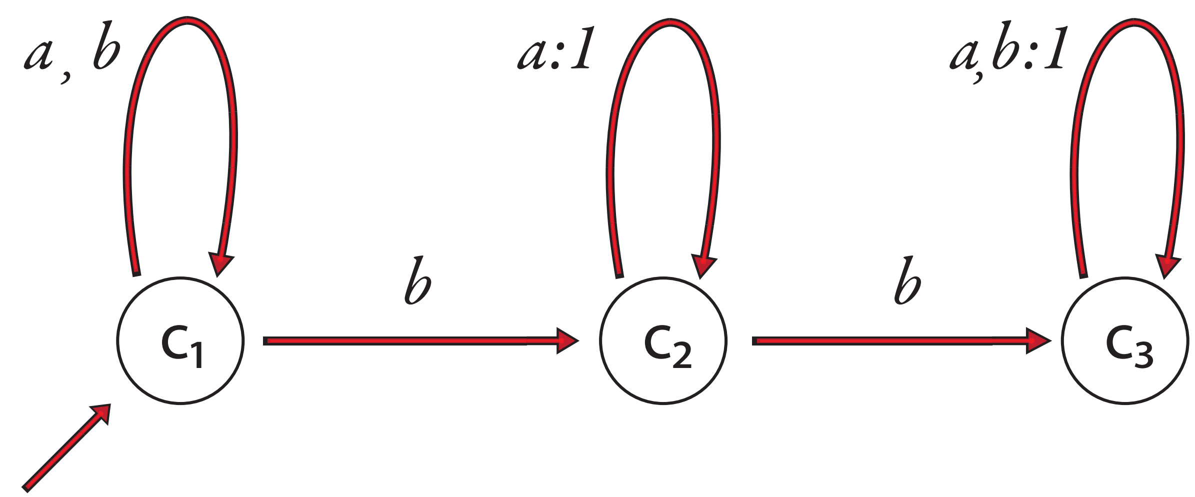
val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of *w*, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\, ba^2ba^3ba^4ba^5\ldots$

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg \operatorname{liminf}(c_3) = \infty$

Input: $w = a_1a_2a_3a_4a_5a_6\ldots.$

$\neg \operatorname{liminf}(c_3) = \infty$ holds

iff

val(c_3) *has a bounded subsequence*

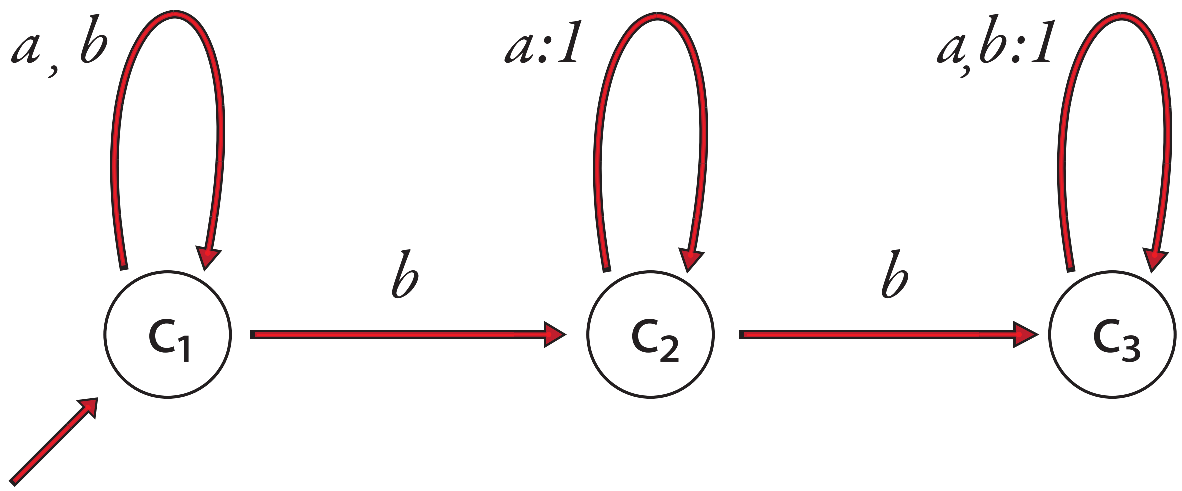
iff

there exist arbitrarily long paths labeled by a prefix of *w*, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\,ba^2ba^3ba^4ba^5\ldots.$

$ababa^2baba^3baba^4ba\ldots.$

An example



initial counter

a:

0	⊤	⊤
⊤	1	⊤
⊤	⊤	1

b:

0	0	⊤
⊤	⊤	0
⊤	⊤	1

Initial valuation:

0	⊤	⊤

Acceptance condition: $\neg\, \liminf(c_3) = \infty$

Input: $w = a_1a_2a_3a_4a_5a_6\ldots.$

✗ $a\, ba^2ba^3ba^4ba^5\ldots.$

$\neg\, \liminf(c_3) = \infty$ holds

iff

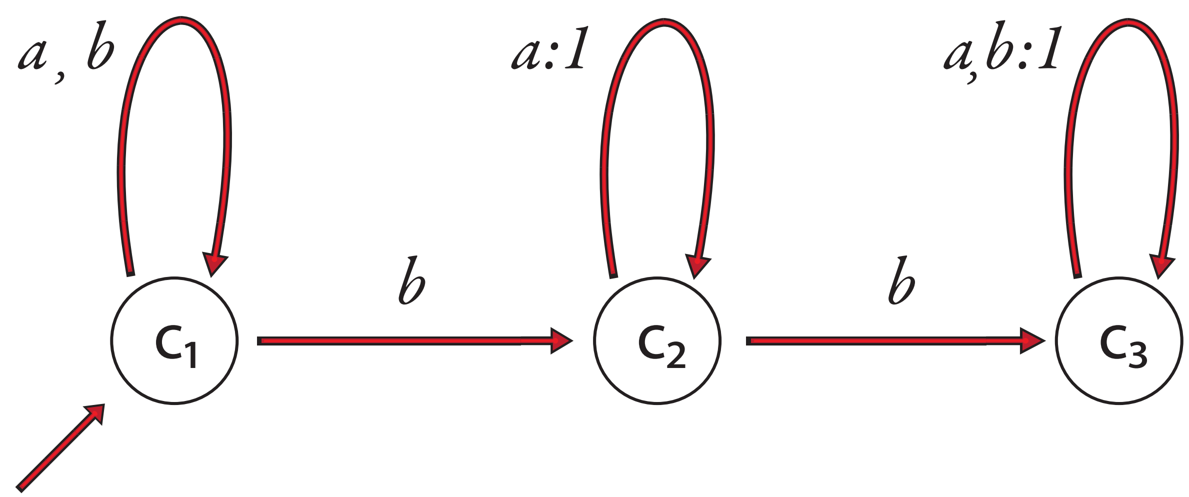
$val(c_3)$ *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of *w*, starting in c_1, ending in c_3 with a *bounded* number of 1's

$ababa^2baba^3baba^4ba\ldots.$ ✓

An example



Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg$ liminf(c_3) = ∞ holds

iff

val(c_3) *has a bounded subsequence*

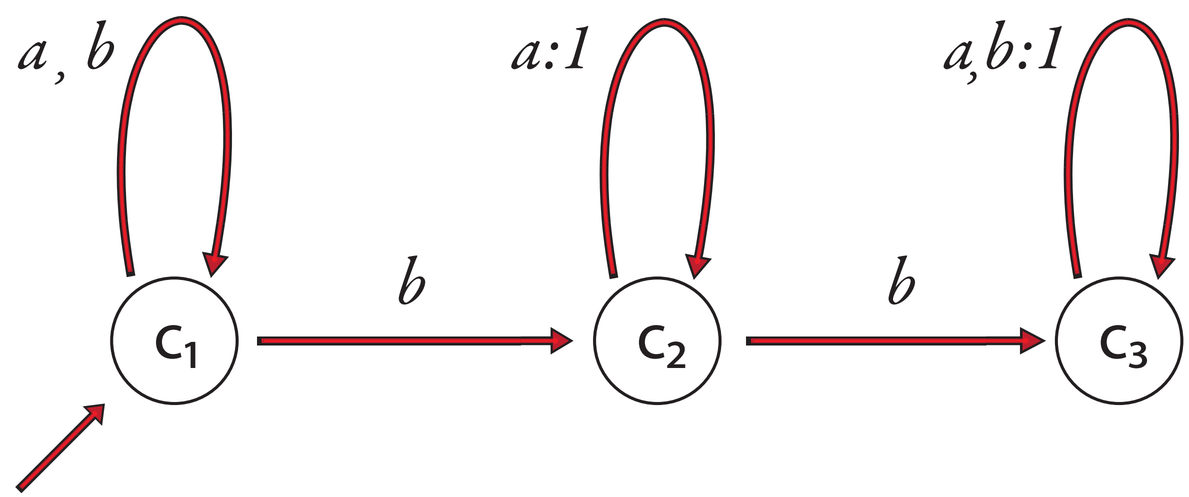
iff

there exist arbitrarily long paths labeled by a prefix of *w*, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\, ba^2ba^3ba^4ba^5\ldots$

$ababa^2baba^3baba^4ba\ldots$ ✓

An example



$a^n b a^{n+1} b$:

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg \liminf(c_3) = \infty$ holds

iff

$val(c_3)$ *has a bounded subsequence*

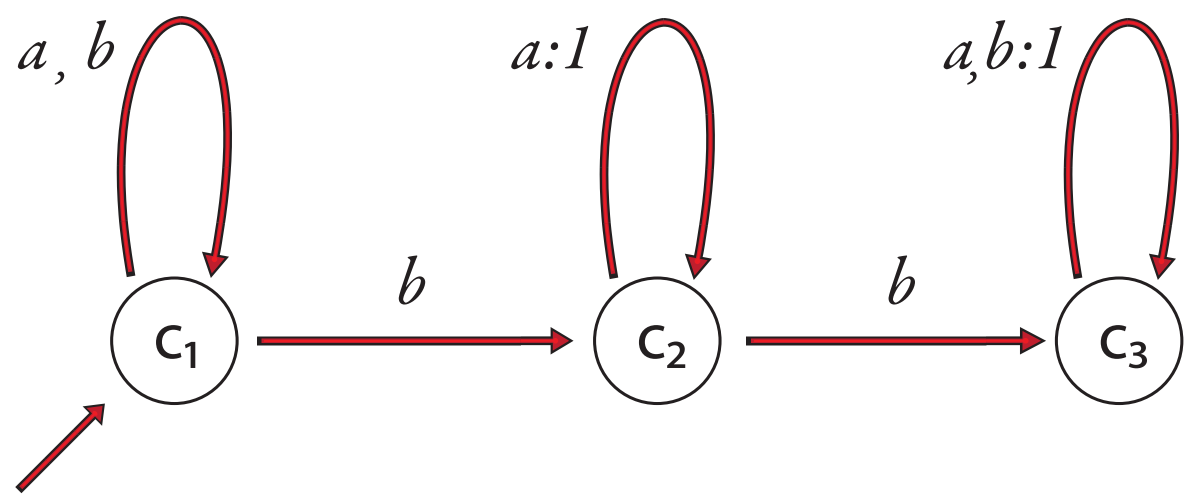
iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\, b a^2 b a^3 b a^4 b a^5 \ldots$

$ababa^2baba^3baba^4ba\ldots$ ✓

An example



$a^n b a^{n+1} b$:

0	0	$n+1$
$\top$	$\top$	$2n+2$
$\top$	$\top$	$2n+3$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

✗ $a\, ba^2ba^3ba^4ba^5\ldots$

$\neg$ liminf(c_3) = ∞ holds

$ababa^2baba^3baba^4ba\ldots$ ✓

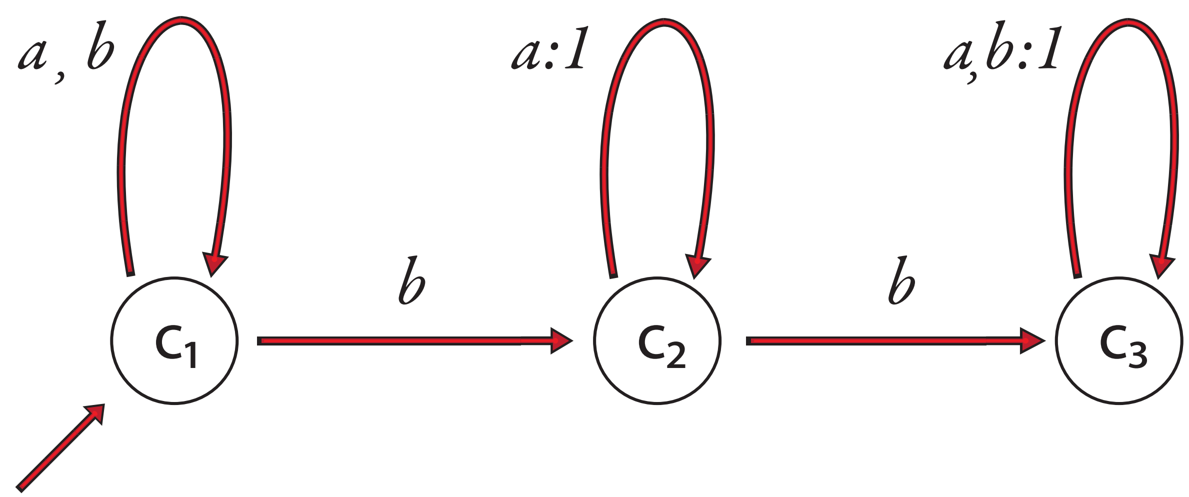
iff

val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

An example



$a^n b a^{n+1} b$:

0	0	$n+1$
⊤	⊤	$2n+2$
⊤	⊤	$2n+3$

$a^n ba\, b$:

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

$\neg$ liminf$(c_3) = \infty$ holds

iff

val(c_3) *has a bounded subsequence*

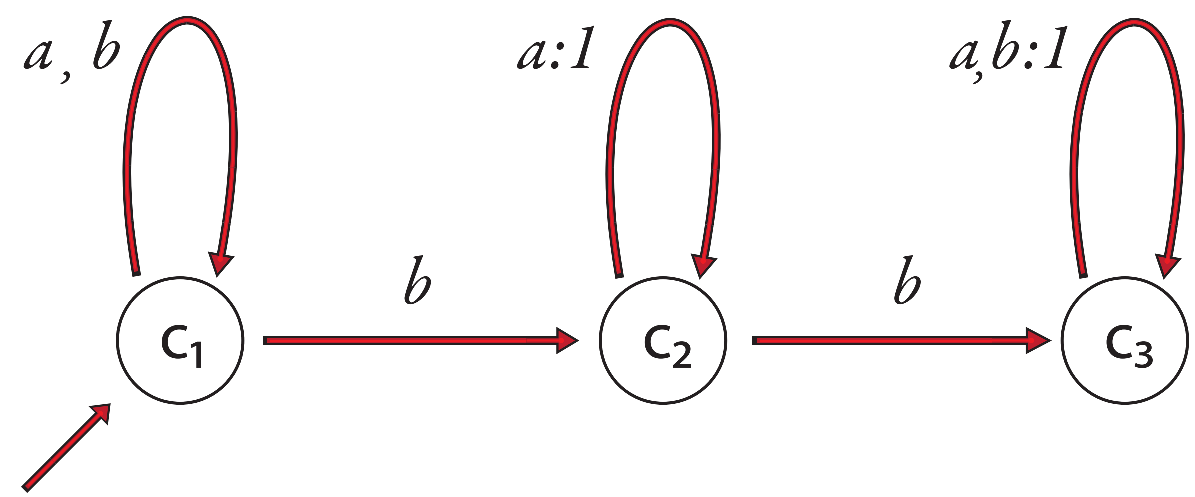
iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\, ba^2ba^3ba^4ba^5 \ldots$

$ababa^2baba^3baba^4ba \ldots$ ✓

An example



$a^n b a^{n+1} b$:

0	0	$n+1$
⊤	⊤	$2n+2$
⊤	⊤	$2n+3$

$a^n ba\, b$:

0	0	1
⊤	⊤	$n+2$
⊤	⊤	$n+3$

Input: $w = a_1a_2a_3a_4a_5a_6\ldots$

$\neg$ liminf(c_3) = ∞ holds

iff

val(c_3) *has a bounded subsequence*

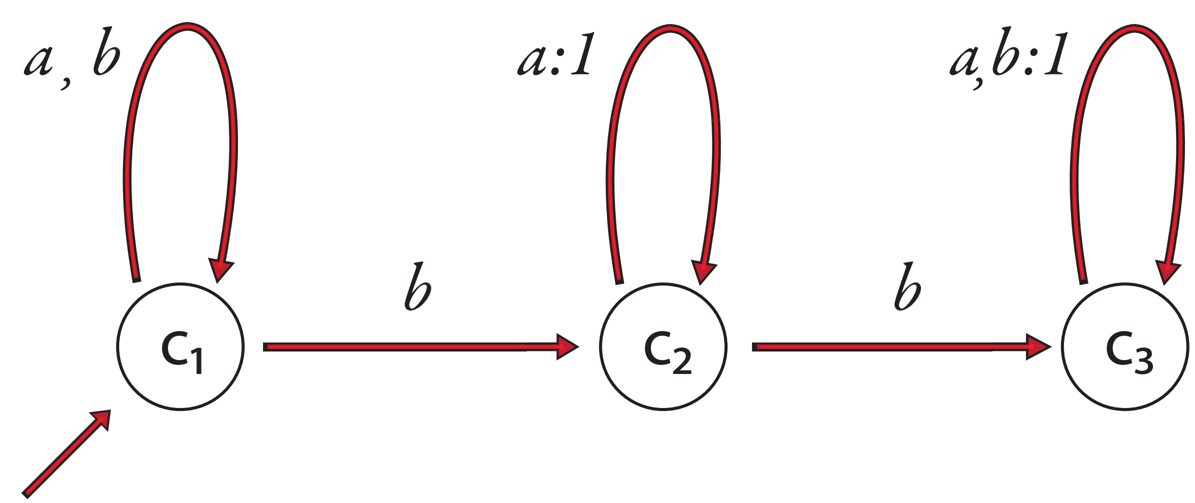
iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\, ba^2ba^3ba^4ba^5\ldots$

$ababa^2baba^3baba^4ba\ldots$ ✓

An example



$a^n b a^{n+1} b$:

0	0	$n+1$
⊤	⊤	$2n+2$
⊤	⊤	$2n+3$

→

0	0	∞
⊤	⊤	∞
⊤	⊤	∞

$a^n ba\, b$:

0	0	1
⊤	⊤	$n+2$
⊤	⊤	$n+3$

Input: $w = a_1a_2a_3a_4a_5a_6\ldots$

$\neg\, \text{liminf}(c_3) = \infty$ holds

iff

$val(c_3)$ has a bounded subsequence

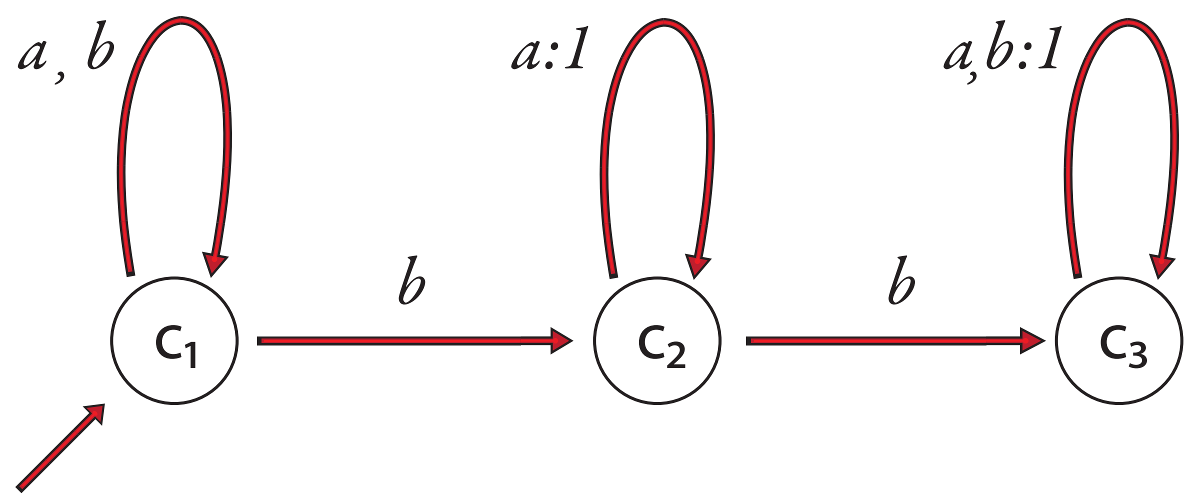
iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

✗ $a\, ba^2ba^3ba^4ba^5\ldots$

$ababa^2baba^3baba^4ba\ldots$ ✓

An example



$a^n b a^{n+1} b$: $\begin{pmatrix} 0 & 0 & n+1 \\ \top & \top & 2n+2 \\ \top & \top & 2n+3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & \infty \\ \top & \top & \infty \\ \top & \top & \infty \end{pmatrix}$

$a^n ba\, b$: $\begin{pmatrix} 0 & 0 & 1 \\ \top & \top & n+2 \\ \top & \top & n+3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ \top & \top & \infty \\ \top & \top & \infty \end{pmatrix}$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

✗ $a\, ba^2ba^3ba^4ba^5 \ldots$

$\neg \operatorname{liminf}(c_3) = \infty$ holds

iff

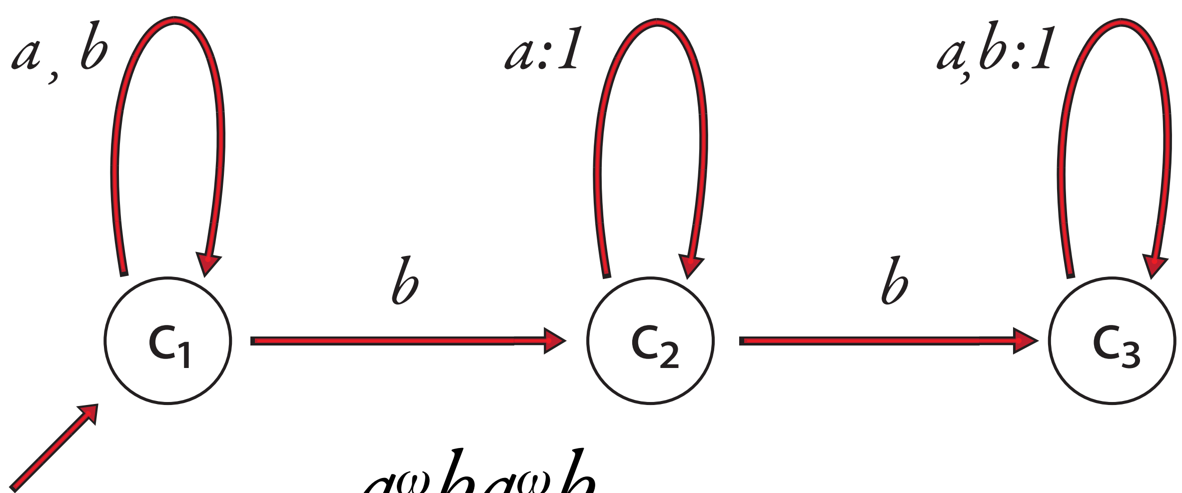
$val(c_3)$ *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

