Deciding Emptiness of min-automata

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joint work with
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Min-automata

deterministic automata with counters
transitions invoke counter operations:

$c{:=}c+1$

$c{:=}\min(d,e)$

acceptance condition is a boolean combination of:

$\liminf(c) = \infty$
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"c tends to ∞"

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which says

„*there exist arbitrarily large (finite) sets X, satisfying* $\varphi(X)$"

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$\mathsf{U}X$ *"X is a block of a's"*

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WMSO + U	$\longleftrightarrow$	WMSO + R
max-automata	$\longleftrightarrow$	min-automata

$$
\begin{array}{ccc}
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Theorem. WMSO+U has the same expressive power as deterministic max-automata.

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max-automata min-automata

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What if we allow both U and R?

WMSO + U + R

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Equivalently: Nesting the quantifiers U and R does not contribute anything to the expressive power of WMSO.

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Theorem. Emptiness of max-automata is decidable.

Theorem. Emptiness of a boolean combination of min- and max-automata is decidable.

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In Example 1, $c:=c+1$ can be written as:

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a a a b b b a a b...

c_0	0	$\top$	1	$\top$	2
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$$
\begin{array}{cccccccccc}
 & a & a & a & b & b & b & a & a & b\ldots \\
c_0 & 0 & \top & 1 & \top & 2 & \top & & & \\
c_1 & \top & 1 & \top & 2 & \top & 2 & & &
\end{array}
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c_0	0	$\top$	1	$\top$	2	$\top$	2		
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	a	a	a	b	b	b	a	a	b...
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$$a\ a\ a\ b\ b\ b\ a\ a\ b \ldots$$

$$\begin{array}{llllllllll} c_0 & 0 & \top & 1 & \top & 2 & \top & 2 & \top & 3 \\ c_1 & \top & 1 & \top & 2 & \top & 2 & \top & 3 & \top \end{array}$$

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$$\begin{array}{lccccccccc} & & a & a & a & b & b & b & a & a\ b\ldots \\ c_0 & 0 & \top & 1 & \top & 2 & \top & 2 & \top & 3\ \top \\ c_1 & \top & 1 & \top & 2 & \top & 2 & \top & 3 & \top\ 3 \end{array}$$

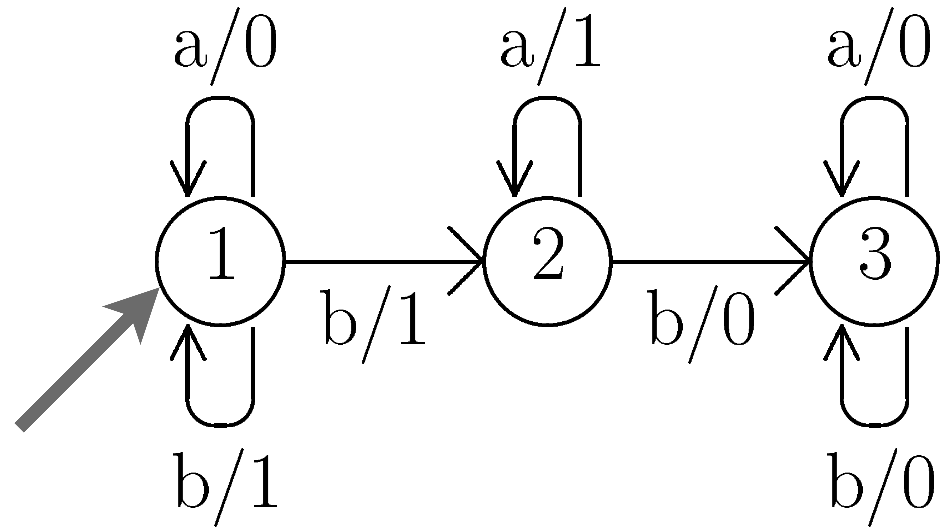
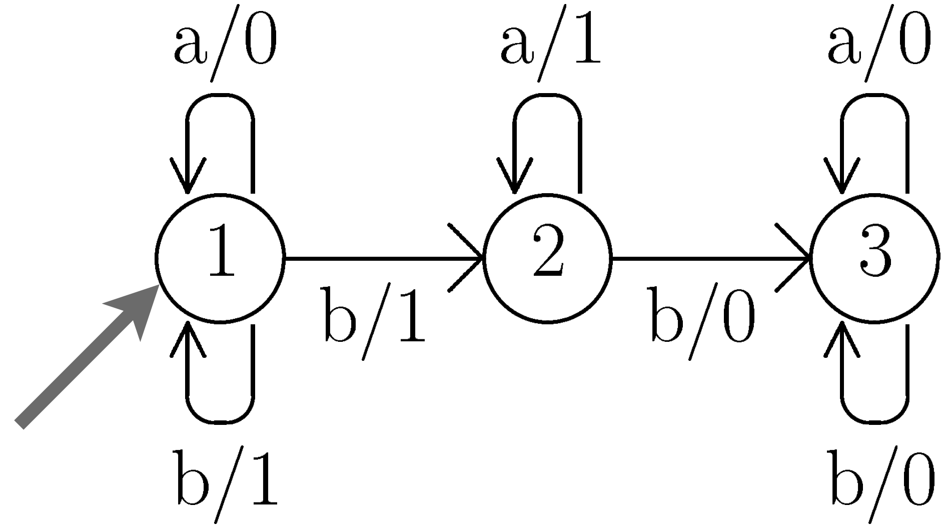


Figure 1: Example of a distance automaton A.



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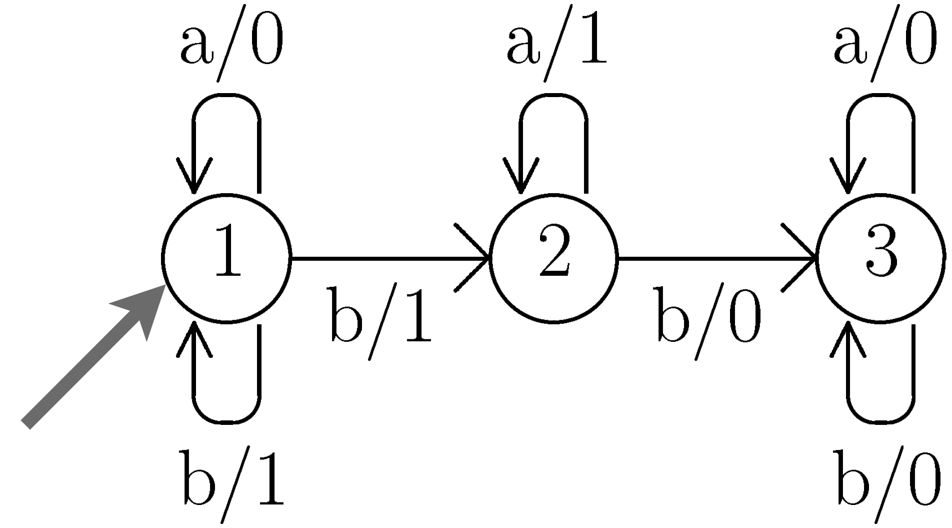
Example of a min-automaton *A.*



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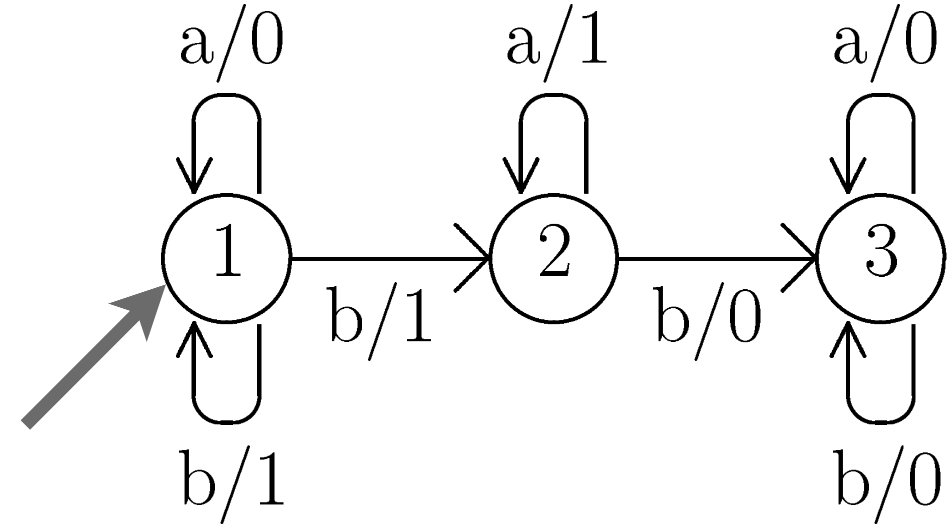
$$a: \begin{bmatrix} 0 & \top & \top \\ \top & 1 & \top \\ \top & \top & 0 \end{bmatrix} \qquad b: \begin{bmatrix} 1 & 1 & \top \\ \top & \top & 0 \\ \top & \top & 0 \end{bmatrix}$$



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$$a: \begin{array}{|ccc|} 0 & \top & \top \\ \top & 1 & \top \\ \top & \top & 0 \end{array} \qquad b: \begin{array}{|ccc|} 1 & 1 & \top \\ \top & \top & 0 \\ \top & \top & 0 \end{array}$$

$$abb: \begin{array}{|ccc|} 2 & 2 & 1 \\ \top & \top & 0 \\ \top & \top & 0 \end{array}$$

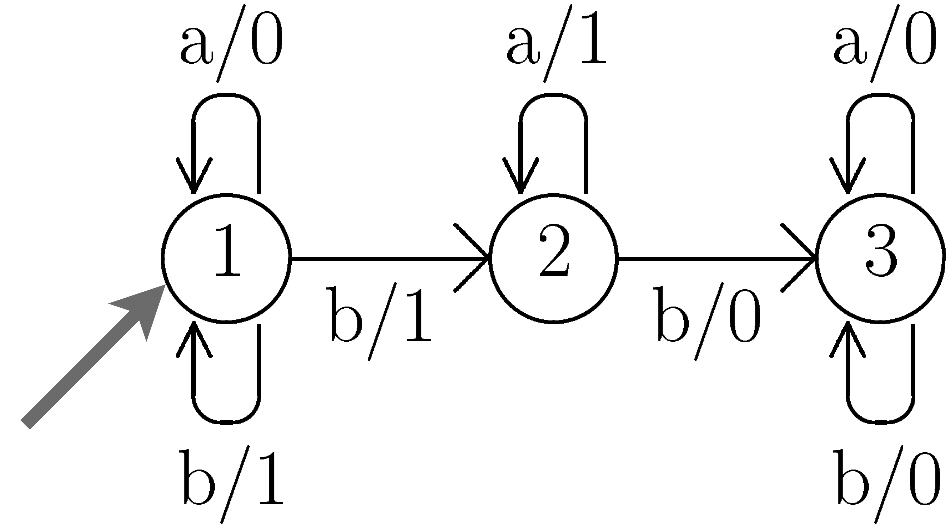

Figure 1: Example of a distance automaton A.

Example of a min-automaton *A*.

$$a: \quad \begin{bmatrix} 0 & \top & \top \\ \top & 1 & \top \\ \top & \top & 0 \end{bmatrix} \qquad b: \quad \begin{bmatrix} 1 & 1 & \top \\ \top & \top & 0 \\ \top & \top & 0 \end{bmatrix}$$

$$abb: \quad \begin{bmatrix} 2 & 2 & 1 \\ \top & \top & 0 \\ \top & \top & 0 \end{bmatrix}$$

Initial valuation: $\begin{bmatrix} 0 & \top & \top \end{bmatrix}$

Acceptance condition: $\neg \ \mathrm{liminf}(c_3) = \infty$

The tropical semiring

The tropical semiring

$$T = \{0, 1, 2, \ldots, \infty, \top\}$$

The tropical semiring

$$T = \{0,1,2,\ldots, \infty, \top\}$$

with operations +, min

ordered by $0<1<2<\ldots<\infty<\top$

where $\top + x = x + \top = \top$

The tropical semiring

$$T = \{0,1,2,\ldots,\ \infty,\ \top\}$$

with operations +, min

ordered by $0<1<2<\ldots<\infty<\top$

where $\top + x = x + \top = \top$

$$T_n = \{0,1,2,\ldots,n,\ \infty,\ \top\}$$

where $n+1=\infty$

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where $\top + x = x + \top = \top$

$$T_n = \{0,1,2,\ldots,n,\ \infty,\ \top\}$$

where $n+1=\infty$

$$\pi_n : T \to T_n$$

maps $n+1$, $n+2$, ... to ∞

is a homomorphism of semirings

The tropical semiring

$$T = \{0,1,2,\ldots, \infty, \top\}$$

with operations +, min

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$$\pi_n : T_m \to T_n \text{ for } m>n$$

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The tropical semiring

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with operations +, min

ordered by $0 < 1 < 2 < \ldots < \infty < \top$

where $\top + x = x + \top = \top$

$M_k T$ – *k by k* matrices over T

with matrix multiplication

$$T_n = \{0, 1, 2, \ldots, n, \infty, \top\}$$

where $n+1=\infty$

$M_k T_n$ – *k by k* matrices over T_n

with matrix multiplication

$$\pi_n : T \to T_n$$

$$\pi_n : T_m \to T_n \text{ for } m > n$$

maps $n+1$, $n+2$, ... to ∞

is a homomorphism of semirings

$$\pi_n : M_k T \to M_k T_n$$

$$\pi_n : T_m \to T_n \text{ for } m > n$$

maps $n+1$, $n+2$, ... to ∞

is a homomorphism of semirings

The tropical semiring

$T = \{0,1,2,\ldots, \infty, \top\}$
with operations +, min
ordered by $0<1<2<\ldots<\infty<\top$
where $\top + x = x + \top = \top$

$M_k T$ – *k by k* matrices over T
with matrix multiplication

$T_n = \{0,1,2,\ldots,n, \infty, \top\}$
where $n+1=\infty$

$M_k T_n$ – *k by k* matrices over T_n
with matrix multiplication

$\pi_n : T \to T_n$

$\pi_n : T_m \to T_n$ for $m>n$

maps $n+1$, $n+2$, ... to ∞

is a homomorphism of semirings

$\pi_n : M_k T \to M_k T_n$

$\pi_n : T_m \to T_n$ for $m>n$

maps $n+1$, $n+2$, ... to ∞

is a homomorphism of semirings

$$\begin{array}{|ccc|} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{array} \xrightarrow{\pi_6} \begin{array}{|ccc|} 3 & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{array} \xrightarrow{\pi_2} \begin{array}{|ccc|} \infty & \infty & 1 \\ 0 & \infty & 1 \\ \infty & \infty & \infty \end{array}$$

Profinite monoid

Profinite monoid

$$S_0 \longleftarrow S_1 \longleftarrow S_2 \longleftarrow S_3 \;\cdots\; S_7 \;\cdots\; S_{32} \;\cdots\; S_{1000} \;\cdots$$

Profinite monoid

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \dashleftarrow S_7 \dashleftarrow S_{32} \dashleftarrow S_{1000} \dashleftarrow \cdots$$

preserve multiplication

Profinite monoid

$$M_3 T_0 \leftarrow M_3 T_1 \leftarrow M_3 T_2 \leftarrow M_3 T_3 \leftarrow \cdots$$

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots\ S_7 \leftarrow \cdots\ S_{32} \leftarrow \cdots\ S_{1000} \leftarrow \cdots$$

preserve multiplication

Profinite monoid

$$S_0 \longleftarrow S_1 \longleftarrow S_2 \longleftarrow S_3 \leftarrow\cdots\; S_7 \leftarrow\cdots\; S_{32} \leftarrow\cdots\; S_{1000} \leftarrow\cdots\cdots$$

preserve multiplication

Profinite monoid

3	∞	1
0	∞	1
2	∞	∞

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots S_7 \leftarrow \cdots S_{32} \leftarrow \cdots S_{1000} \leftarrow \cdots\cdots$$

preserve multiplication

Profinite monoid

$$\begin{bmatrix} \infty & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{bmatrix} \quad \begin{bmatrix} 3 & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{bmatrix}$$

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots S_7 \leftarrow \cdots S_{32} \leftarrow \cdots S_{1000} \leftarrow \cdots$$

preserve multiplication

Profinite monoid

$$\begin{matrix} \infty & \infty & 1 \\ 0 & \infty & 1 \\ \infty & \infty & \infty \end{matrix} \qquad \begin{matrix} \infty & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{matrix} \qquad \begin{matrix} 3 & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{matrix}$$

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \dashleftarrow S_7 \dashleftarrow S_{32} \dashleftarrow S_{1000} \dashleftarrow \cdots$$

preserve multiplication

Profinite monoid

$$\begin{bmatrix} \infty & \infty & \infty \\ 0 & \infty & \infty \\ \infty & \infty & \infty \end{bmatrix} \quad \begin{bmatrix} \infty & \infty & 1 \\ 0 & \infty & 1 \\ \infty & \infty & \infty \end{bmatrix} \quad \begin{bmatrix} \infty & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{bmatrix} \quad \begin{bmatrix} 3 & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{bmatrix}$$

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots\; S_7 \leftarrow \cdots\; S_{32} \leftarrow \cdots\; S_{1000} \leftarrow \cdots$$

preserve multiplication

Profinite monoid

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots S_7 \leftarrow \cdots S_{32} \leftarrow \cdots S_{1000} \leftarrow \cdots\cdots$$

preserve multiplication

Profinite monoid

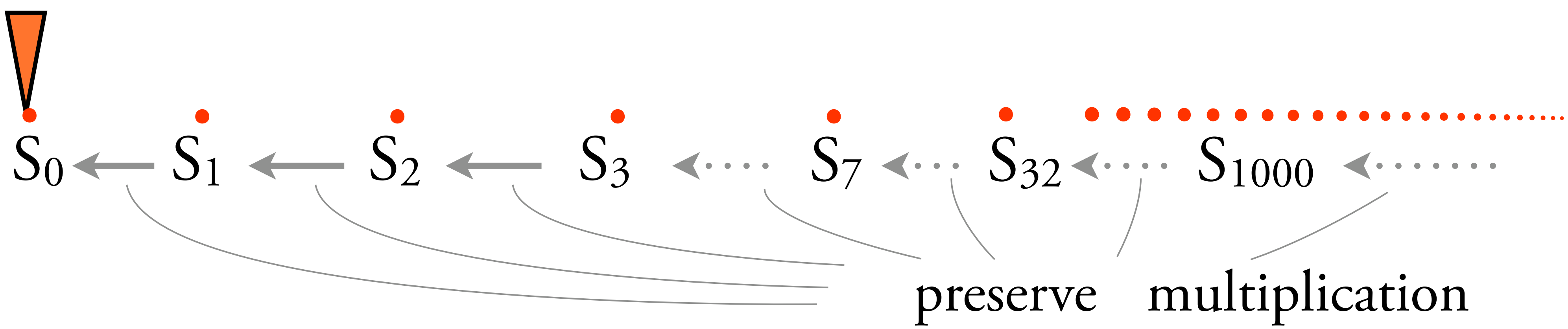
$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots S_7 \leftarrow \cdots S_{32} \leftarrow \cdots S_{1000} \leftarrow \cdots$$

preserve multiplication

Profinite monoid

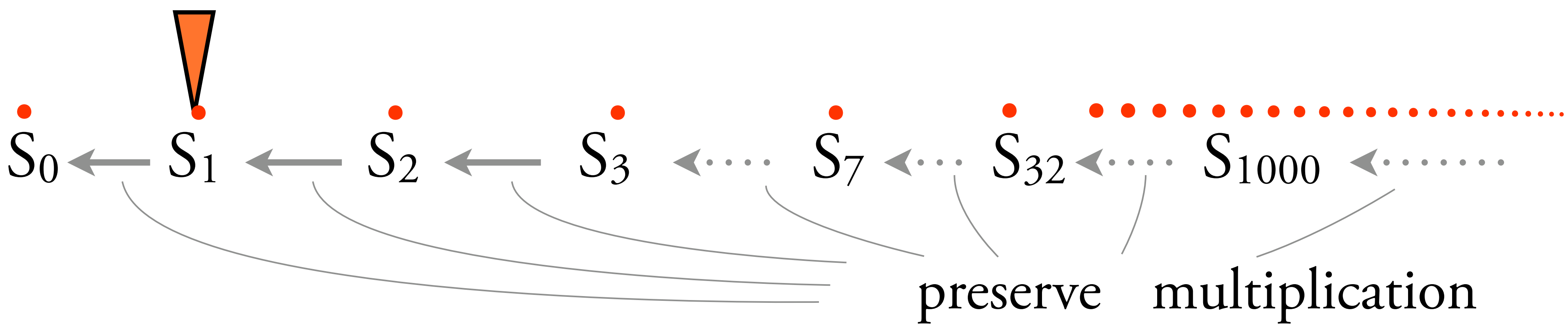
0 -approximation:

$$\begin{matrix} \infty & \infty & \infty \\ 0 & \infty & \infty \\ \infty & \infty & \infty \end{matrix}$$



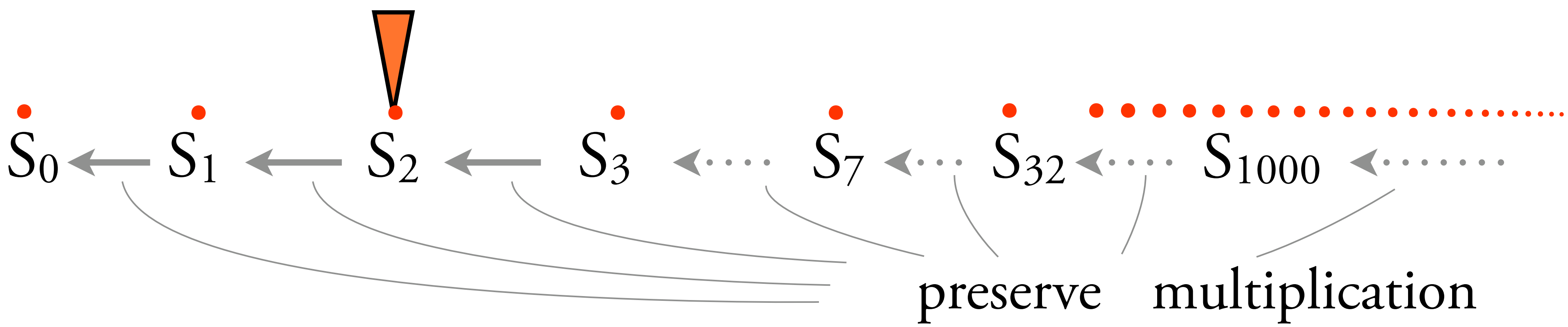
Profinite monoid

1 -approximation: $\begin{array}{ccc} \infty & \infty & 1 \\ 0 & \infty & 1 \\ \infty & \infty & \infty \end{array}$



Profinite monoid

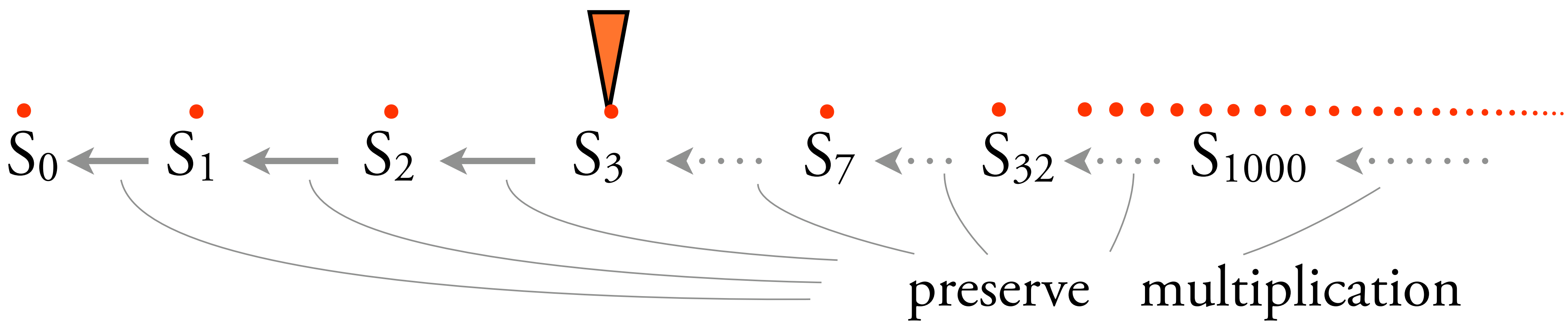
2 -approximation: $\begin{matrix} \infty & \infty & 1 \\ 0 & \infty & 1 \\ 2 & \infty & \infty \end{matrix}$



Profinite monoid

3 -approximation:

3	∞	1
0	∞	1
2	∞	∞



Profinite monoid

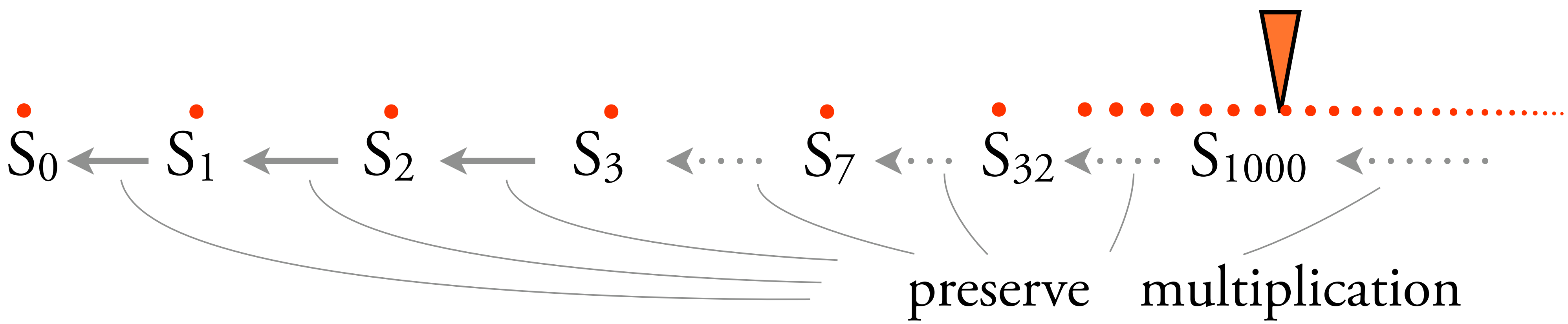
7 -approximation: $\begin{array}{ccc} 3 & \infty & 1 \\ 0 & \infty & 1 \\ 2 & 7 & \infty \end{array}$

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots S_7 \leftarrow \cdots S_{32} \leftarrow \cdots S_{1000} \leftarrow \cdots$$

preserve multiplication

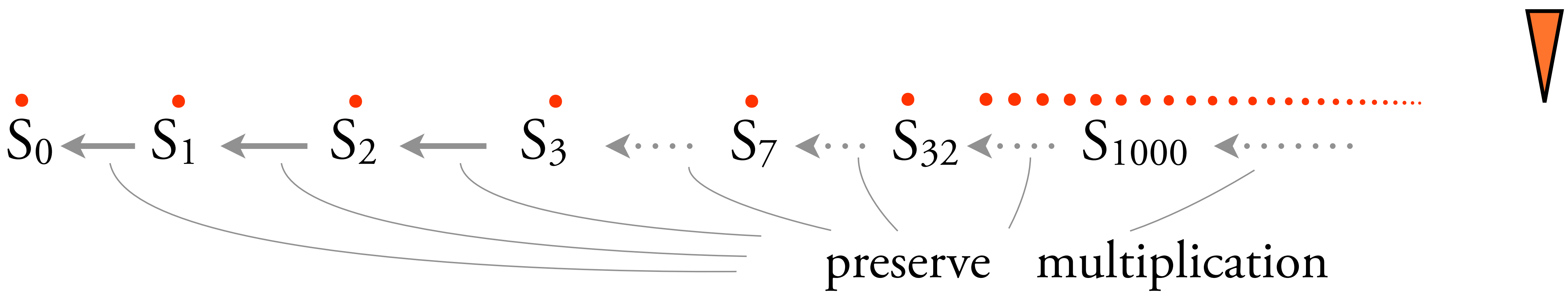
Profinite monoid

1000 -approximation: $\begin{array}{ccc} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{array}$



Profinite monoid

3	32	1
0	11	1
2	7	∞



Profinite monoid

3	32	1
0	11	1
2	7	∞

$$S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow S_3 \leftarrow \cdots S_7 \leftarrow \cdots S_{32} \leftarrow \cdots S_{1000} \leftarrow \cdots$$

preserve multiplication

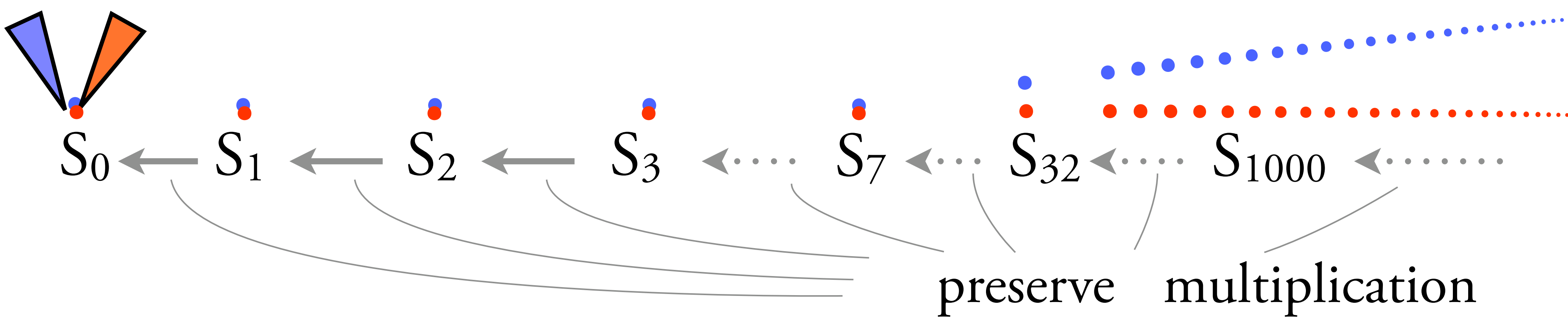
Metric

Two elements are close if only an approx. with high threshold can distinguish them

Profinite monoid

3	50	1
0	11	1
2	7	∞

3	32	1
0	11	1
2	7	∞

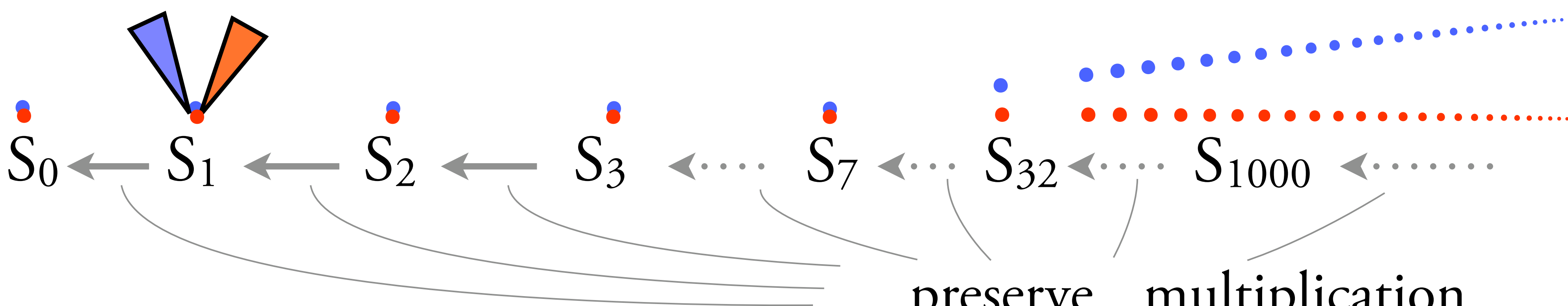


Metric

Two elements are close if only an approx. with high threshold can distinguish them

Profinite monoid

$$\begin{array}{ccc} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{array} \qquad \begin{array}{ccc} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{array}$$



Metric

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2	7	∞

3	32	1
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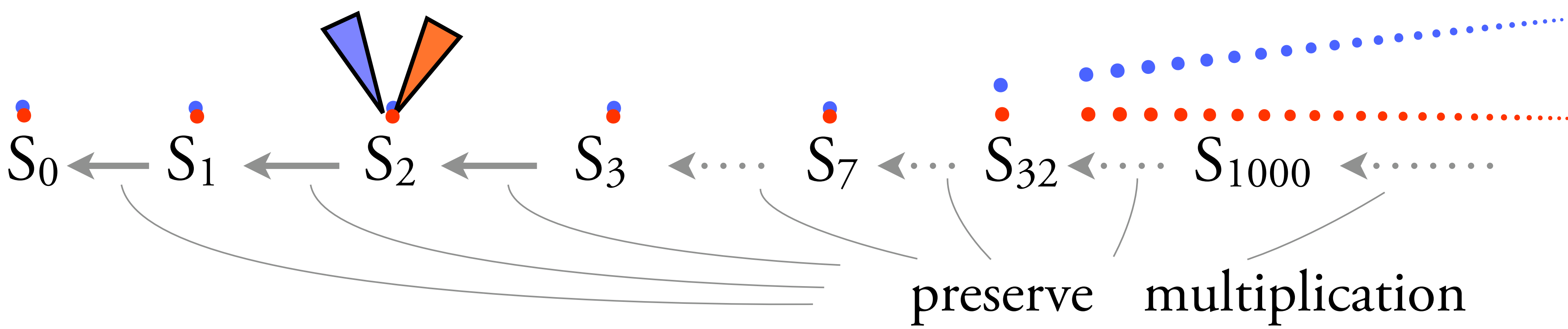
2	7	∞

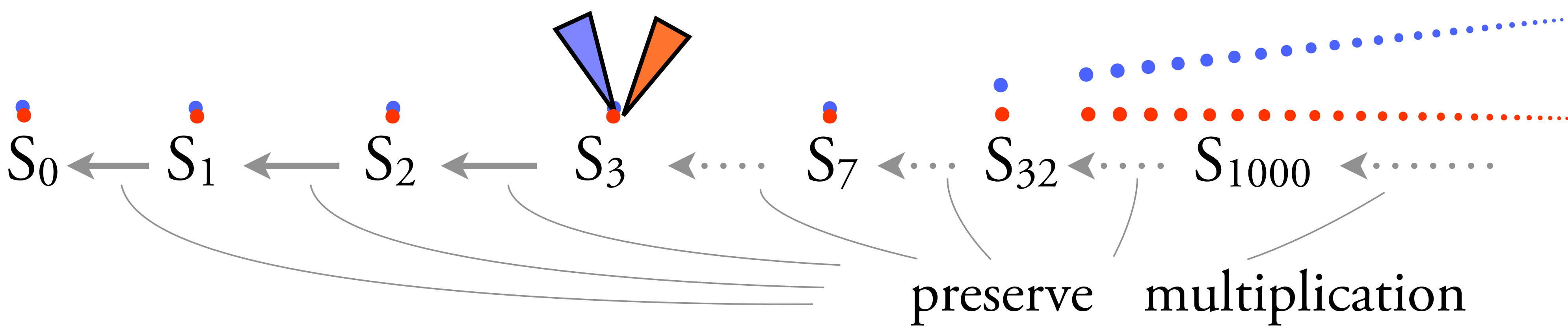


Metric

Two elements are close if only an approx. with high threshold can distinguish them

Profinite monoid

$$\begin{matrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix} \qquad \begin{matrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix}$$



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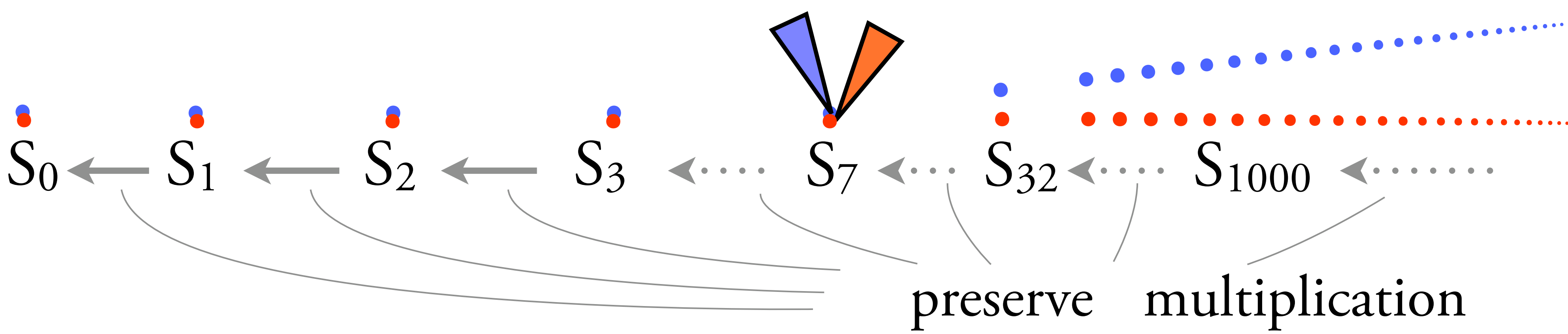

Metric

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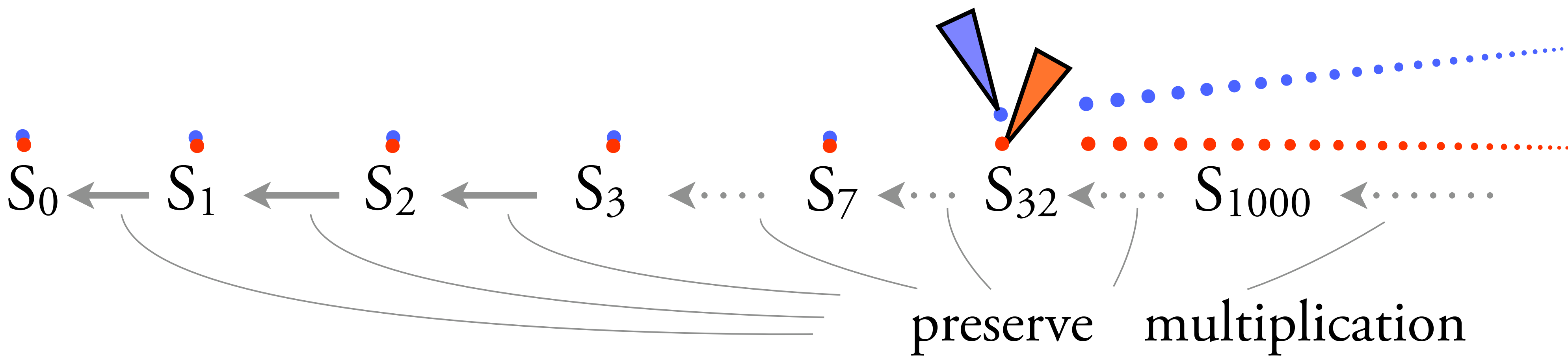

Metric

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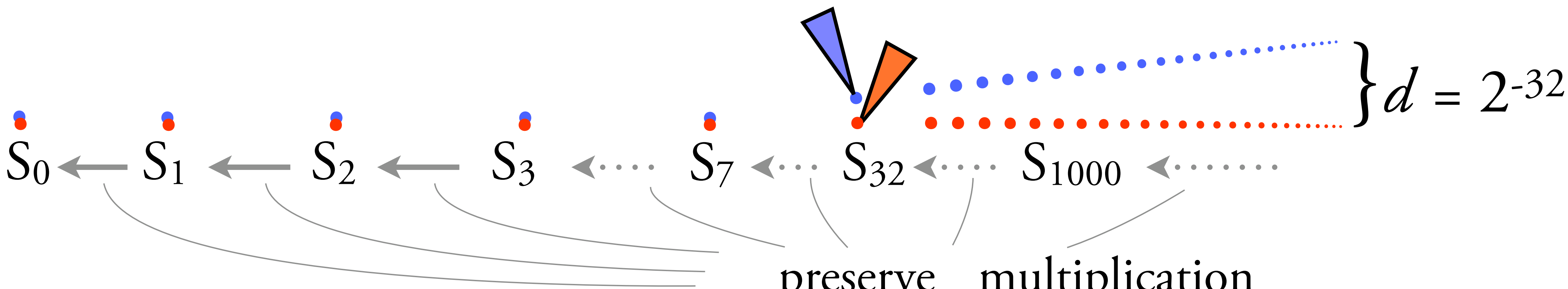
Profinite monoid

$$\begin{matrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix} \qquad \begin{matrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix}$$



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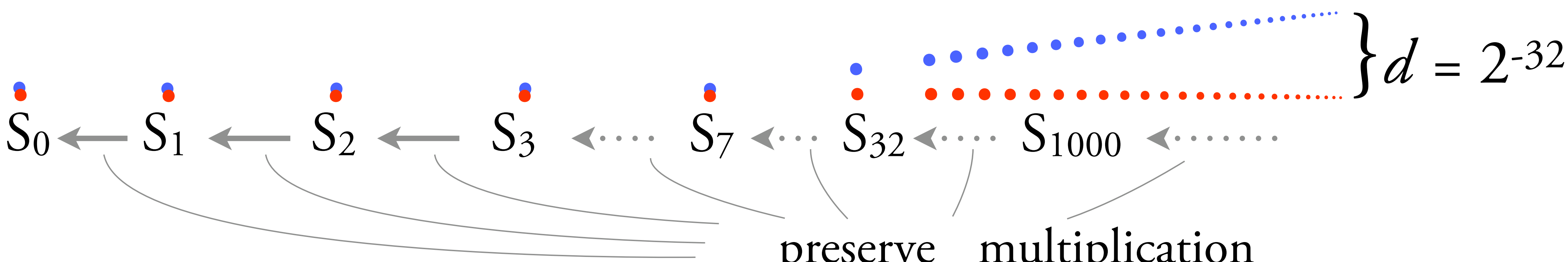
$$\begin{matrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix} \qquad\qquad \begin{matrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix}$$



Metric

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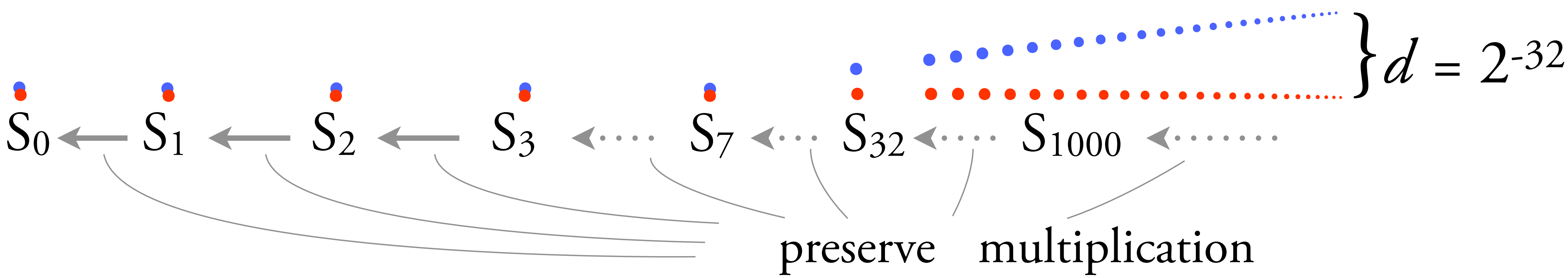


Metric

Two elements are close if only an approx. with high threshold can distinguish them

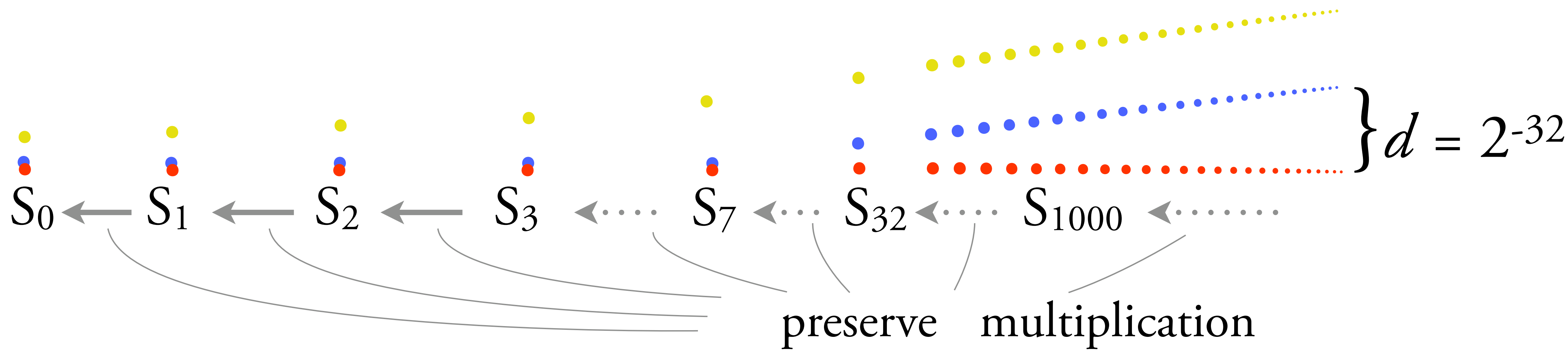
Multiplication

The n-approximation of $x \cdot y$ is the product of their n-approximations.

we again obtain a sequence consistent with the mappings

Profinite monoid

$$\begin{bmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{bmatrix} \cdot \begin{bmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{bmatrix} = \begin{bmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{bmatrix}$$



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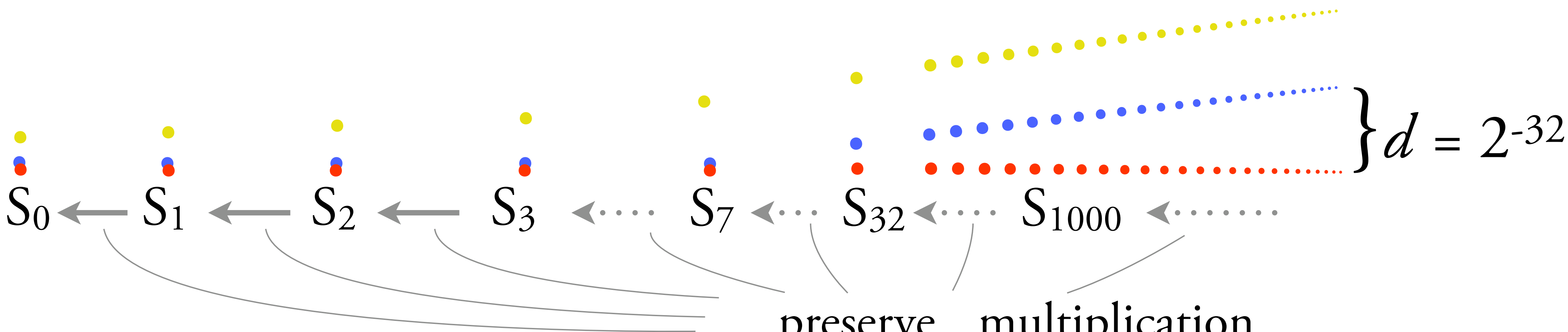


Metric

Two elements are close if only an approx. with high threshold can distinguish them

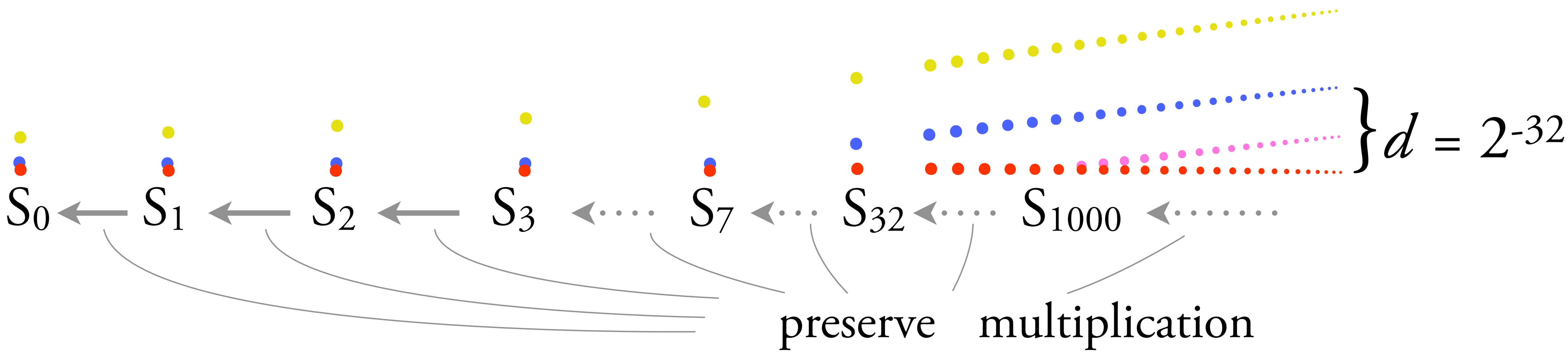
Multiplication *is continuous*

The n-approximation of $x{\cdot}y$ is the product of their n-approximations.

we again obtain a sequence consistent with the mappings

Profinite monoid

$$\begin{pmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} \cdot \begin{pmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} = \begin{pmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{pmatrix}$$



Metric

Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*

The n-approximation of $x{\cdot}y$ is the product of their n-approximations.

we again obtain a sequence consistent with the mappings

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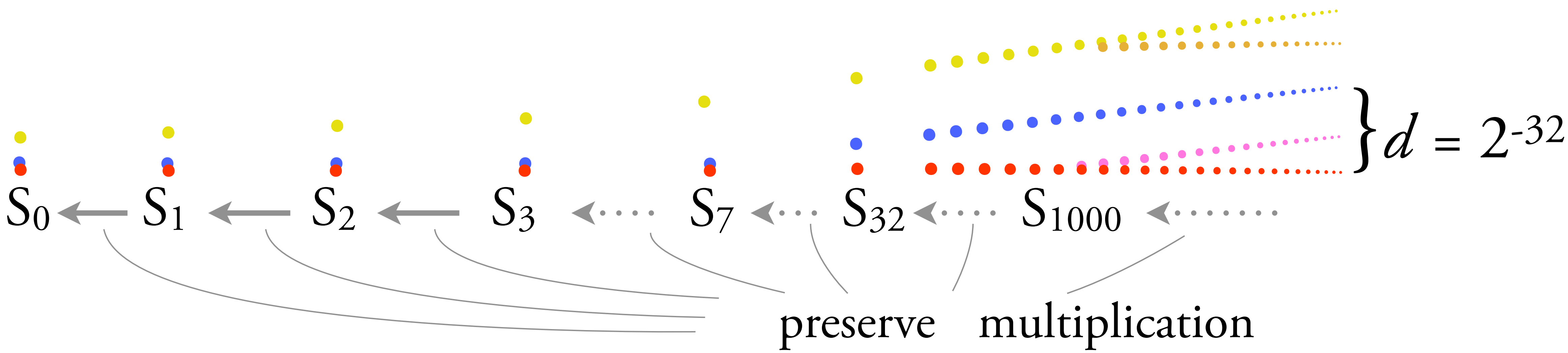
$$\begin{pmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} \cdot \begin{pmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} = \begin{pmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{pmatrix}$$



Metric

Two elements are close if only an approx. with high threshold can distinguish them

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The n-approximation of $x{\cdot}y$ is the product of their n-approximations.