$ababa^2baba^3baba^4ba \ldots$ ✓

An example



$a^\omega b a^\omega b$

$a^n b a^{n+1} b$:

0	0	$n+1$
⊤	⊤	$2n+2$
⊤	⊤	$2n+3$

→

0	0	∞
⊤	⊤	∞
⊤	⊤	∞

$a^n ba\, b$:

0	0	1
⊤	⊤	$n+2$
⊤	⊤	$n+3$

→

0	0	1
⊤	⊤	∞
⊤	⊤	∞

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \dots$

✗ $a\, ba^2ba^3ba^4ba^5\dots$

$\neg$ liminf(c_3) = ∞ holds

$ababa^2baba^3baba^4ba\dots$ ✓

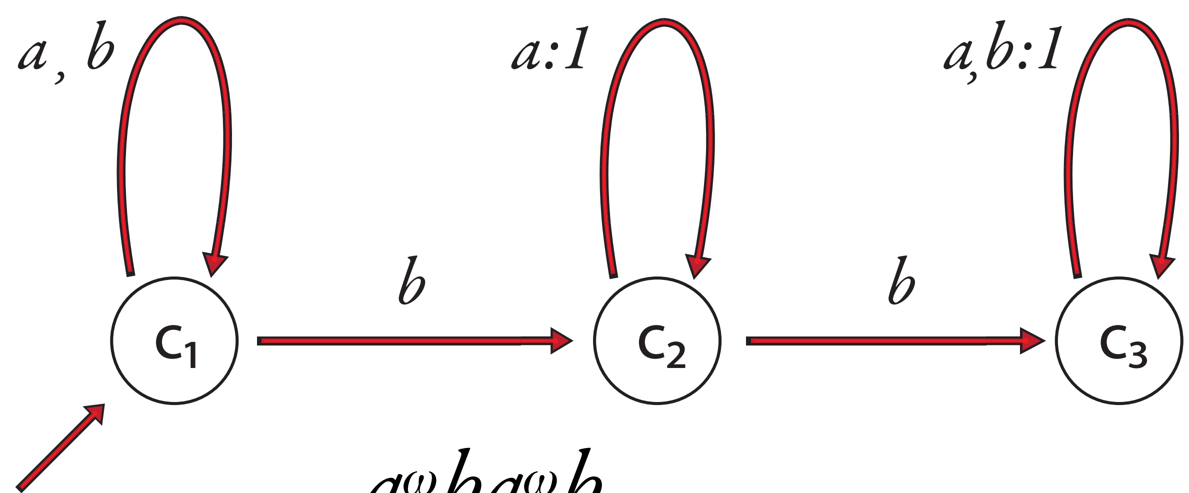
iff

val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of *w*, starting in c_1, ending in c_3 with a *bounded* number of 1's

An example



$a^n b a^{n+1} b$: $\begin{pmatrix} 0 & 0 & n+1 \\ \top & \top & 2n+2 \\ \top & \top & 2n+3 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 0 & 0 & \infty \\ \top & \top & \infty \\ \top & \top & \infty \end{pmatrix}}_{a^\omega b a^\omega b}$

$a^n ba\, b$: $\begin{pmatrix} 0 & 0 & 1 \\ \top & \top & n+2 \\ \top & \top & n+3 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ \top & \top & \infty \\ \top & \top & \infty \end{pmatrix}}_{a^\omega bab}$

Input: $w = a_1 a_2 a_3 a_4 a_5 a_6 \ldots$

✗ $a\, ba^2ba^3ba^4ba^5\ldots$

$\neg$ liminf(c_3) = ∞ holds

$ababa^2baba^3baba^4ba\ldots$ ✓

iff

val(c_3) *has a bounded subsequence*

iff

there exist arbitrarily long paths labeled by a prefix of w, starting in c_1, ending in c_3 with a *bounded* number of 1's

Plan

1. Introduction to the problem
2. Reduce emptiness of min-automata to the *finite section problem*, via a Ramsey-type theorem
3. Solve the finite section problem using Simon's factorization theorem

Plan

✓ 1. Introduction to the problem

2. Reduce emptiness of min-automata to the *finite section problem*, via a Ramsey-type theorem

3. Solve the finite section problem using Simon's factorization theorem

The tropical semiring

The tropical semiring

$$T = \{0, 1, 2, \ldots, \infty, \top\}$$

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$$T = \{0, 1, 2, \ldots, \infty, \top\}$$

with operations +, min

ordered by $0 < 1 < 2 < \ldots < \infty < \top$

where $\top + x = x + \top = \top$

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$M_k T$ – *k by k* matrices over T

with matrix multiplication

3	32	$\top$
$\top$	11	1
2	7	∞

The tropical semiring

$T = \{0,1,2,\ldots, \infty, \top\}$
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Topology

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with matrix multiplication

3	32	$\top$
$\top$	11	1
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Topology

1 2 3 4 5 6 7 8 9 ∞ $\top$

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Topology

1 2 3 4 5 6 7 8 9 ∞ $\top$

$$d(m,n)=|1/m-1/n|$$

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$d(m,n) = |1/m - 1/n|$

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Topology

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$d(m,n)=|1/m-1/n|$

product topology on $T^{k \times k}$

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$M_k T$ – *k by k* matrices over T

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2 3 4 5 6 7 8 9 ∞ ⊤

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$d(M,N)=\max_{i,j}|M[i,j]-N[i,j]|$

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$\mathrm{M}_k T$ – *k by k* matrices over *T*

with matrix multiplication

3	32	⊤
⊤	11	1
2	7	∞

Topology

2 3 4 5 6 7 8 9 ∞ ⊤

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product topology on $T^{k\mathrm{x}k}$

$d(M,N)=\mathrm{max}_{i,j}|M[i,j]-N[i,j]|$

3	1	0
5	1	1
2	7	*10*

3	38	⊤
4	11	1
2	7	*20*

3	32	⊤
⊤	11	1
2	7	*35*

3	32	⊤
⊤	11	1
2	7	*45*

3	32	⊤
⊤	11	1
2	7	*59*

… →

3	32	⊤
⊤	11	1
2	7	∞

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profinite semigroup

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profinite semigroup

- compact space

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product topology on $T^{k\mathrm{x}k}$

$d(M,N)=\max_{i,j}|M[i,j]-N[i,j]|$

profinite semigroup

- compact space
- matrix multiplication is continuous

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2	7	∞

Topology

2 3 4 5 6 7 8 9 ∞ $\top$

$d(m,n)=|1/m-1/n|$

product topology on $T^{k\times k}$

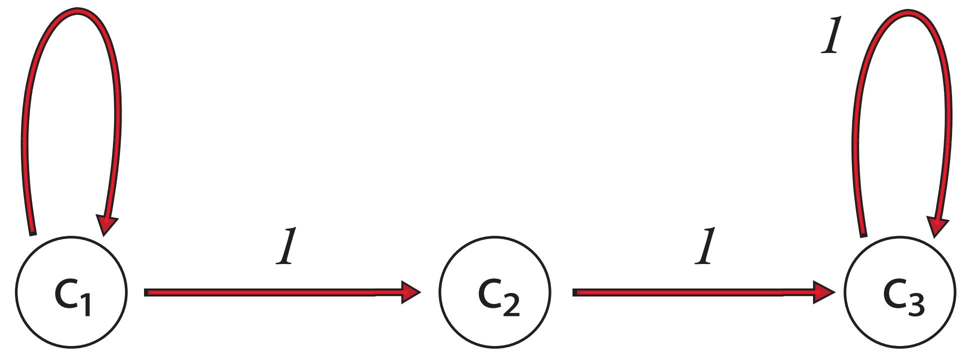
$d(M,N)=\max_{i,j}|M[i,j]-N[i,j]|$

profinite semigroup

- compact space
- matrix multiplication is continuous
- naturally equipped with the ω-power

ω-power

ω-power



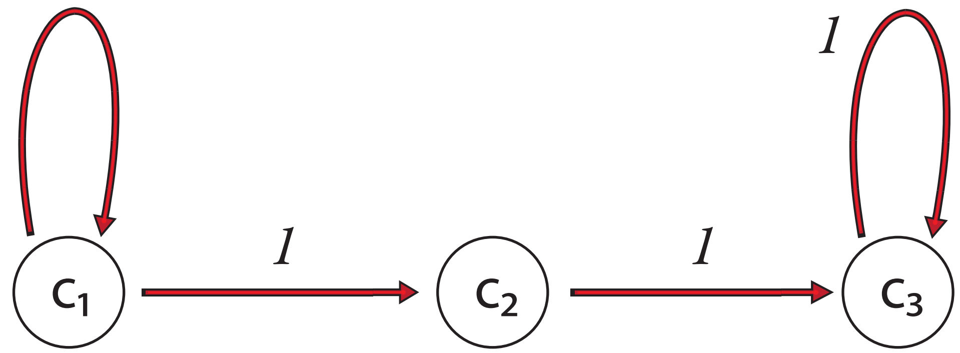
x:

0	1	$\top$
$\top$	$\top$	1
$\top$	$\top$	1

x^{ω}:

0	1	2
$\top$	$\top$	∞
$\top$	$\top$	∞

ω-power



x:

0	1	$\top$
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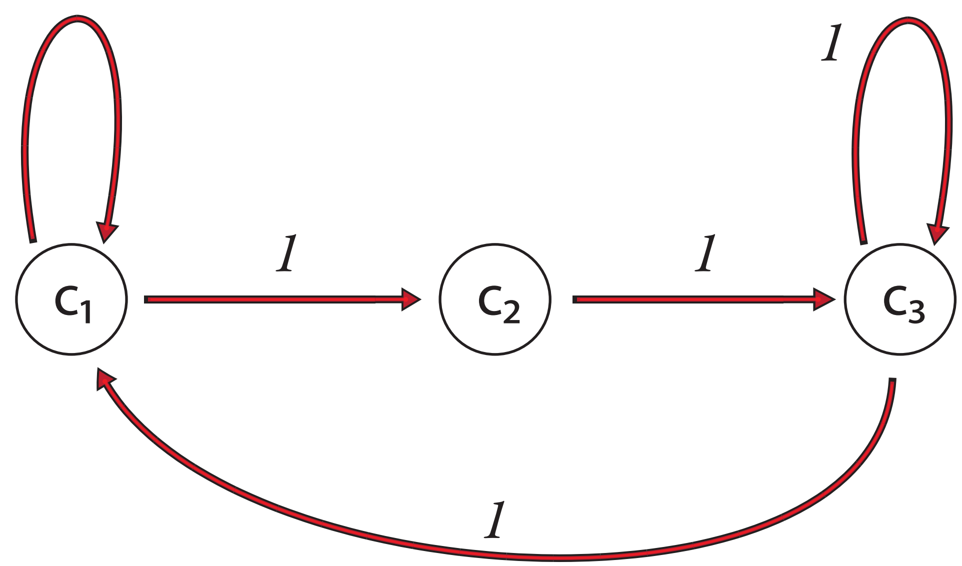


y:

0	1	$\top$
$\top$	$\top$	1
1	$\top$	1

y^{ω}:

0	1	2
2	3	4
1	2	3

ω-power

x:

0	1	⊤
⊤	⊤	1
⊤	⊤	1

x^ω:

0	1	2
⊤	⊤	∞
⊤	⊤	∞

y:

0	1	⊤
⊤	⊤	1
1	⊤	1

y^ω:

0	1	2
2	3	4
1	2	3

ω-power continuous

x:

0	1	⊤
⊤	⊤	1
⊤	⊤	1

x^{ω}:

0	1	2
⊤	⊤	∞
⊤	⊤	∞

y:

0	1	⊤
⊤	⊤	1
1	⊤	1

y^{ω}:

0	1	2
2	3	4
1	2	3

ω-power continuous

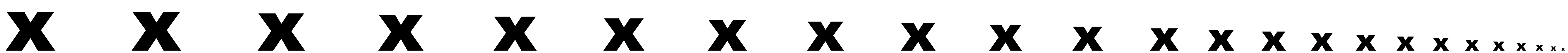
x: $\begin{pmatrix} 0 & 1 & \top \\ \top & \top & 1 \\ \top & \top & 1 \end{pmatrix}$ x^{ω}: $\begin{pmatrix} 0 & 1 & 2 \\ \top & \top & \infty \\ \top & \top & \infty \end{pmatrix}$

$$s^{\omega} = lim_{n \to \infty}\ s^{n!}$$

y: $\begin{pmatrix} 0 & 1 & \top \\ \top & \top & 1 \\ 1 & \top & 1 \end{pmatrix}$ y^{ω}: $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$

Ramsey Theorem

for compact spaces



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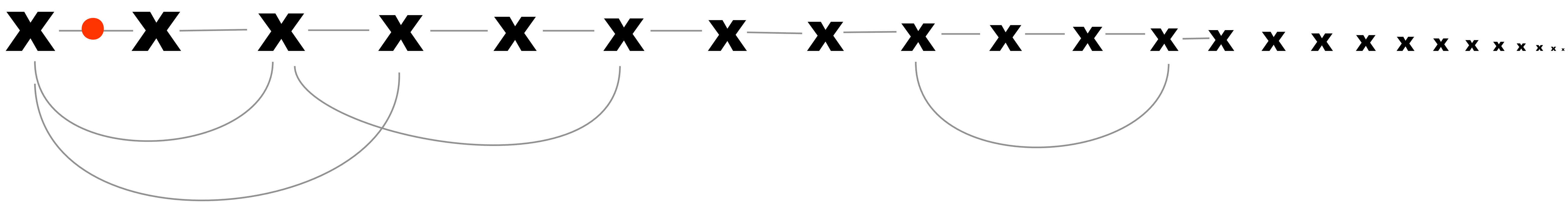

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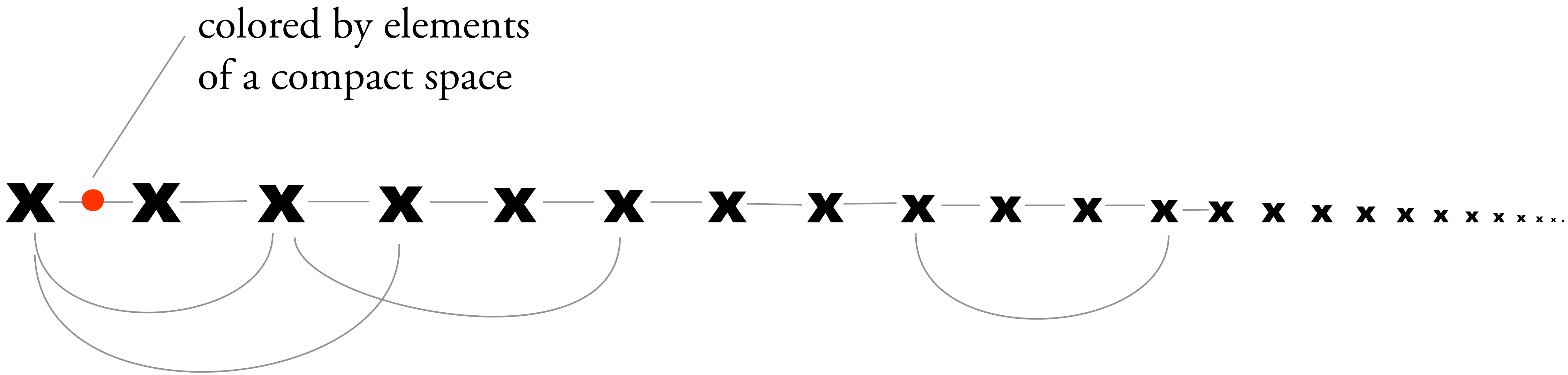

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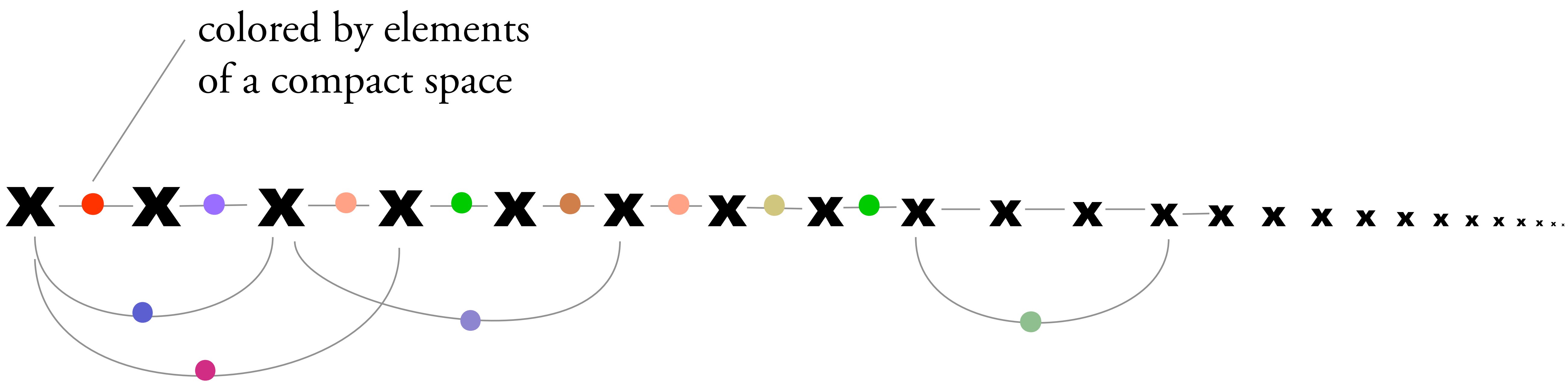

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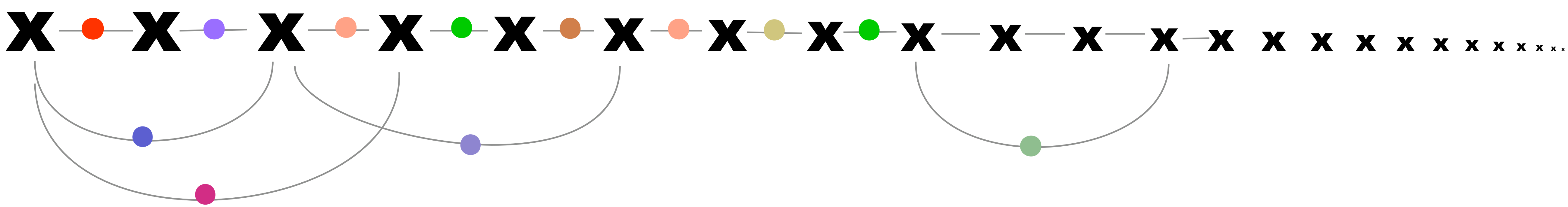

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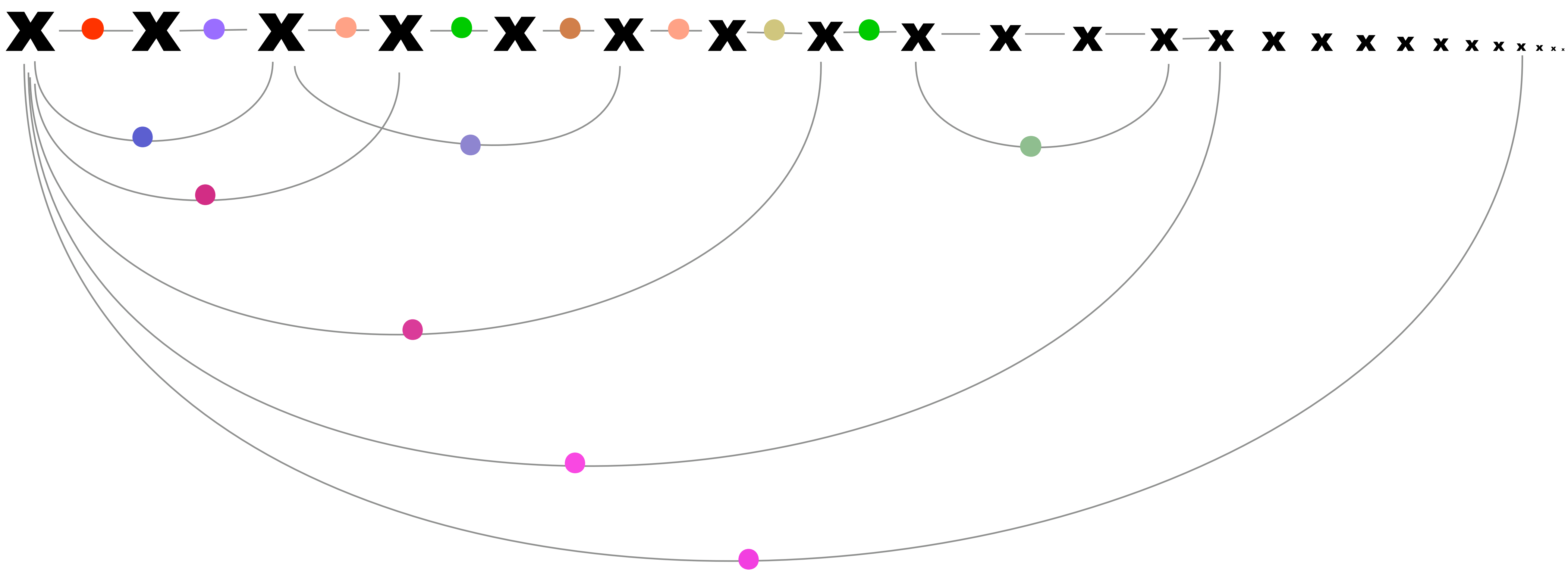

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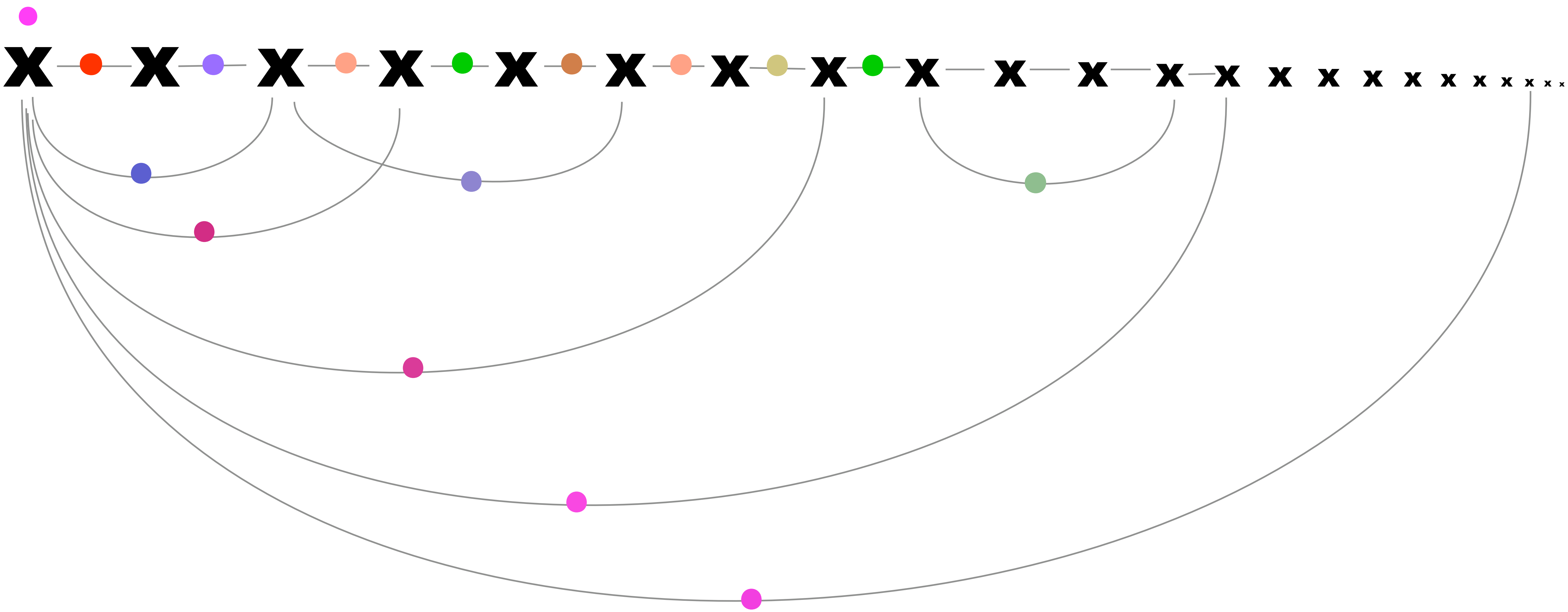