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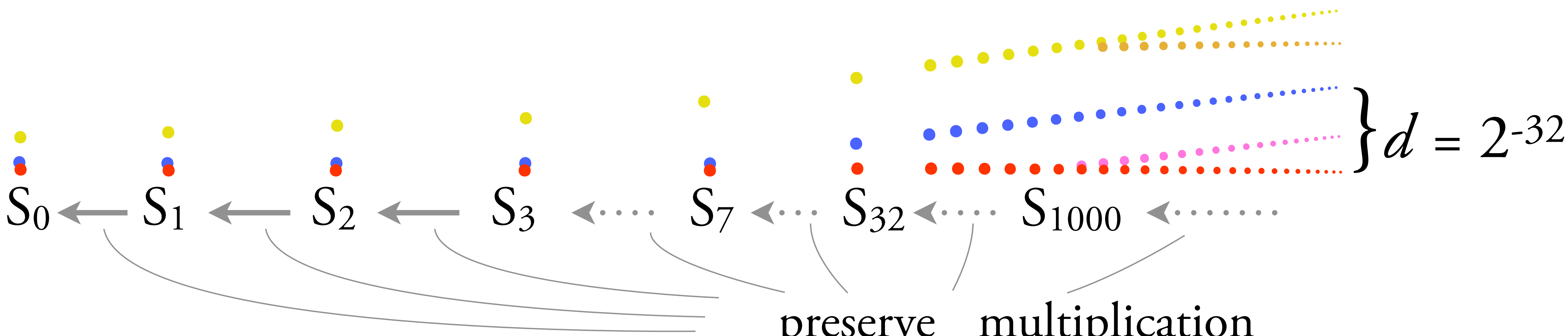


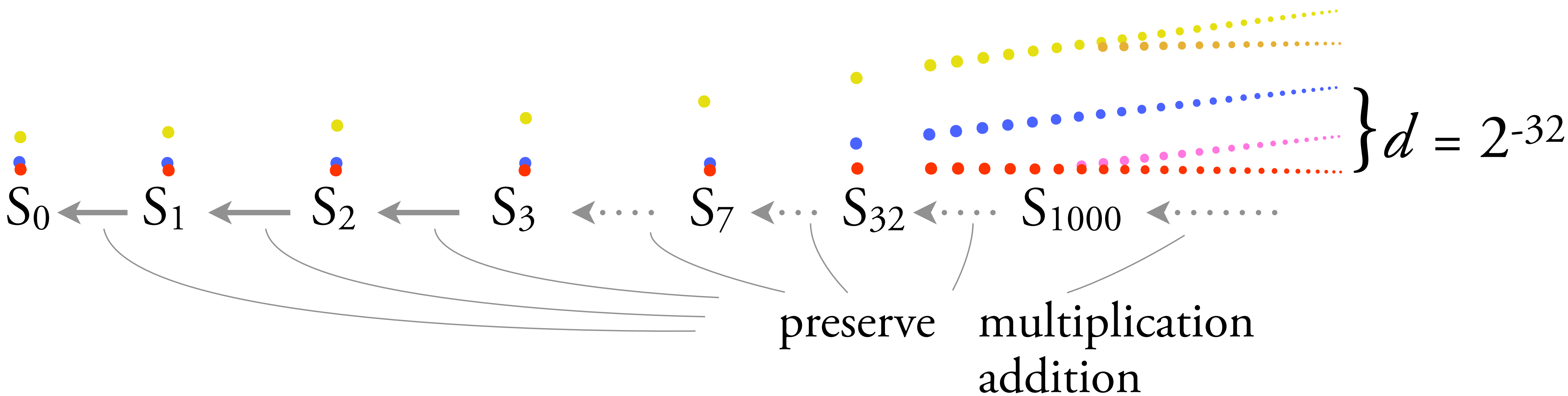
Metric

Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*

Profinite monoid

$$\begin{pmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} \cdot \begin{pmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} = \begin{pmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{pmatrix}$$

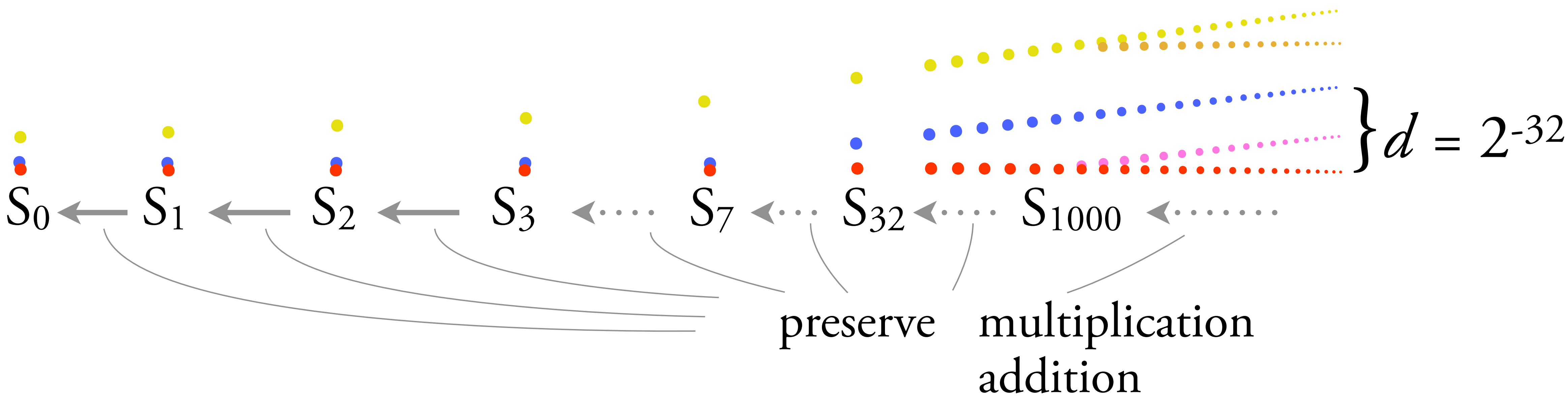


Metric
  Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*

Profinite monoid

$$\begin{matrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix} \cdot \begin{matrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{matrix} = \begin{matrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{matrix}$$



Metric

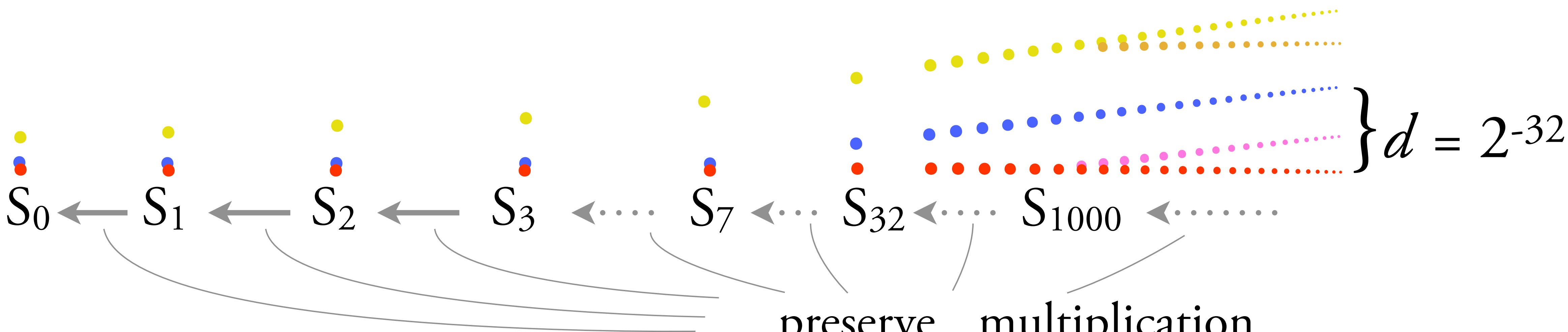
Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*

addition

Profinite monoid

$$\begin{pmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} \cdot \begin{pmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} = \begin{pmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{pmatrix}$$



preserve multiplication
addition
idempotent power (ω)

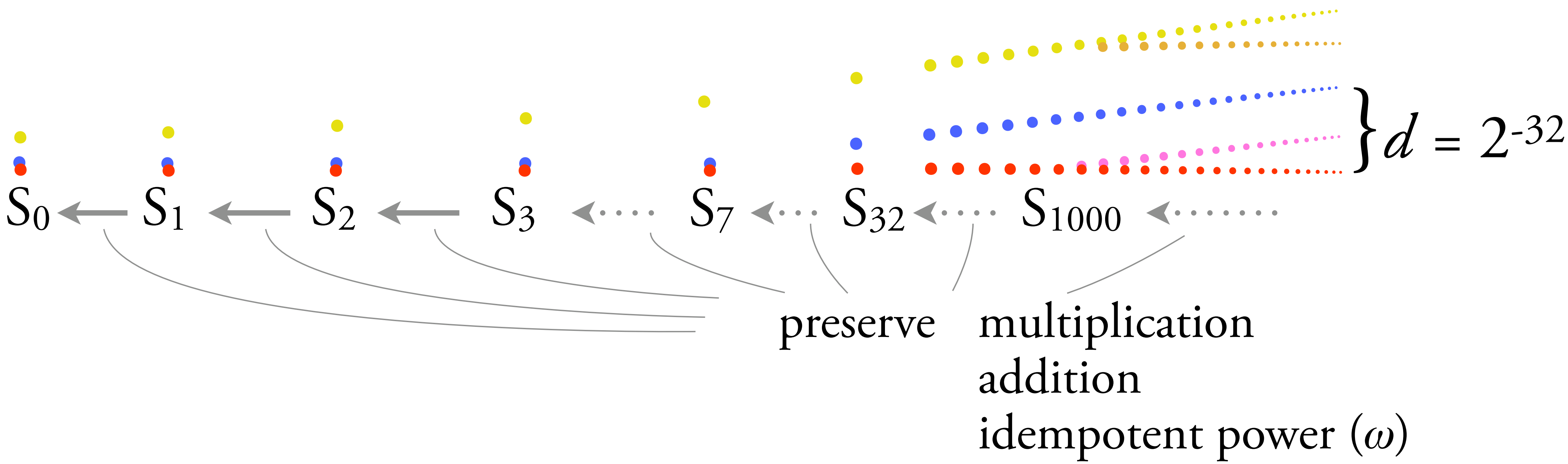
Metric

Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*
addition

Profinite monoid

$$\begin{pmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} \cdot \begin{pmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} = \begin{pmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{pmatrix}$$



Metric

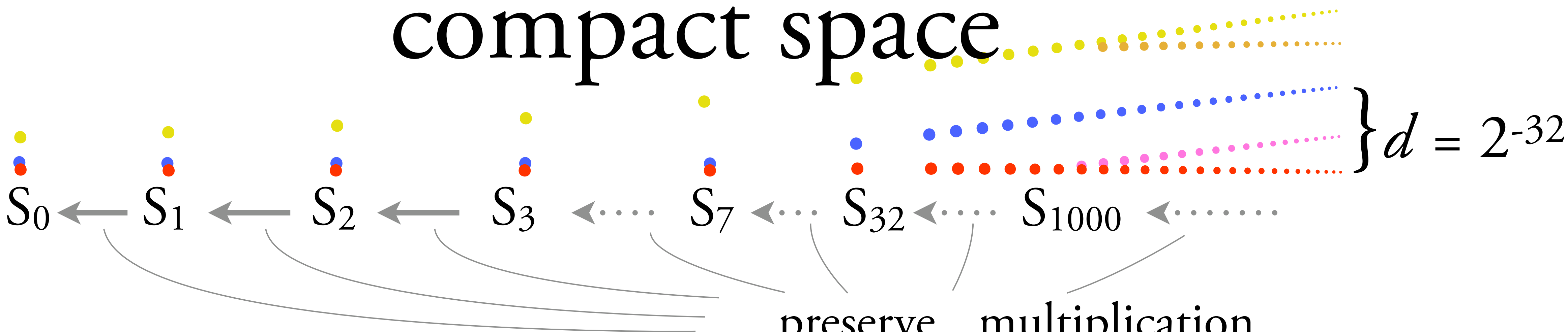
Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*
addition
ω-power

Profinite monoid

$$\begin{pmatrix} 3 & 50 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} \cdot \begin{pmatrix} 3 & 32 & 1 \\ 0 & 11 & 1 \\ 2 & 7 & \infty \end{pmatrix} = \begin{pmatrix} 3 & 8 & 4 \\ 3 & 8 & 4 \\ 5 & 18 & 3 \end{pmatrix}$$

compact space



Metric

Two elements are close if only an approx. with high threshold can distinguish them

Multiplication *is continuous*
addition
ω-power

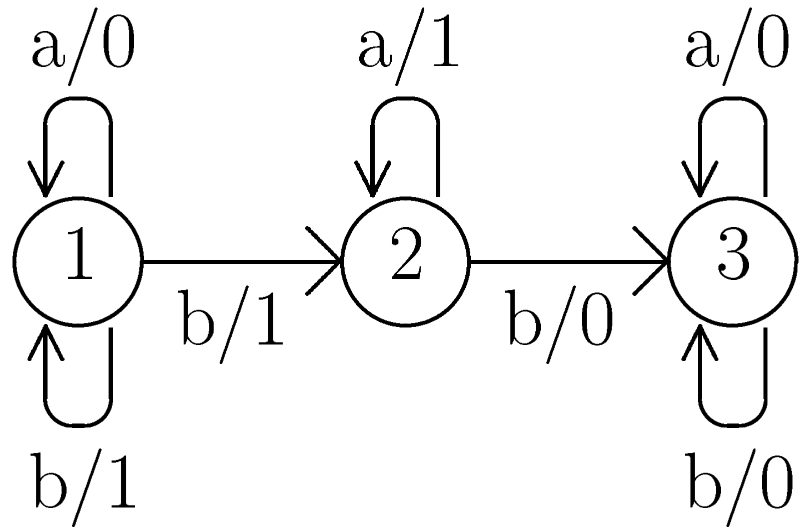


Figure 1: Example of a distance automaton A.

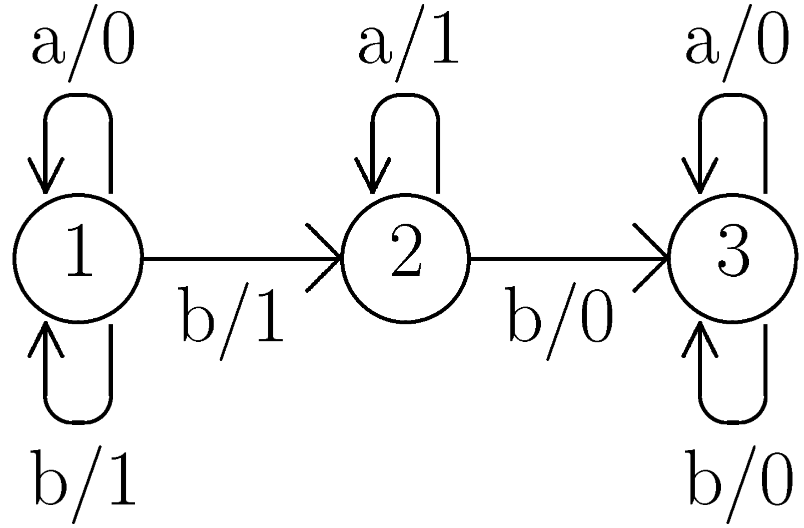


Figure 1: Example of a distance automaton A.

Example of a min-automaton *A.*

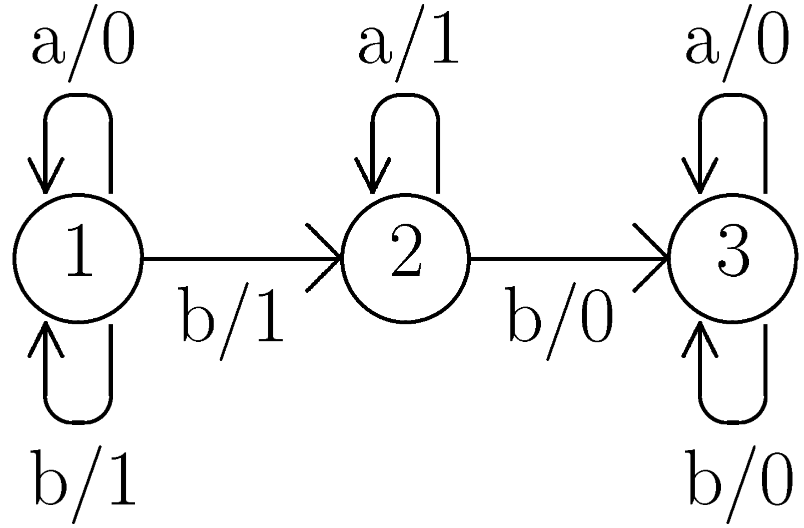


Figure 1: Example of a distance automaton A.

Example of a min-automaton *A*.

$$a: \quad \begin{bmatrix} 0 & \top & \top \\ \top & 1 & \top \\ \top & \top & 0 \end{bmatrix} \qquad b: \quad \begin{bmatrix} 1 & 1 & \top \\ \top & \top & 0 \\ \top & \top & 0 \end{bmatrix}$$

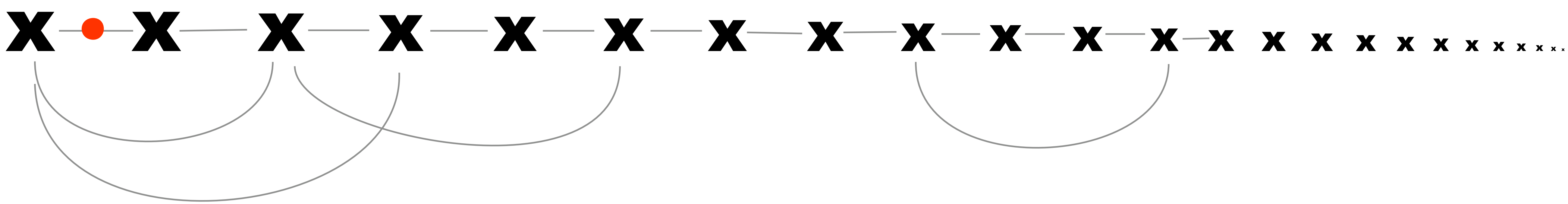
$$b^{\omega}: \quad \begin{bmatrix} \infty & \infty & 1 \\ \top & \top & 0 \\ \top & \top & 0 \end{bmatrix}$$

Ramsey Theorem

for compact spaces

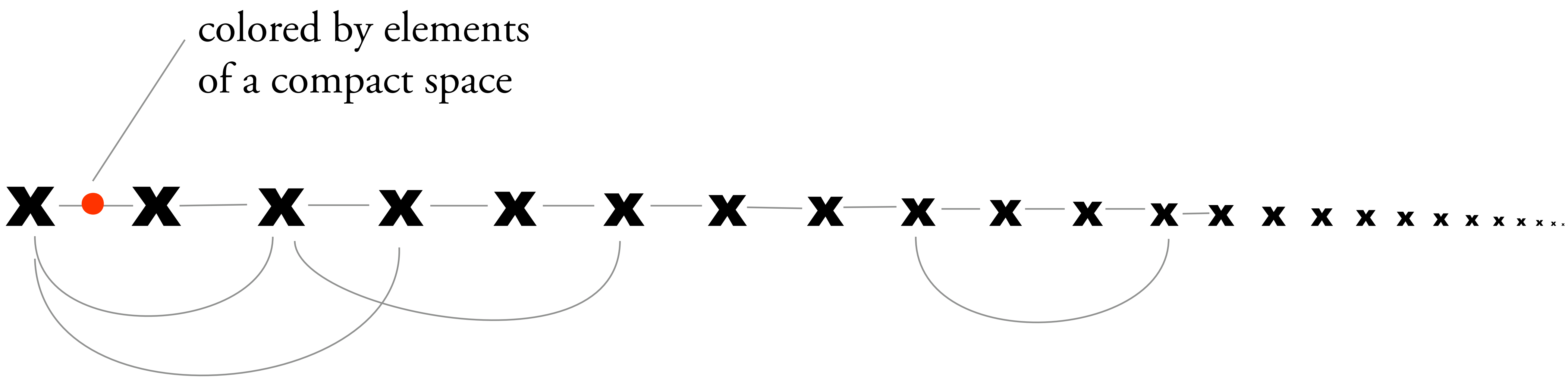
Ramsey Theorem

for compact spaces



Ramsey Theorem

for compact spaces



Ramsey Theorem

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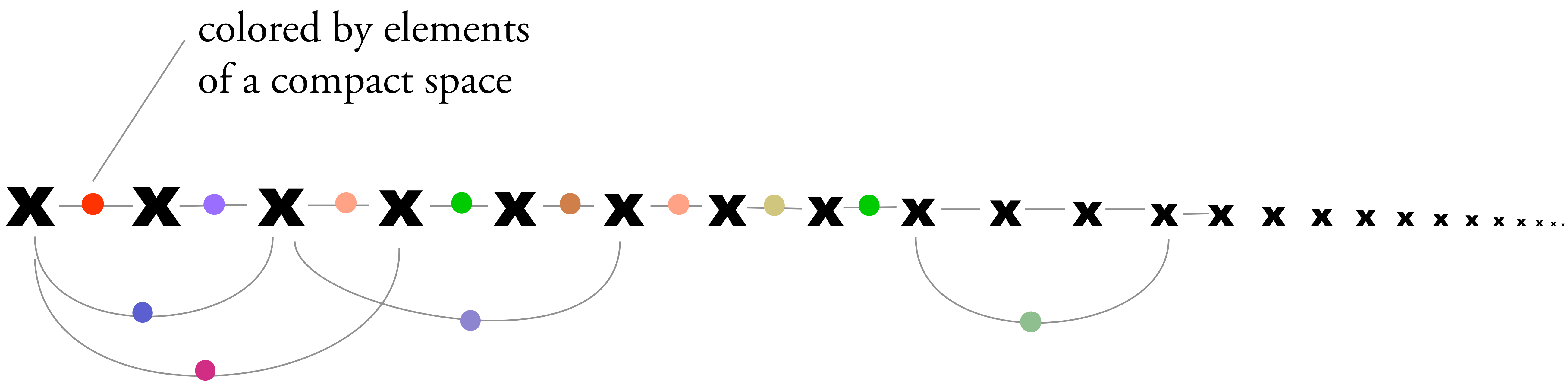
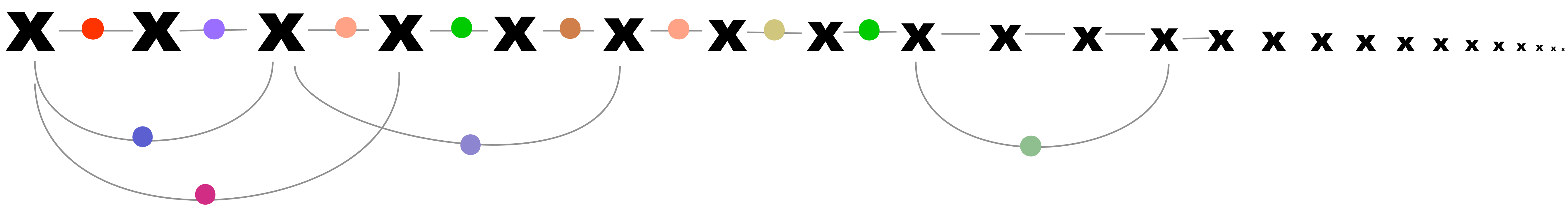