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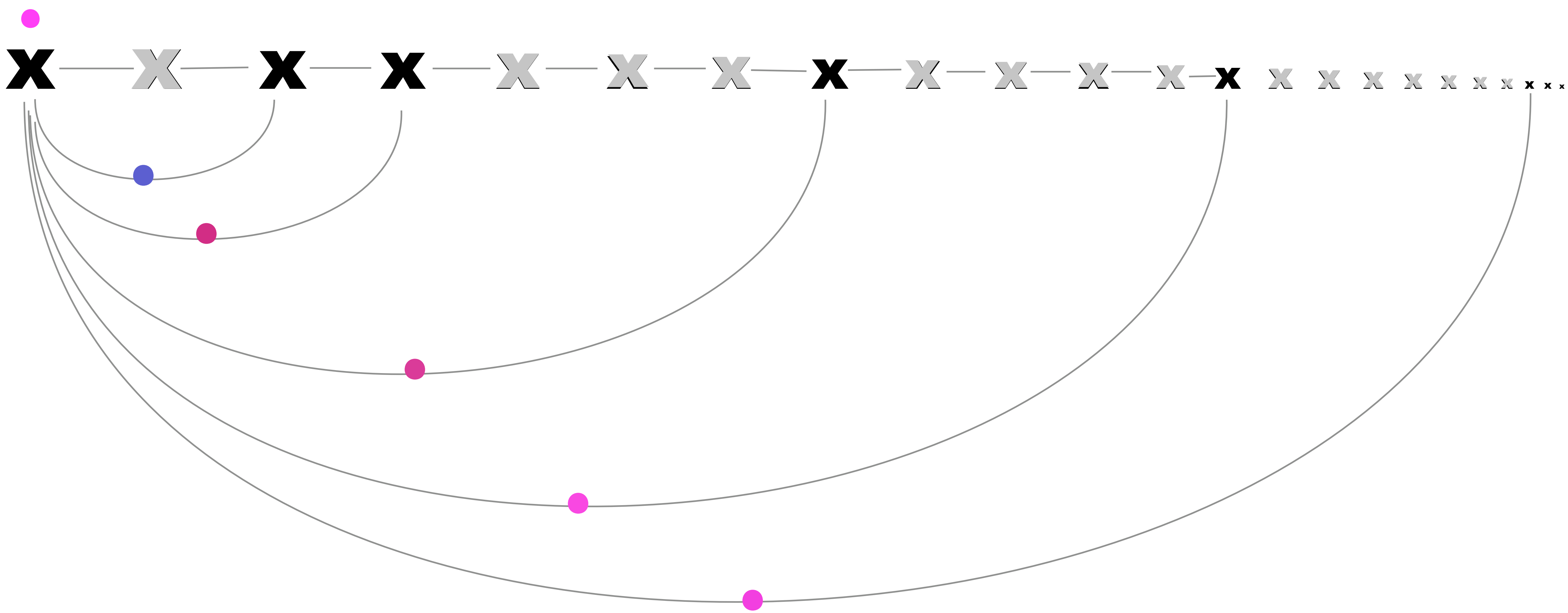

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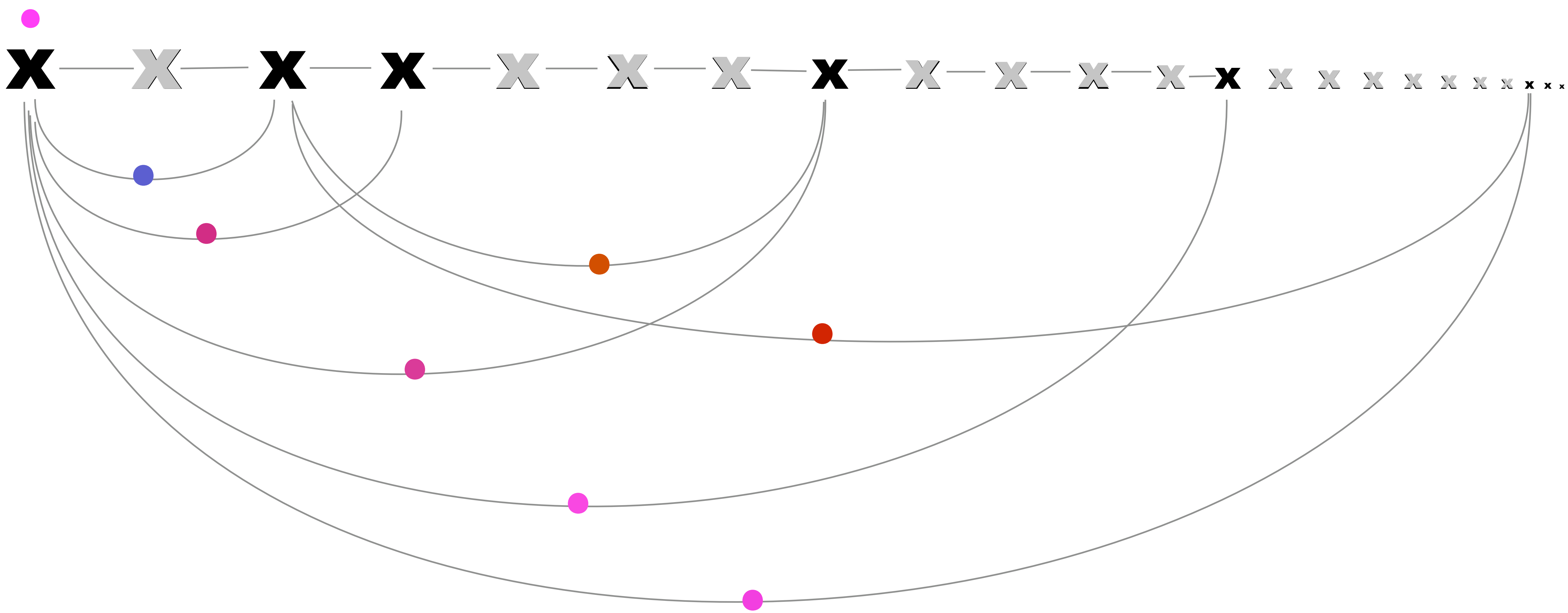

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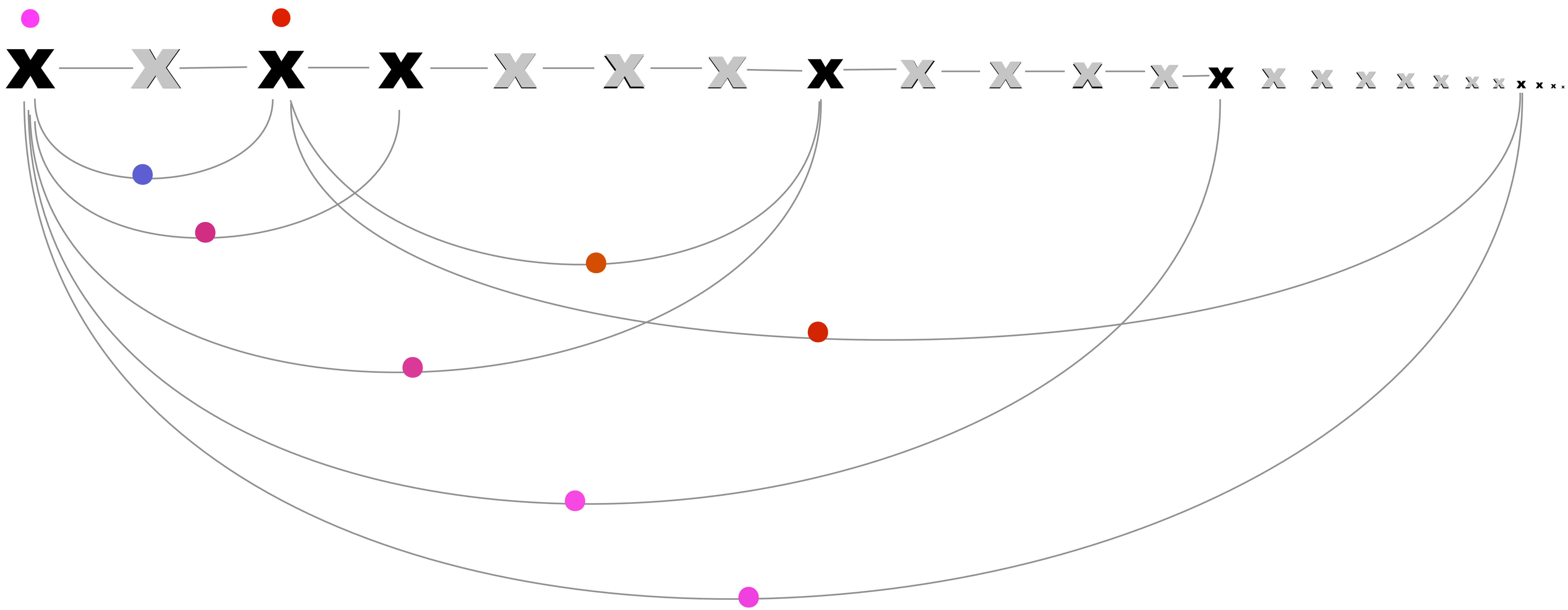

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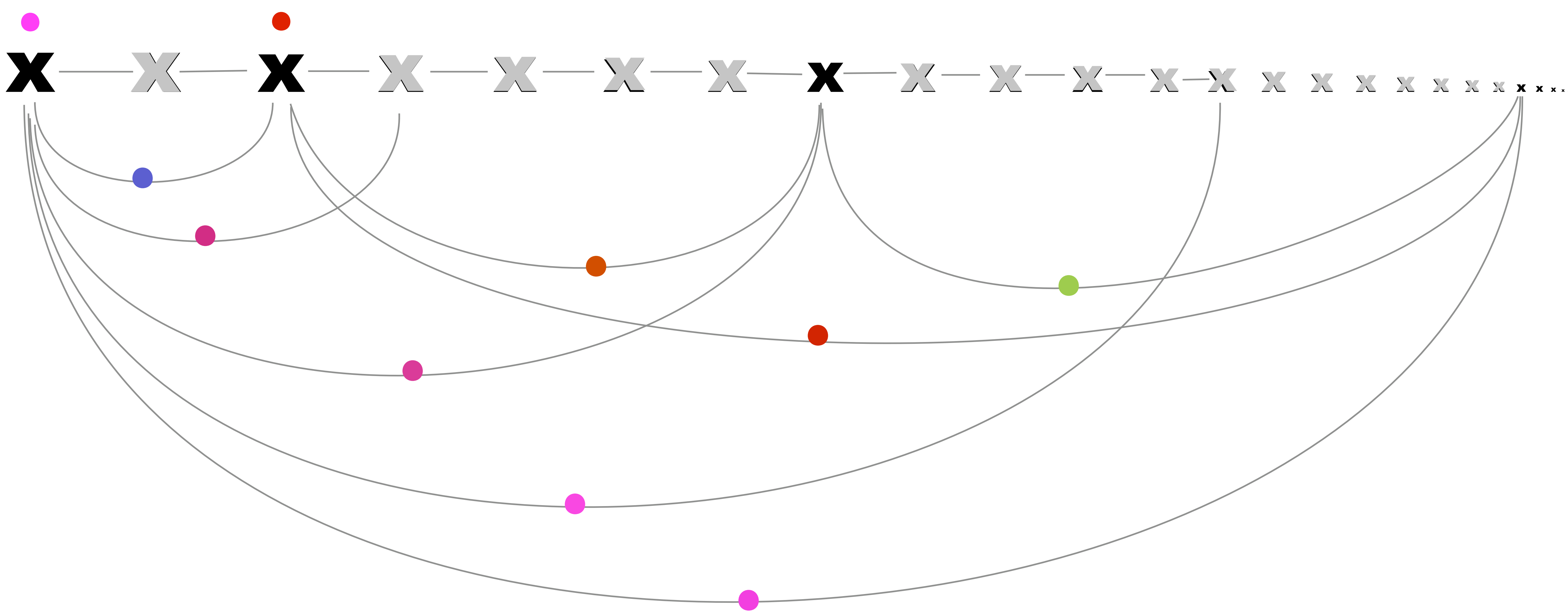

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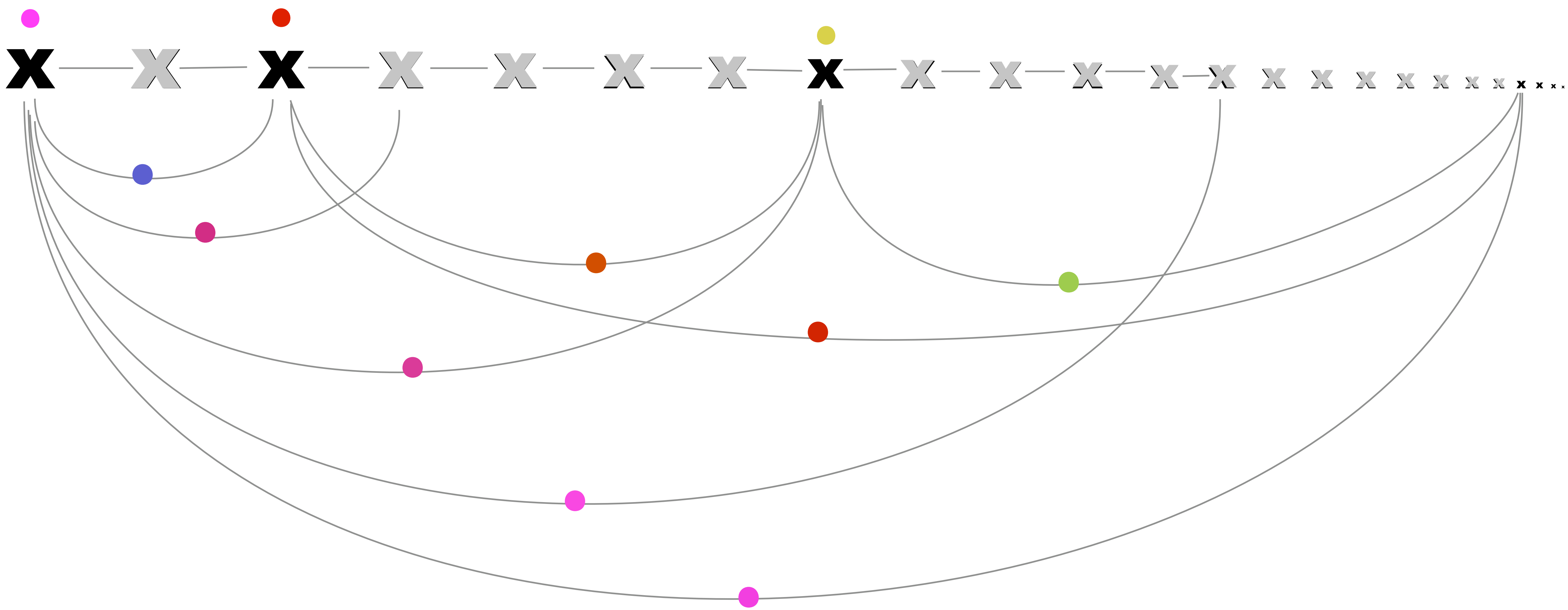

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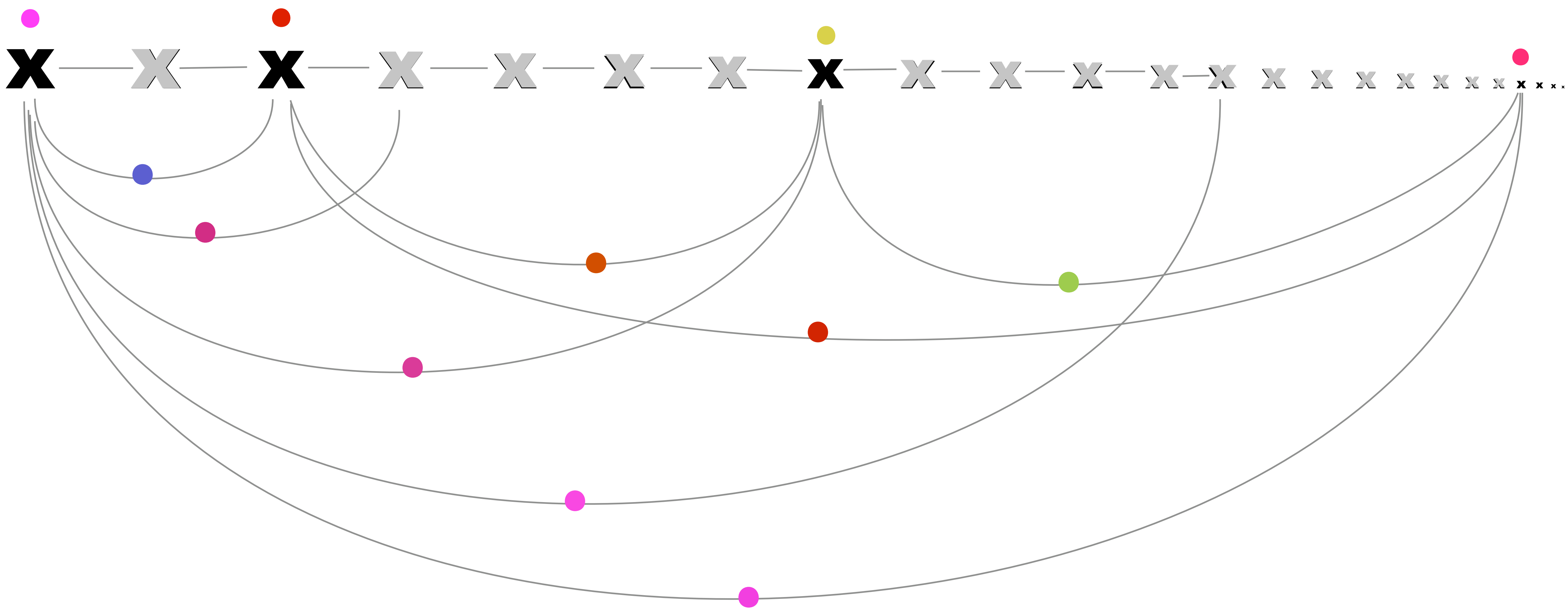

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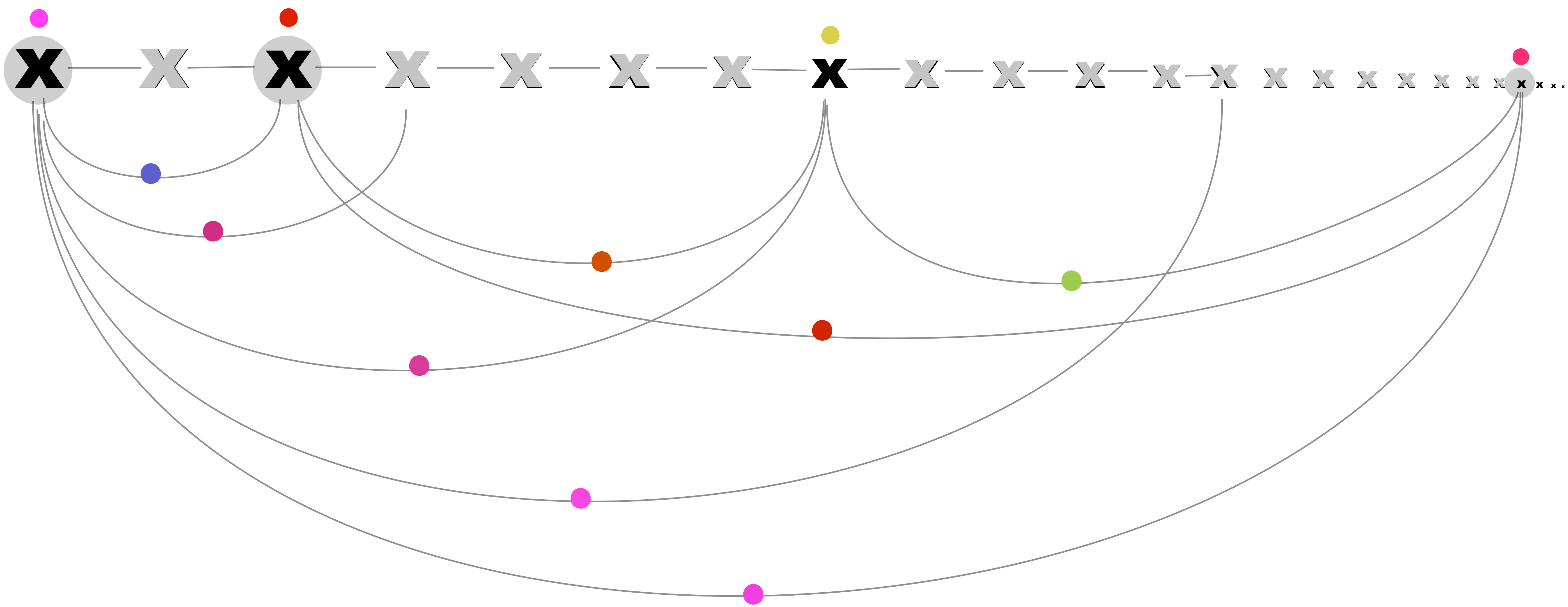