Ramsey Theorem

for compact spaces



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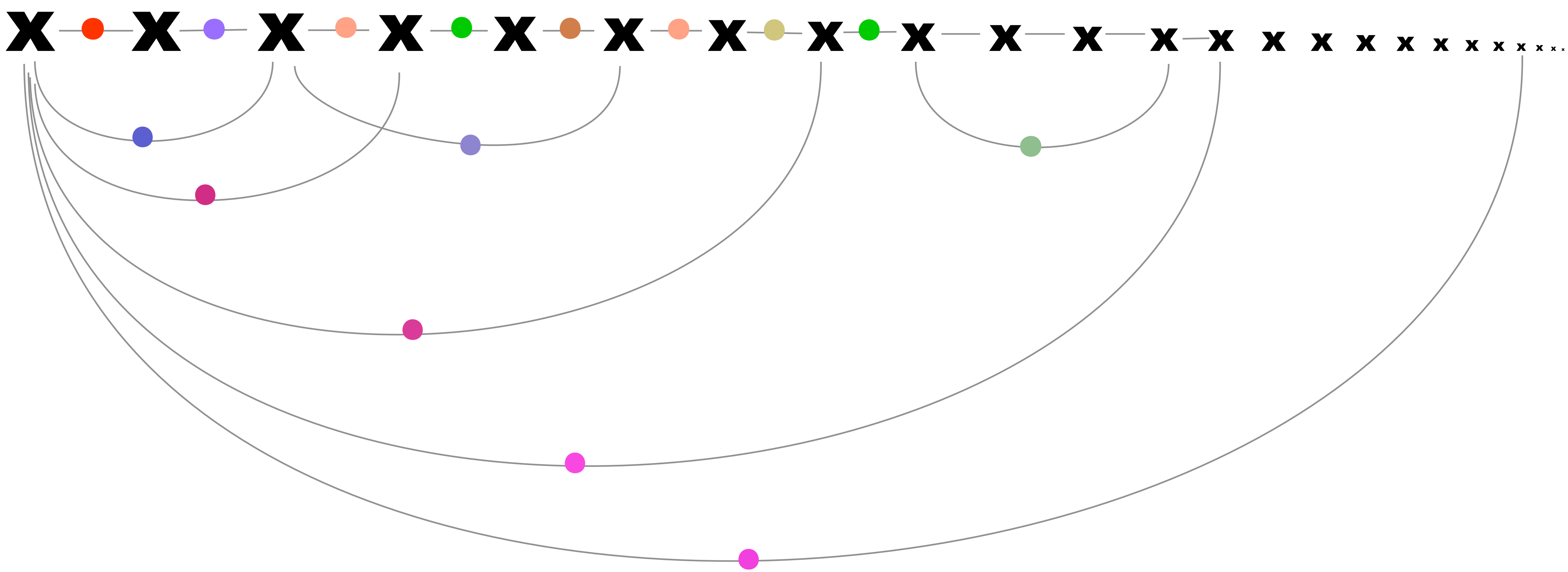

Ramsey Theorem

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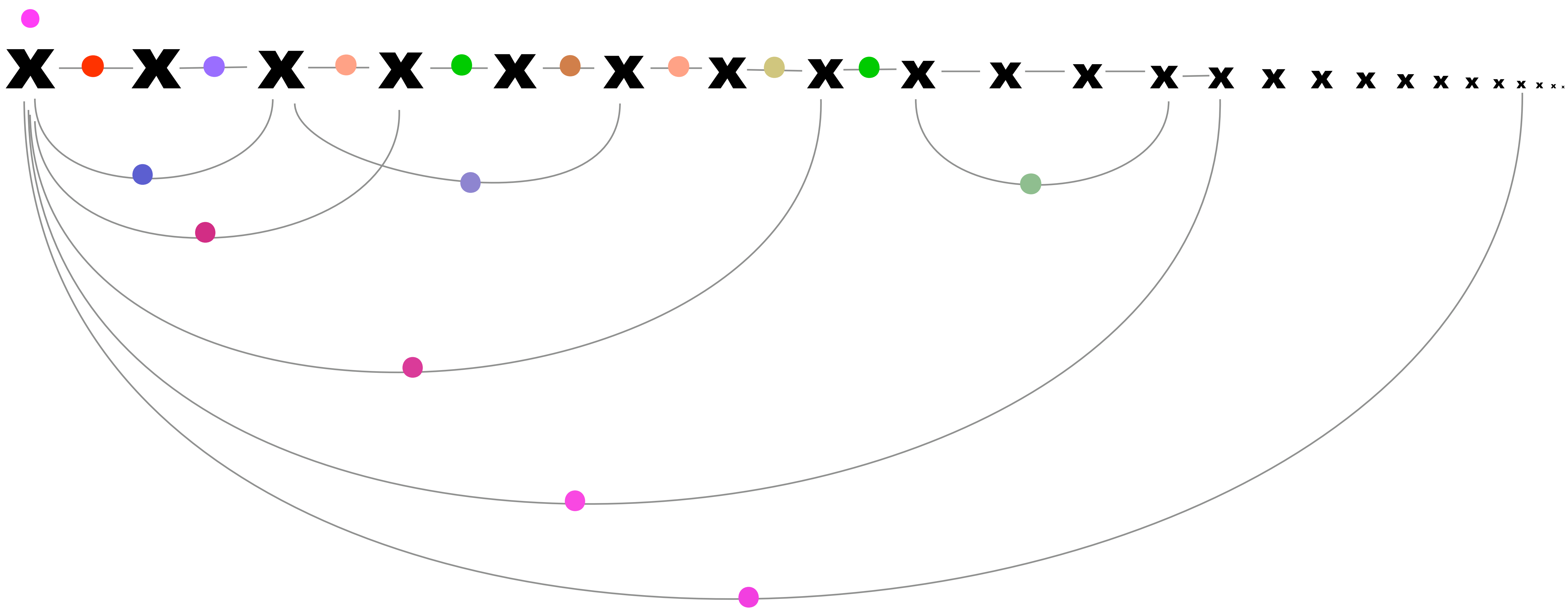

Ramsey Theorem

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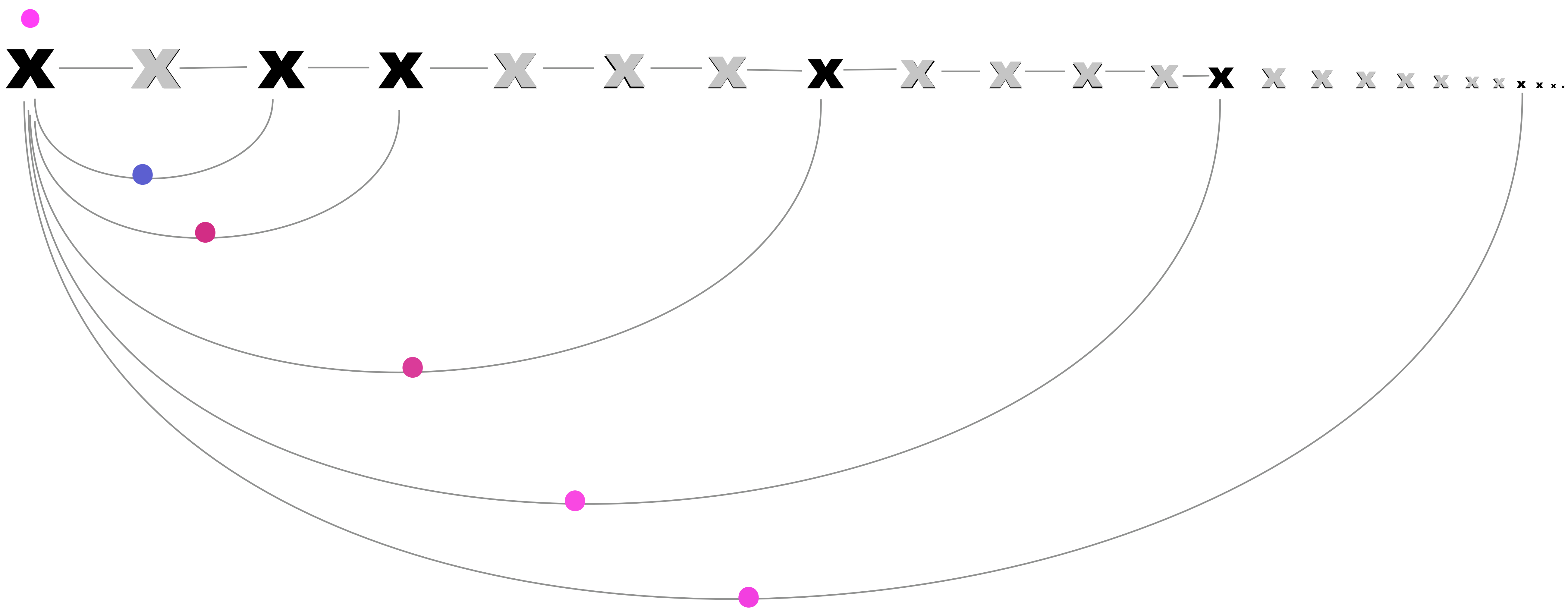

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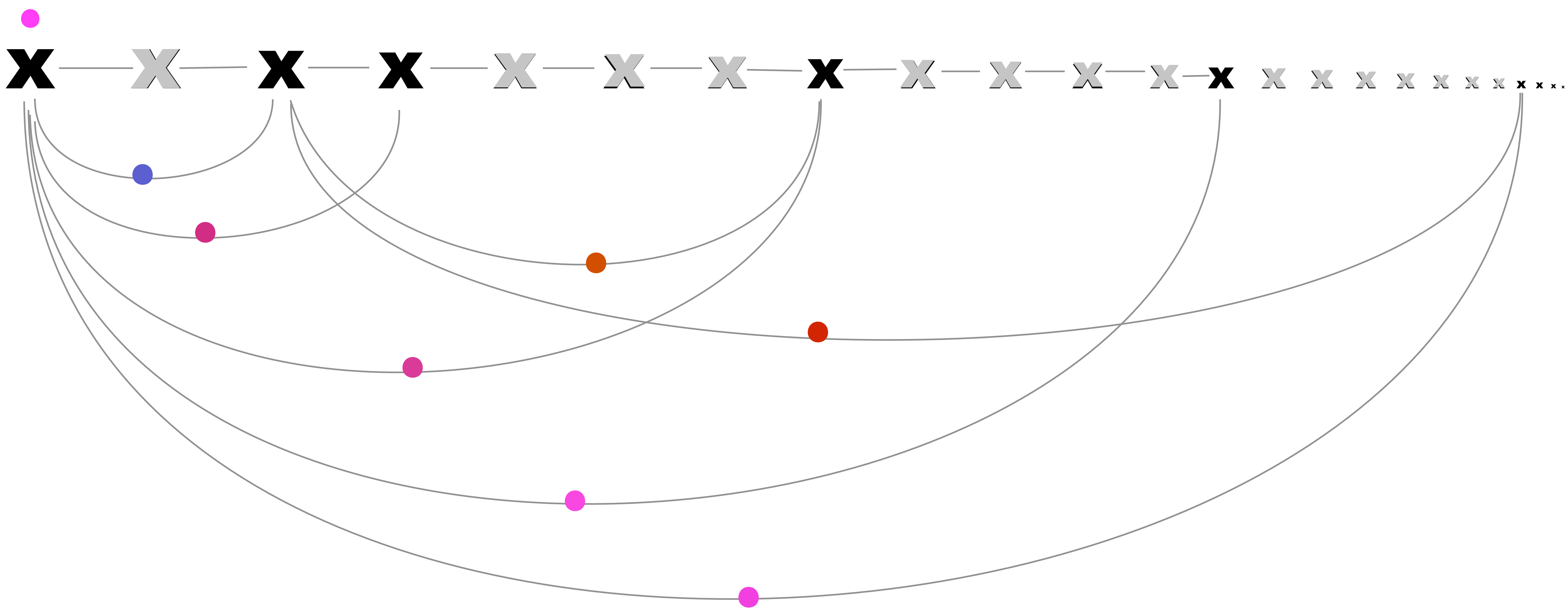

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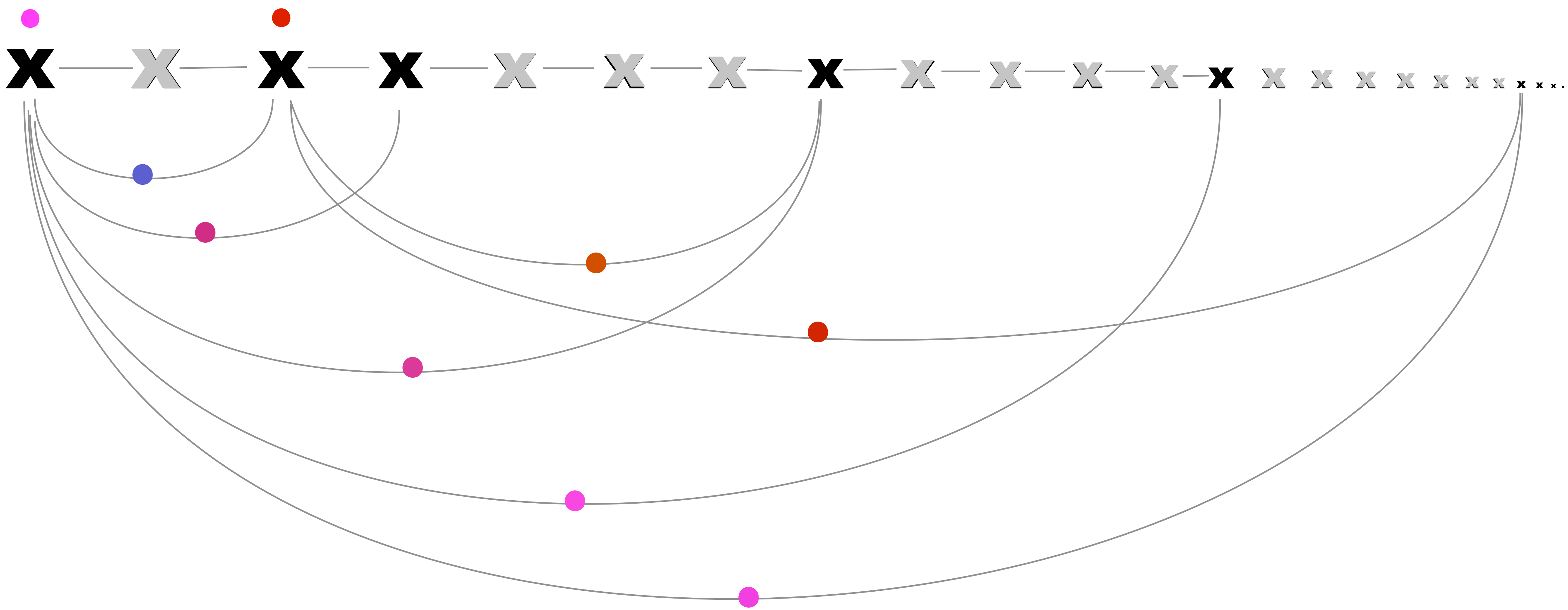

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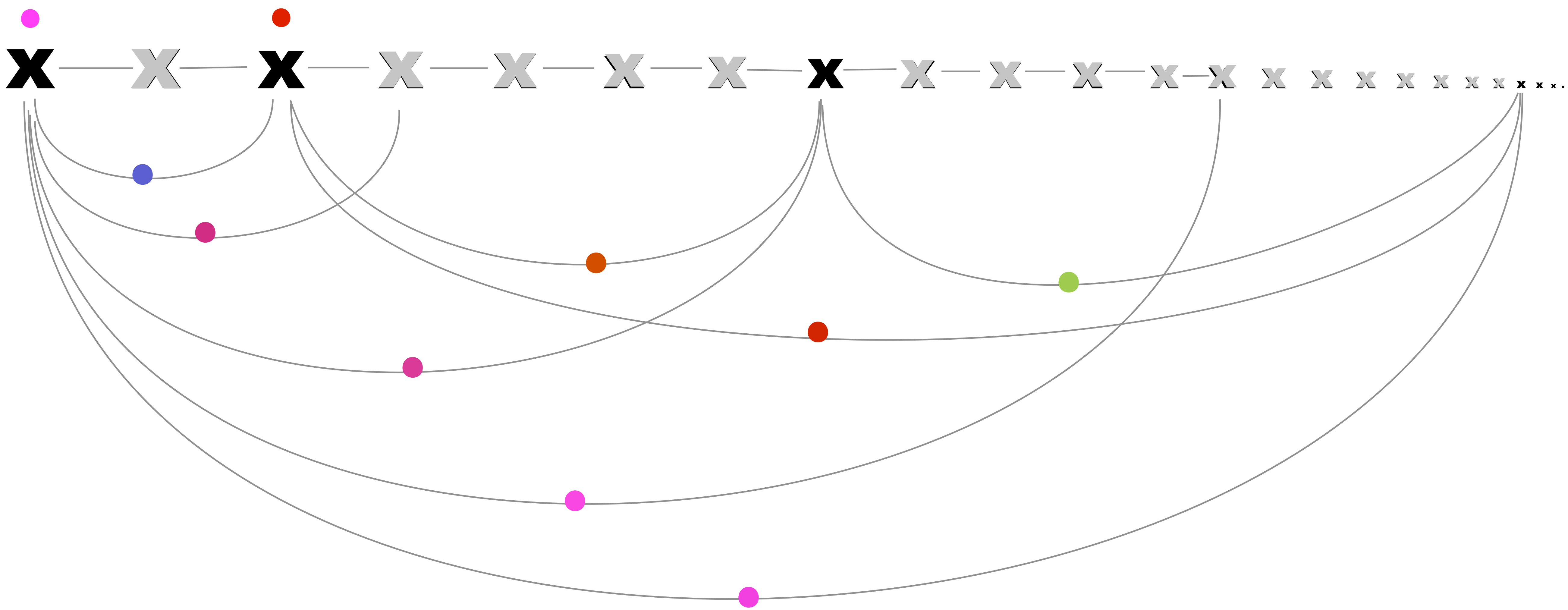

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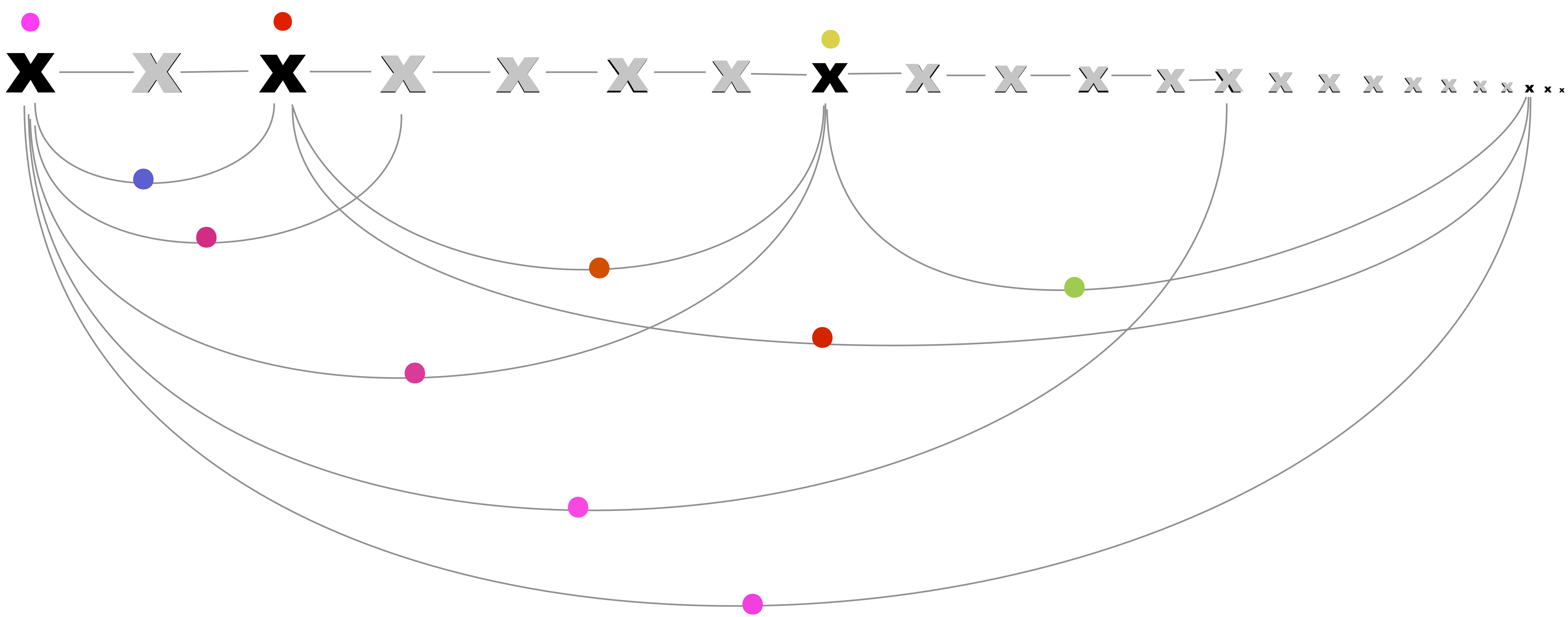

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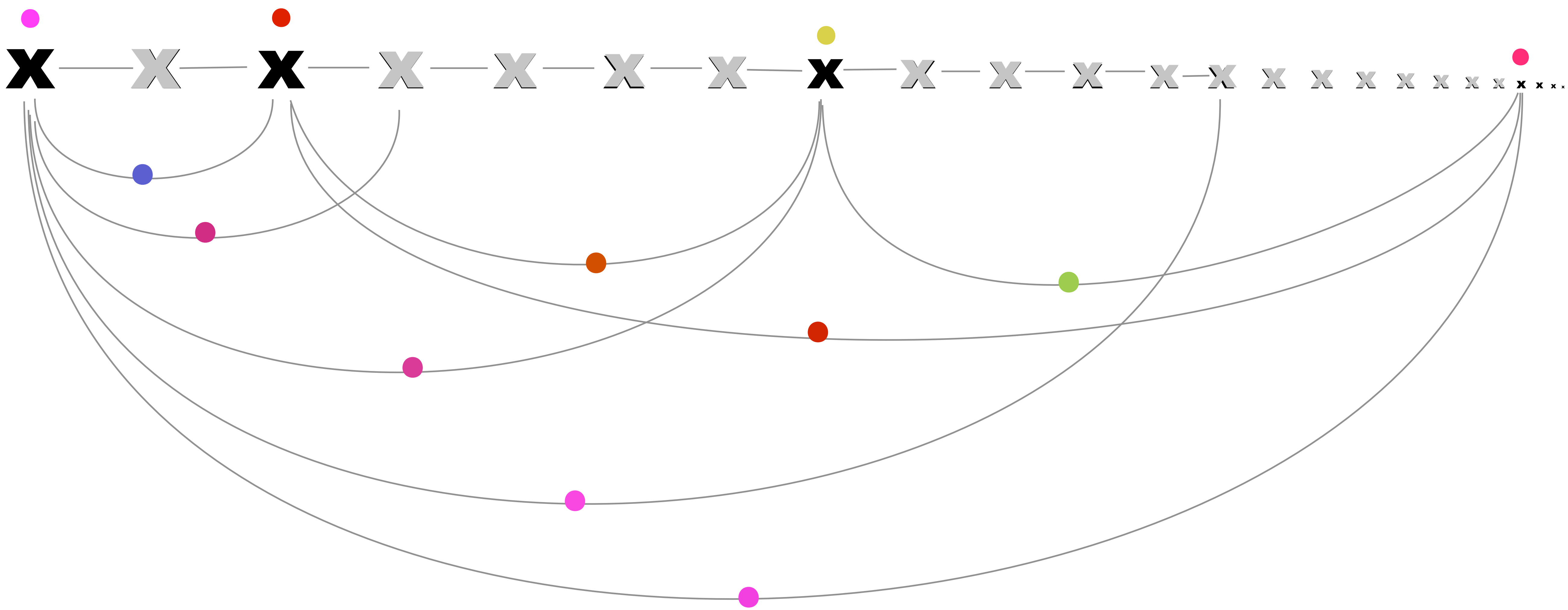

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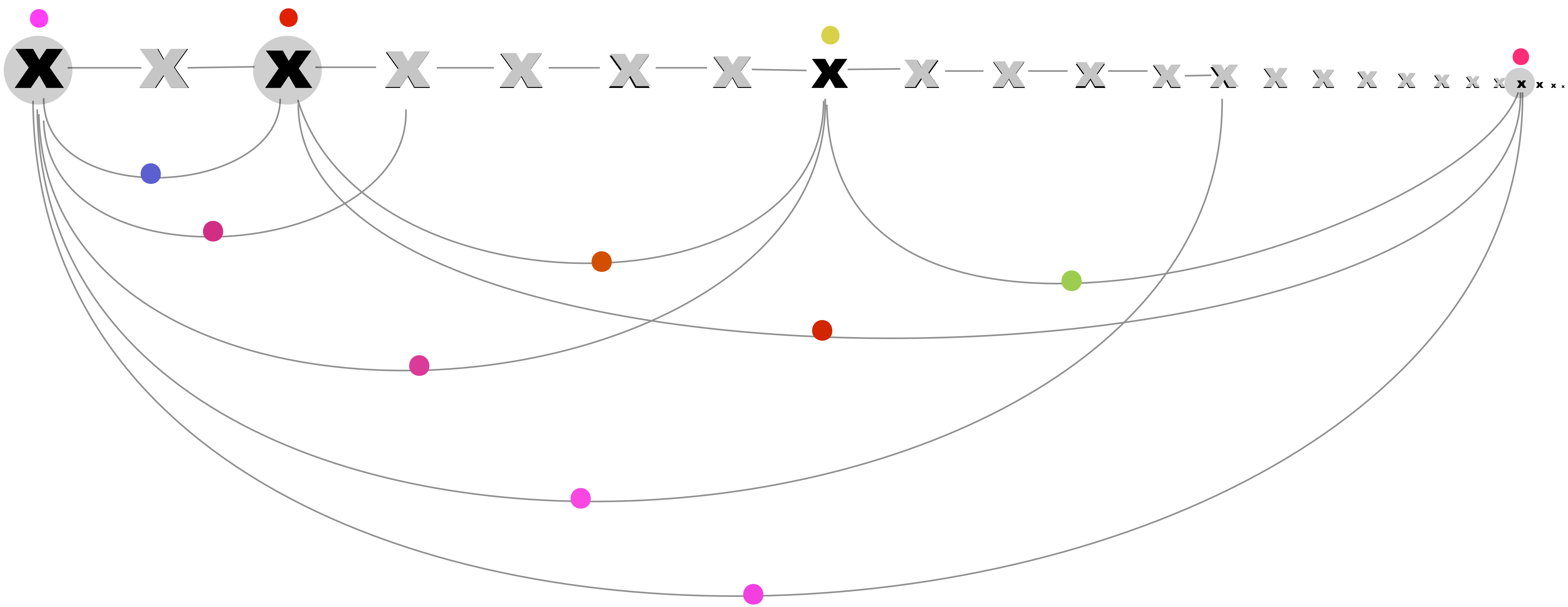

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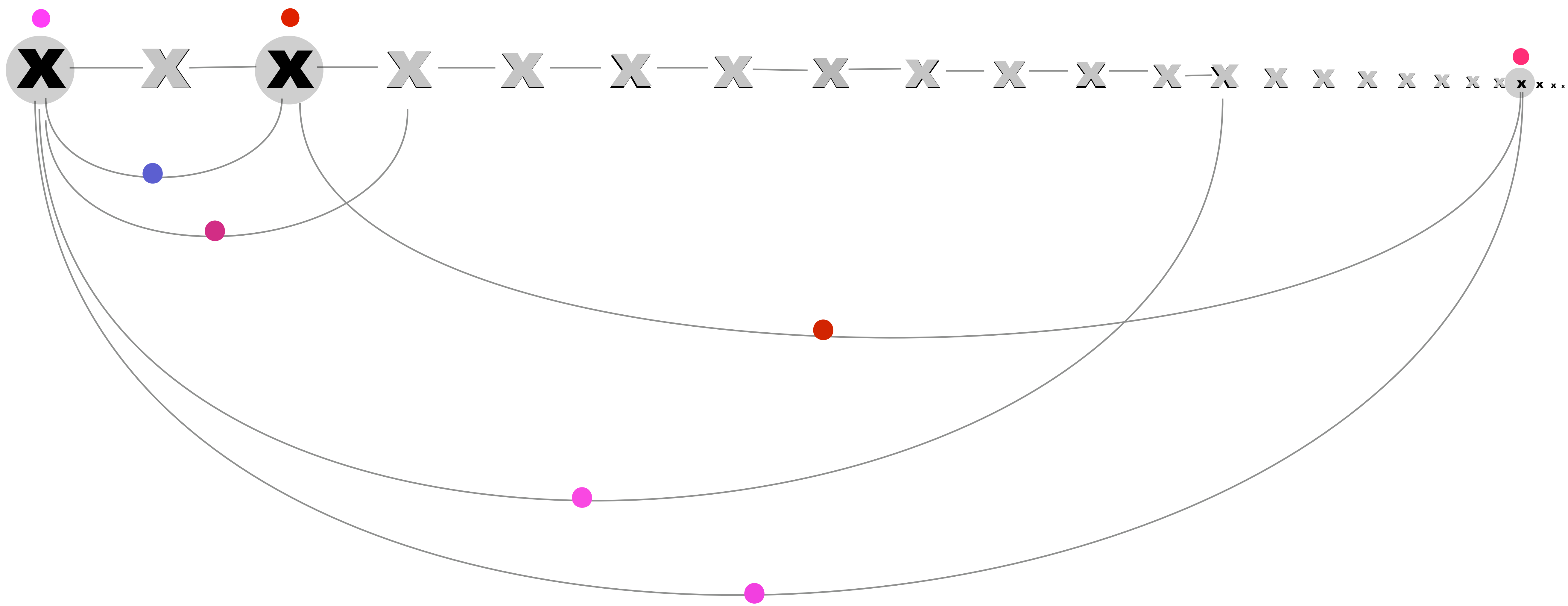

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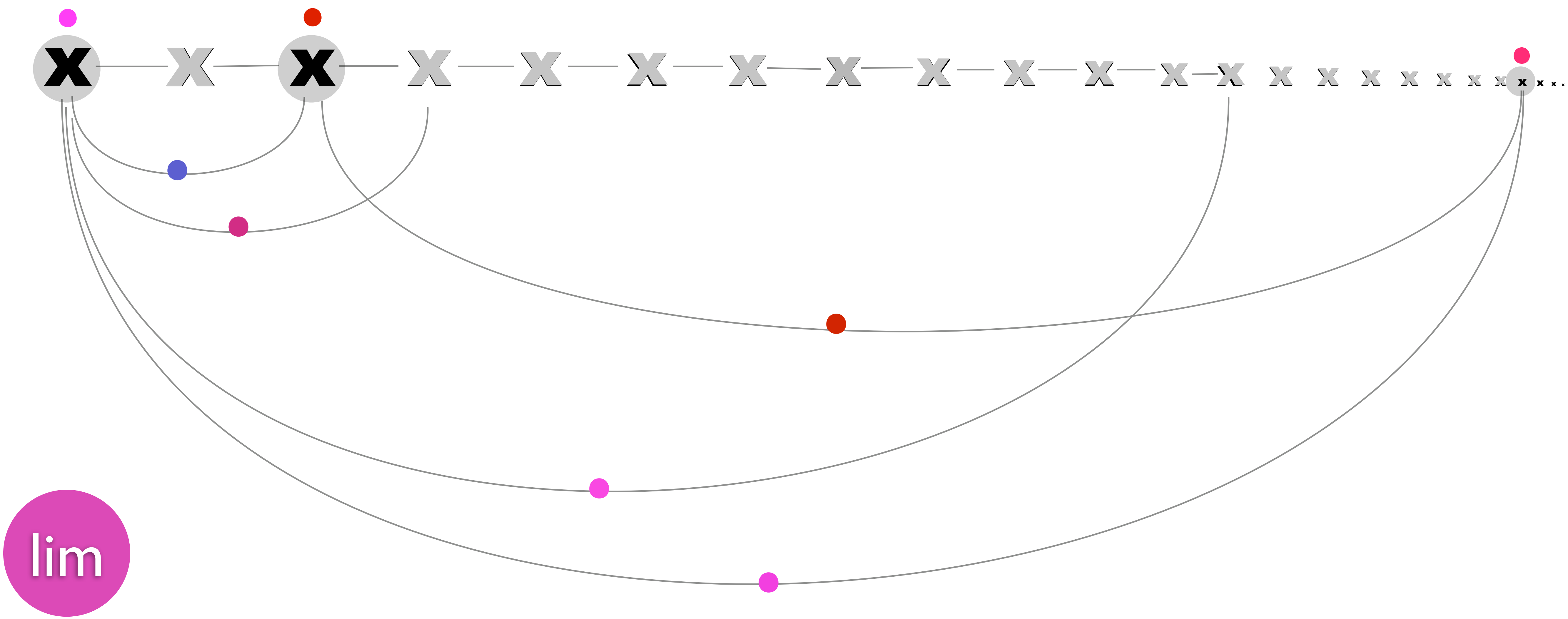

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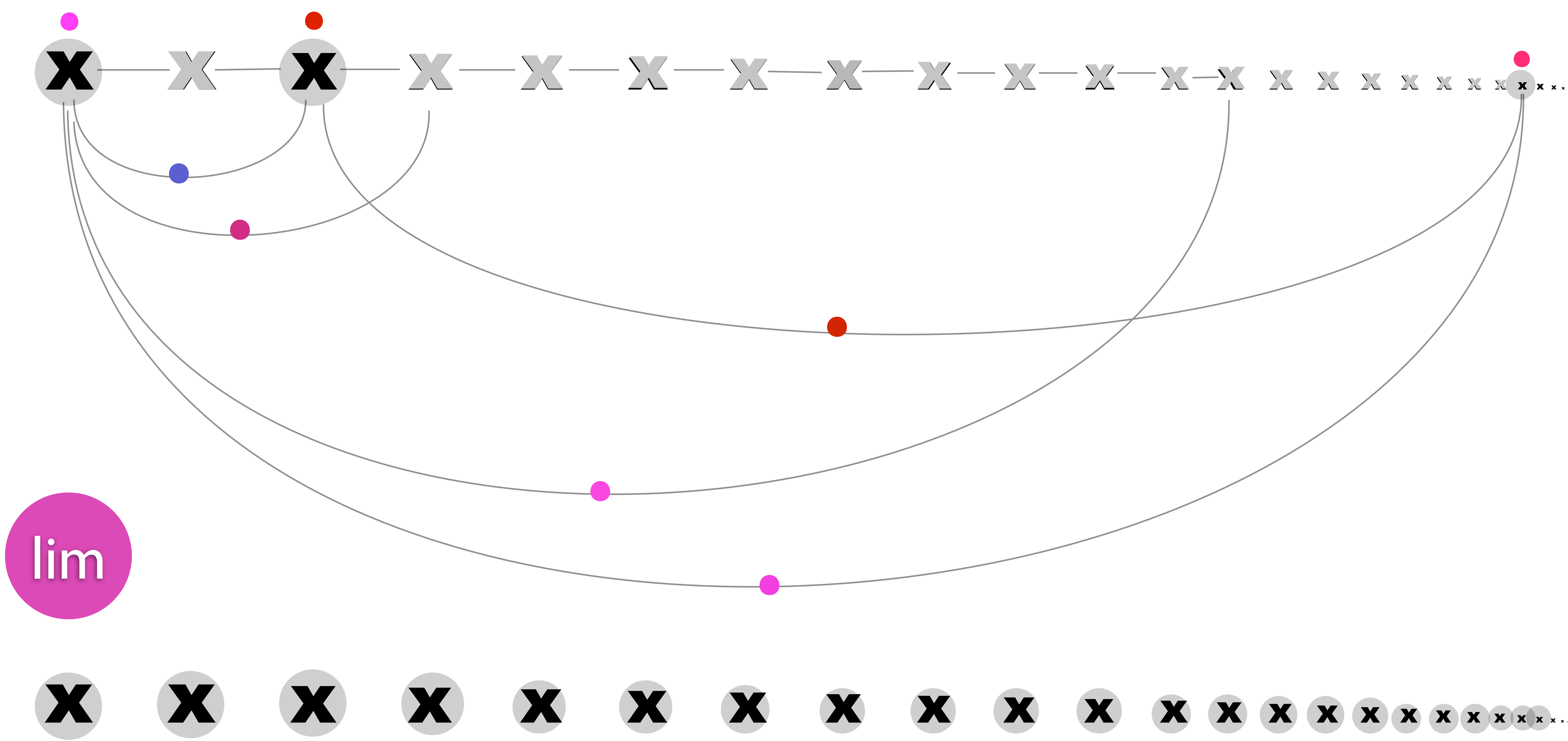

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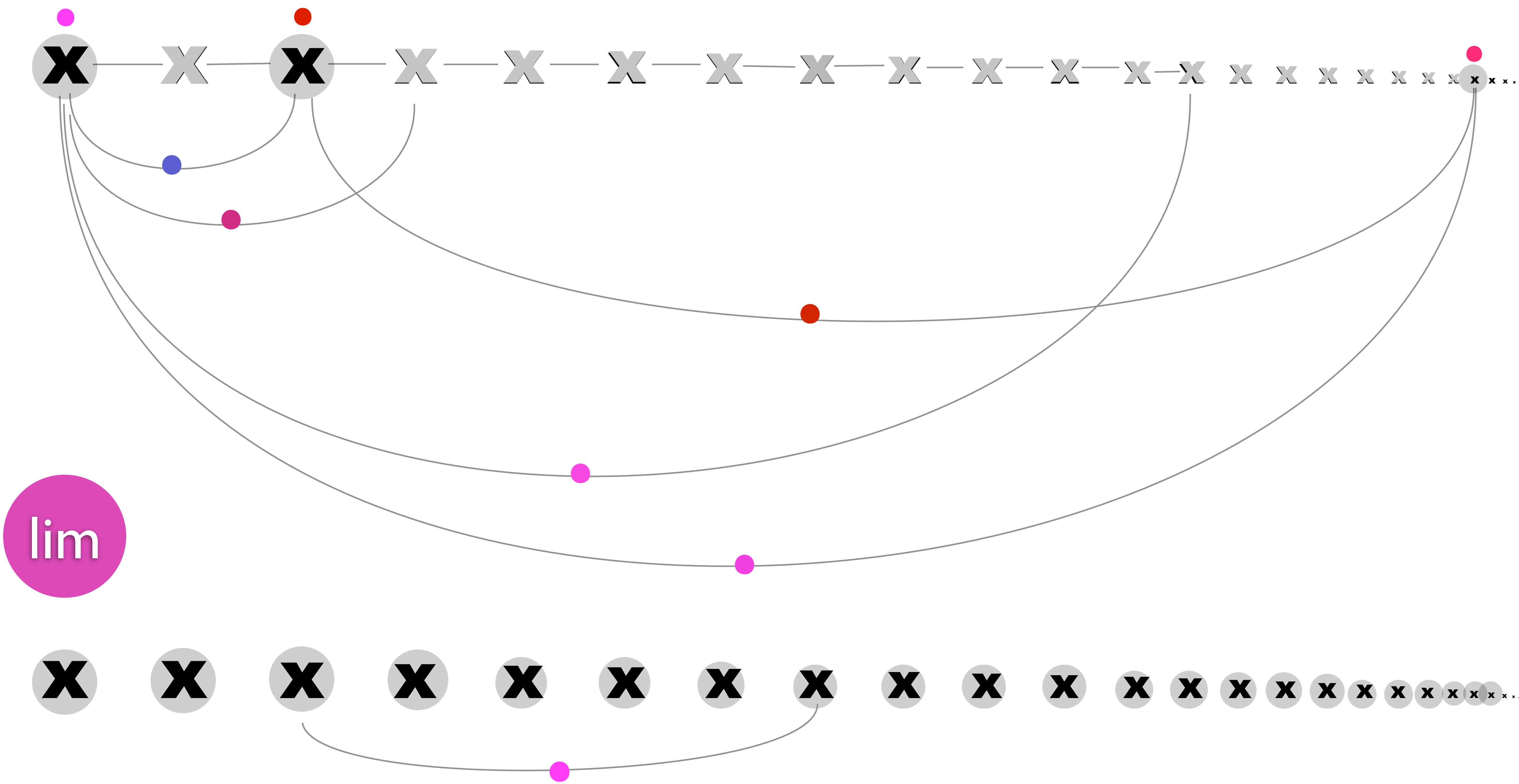

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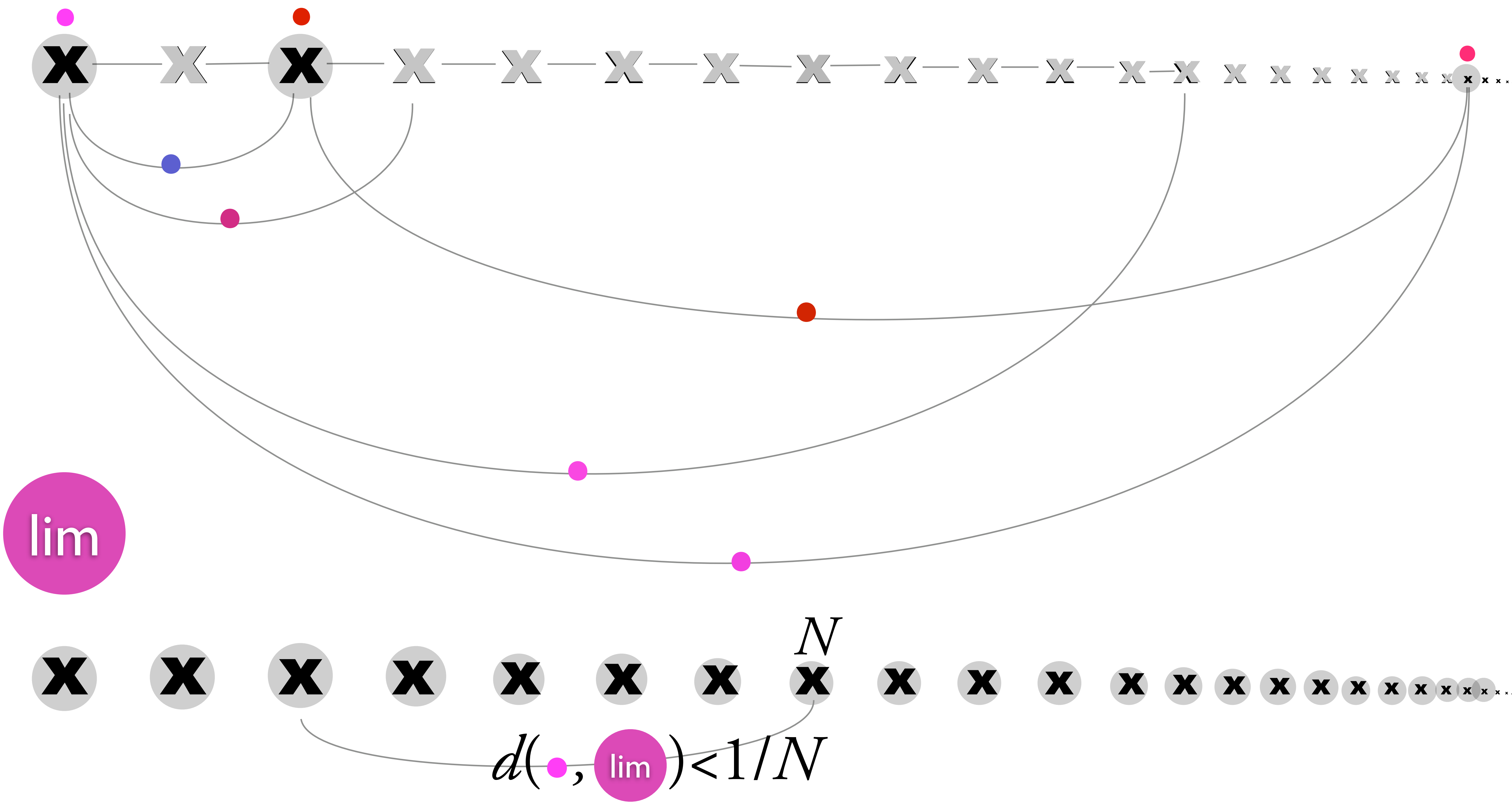

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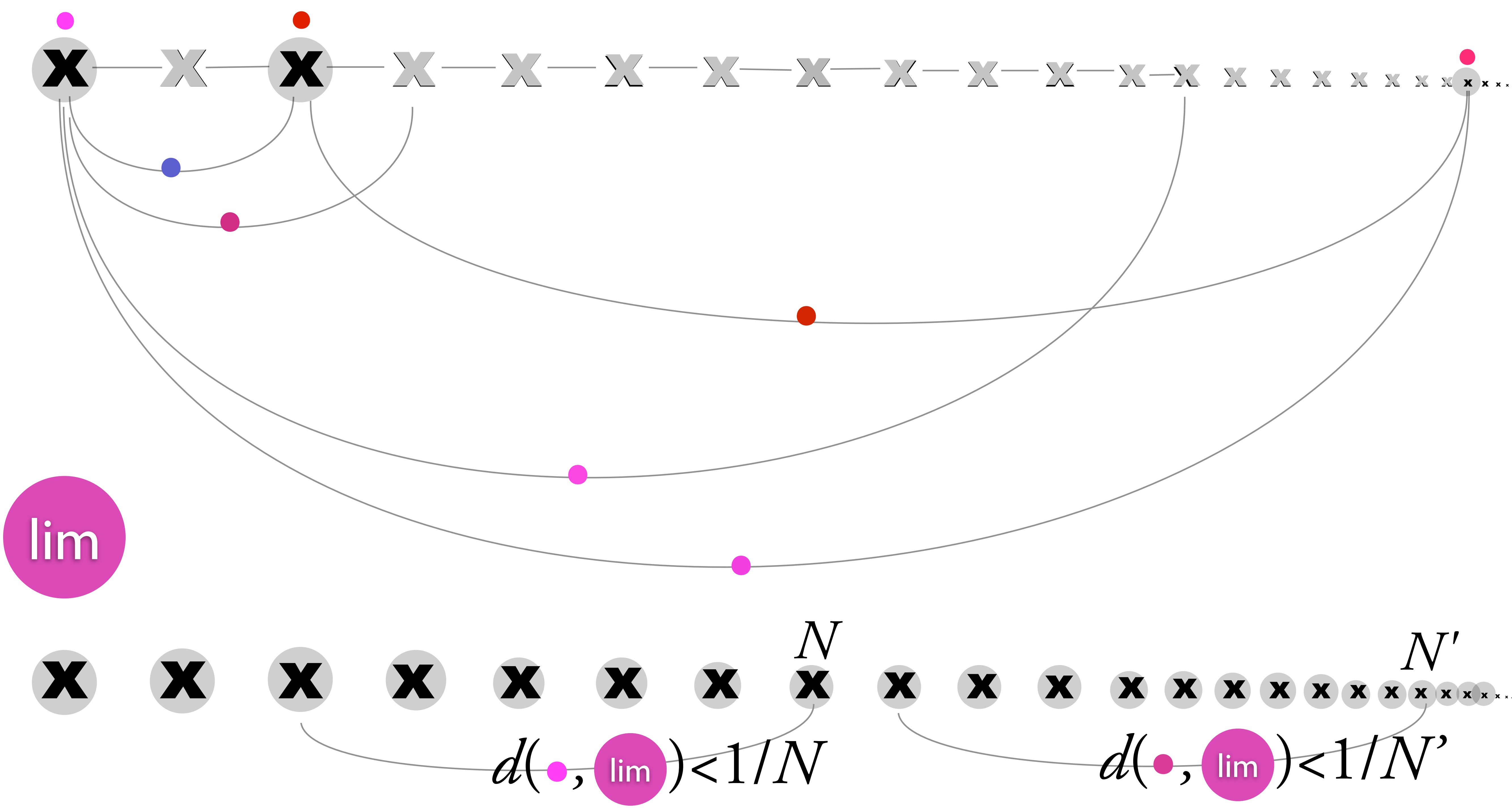
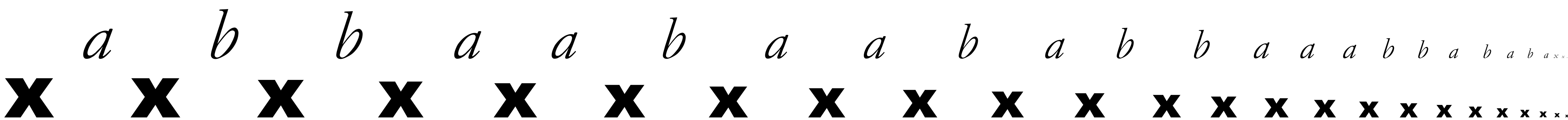


Emptiness of min-automata

$a \quad b \quad b \quad a \quad a \quad b \quad a \quad a \quad b \quad a \quad b \quad b \quad a \quad a \quad a \quad b \quad b \quad a \quad b \quad a \quad b \quad a \ldots$

Emptiness of min-automata



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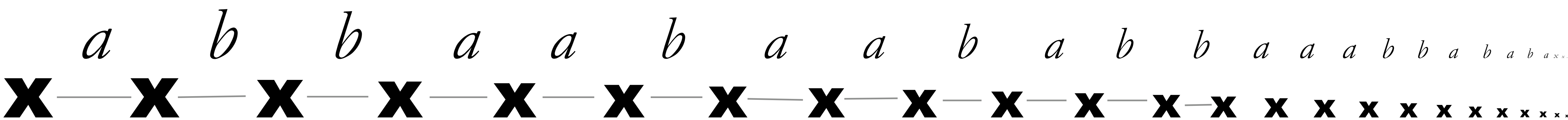