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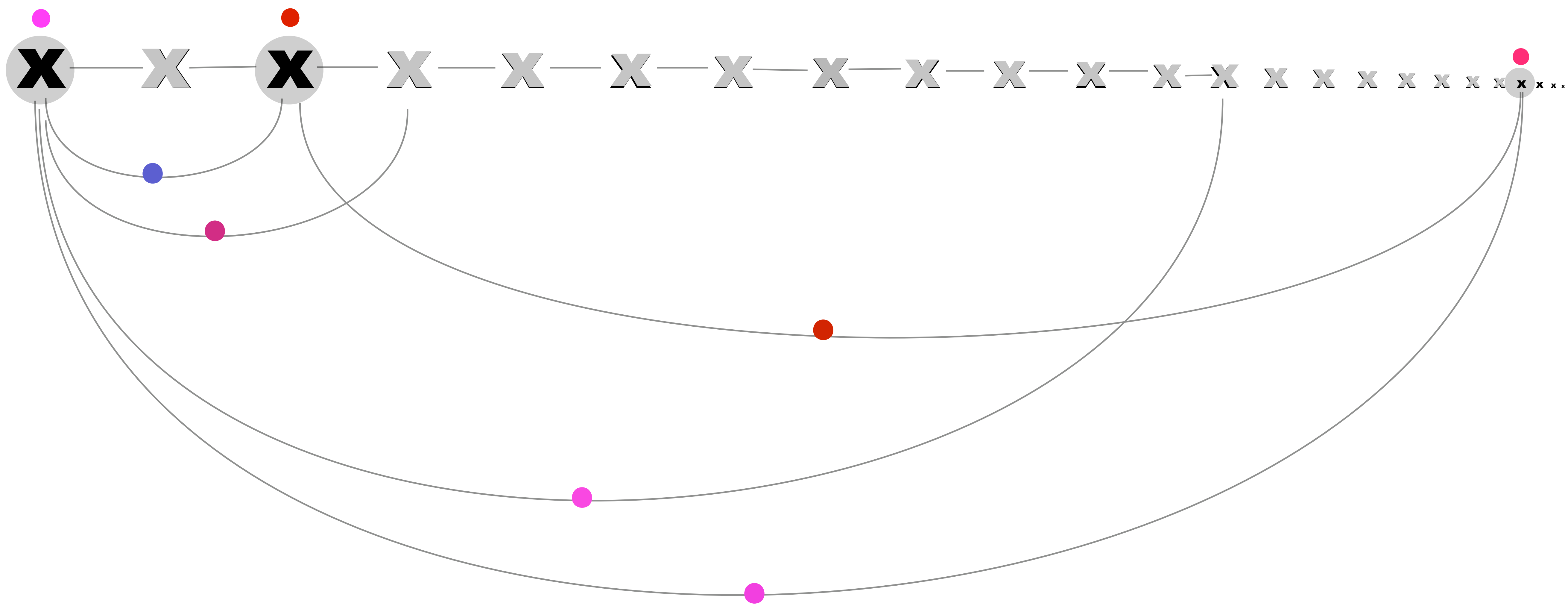

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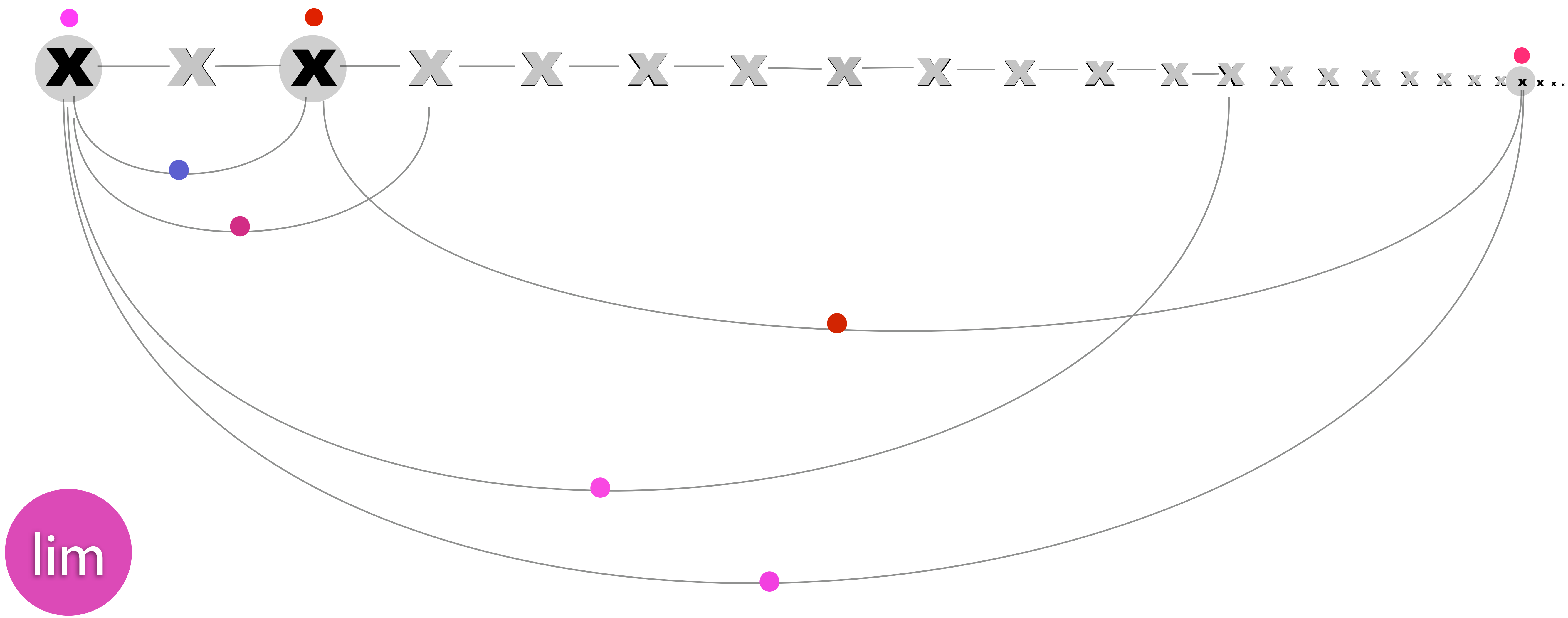

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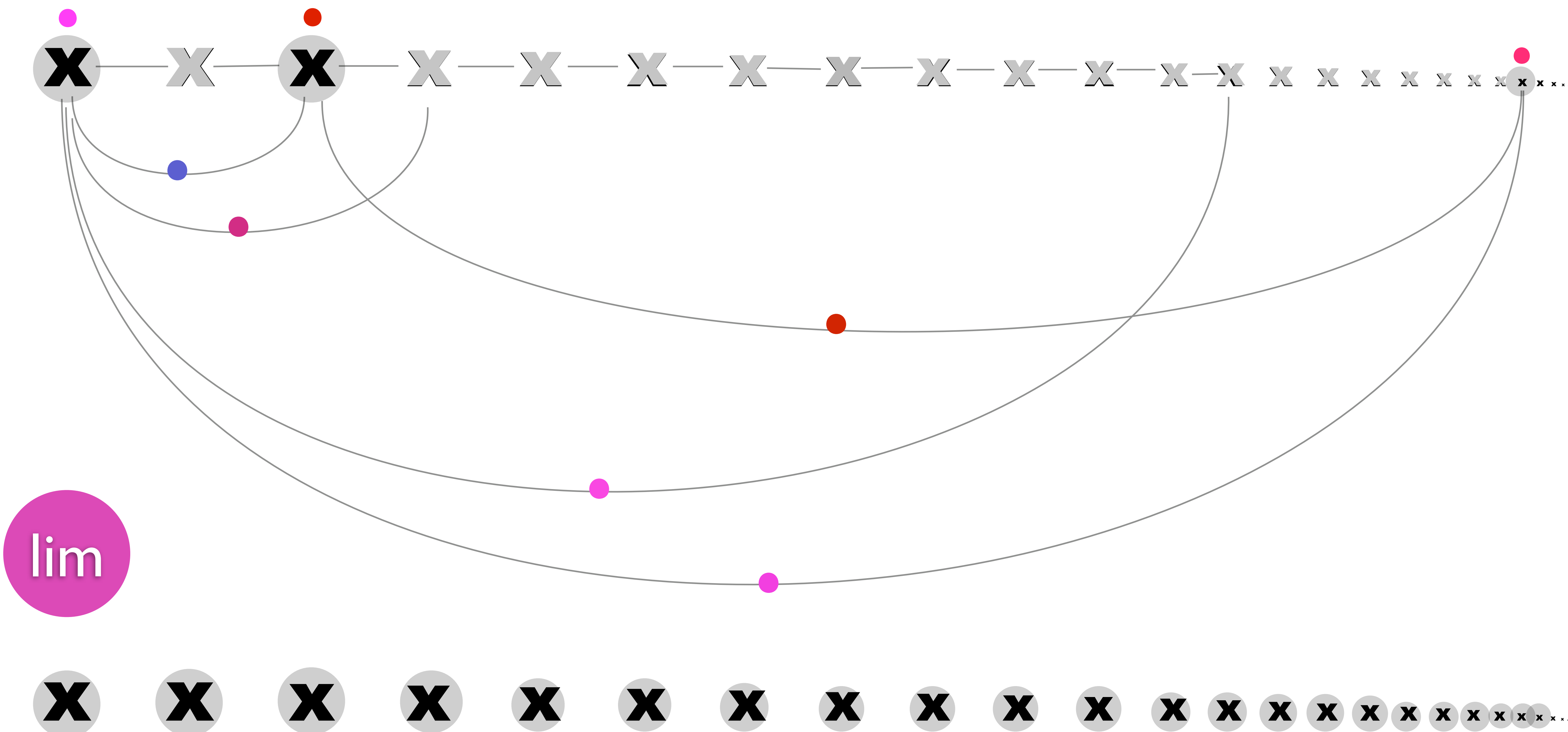

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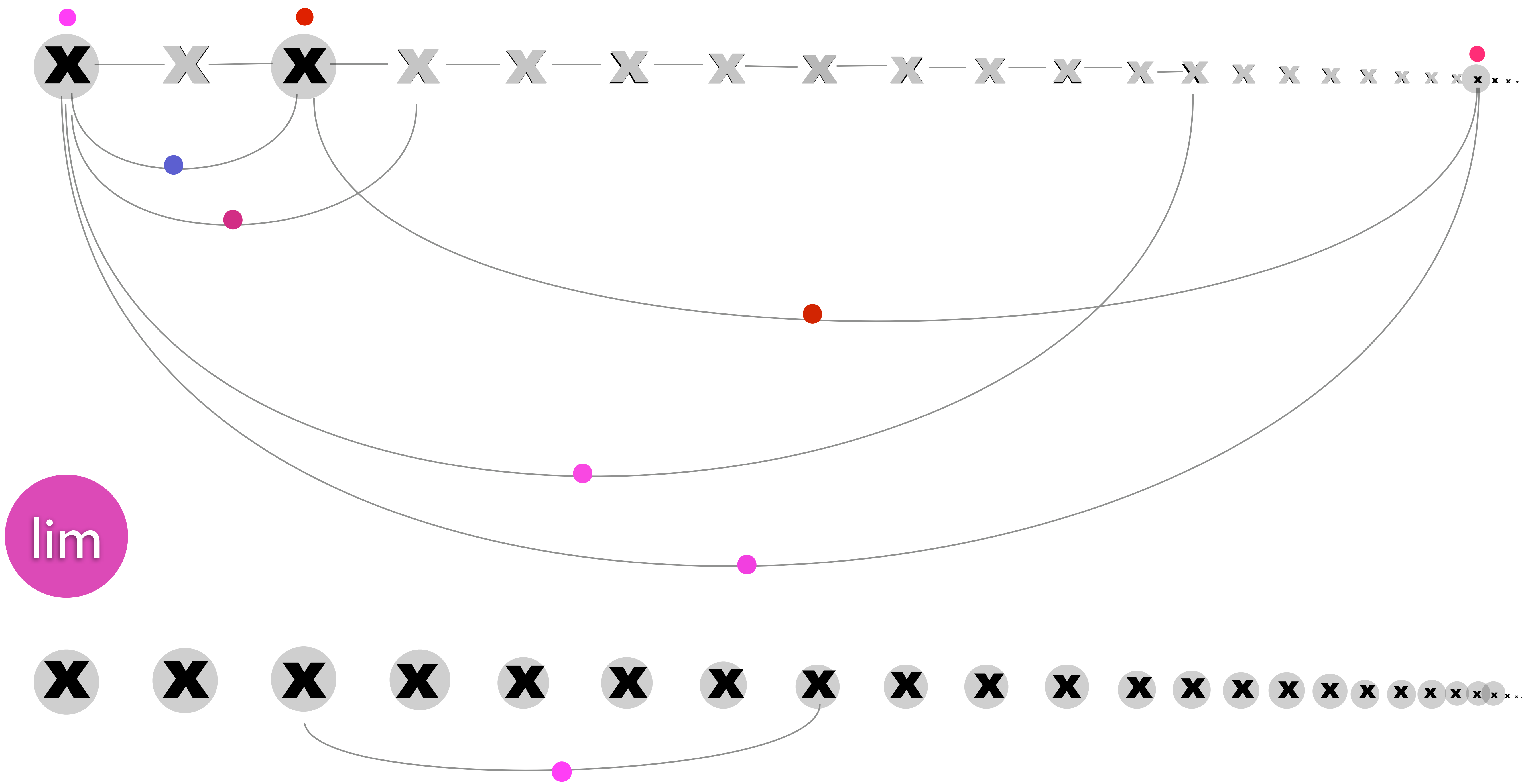

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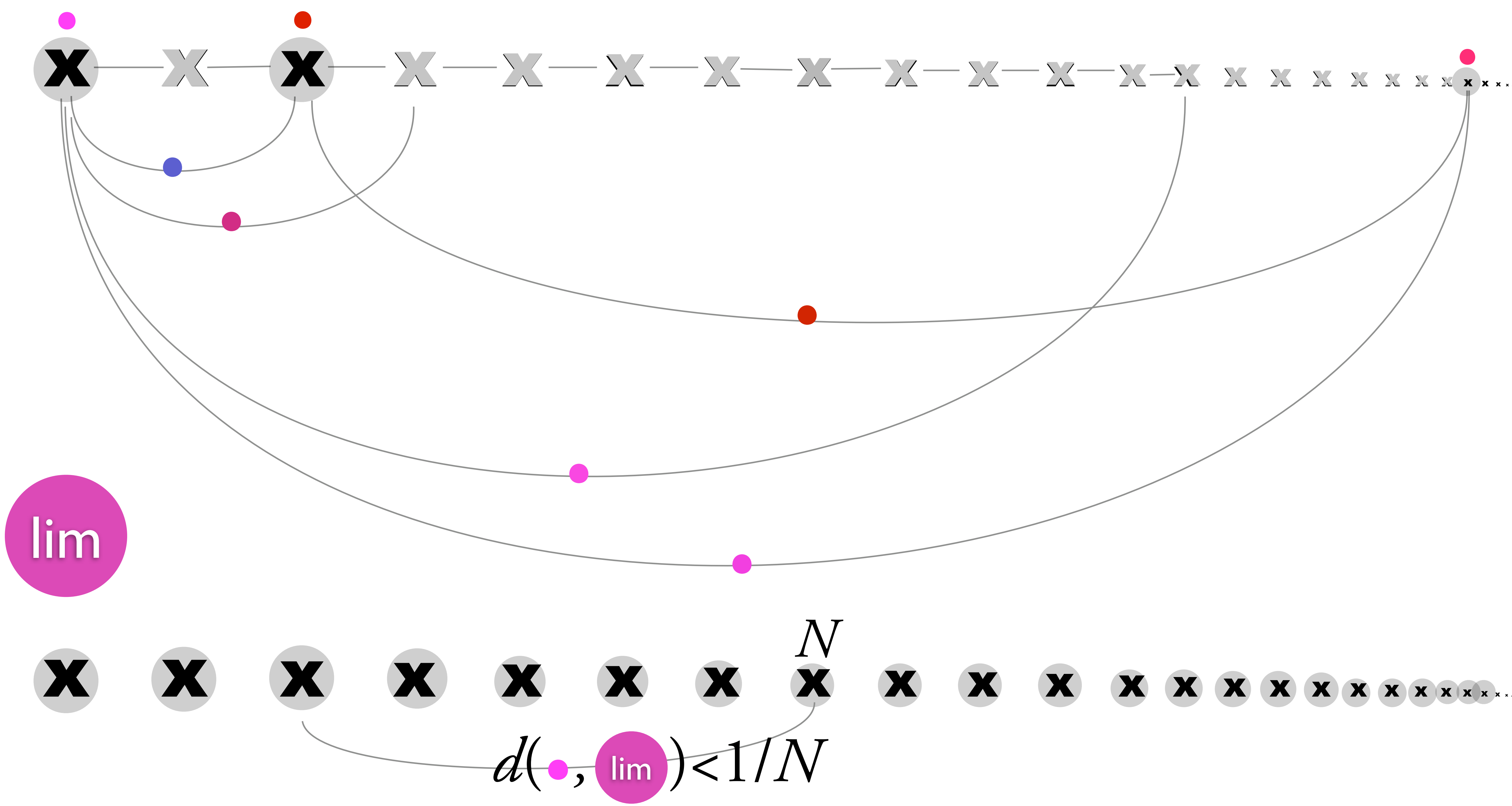

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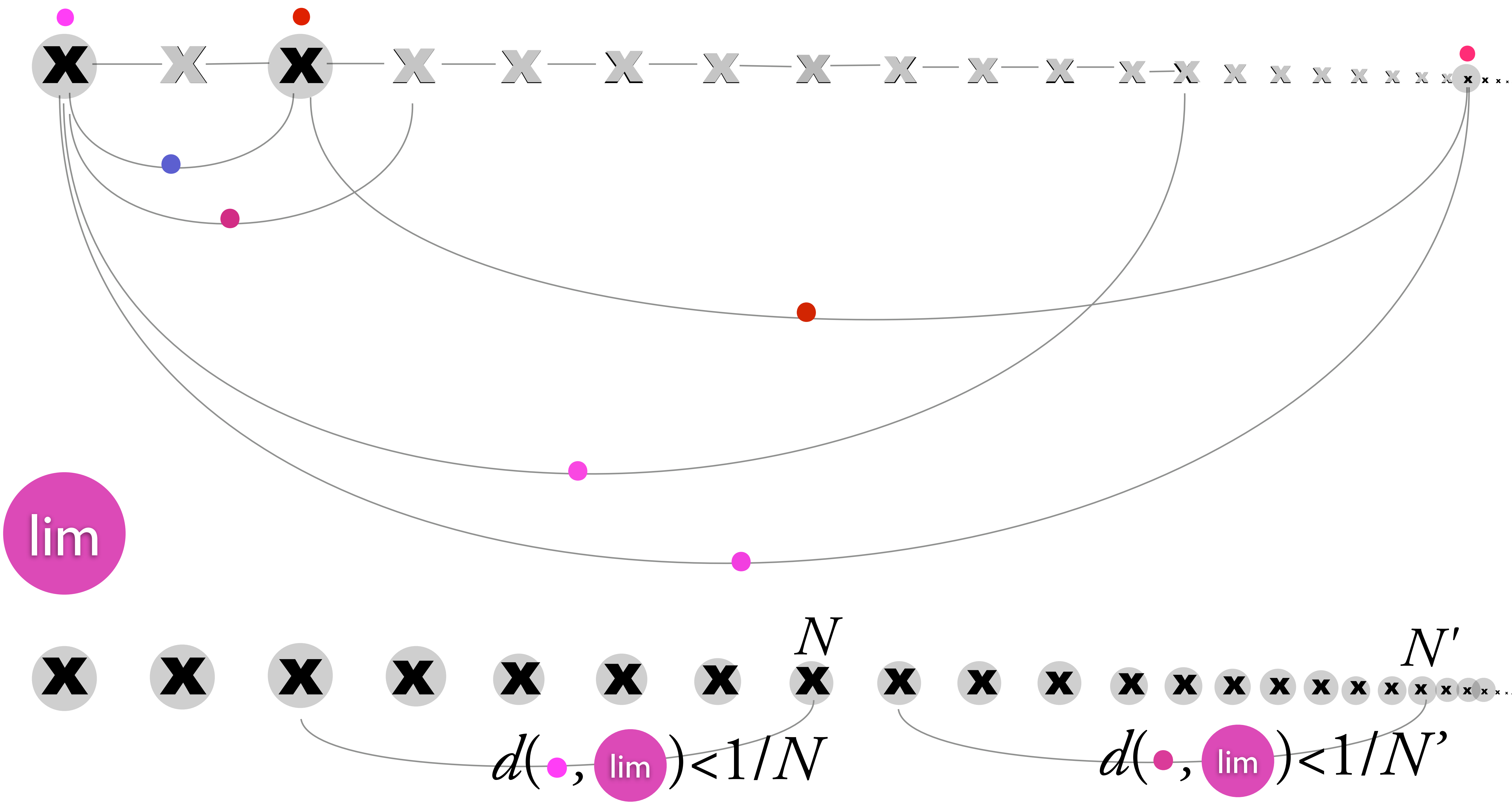
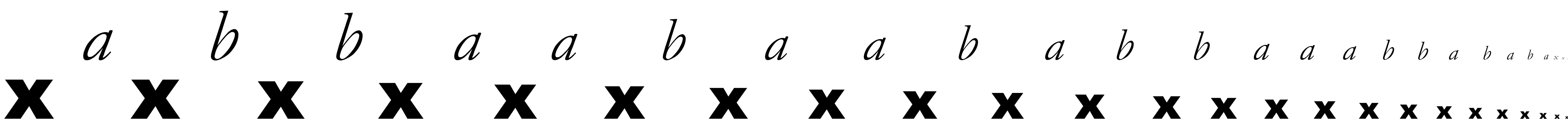


The reduction

$a \quad b \quad b \quad a \quad a \quad b \quad a \quad a \quad b \quad a \quad b \quad b \quad a \quad a \quad a \quad b \quad b \quad a \quad b \quad a \quad b \quad a \; x \; x$

The reduction



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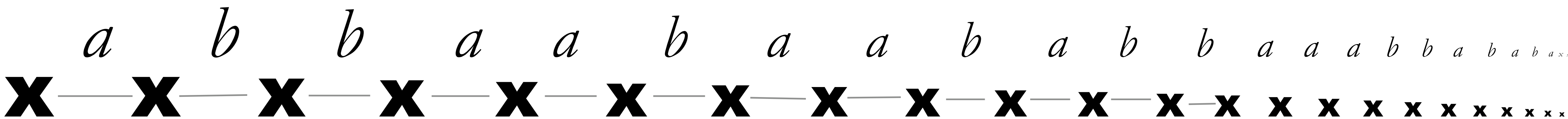

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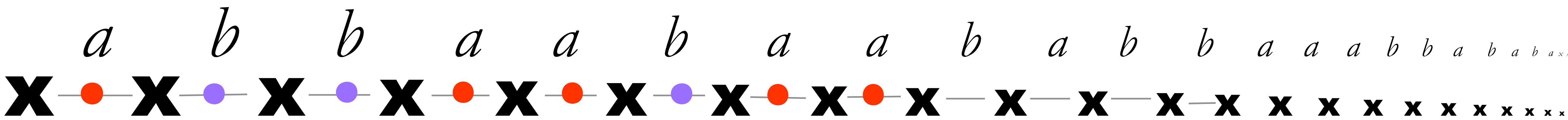