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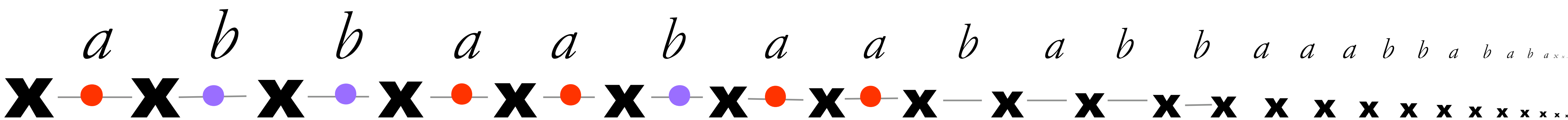

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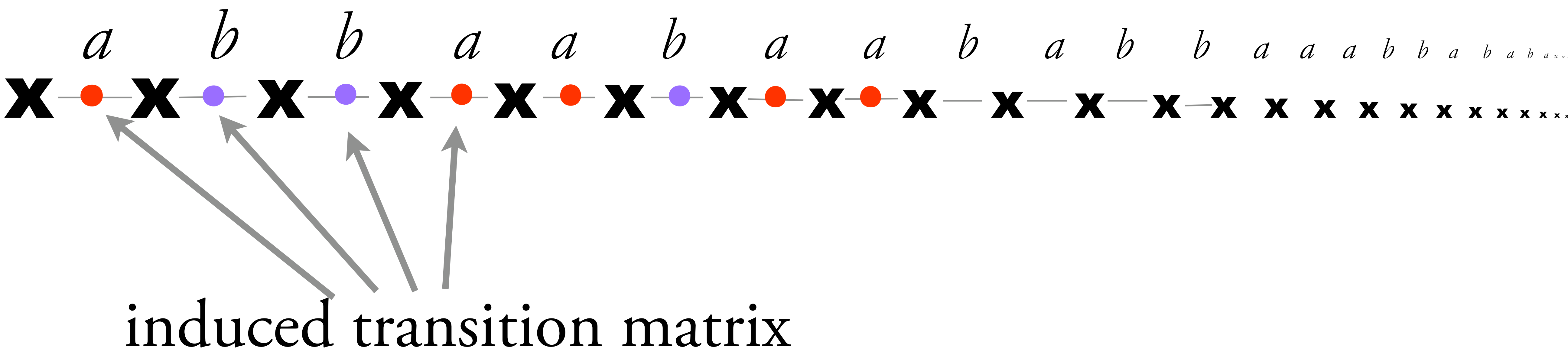

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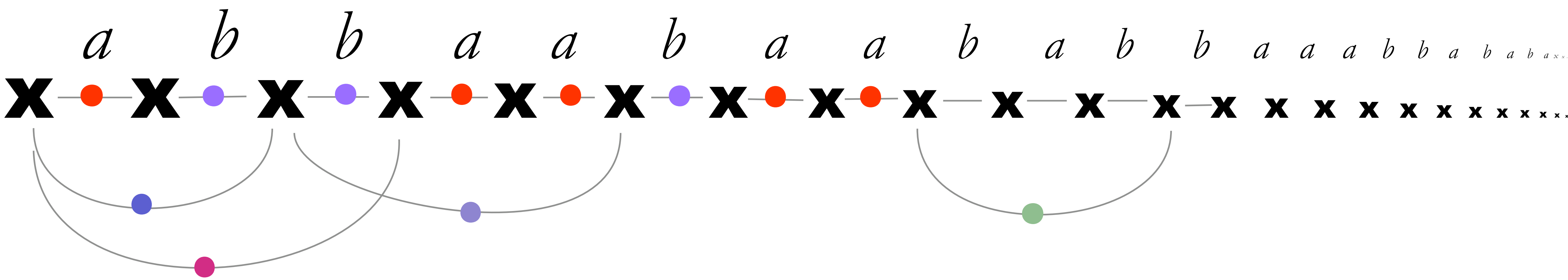

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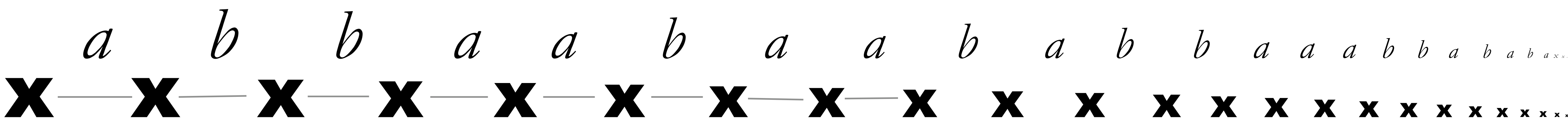
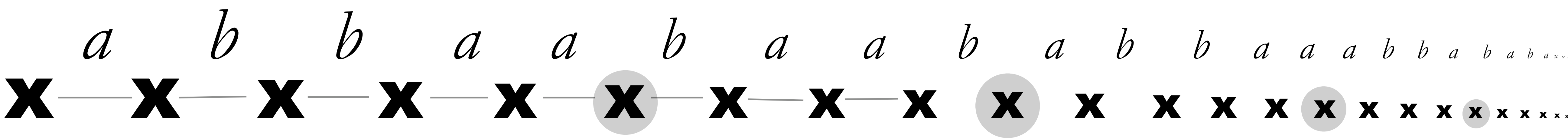


lim

Emptiness of min-automata



lim

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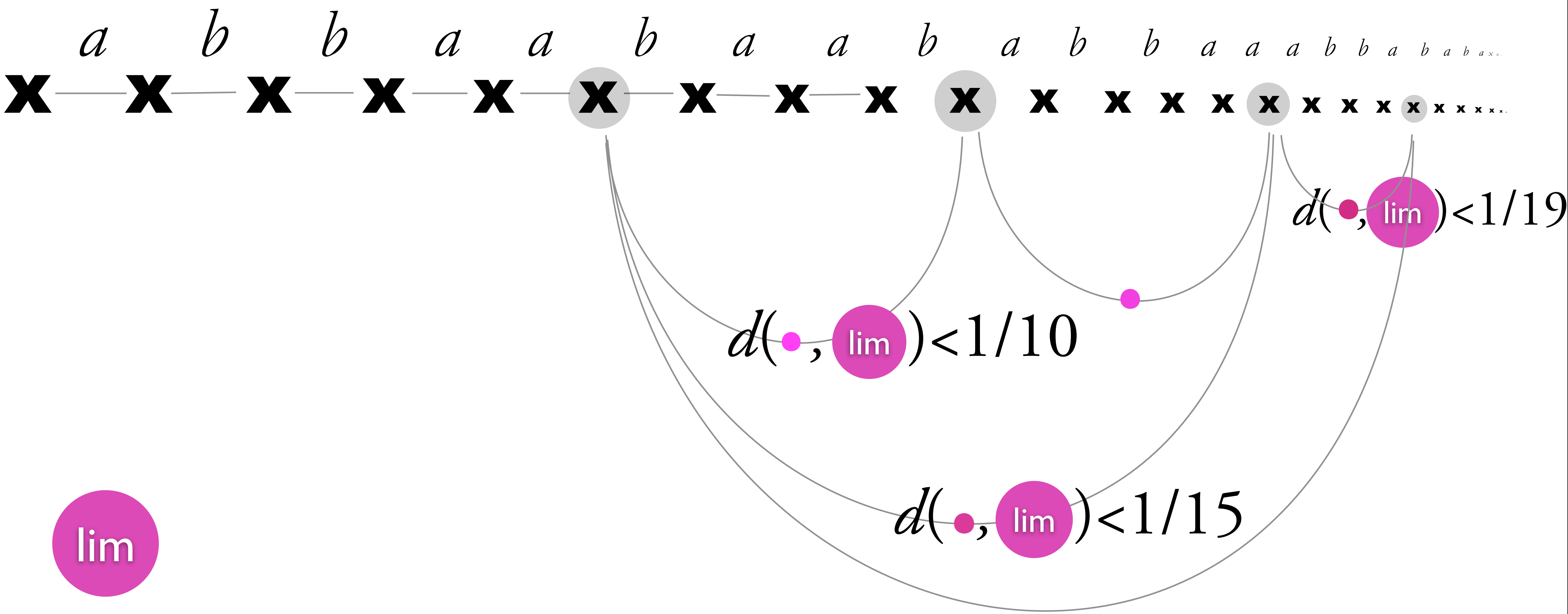

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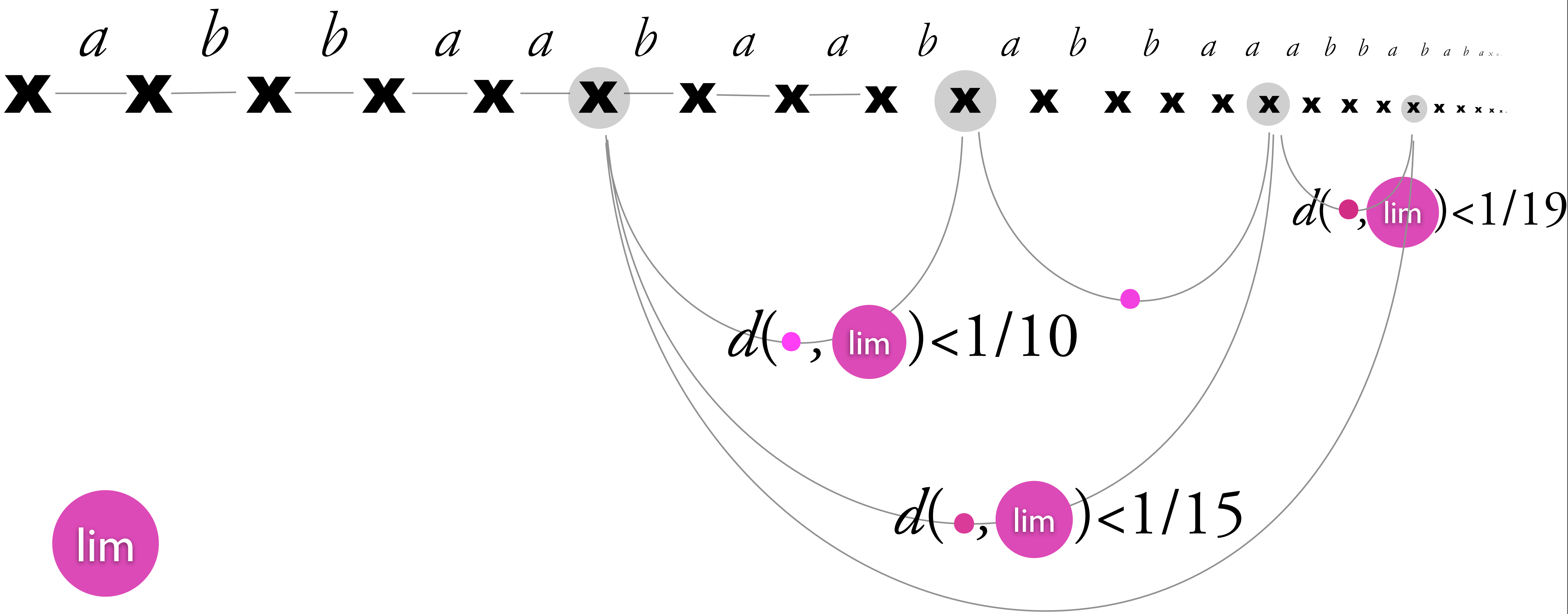
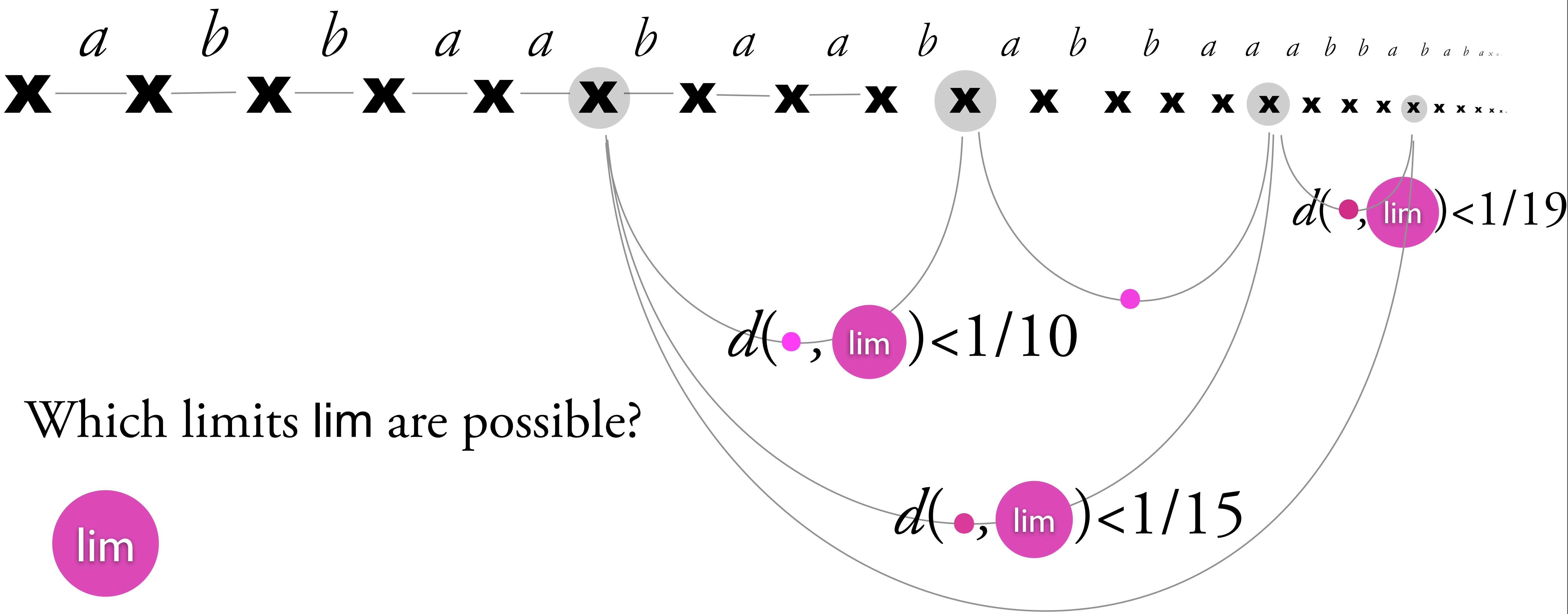


Counter c does *not* converge to ∞

iff exists a counter d such that

$\lim[d,d]=0$ *and* $\lim[d,c]<\infty$.

Emptiness of min-automata



Which limits lim are possible?

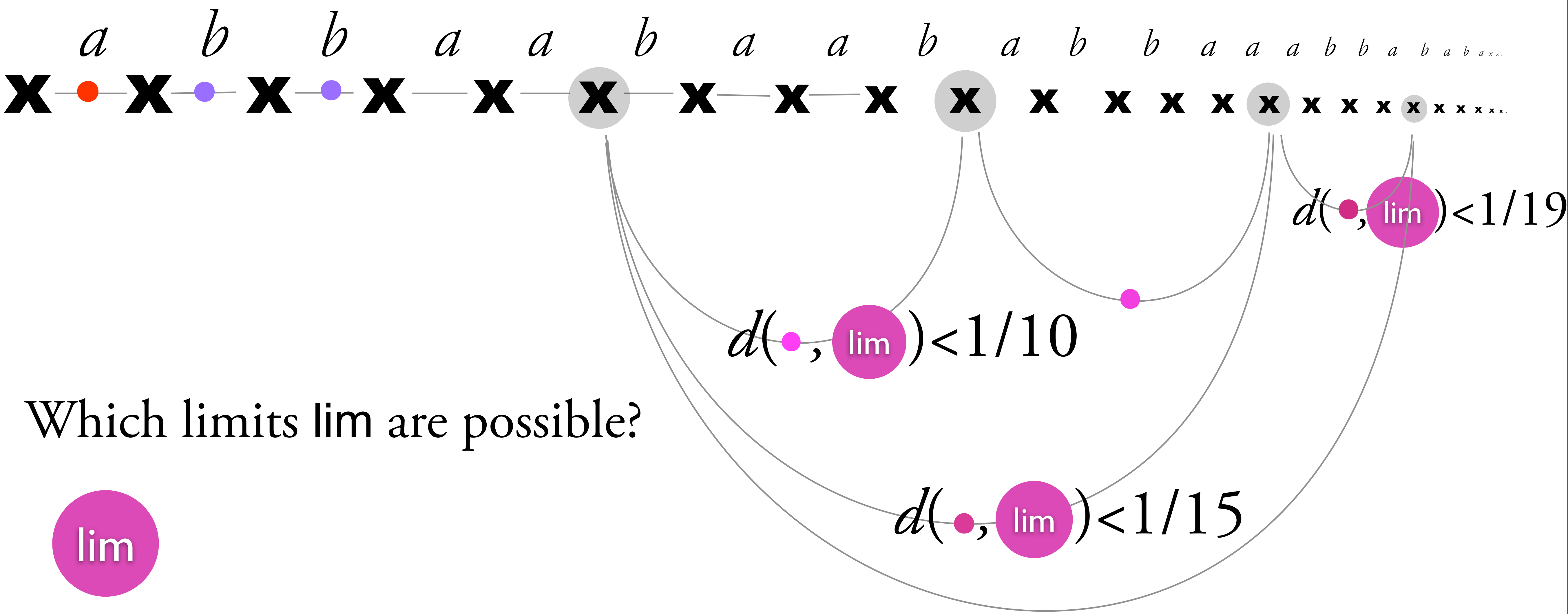
lim

Counter c does *not* converge to ∞

iff exists a counter d such that

$\text{lim}[d,d] = 0$ *and* $\text{lim}[d,c] < \infty$.

Emptiness of min-automata



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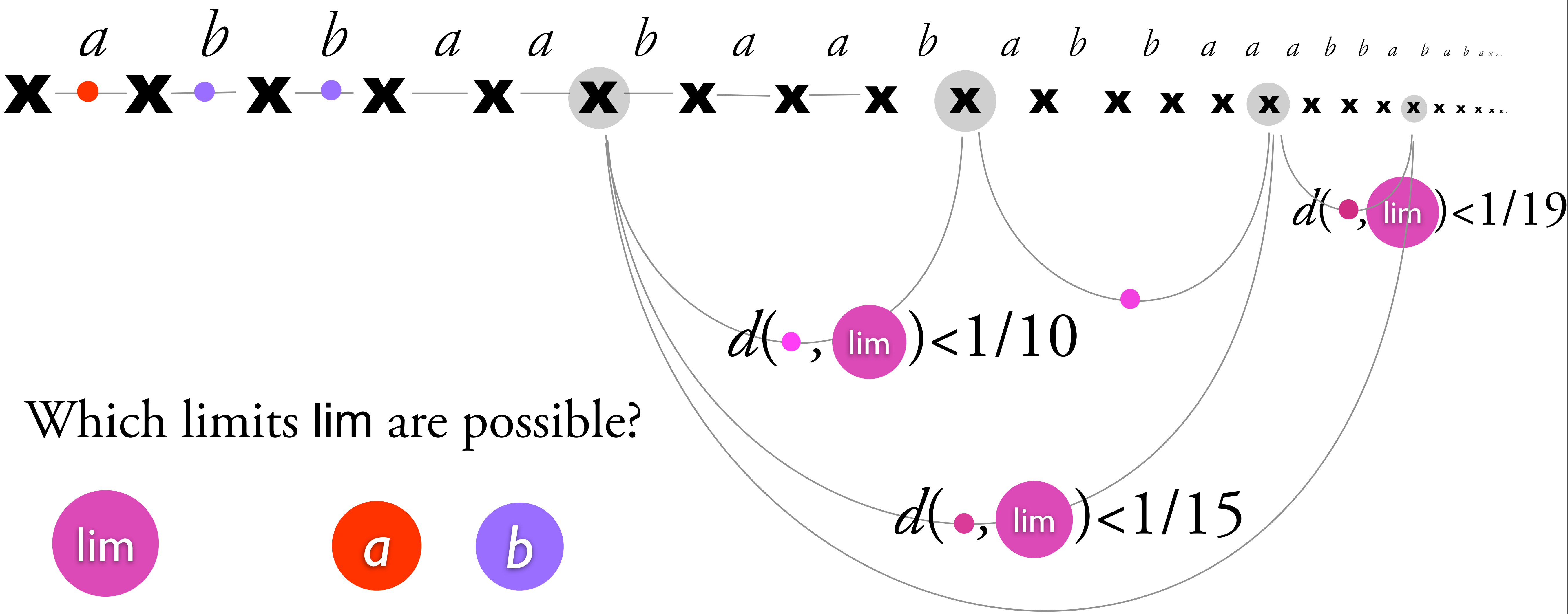

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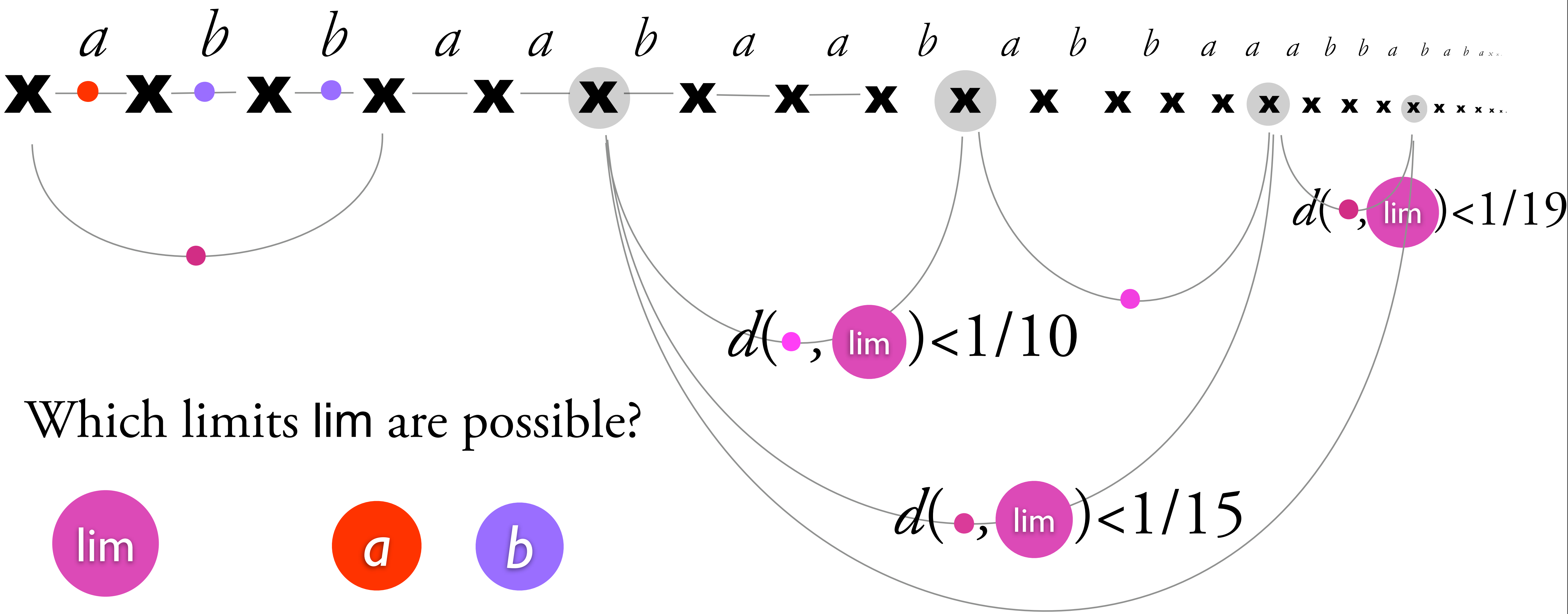

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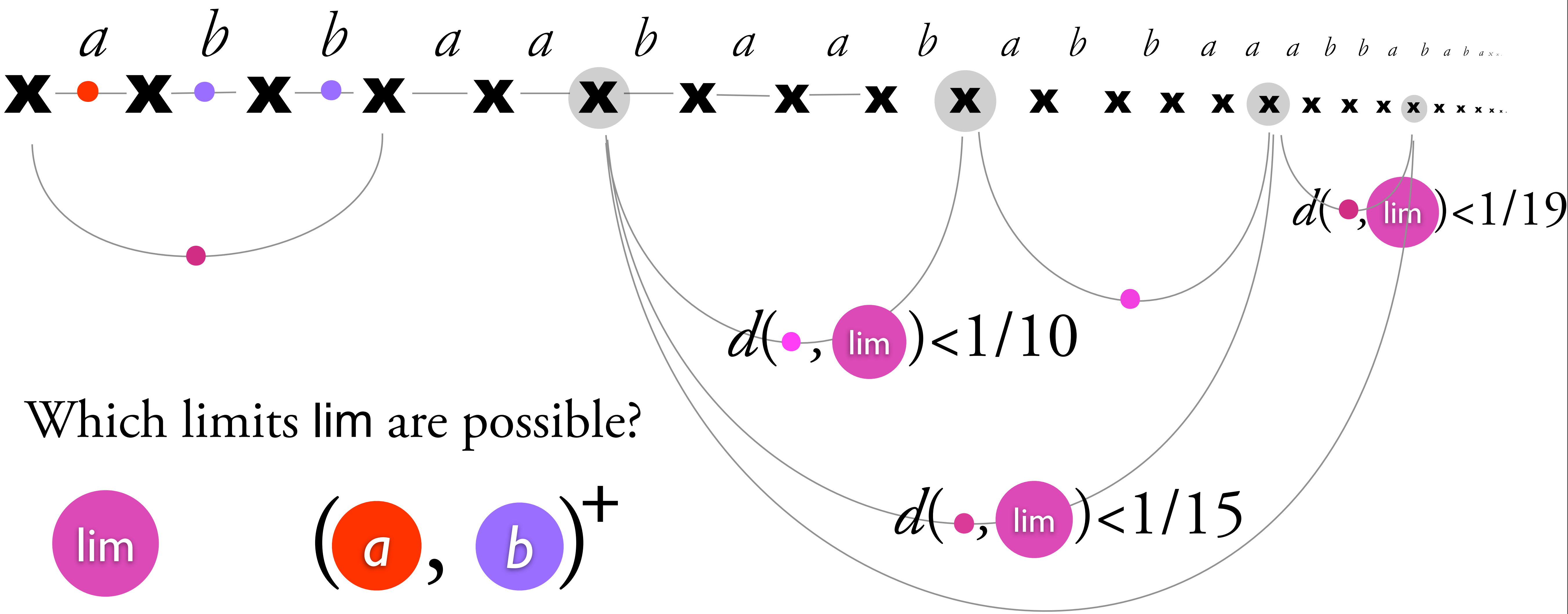
Emptiness of min-automata



Which limits lim are possible?