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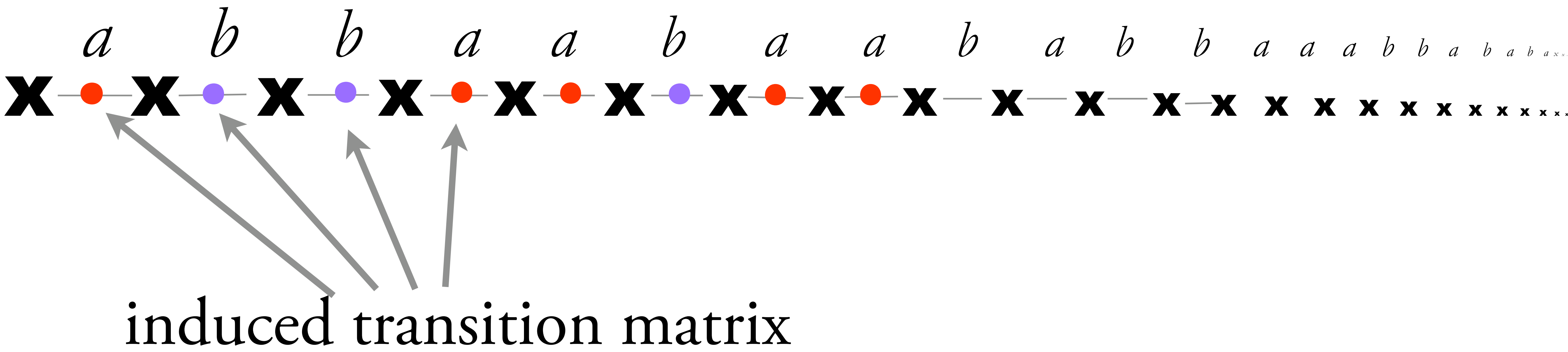

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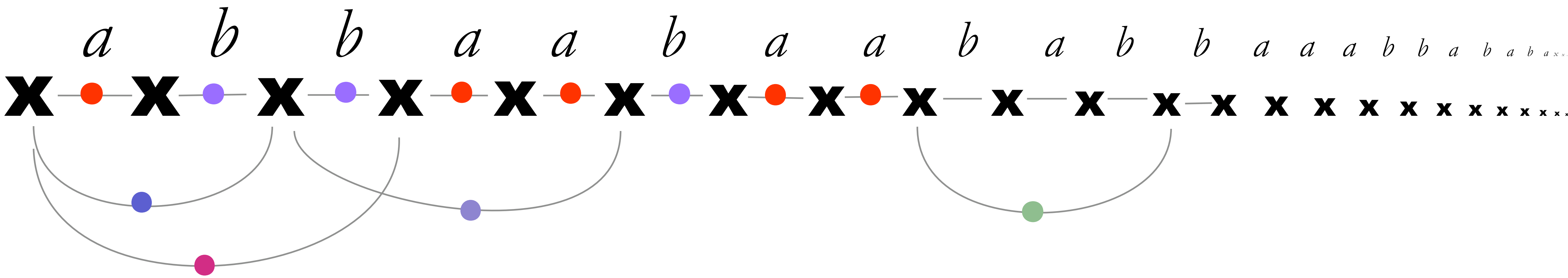

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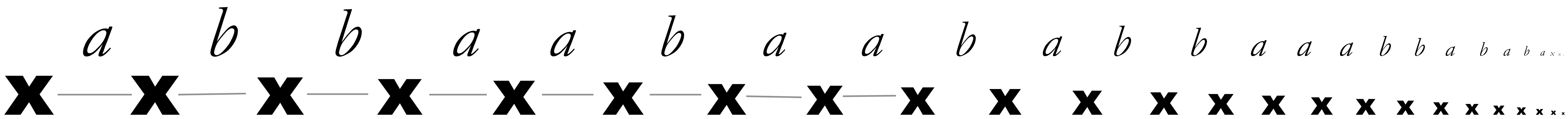
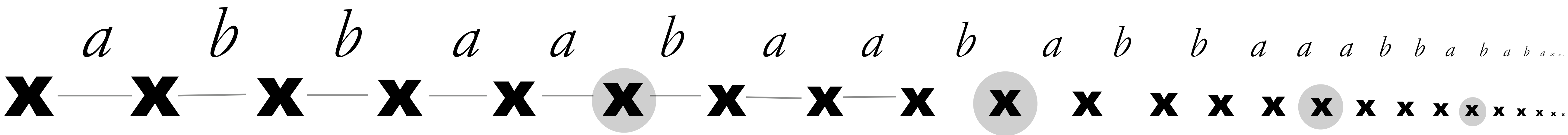


lim

The reduction



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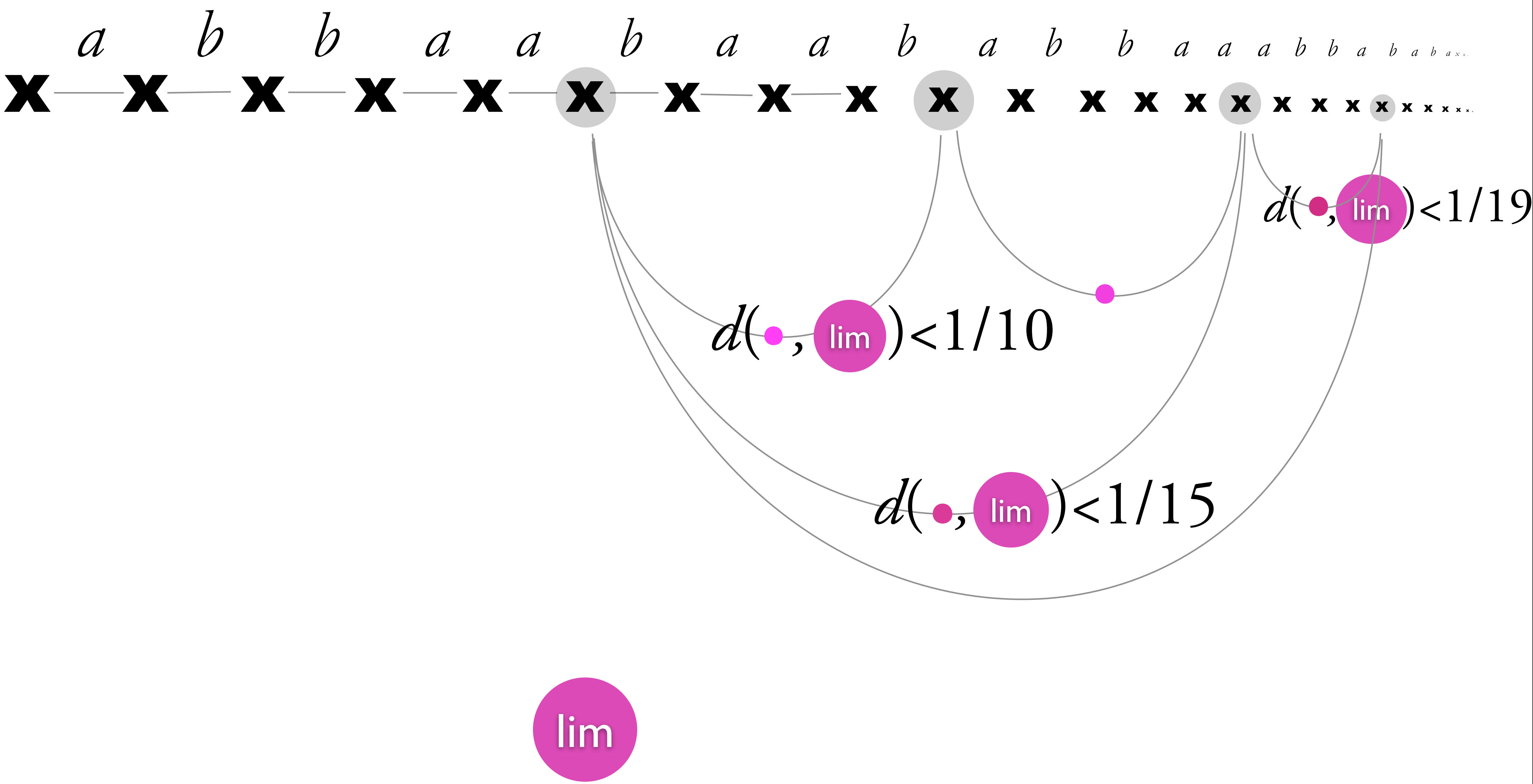

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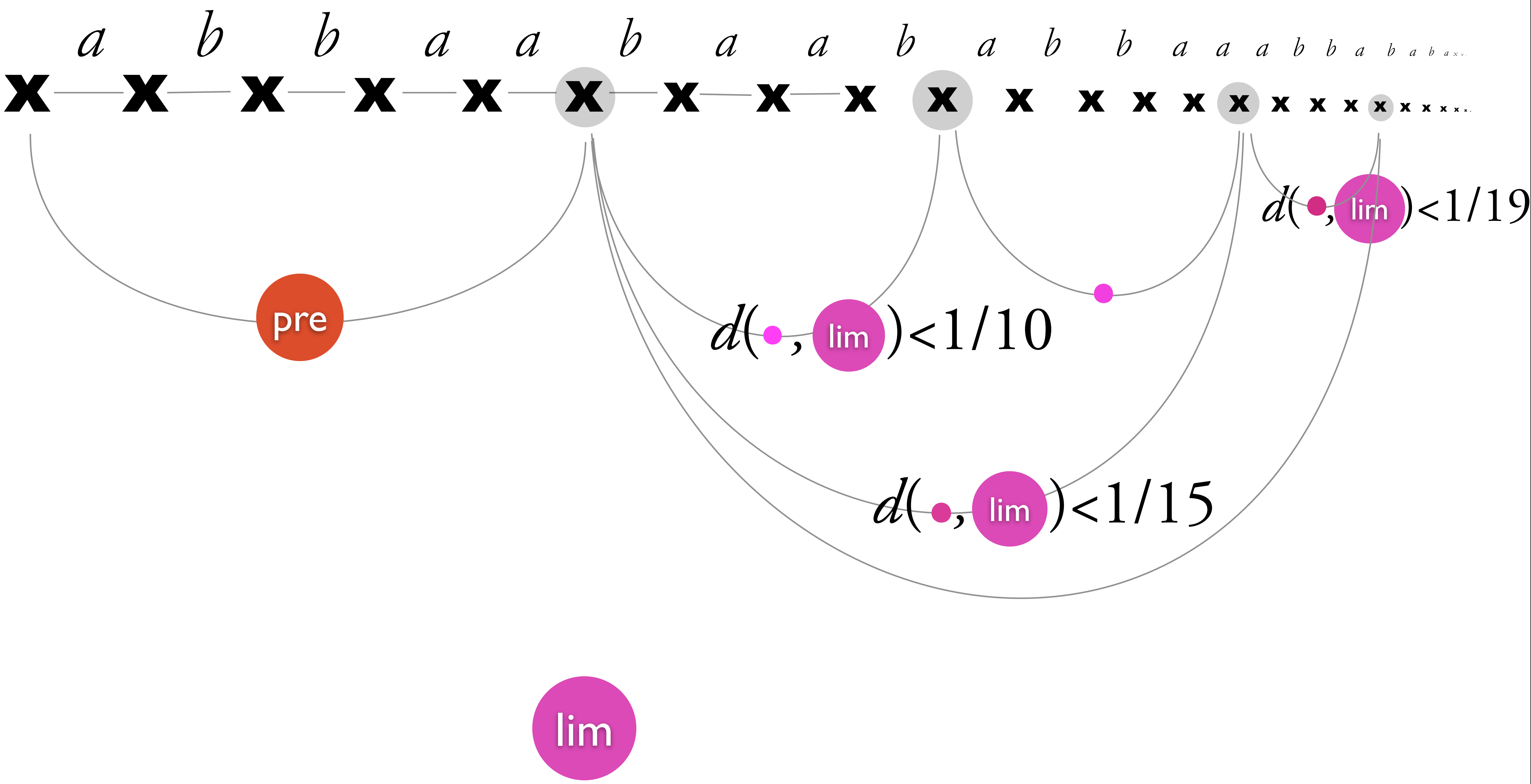

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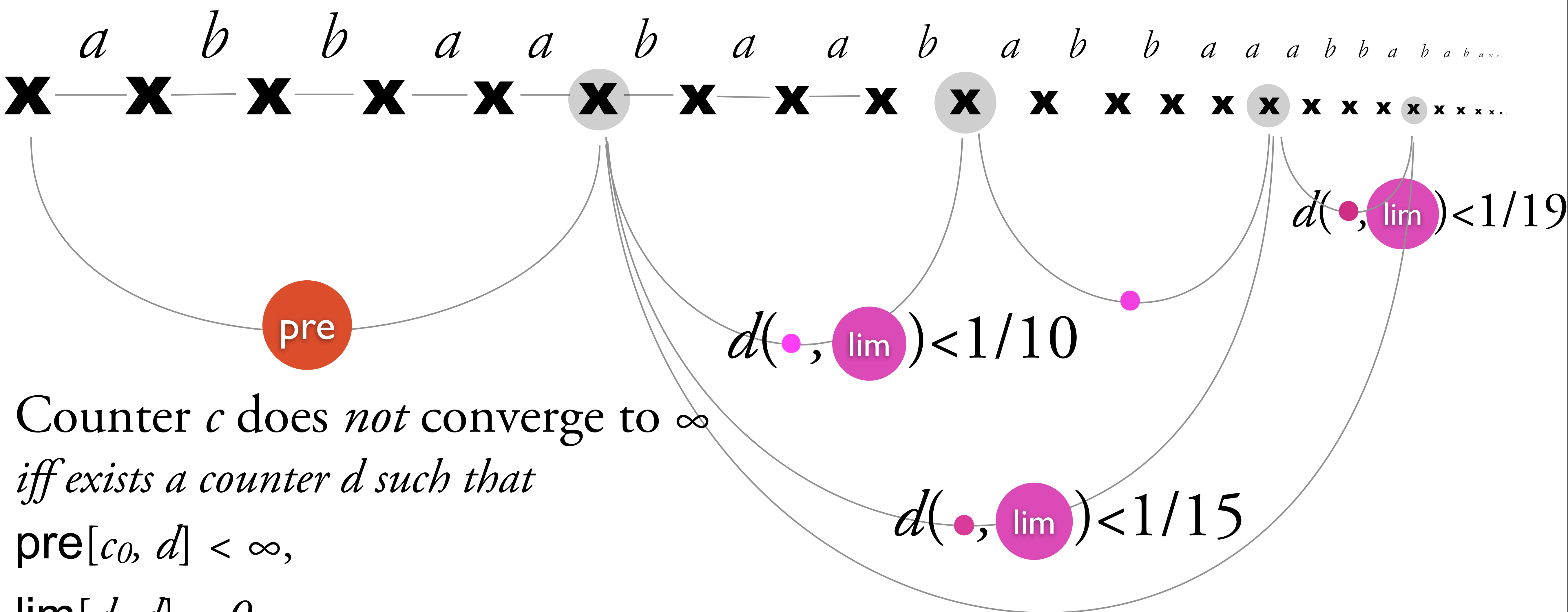


Counter c does *not* converge to ∞

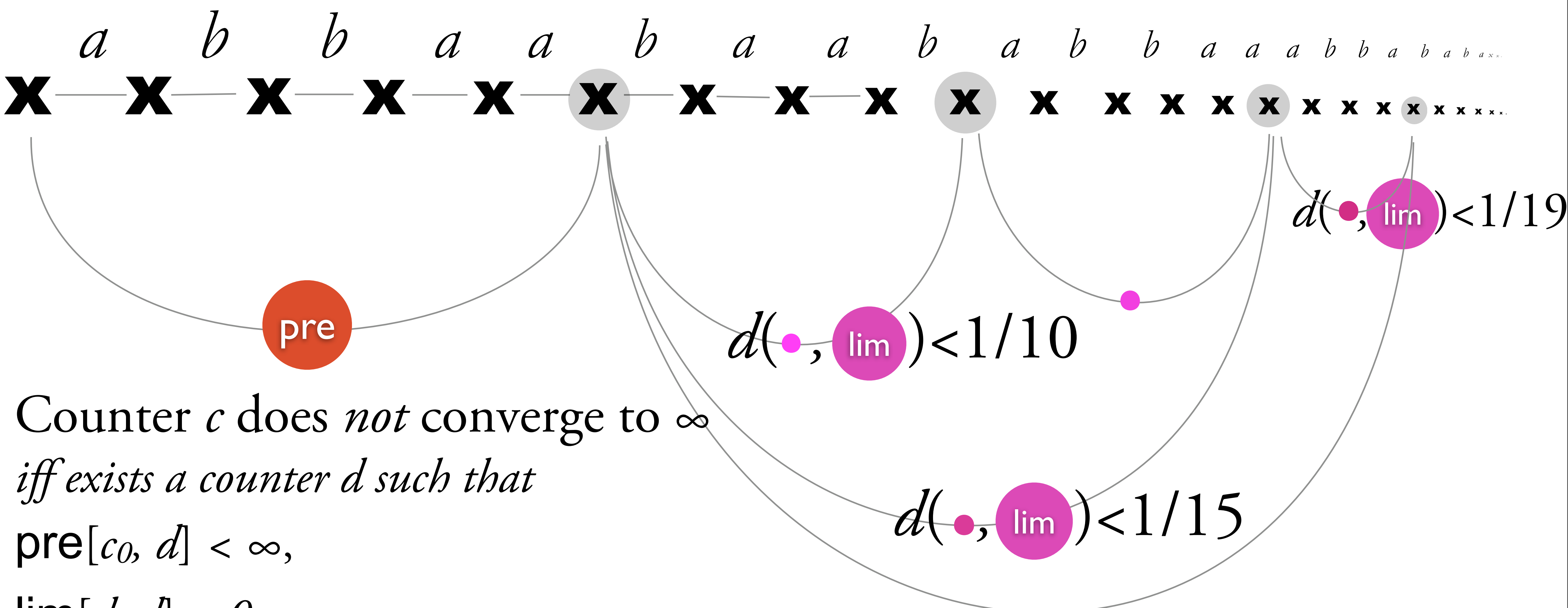
iff exists a counter d such that

pre$[c_0, d] < \infty$,

lim$[d, d] = 0$,

lim$[d, c] < \infty$.

The reduction



Counter c does *not* converge to ∞

iff exists a counter d such that

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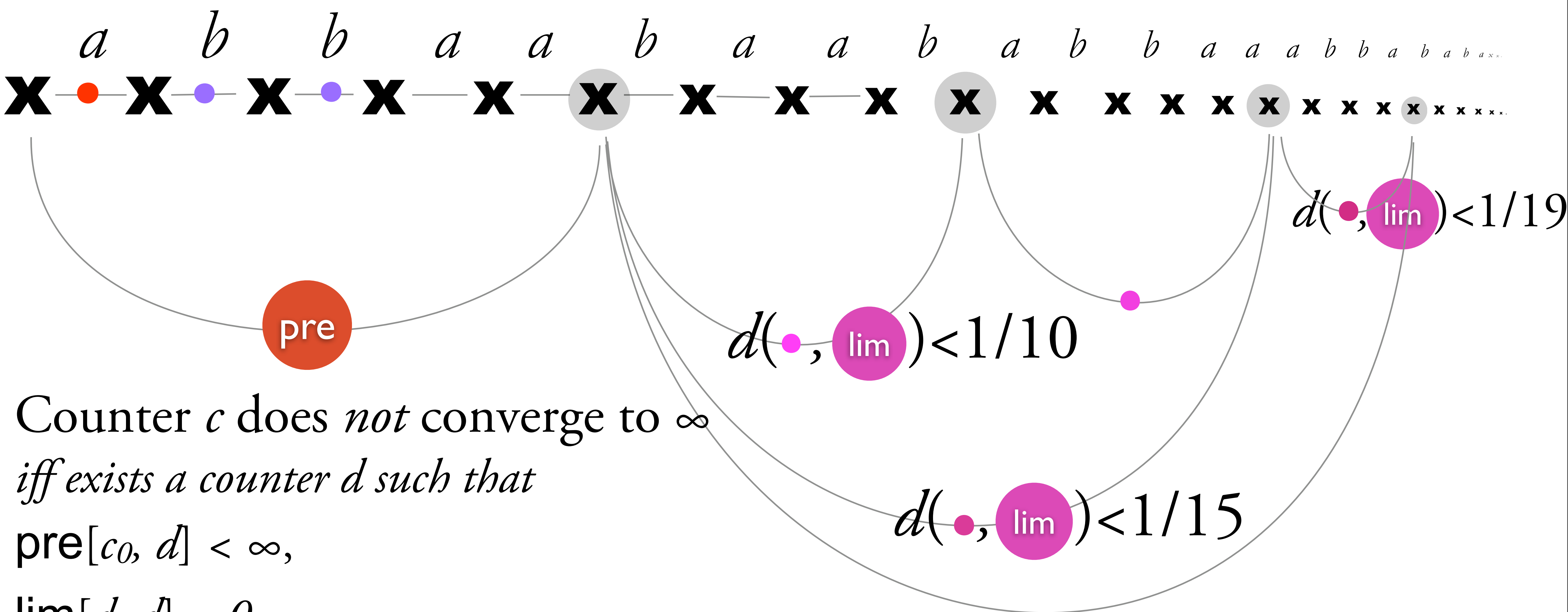
Which limits lim are possible?

The reduction



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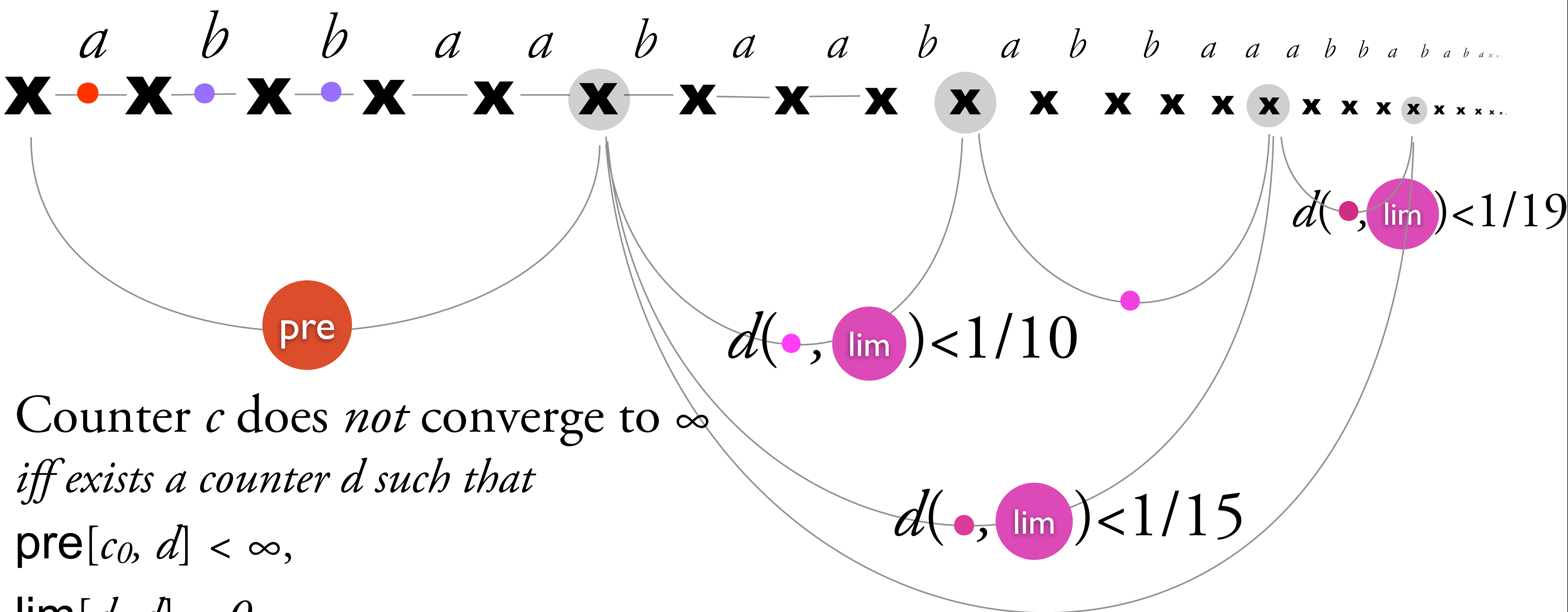

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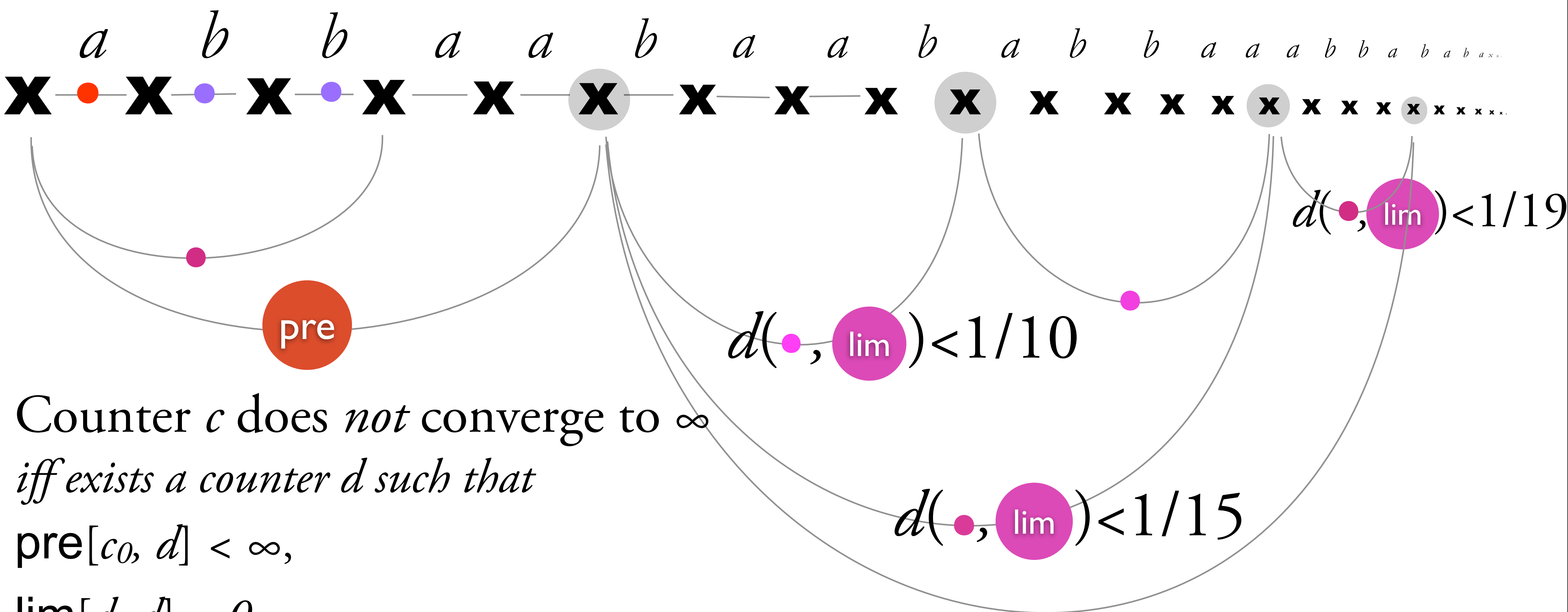

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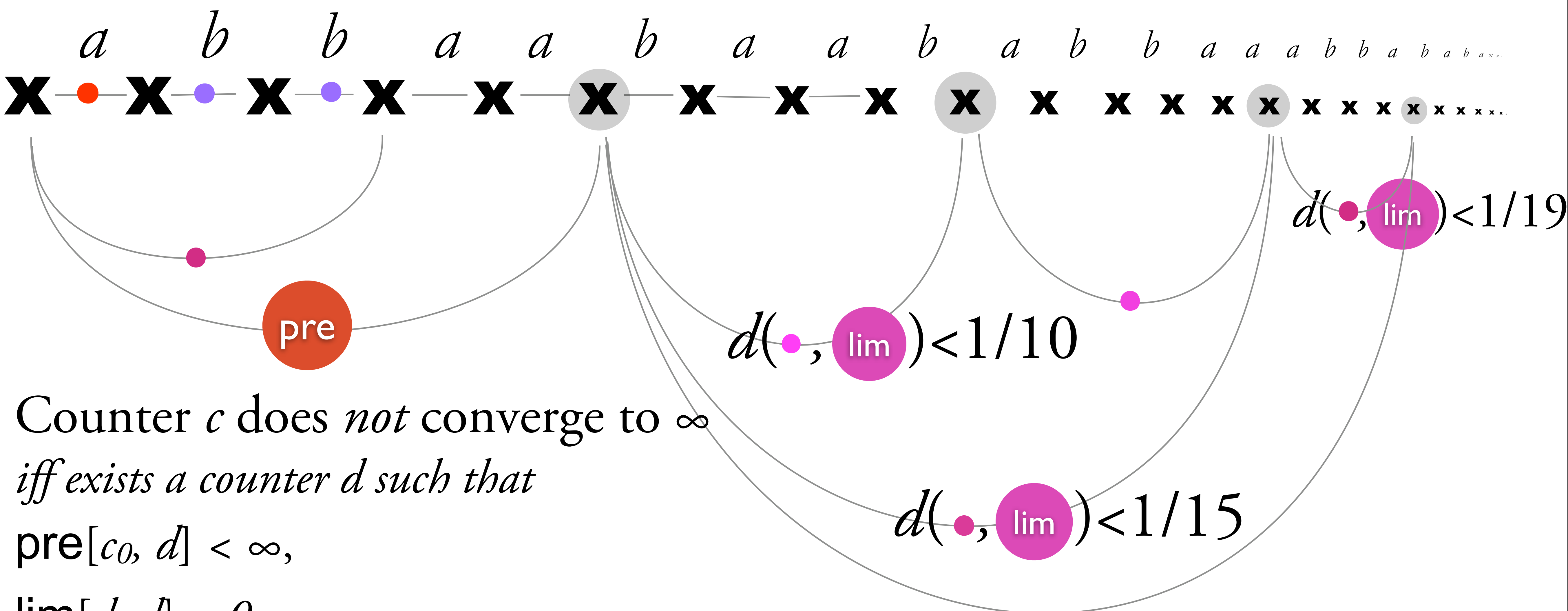

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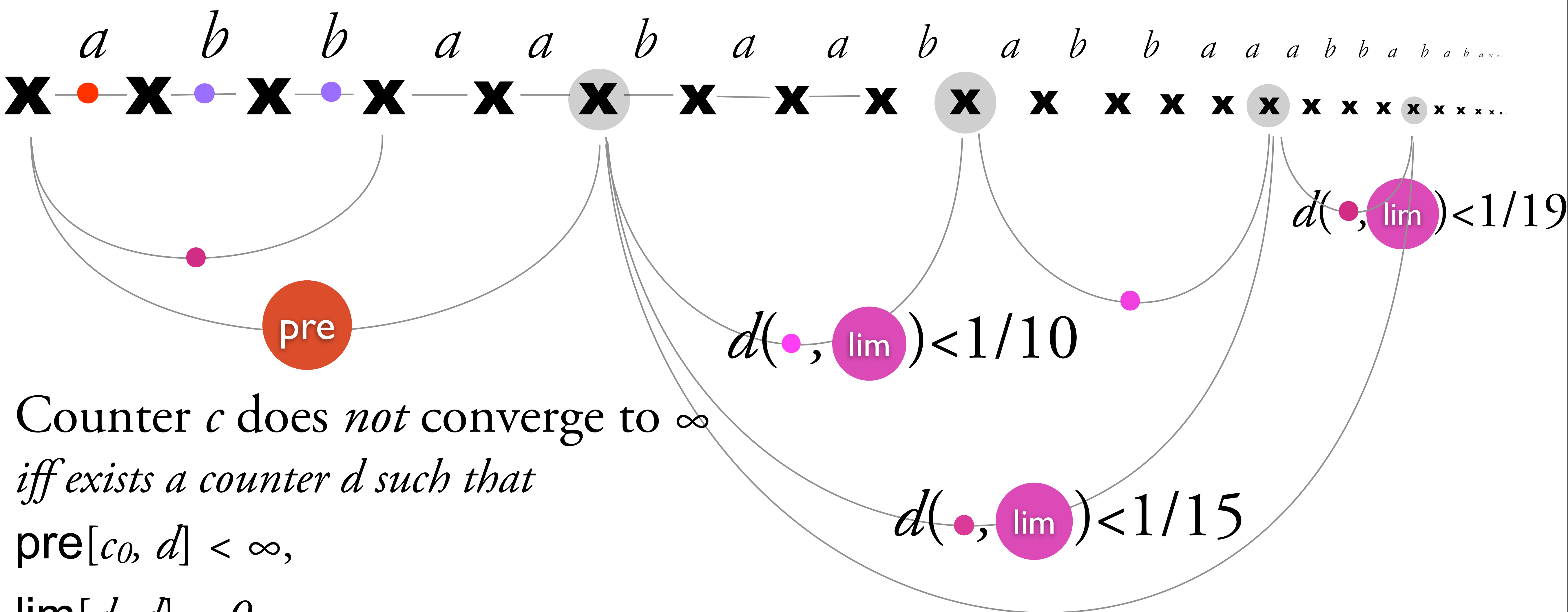

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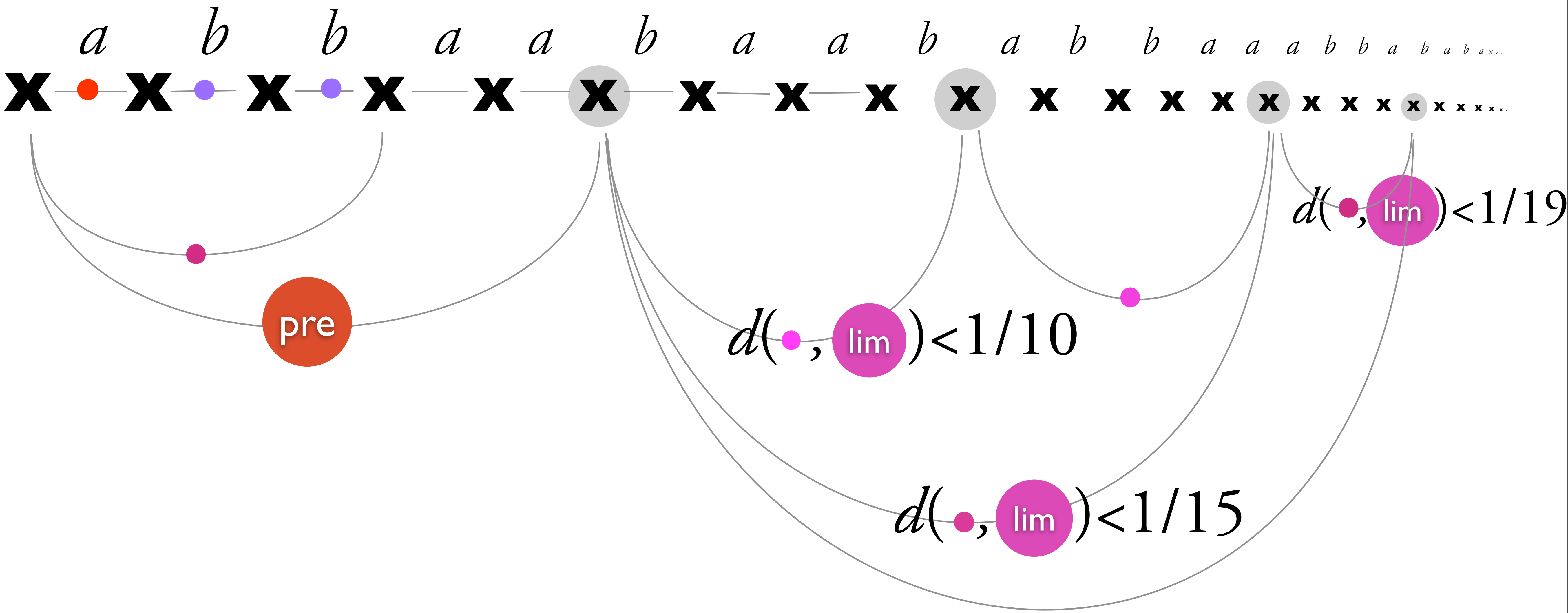
Which limits lim are possible?

$$\mathrm{lim} \in \overline{(a, b)^+}$$

The reduction



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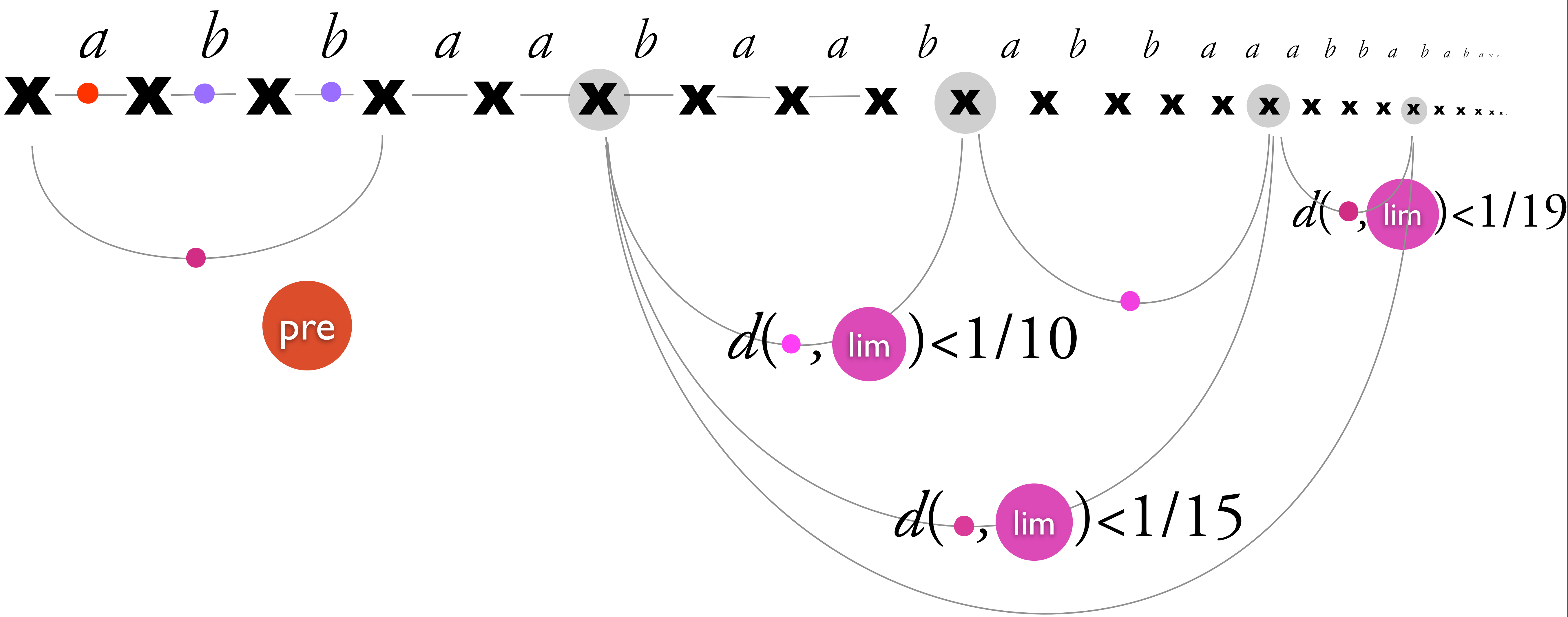

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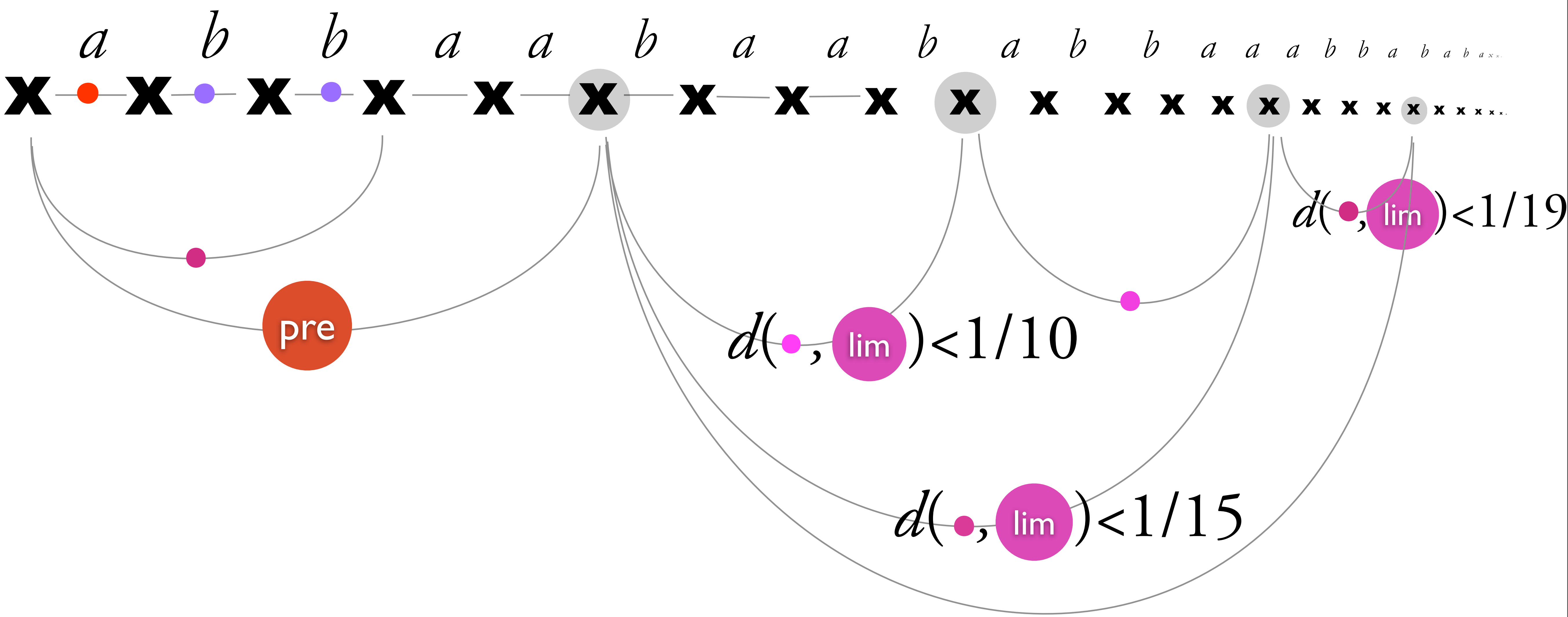

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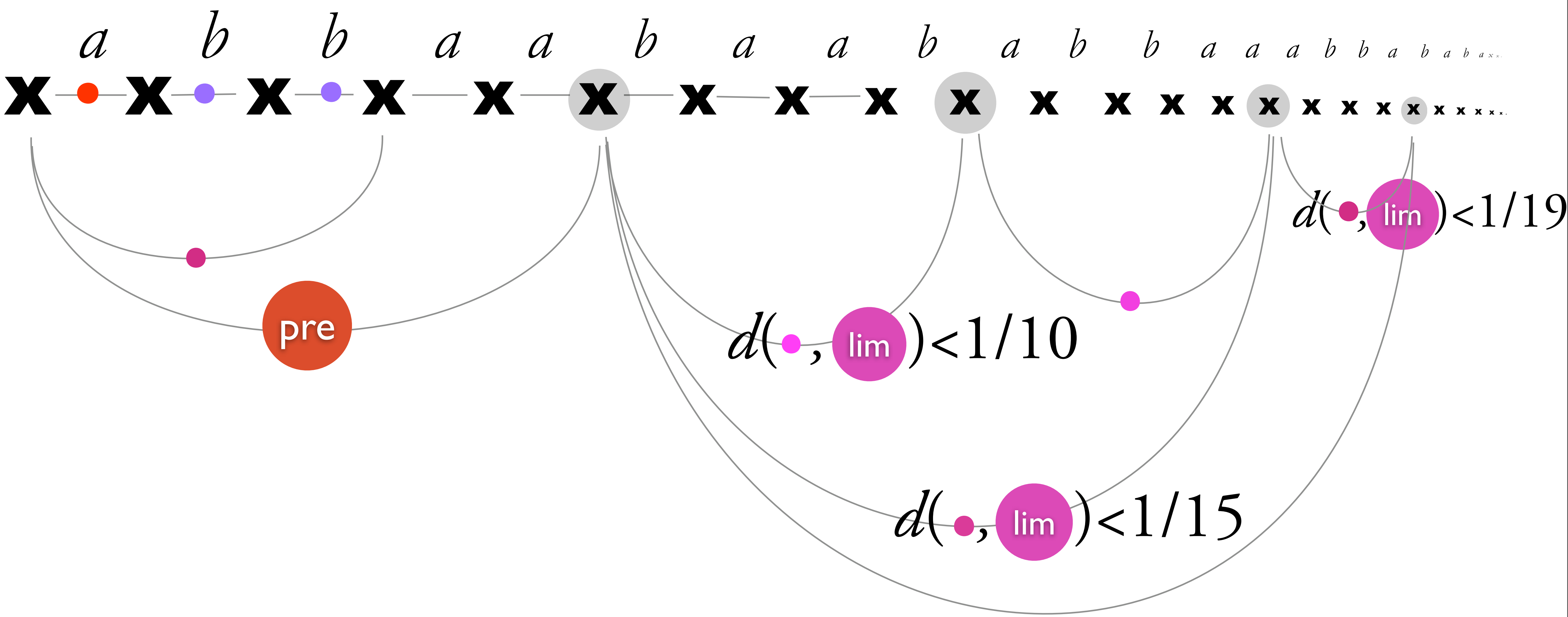

Which limits lim are possible?

$\overline{(a, b)^+}$

Finite section problem: given tropical matrices a b

decide whether there exists $\mathrm{lim} \in \overline{(a, b)^+}$ with $\mathrm{lim}[d, c] = \infty$.

Plan

✓ 1. Introduction to the problem

2. Reduce emptiness of min-automata to the *finite section problem*, via a Ramsey-type theorem

3. Solve the finite section problem using Simon's Factorization Theorem

Plan

✓ 1. Introduction to the problem

✓ 2. Reduce emptiness of min-automata to the *finite section problem*, via a Ramsey-type theorem

3. Solve the finite section problem using Simon's Factorization Theorem

Simon's Factorization Theorem

for semigroups with stabilization

Simon's Factorization Theorem

for semigroups with stabilization

semigroup with stabilization

Simon's Factorization Theorem

for semigroups with stabilization

$$(S , \cdot , {}^{\#})$$

semigroup with stabilization

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S\, ,\, \cdot}\, ,\ ^{\#})$$

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- $s^{\#} = (s^{n})^{\#}$ *for* n=1,2,3,...