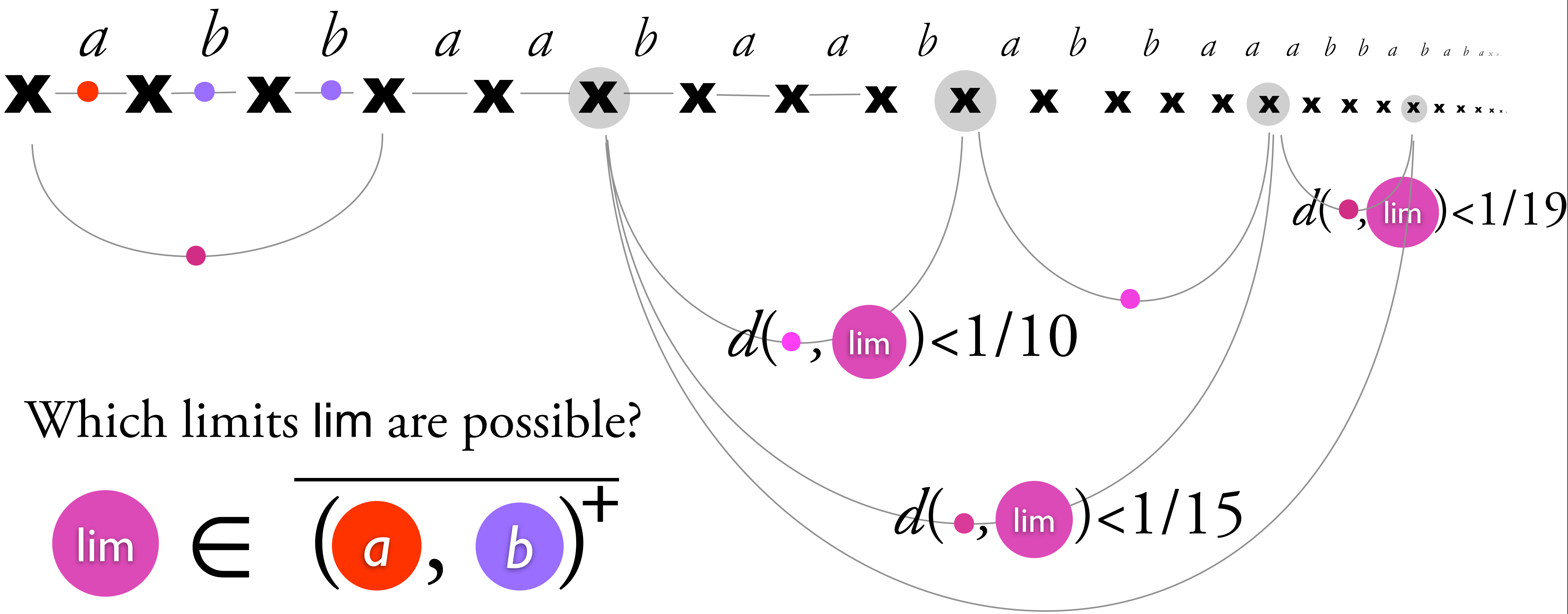
lim $\quad (a, b)^+$

Counter c does *not* converge to ∞

iff exists a counter d such that

lim$[d,d] = 0$ *and* lim$[d,c] < \infty$.

Emptiness of min-automata



Which limits lim are possible?

$$\mathrm{lim} \in \overline{(a, b)^{+}}$$

Counter c does *not* converge to ∞

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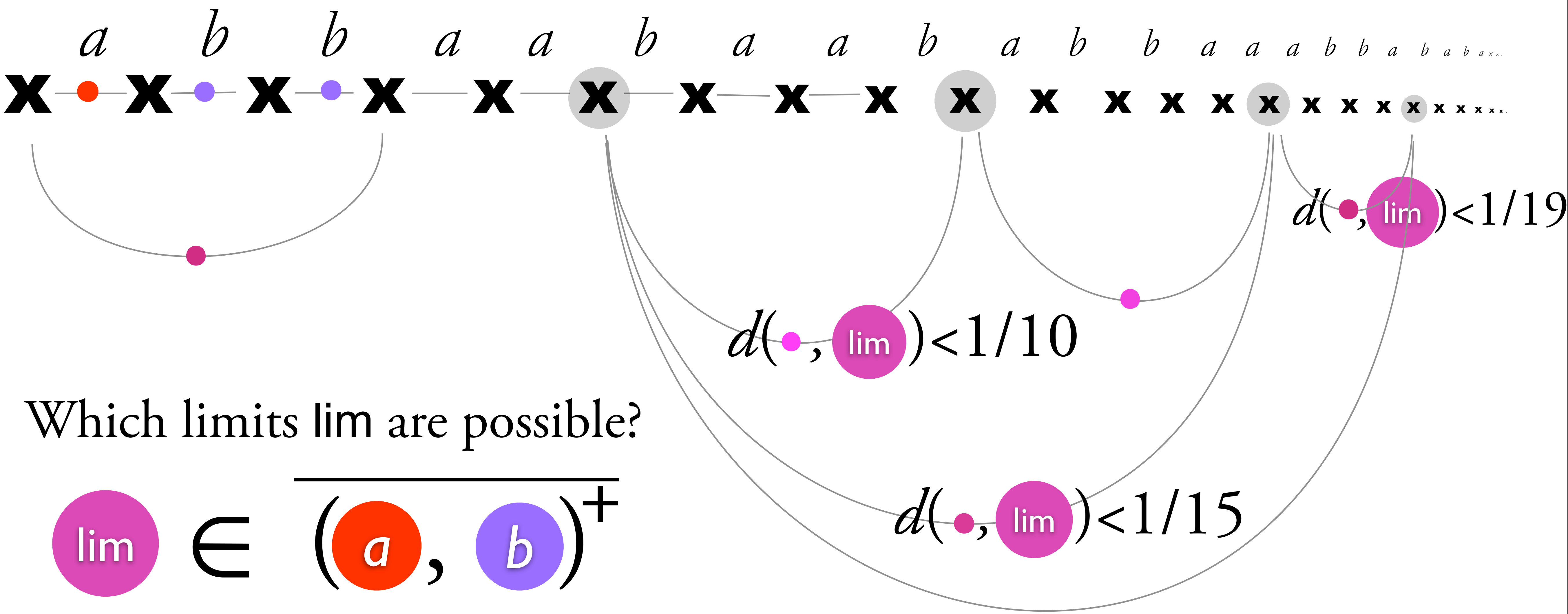

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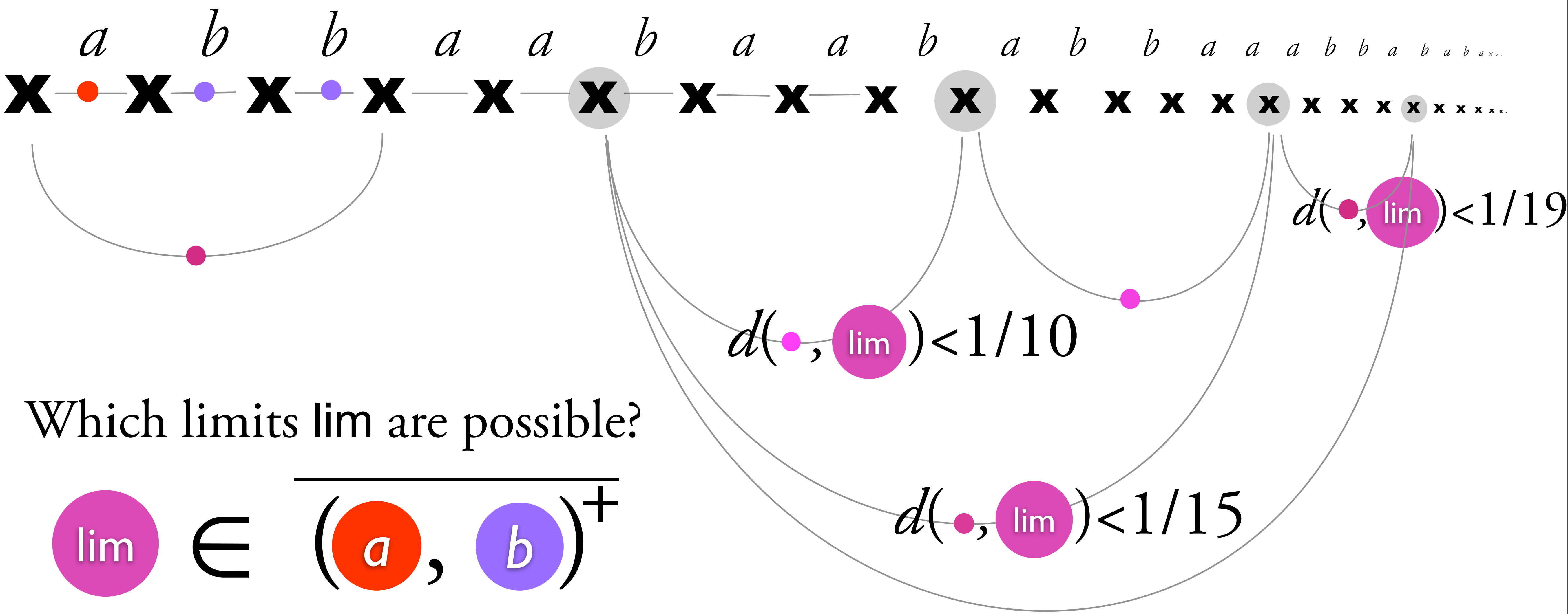
Emptiness of min-automata



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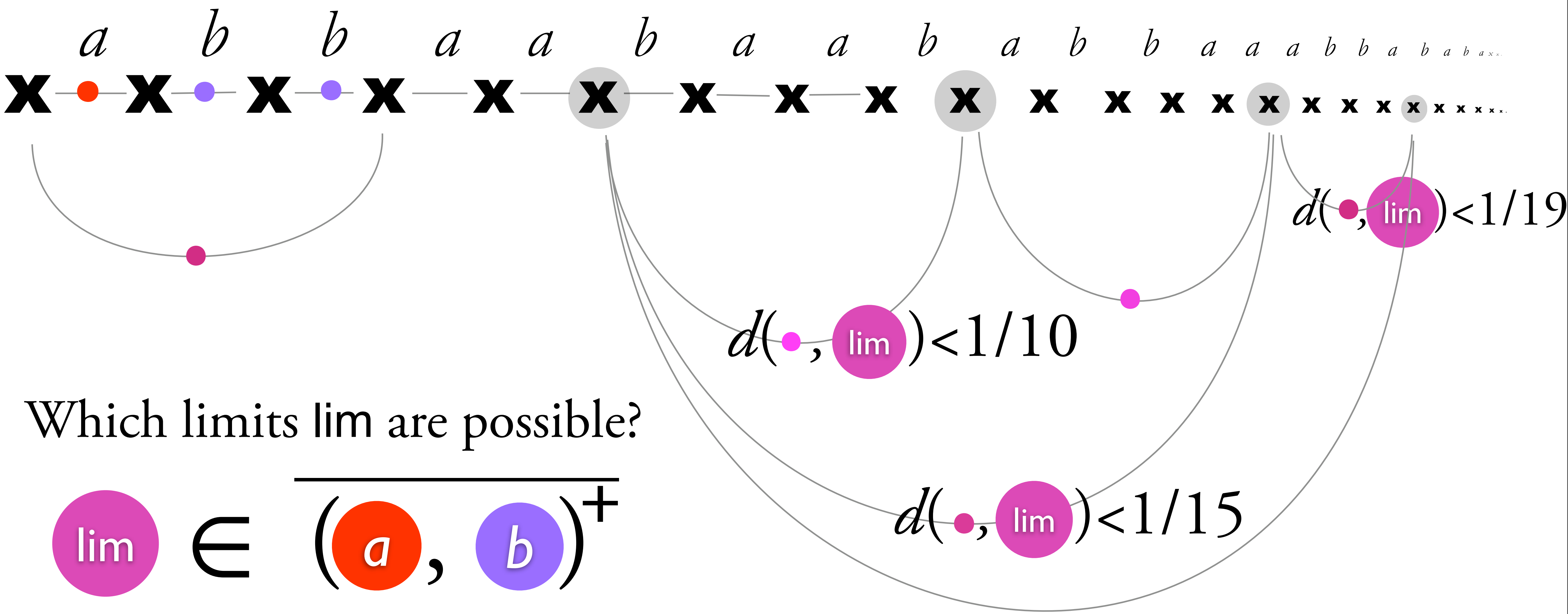
Theorem.

Emptiness of min-automata



Which limits lim are possible?

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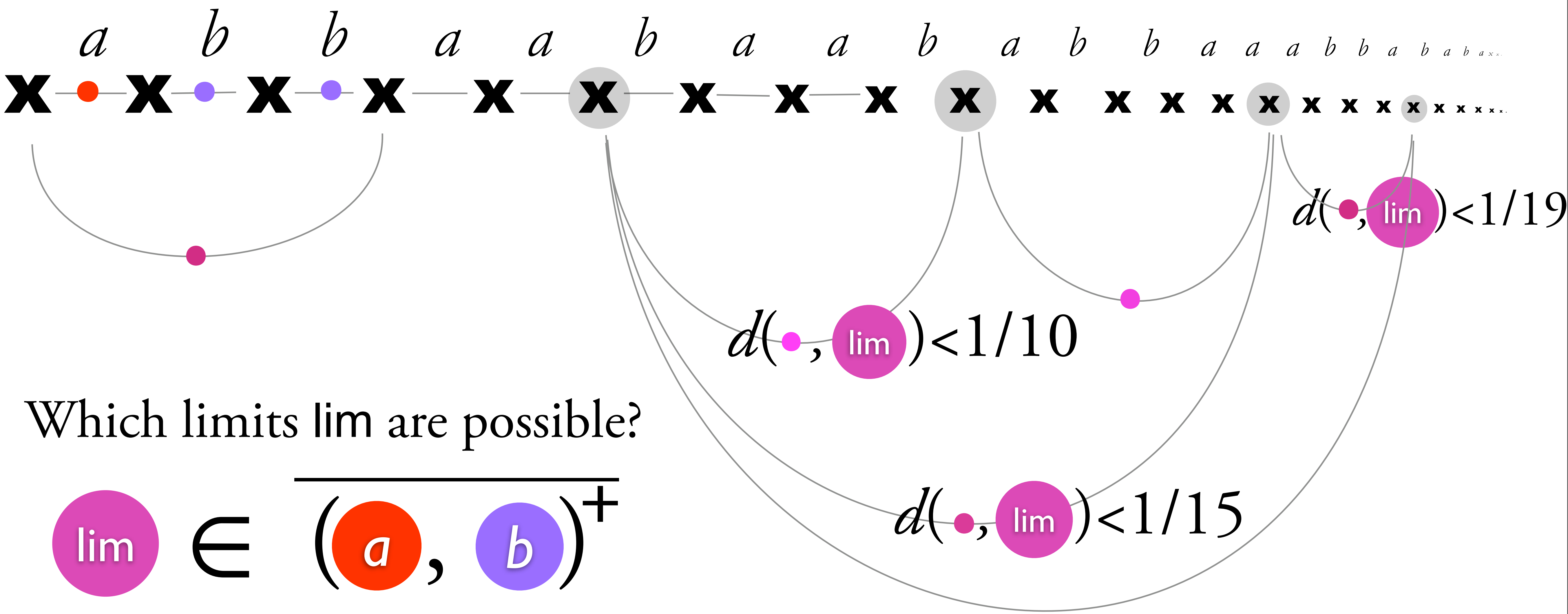
Theorem. $\overline{(a, b)^+}$

Emptiness of min-automata

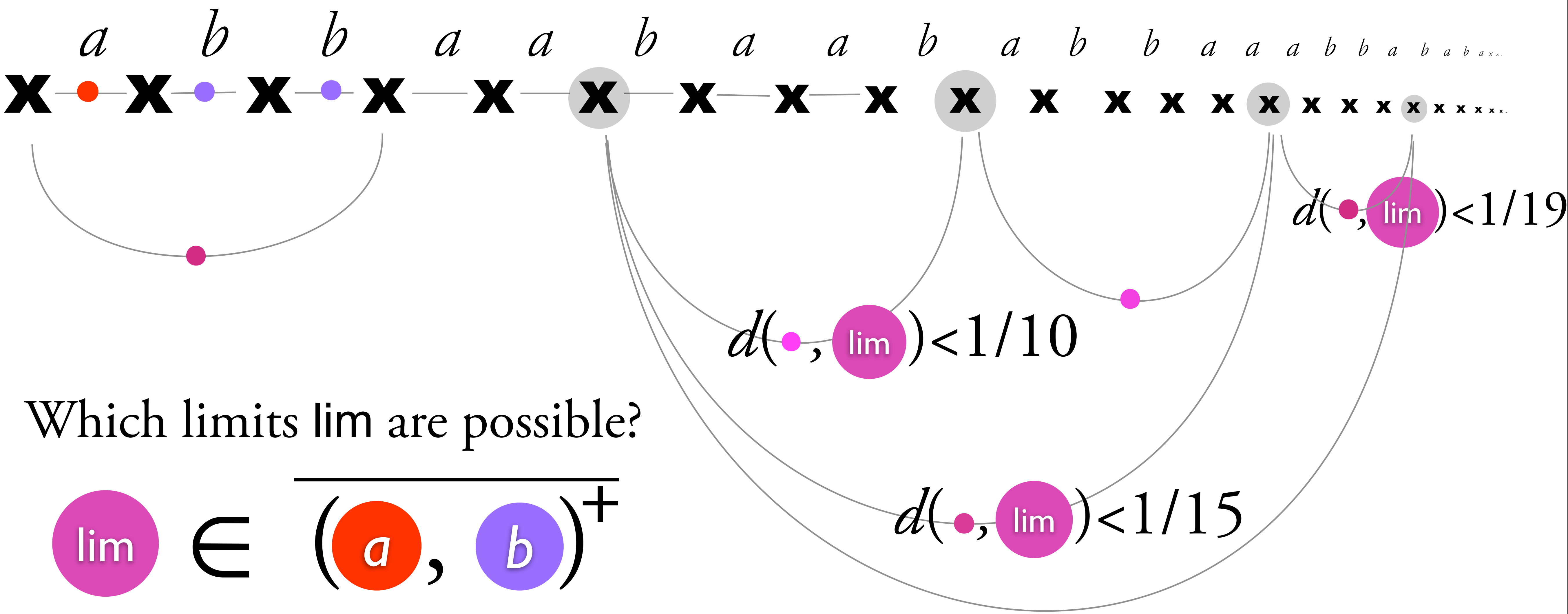


Which limits lim are possible?

$$\text{lim} \in \overline{(a, b)^{+}}$$

Theorem. $\overline{(a, b)^{+}} =$

Emptiness of min-automata



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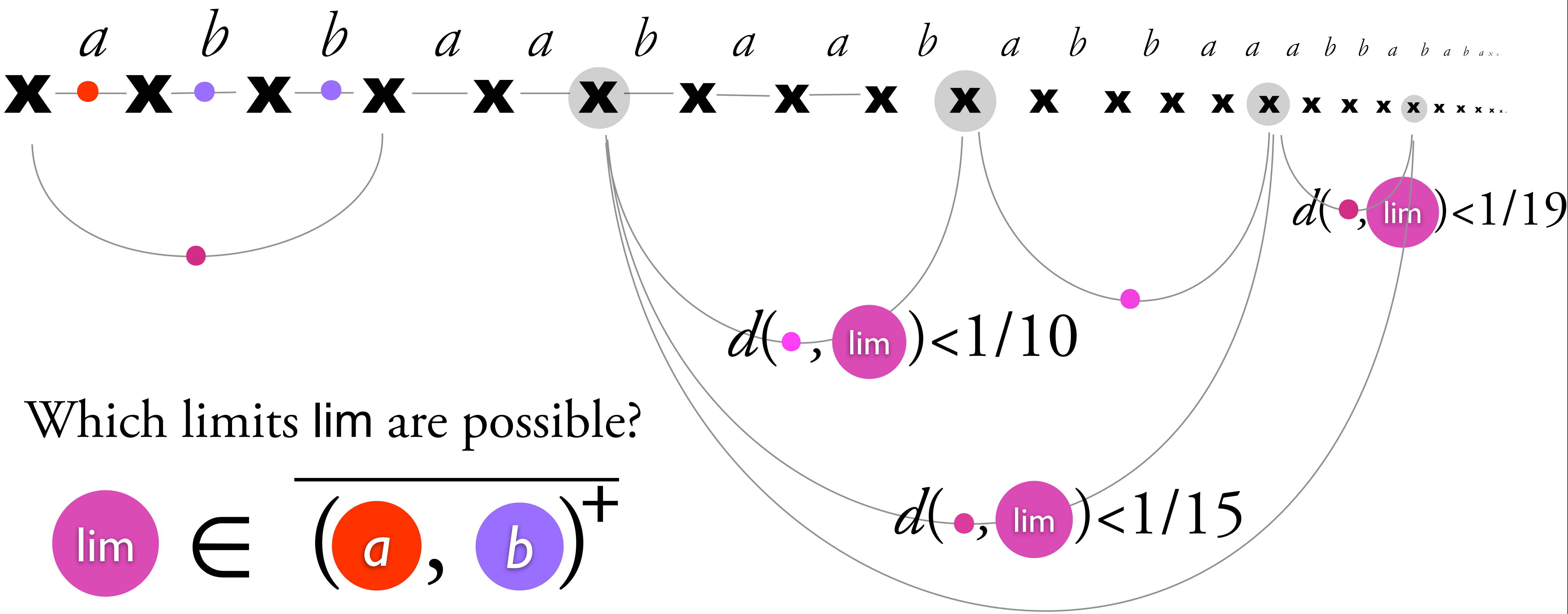
Theorem. $\overline{(a, b)^{+}} = a \quad b$

Emptiness of min-automata

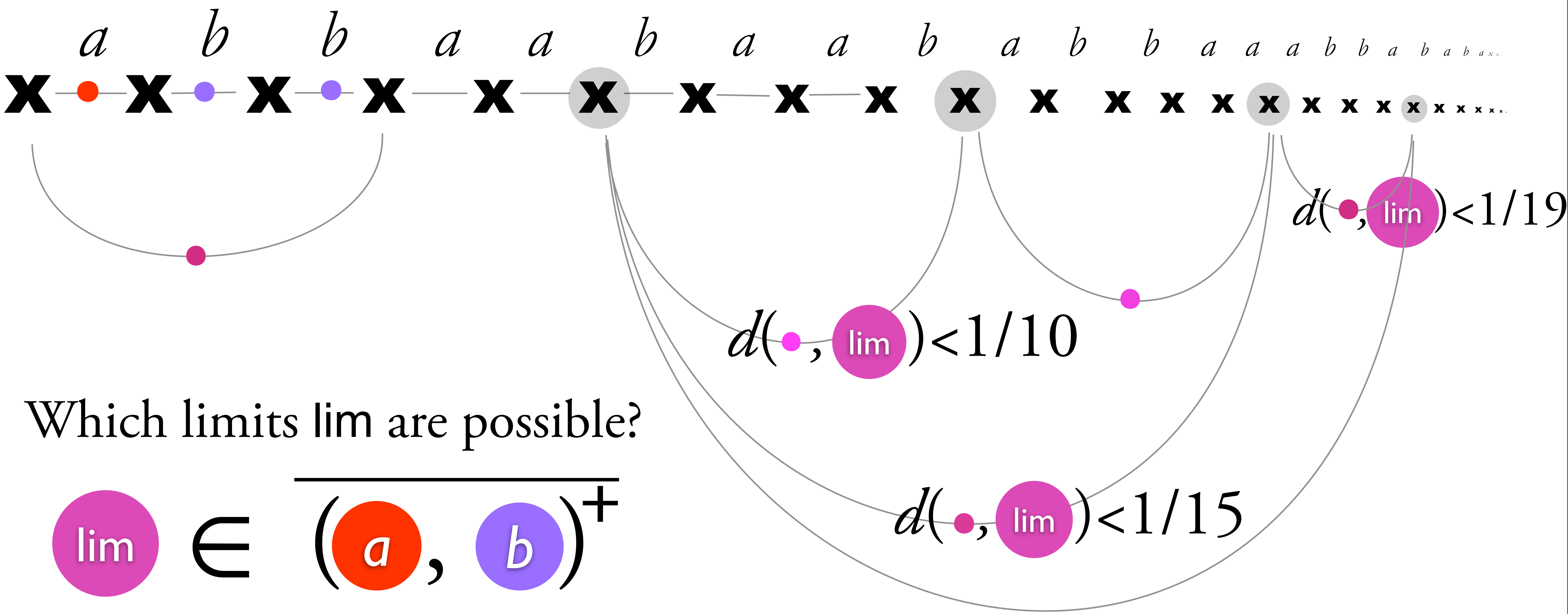


Which limits lim are possible?

$$\text{lim} \in \overline{(a, b)^+}$$

Theorem. $$\overline{(a, b)^+} = (a, b)^{+,\omega}$$

Emptiness of min-automata



Which limits lim are possible?

$$\text{lim} \in \overline{(a, b)^{+}}$$

Theorem. $$\overline{(a, b)^{+}} = (a, b)^{+,\omega}$$

if a, b are matrices over the (min,+)-semiring.