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$$\underbrace{(\,S\,,\,\cdot\,}_{\text{semigroup}},\ {}^{\#}\,)$$

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Example (infinite)

$(\{0,1,2,\ldots,\infty\},\ +,\ {}^{\omega}\ )$,

$0^{\omega} = 0, \quad 1^{\omega} = 2^{\omega} = \ldots = \infty$

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for semigroups with stabilization

$$(\underbrace{S , \cdot}, \, ^{\#})$$

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Example (finite)
$T_N = (\{0,1,2,\ldots,N,\infty\}, +, {}^{\#})$,
$N+1 = N, \quad 0^{\#} = 0, \quad x^{\#} = \infty^{\#} = \infty$

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$N{+}1, N{+}2, \ldots \rightarrow N$

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for semigroups with stabilization

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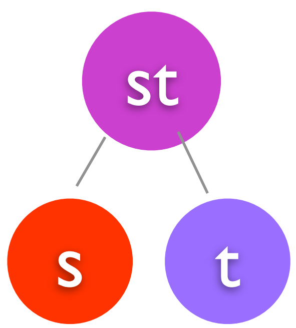
Simon's Factorization Theorem

for semigroups with stabilization

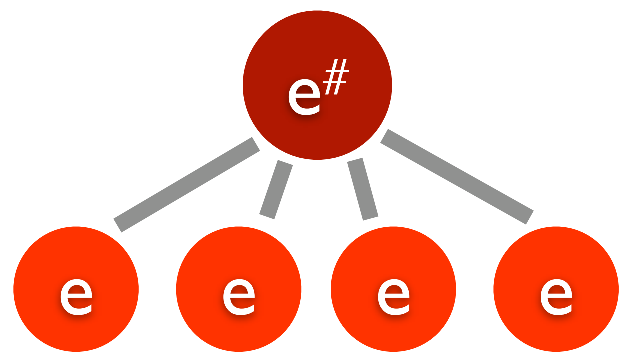
Factorization tree of word $w \in S^+$

Use the two rules to construct tree:

binary rule



idempotent rule



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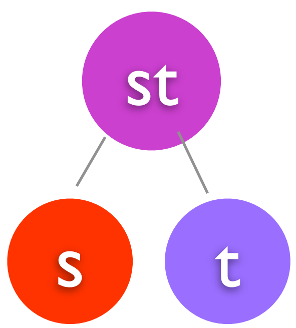

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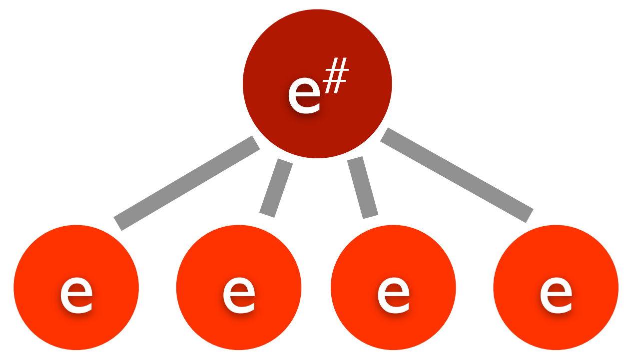

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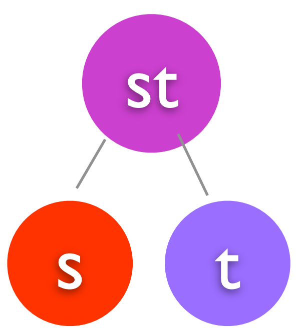

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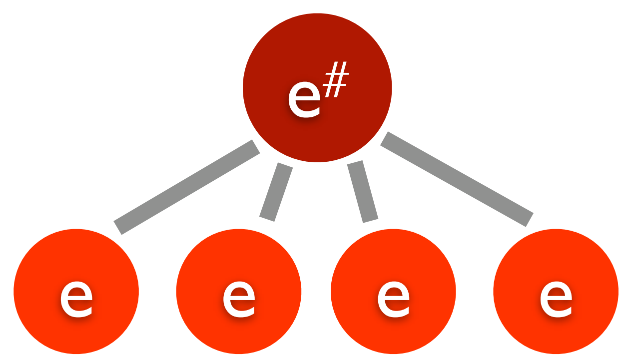

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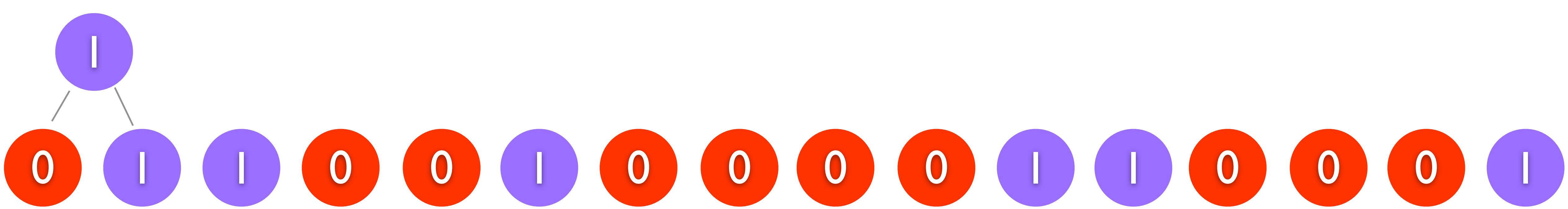

Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^+$

Use the two rules to construct tree:

binary rule — idempotent rule

Example (finite)

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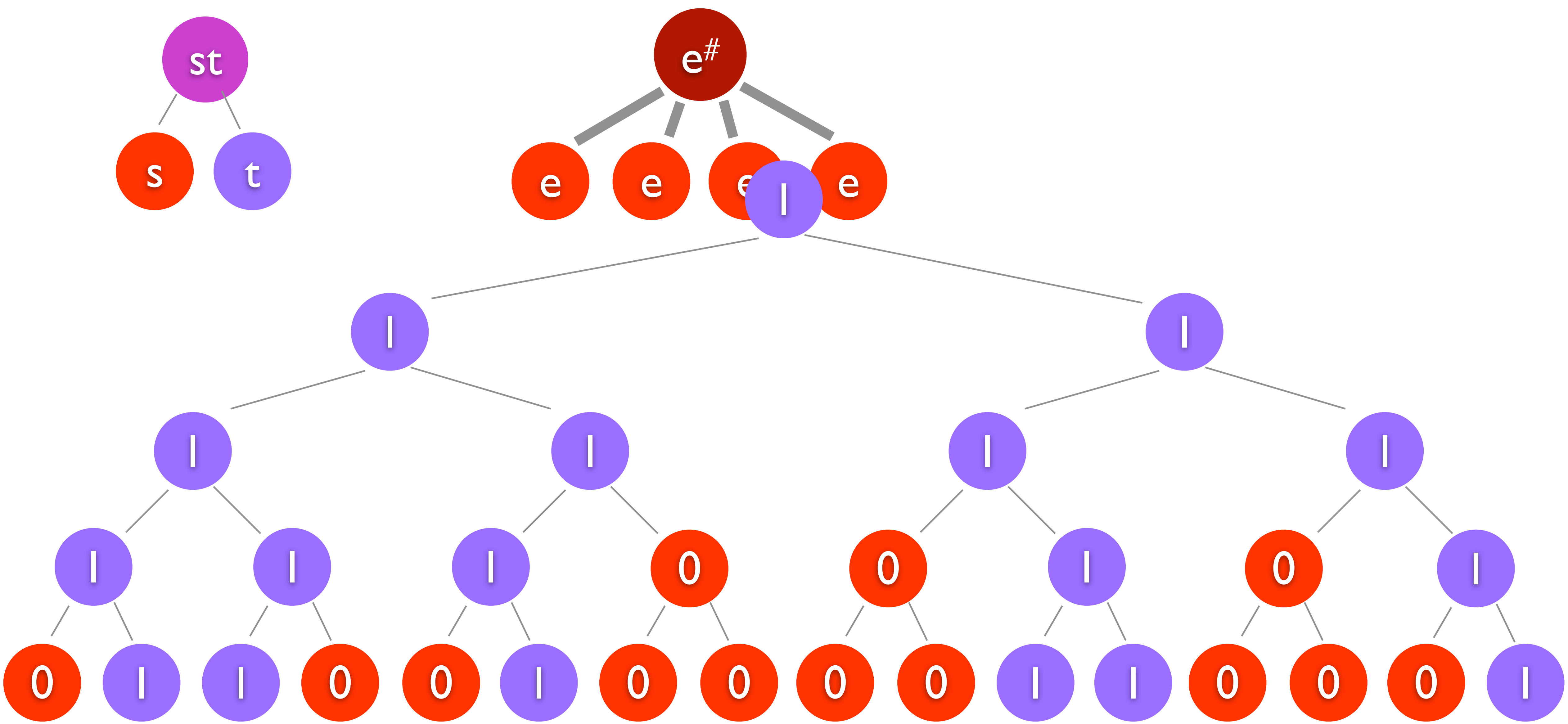


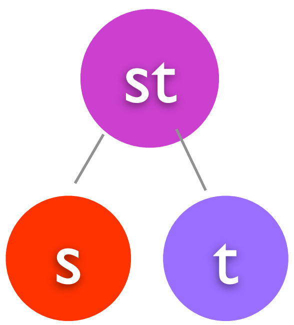
Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^{+}$

Use the two rules to construct tree:

binary rule



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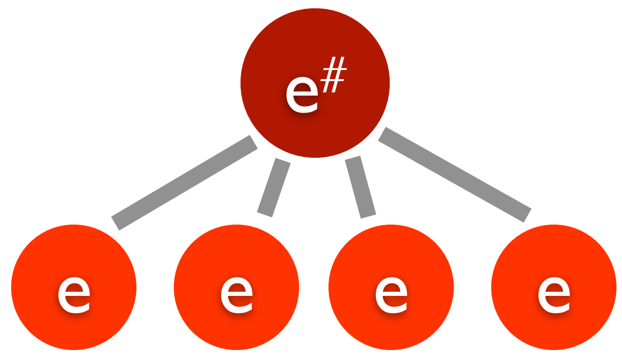

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Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^+$

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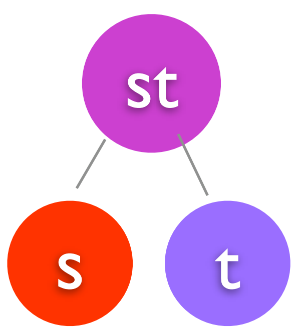

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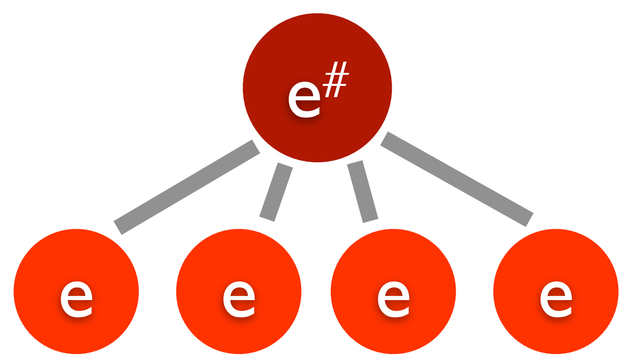

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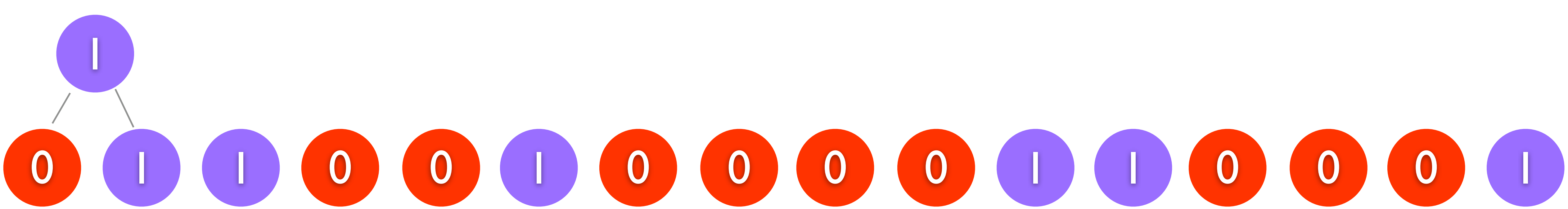


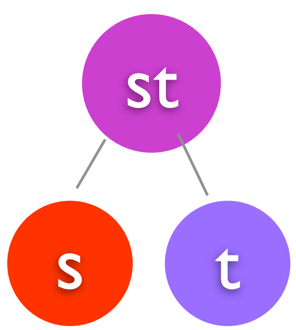
Simon's Factorization Theorem

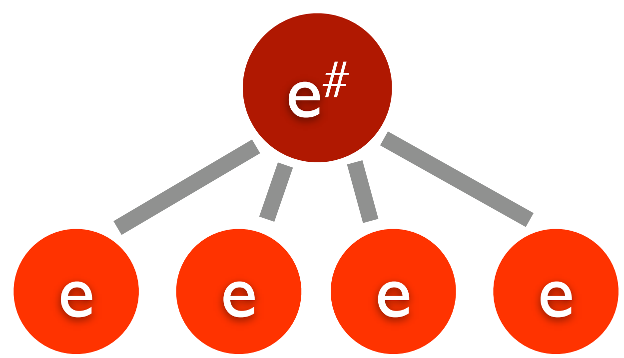
for semigroups with stabilization

Factorization tree of word $w \in S^+$

Use the two rules to construct tree:

binary rule　　　idempotent rule





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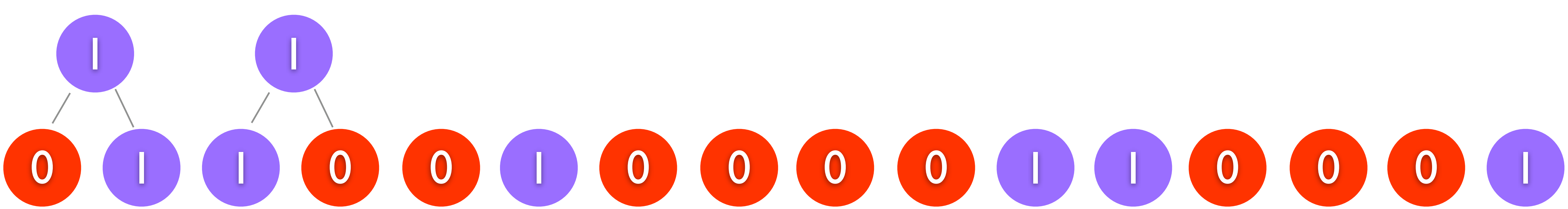

Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^+$

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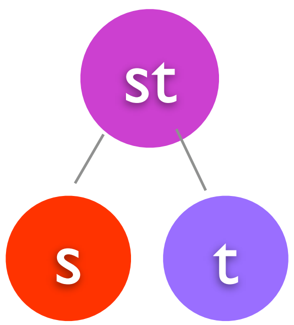

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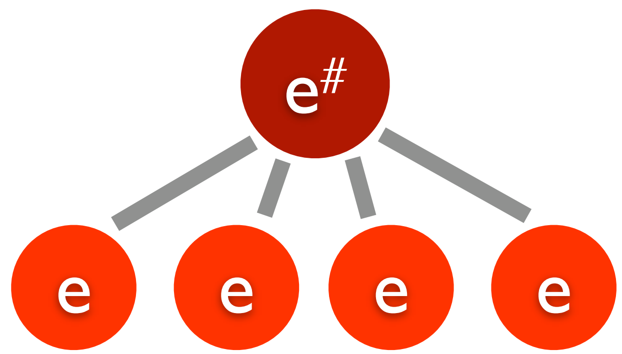

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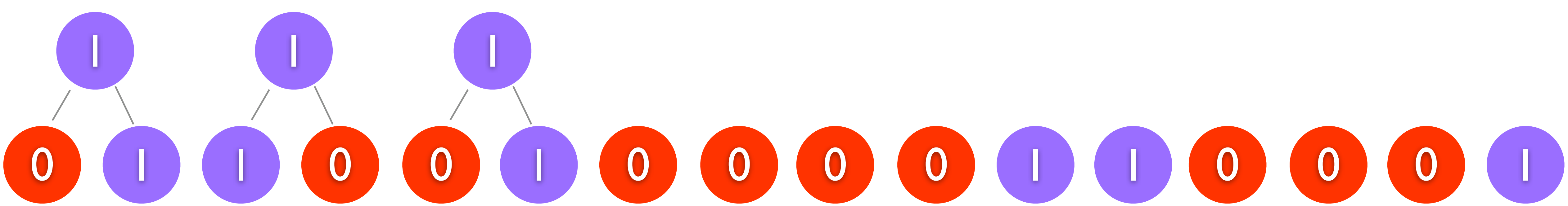

Simon's Factorization Theorem

for semigroups with stabilization

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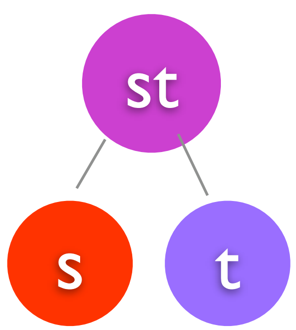

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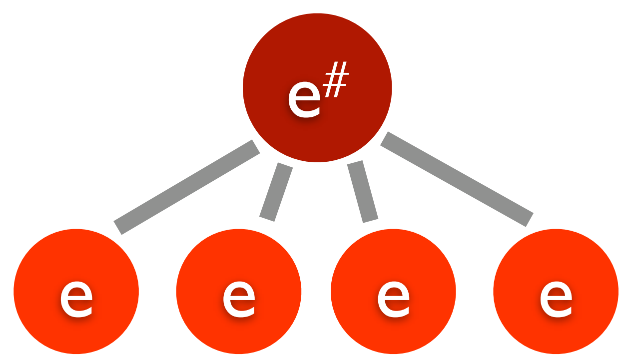

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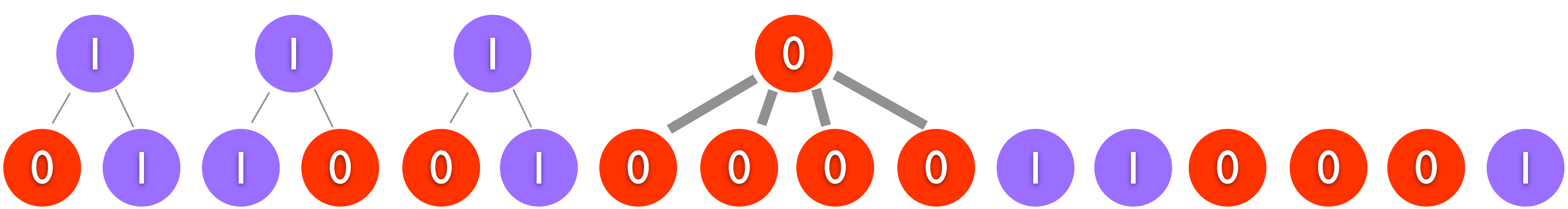

Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^+$

Use the two rules to construct tree:

binary rule | idempotent rule

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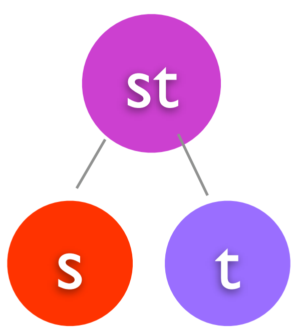
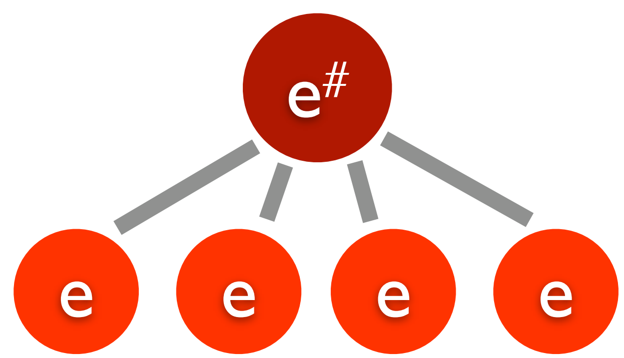
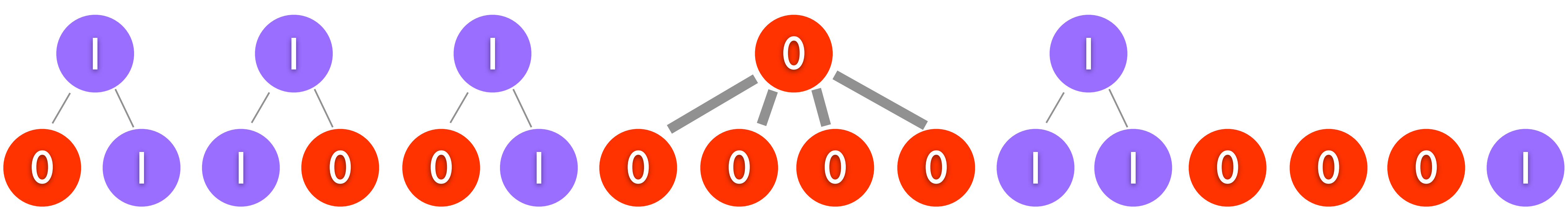






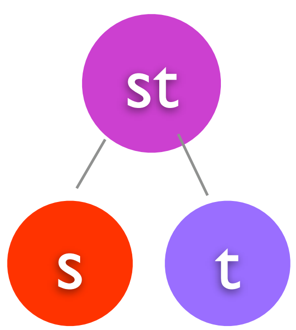
Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^+$

Use the two rules to construct tree:

binary rule



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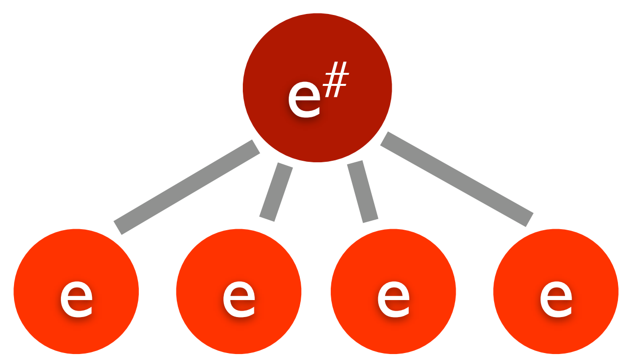

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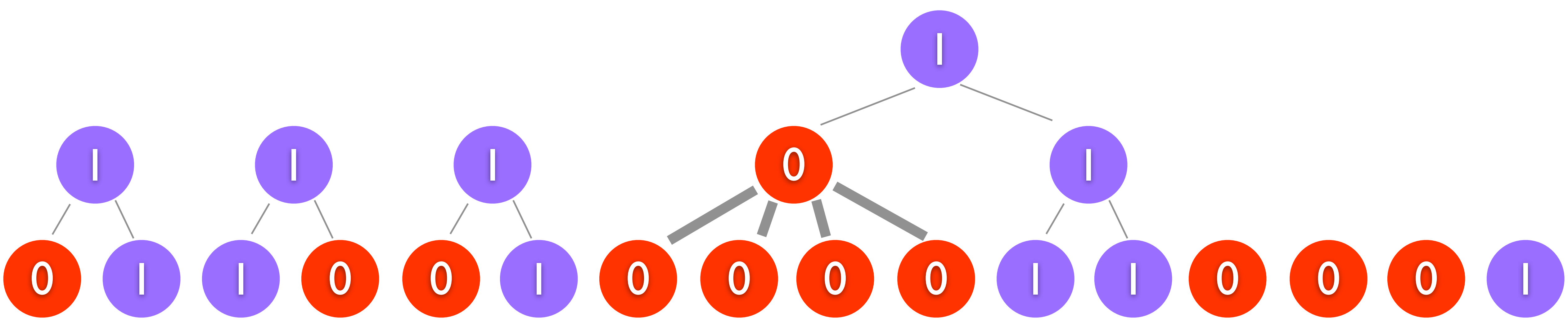

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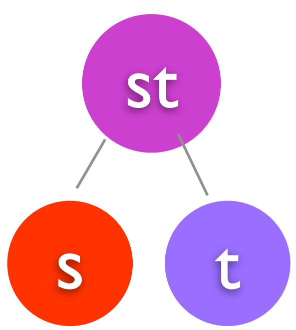

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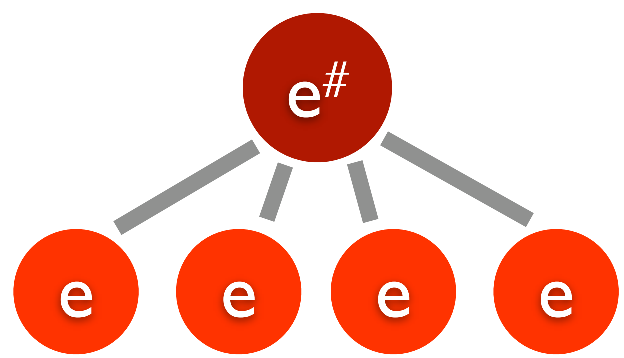

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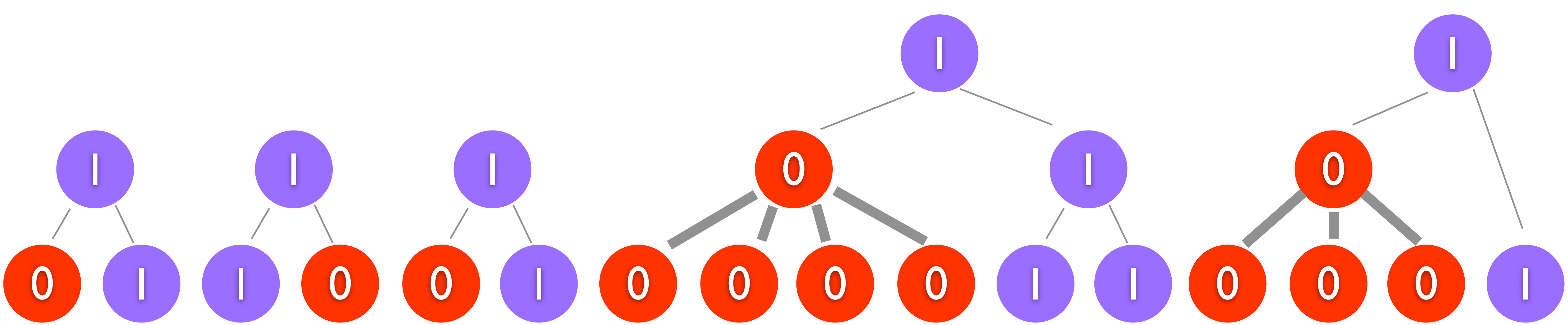

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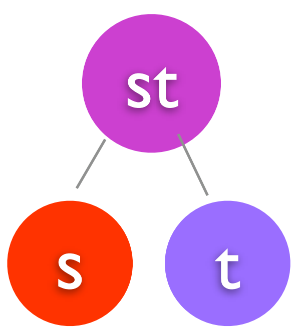

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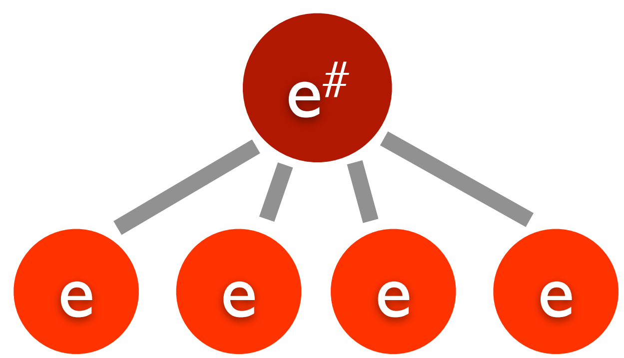

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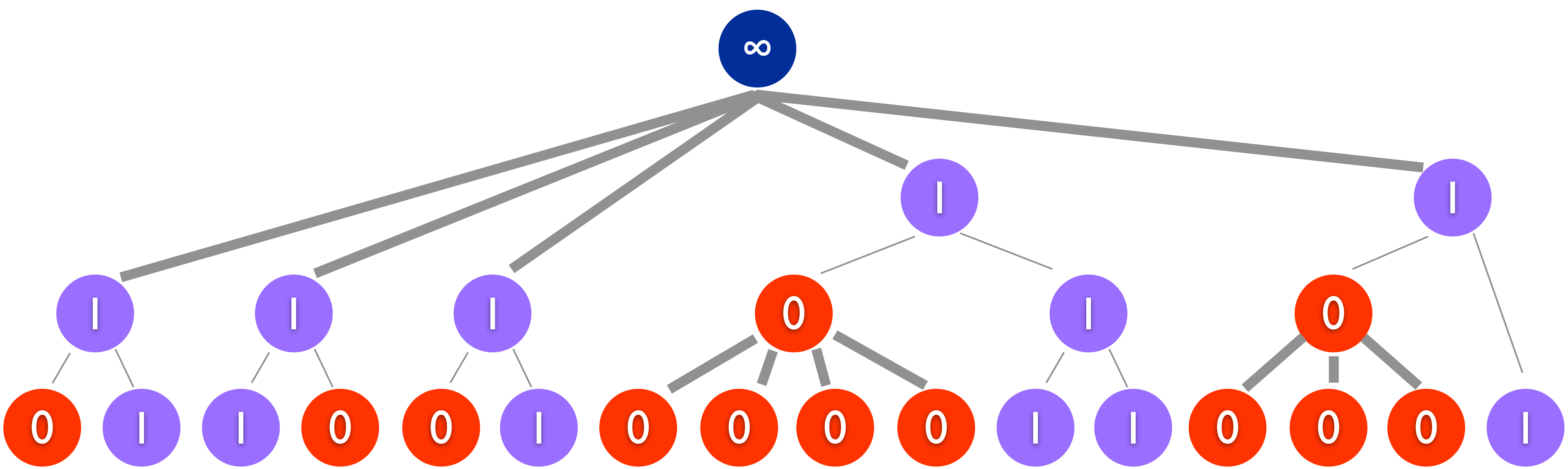

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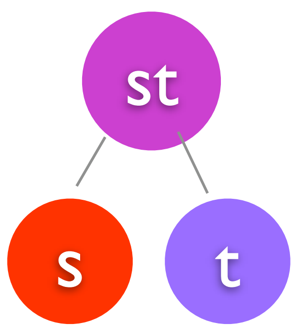

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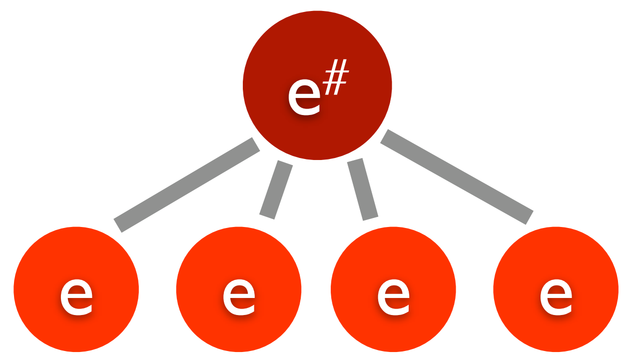

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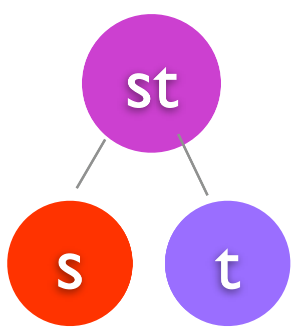

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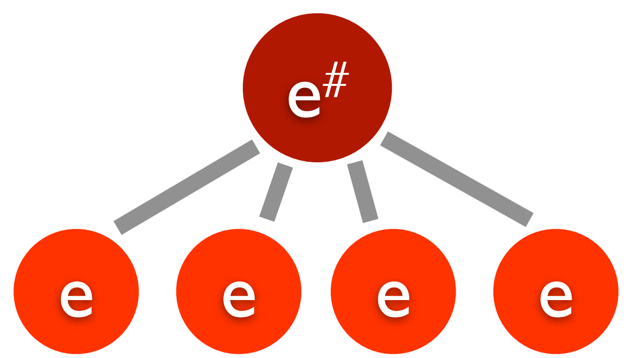

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Theorem. For any finite stabilization semigroup S and word $w \in S^+$ there exists a factorization tree over w of height $\leq 9|S|^2$.

More examples
of semigroups with stabilization

More examples
of semigroups with stabilization

Example (infinite)
$(\mathrm{M}_k T, \cdot, {}^\omega)$

More examples
of semigroups with stabilization

Example (infinite)
$(\mathrm{M}_k T, \cdot, {}^{\omega})$

Example (finite)
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More examples
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Example (infinite)
$(\mathrm{M}_k T, \cdot, {}^{\omega})$

$N{+}1, N{+}2, \ldots \to N$
$\longrightarrow$
α_N

Example (finite)
$(\mathrm{M}_k T_N, \cdot, {}^{\#})$

More examples
of semigroups with stabilization

Example (infinite)
$(M_k T, \cdot, ^{\omega})$

$N{+}1, N{+}2, \ldots \to N$

α_N

Example (finite)
$(M_k T_N, \cdot, ^{\#})$

y:

0	1	$\top$
$\top$	$\top$	1
1	$\top$	1

$\alpha_3(y)$:

0	1	$\top$
$\top$	$\top$	1
1	$\top$	1

y^{ω}:

0	1	2
2	3	4
1	2	3

$\alpha_3(y)^{\#} = \alpha_3(y^{\omega})$:

0	1	2
2	3	3
1	2	3

Lemma.

Lemma. Let a, b be matrices over the (min,+)-semiring.

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Let w be a word over a, b and let r $\in M_k T$ be the "real" product of w and let s $\in M_k T_N$ be the result of a factorization tree w.r.t α_N.

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a b b a a b a a a a b b a a a b

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a b b a a b a a a a b b a a a b = r

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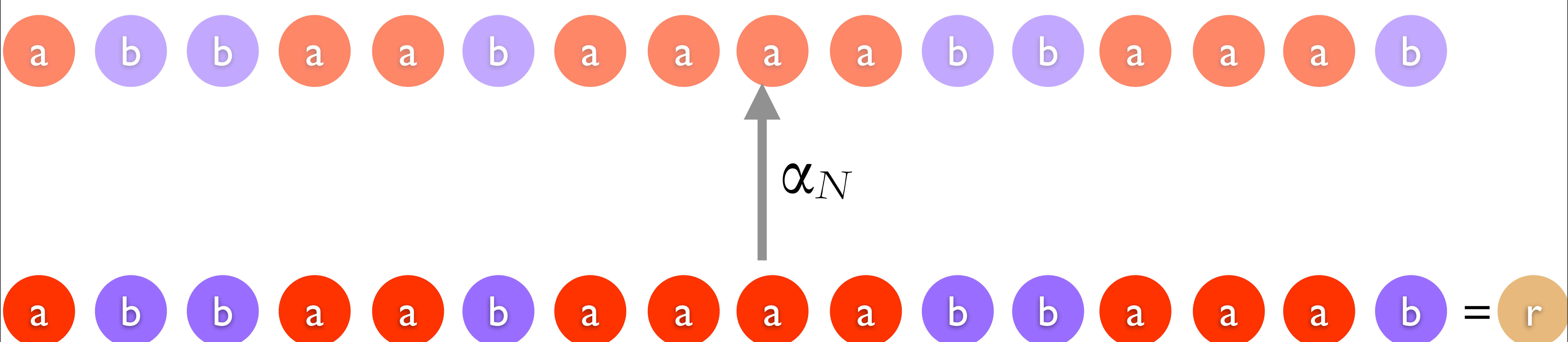

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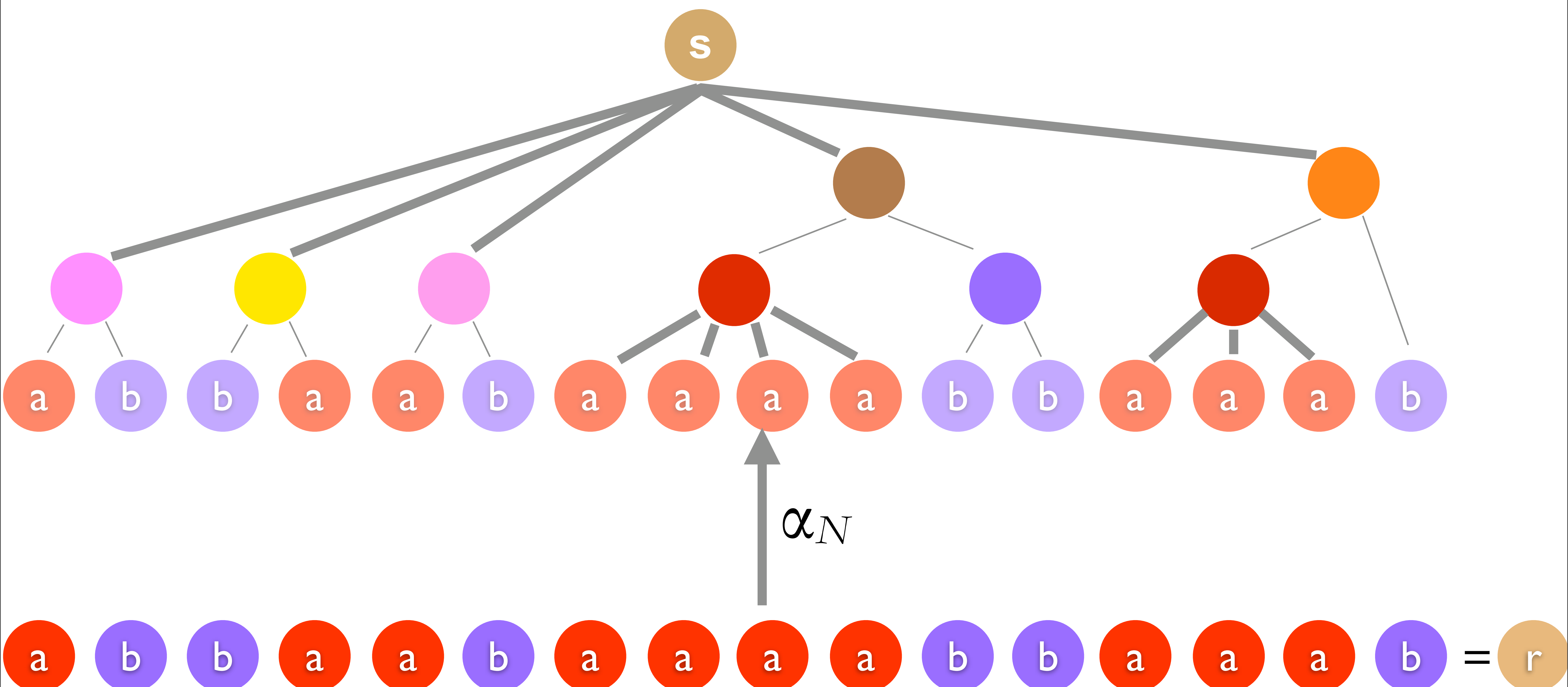
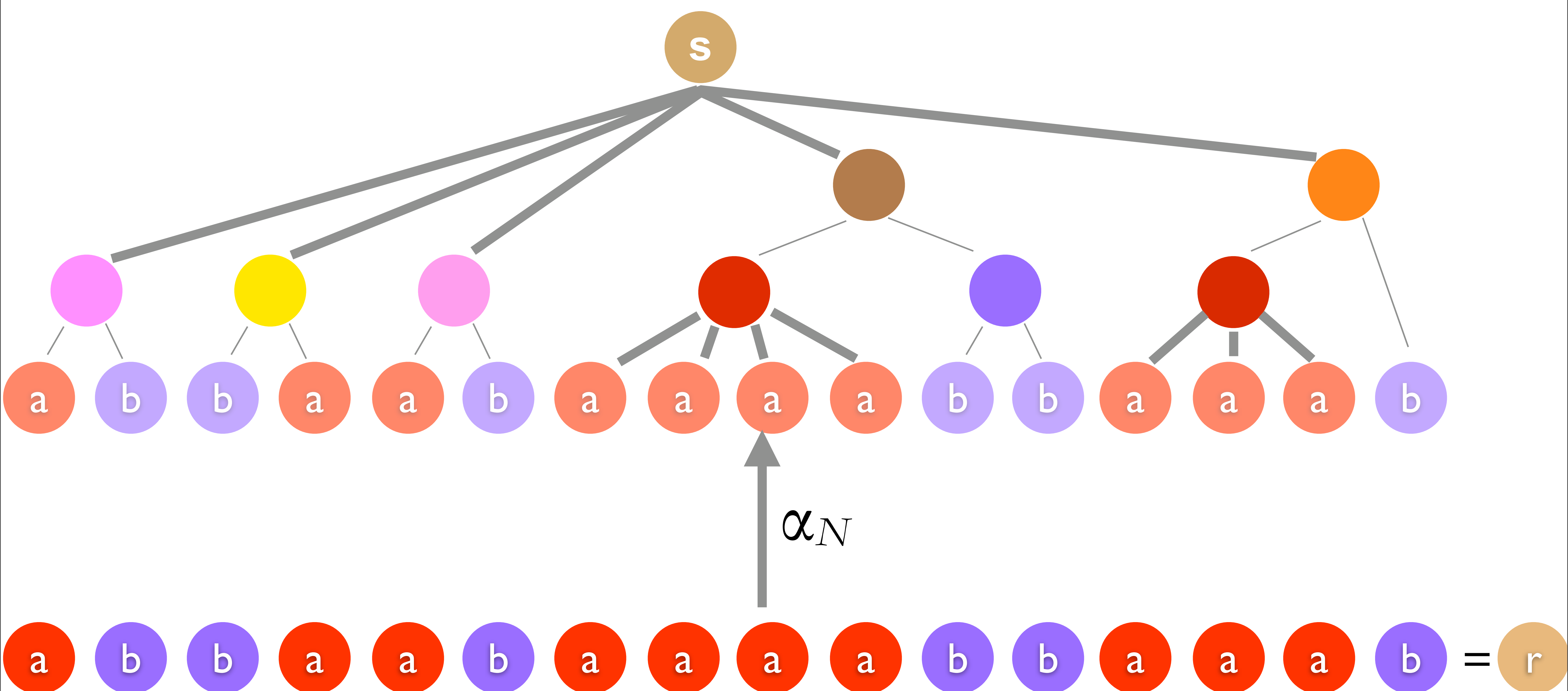


Lemma. Let a, b be matrices over the (min,+)-semiring.

Let w be a word over a, b and let $r \in M_k T$ be the "real" product of w and let $s \in M_k T_N$ be the result of a factorization tree w.r.t α_N.

Then: s r

- agree on values $\{0,1,\ldots,N\text{-}1,\top\}$
- if $s[i,j] = N$ then $N \leq r[i,j] \leq 2^h$



Theorem.

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Then

Theorem. Let *a*, *b* be matrices over the (min,+)-semiring.

Then *a* *b*

Theorem. Let a, b be matrices over the (min,+)-semiring.

Then $(a, b)^+$

Theorem. Let a, b be matrices over the (min,+)-semiring.

Then $\overline{(a, b)^{+}}$

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Then $\overline{(a, b)^{+}} =$

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Then $$\overline{(a, b)^{+}} = a \quad b$$

Theorem. Let a, b be matrices over the (min,+)-semiring.

Then $$\overline{(a, b)^+} = (a, b)^{+,\omega}$$

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$$\overline{(a, b)^{+}} \;\supseteq\; (a, b)^{+,\omega} \quad ✓$$

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$$\overline{(a, b)^{+}} \;\subseteq\; (a, b)^{+,\omega} \quad ?$$

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$$x, \in |x| < N$$

Theorem. Let a, b be matrices over the (min,+)-semiring.

Then $$\alpha_N\left(\overline{(a, b)^+}\right) = \alpha_N\left(((a, b))^{+,\omega}\right)$$

$\supseteq$ ✓

$\subseteq$?

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$\supseteq$ ✓ $\subseteq$?

$x, \in \overline{(a,b)^+}, |x| < N$

Let $r_1, r_2, r_3, \ldots \in (a, b)^+$

such that $x = \lim r_n$

Theorem. Let a, b be matrices over the (min,+)-semiring.

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$x, \in$ $|x| < N$

Let $r_1, r_2, r_3, \ldots \in (a,\, b)^{+}$

such that $x = \lim r_n$

Wlog, we can assume that

- $r_n[i,j] = x[i,j]$ if $x[i,j] \neq \infty$
- $r_n[i,j] > n$ if $x[i,j] = \infty$

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consider $\alpha_N : M_k T \to M_k T_N$

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finite semigroup
with stabilization
every word has
fact. forest of height h

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$$\alpha_N\left(\overline{((a, b))^{+}}\right) \overset{\supseteq\ \checkmark}{\underset{\subseteq\ ?}{=}} \alpha_N\left(((a, b))^{+,\omega}\right)$$

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Let $r_1, r_2, r_3, \ldots \in ((a, b))^{+}$

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$r = abbbbab....aaab$

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$r = abbbbab\ldots.aaab$

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result of fact. forest of height h

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finite semigroup with stabilization every word has fact. forest of height h

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finite semigroup with stabilization every word has fact. forest of height h

r = *abbbbab....aaab*

$s \in M_k T_N$ has no N's!!!

result of fact. forest of height h

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finite semigroup with stabilization every word has fact. forest of height h

$r = abbbbab\ldots.aaab$

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result of fact. forest of height h

$\alpha_N(x) = s$

- agree on values $\{0,1,\ldots,N\text{-}1,\top\}$
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Theorem. Let a, b be matrices over the (min,+)-semiring.

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finite semigroup with stabilization every word has fact. forest of height h

$r = abbbbab....aaab$

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result of fact. forest of height h

- agree on values $\{0,1,\ldots,N\text{-}1,\top\}$
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$$\alpha_N(x) = s \in \alpha_N\left((a, b)^{+,\omega}\right)$$

Plan

✓ 1. Introduction to the problem

✓ 2. Reduce emptiness of min-automata to the *finite section problem*, via a Ramsey-type theorem

3. Solve the finite section problem using Simon's Factorization Theorem

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✓ Thank you for your attention!