Simon's Factorization Theorem

for semigroups with stabilization

Simon's Factorization Theorem

for semigroups with stabilization

semigroup with stabilization

Simon's Factorization Theorem

for semigroups with stabilization

$$(S , \cdot , {}^{\#})$$

semigroup with stabilization

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S, \cdot}, {}^{\#})$$

semigroup with stabilization

Simon's Factorization Theorem

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Simon's Factorization Theorem

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$$(\underbrace{S\,,\,\cdot}\,,\,{}^{\#})$$

semigroup with stabilization

- $s^{\#} = (s^{n})^{\#}$ *for* n=1,2,3,...

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S\,,\,\cdot}_{\text{semigroup with stabilization}}\,,\ {}^{\#})$$

semigroup with stabilization

- $s^{\#} = (s^{n})^{\#}$ *for* n=1,2,3,...
- $(s\ t)^{\#}\, s = s\,(t\ s)^{\#}$

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S\ ,\ \cdot}_{}\ ,\ {}^{\#})$$

semigroup with stabilization

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- $(s\ t)^{\#}\, s = s\ (t\ s)^{\#}$
- $s^{\#} s^{\#} = s^{\#}$

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S\, , \,\cdot}_{} \, , \; ^{\#})$$

semigroup with stabilization

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- $s^{\#}\, s^{\#} = s^{\#}$
- $e^{\#}\, e = e^{\#}$ *if e is idempotent*

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$$(\underbrace{S , \cdot}_{\text{semigroup with stabilization}} , \ {}^{\#})$$

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Simon's Factorization Theorem

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$\cancel{e = e^{\#}}$ ~~*if e is idempotent*~~

Example 1 (infinite)

$(\{0,1,2,\ldots,\infty\}, +, {}^{\omega}),$

$0^{\omega} = 0, \quad 1^{\omega} = 2^{\omega} = \ldots = \infty$

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S , \cdot}_{} , {}^{\#})$$

semigroup with stabilization

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Example 1 (infinite)
$(\{0,1,2,\ldots,\infty\}, +, {}^{\omega})$,
$0^{\omega} = 0, \quad 1^{\omega} = 2^{\omega} = \ldots = \infty$

Example 2 (finite)
$(\{0,1,\infty\}, +, {}^{\#})$,
$1+1=1, \quad 0^{\#} = 0, \quad 1^{\#} = \infty^{\#} = \infty$

Simon's Factorization Theorem

for semigroups with stabilization

$$(\underbrace{S \,,\, \cdot}_{} \,,\, ^{\#})$$

semigroup with stabilization

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Example 1 (infinite)
$(\{0,1,2,\ldots,\infty\}, +, {}^{\omega})$,
$0^{\omega} = 0, \quad 1^{\omega} = 2^{\omega} = \ldots = \infty$

$$\xrightarrow{1,2,3\ldots \to 1}$$

Example 2 (finite)
$(\{0,1,\infty\}, +, {}^{\#})$,
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Simon's Factorization Theorem

for semigroups with stabilization

Example 2 (finite)

$(\{0,1,\infty\}, +, ^{\#})$,

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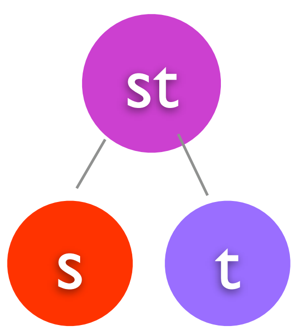
Simon's Factorization Theorem

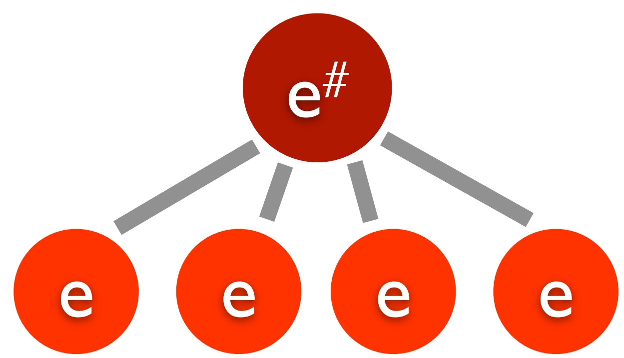
for semigroups with stabilization

Factorization tree of word $w \in S^+$

Use the two rules to construct tree:

binary rule idempotent rule





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for semigroups with stabilization

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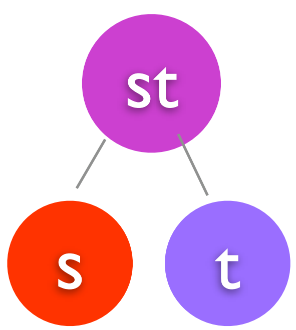
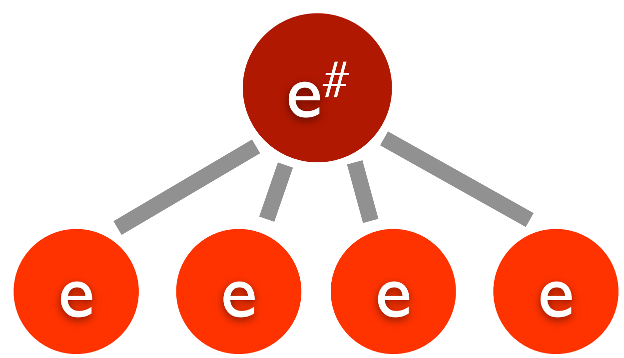


idempotent rule



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Simon's Factorization Theorem

for semigroups with stabilization

Factorization tree of word $w \in S^+$

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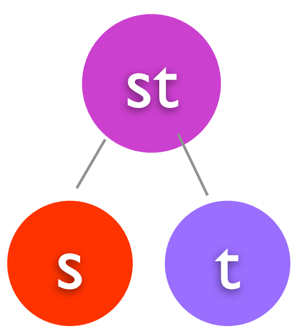

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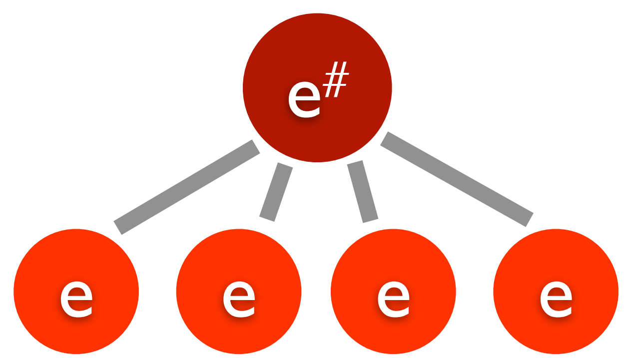

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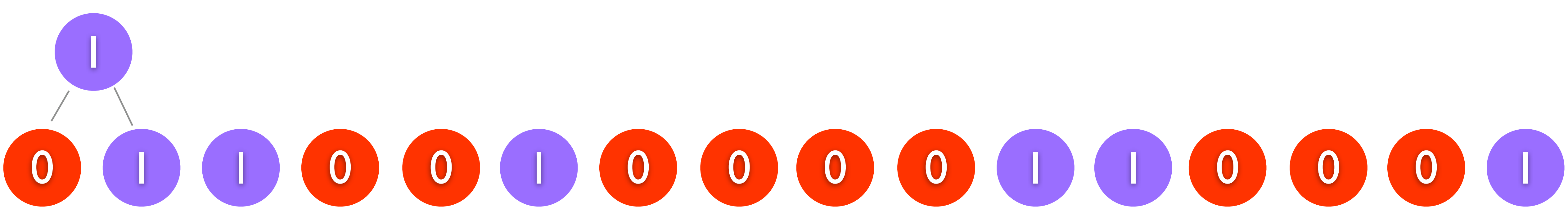

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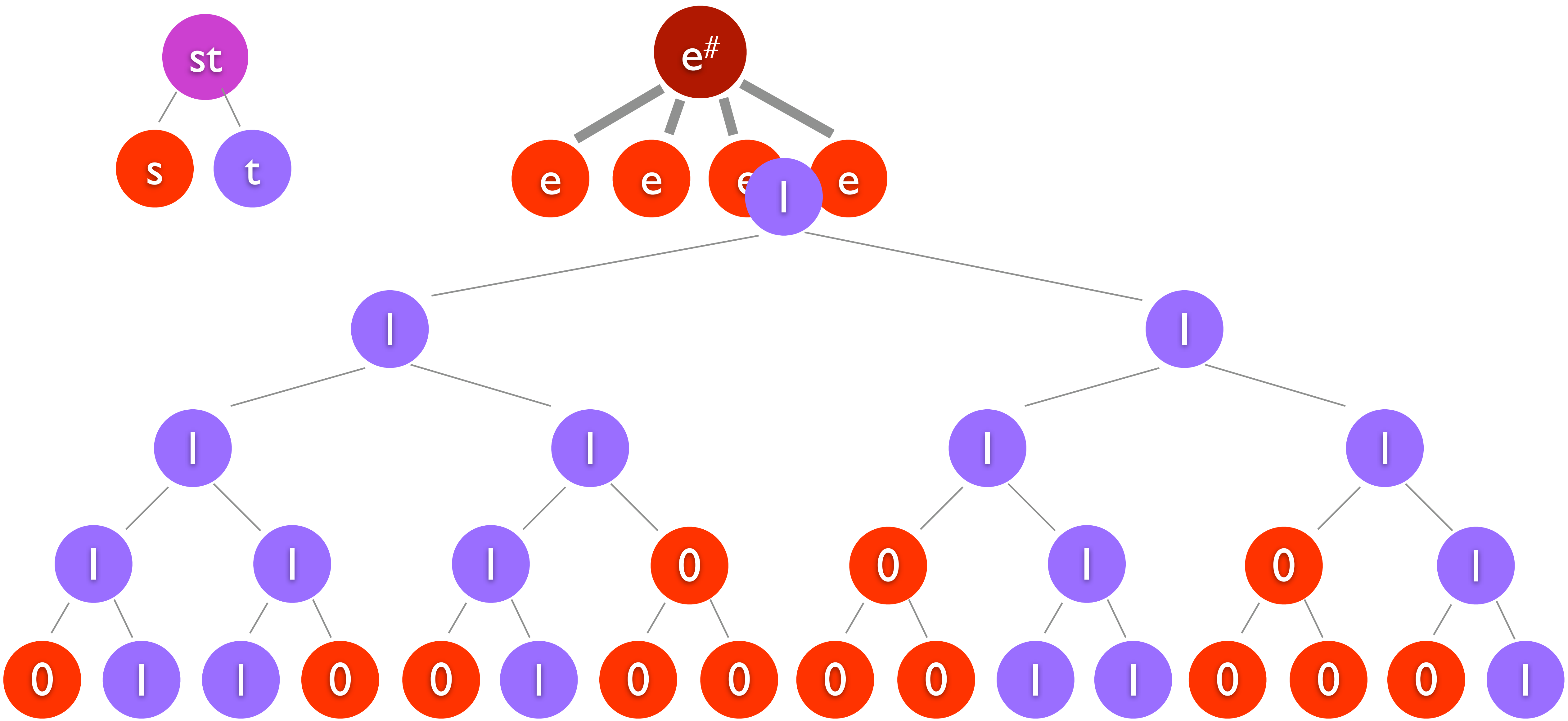

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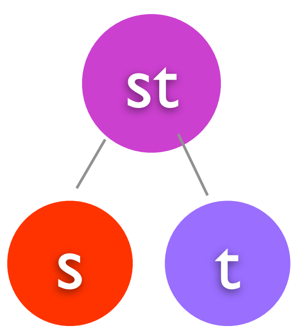

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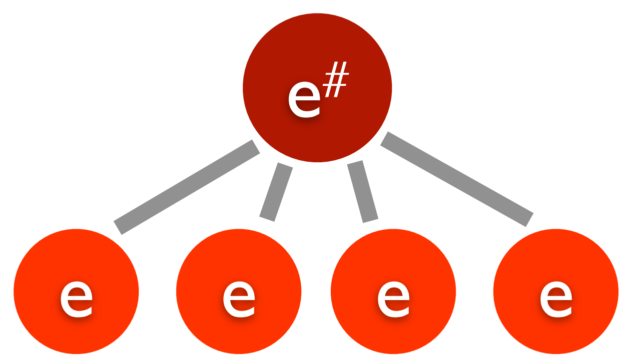

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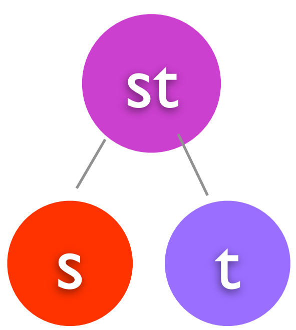

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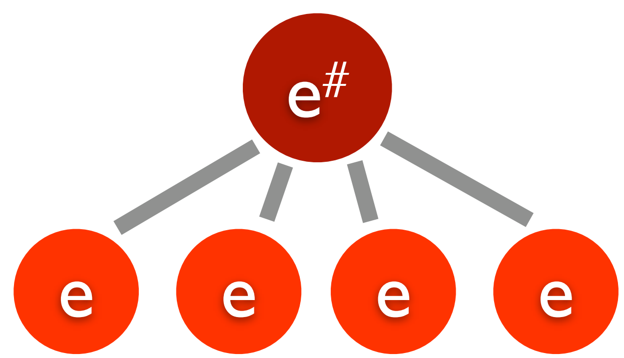

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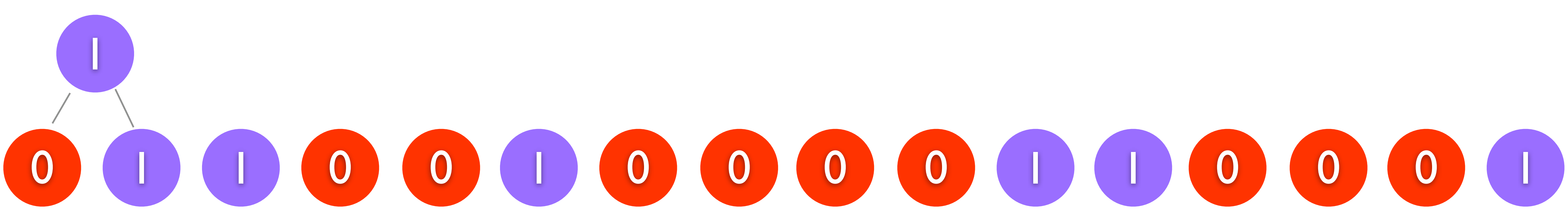

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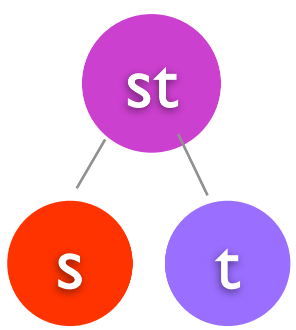

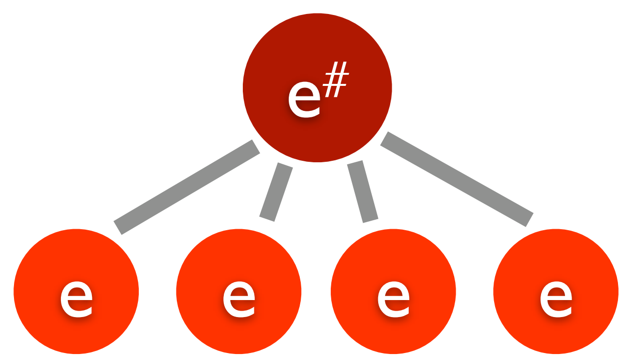

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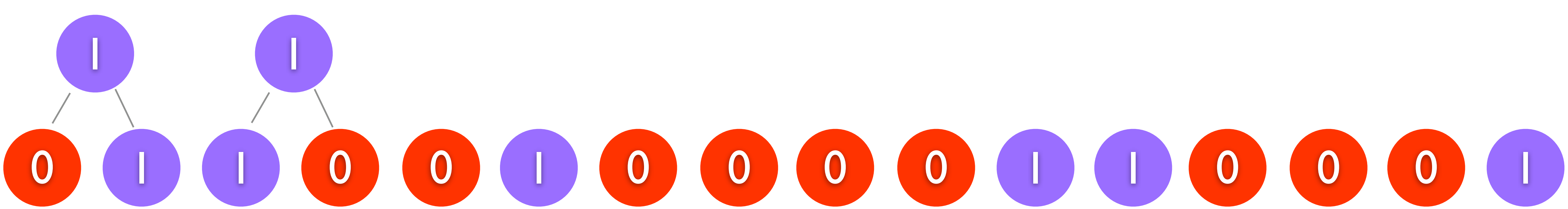

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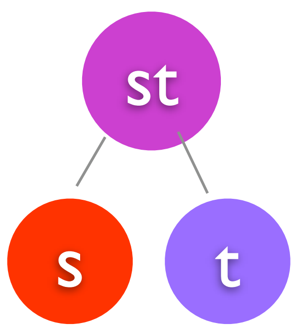

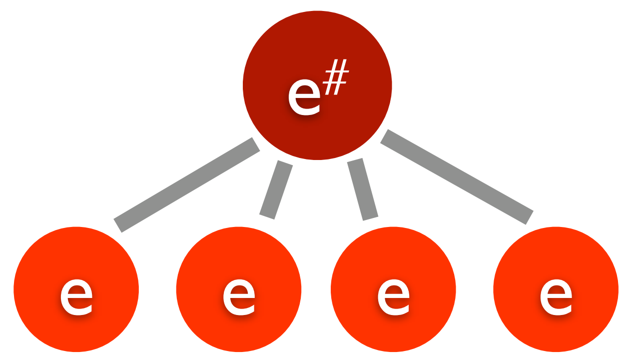

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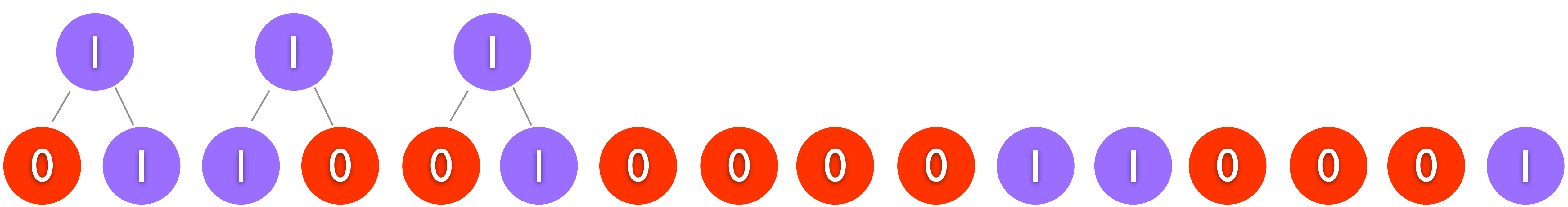

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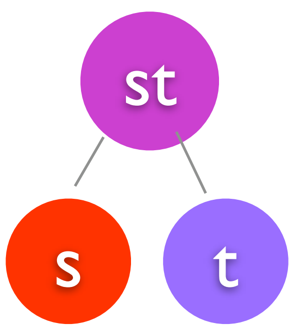

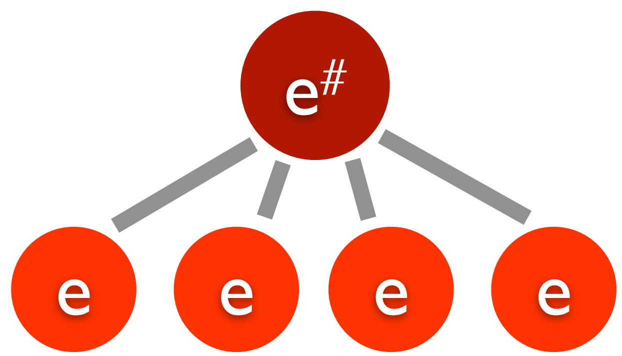

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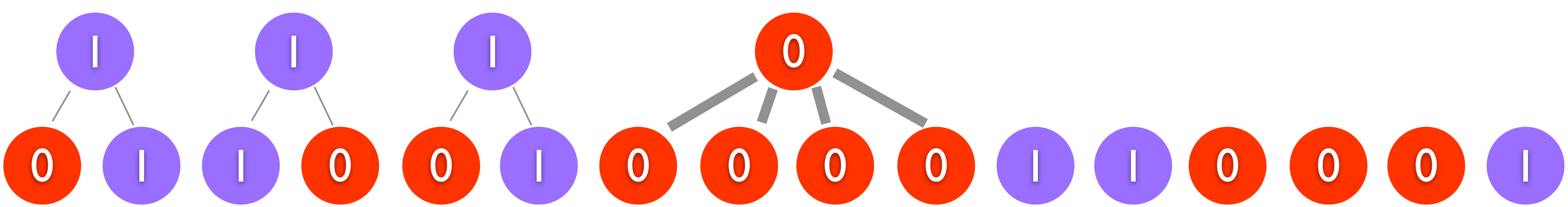

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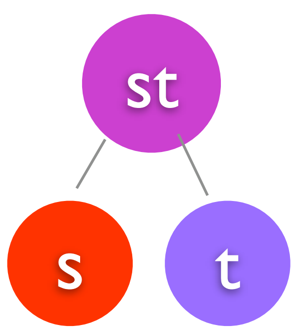

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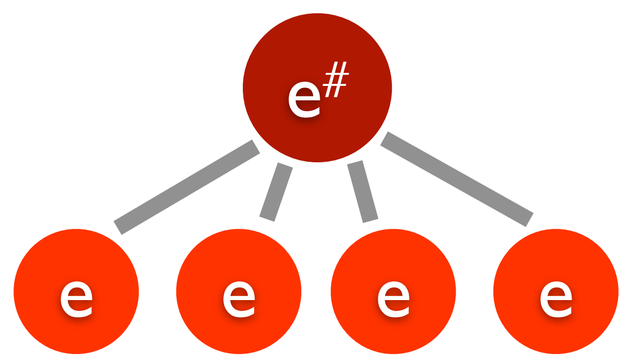

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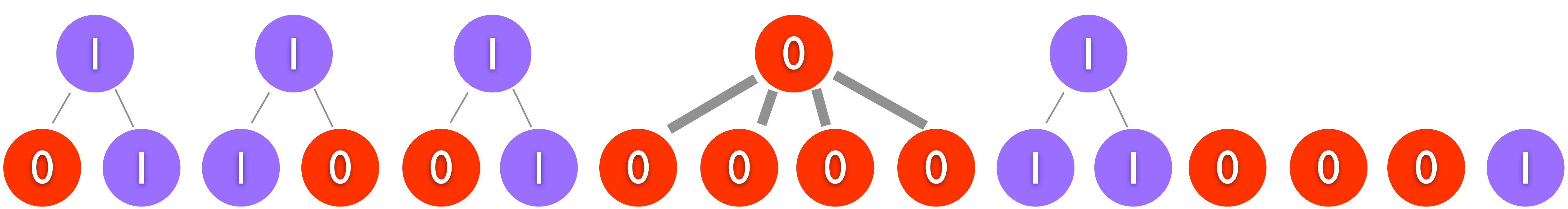

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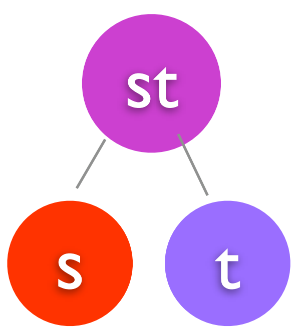

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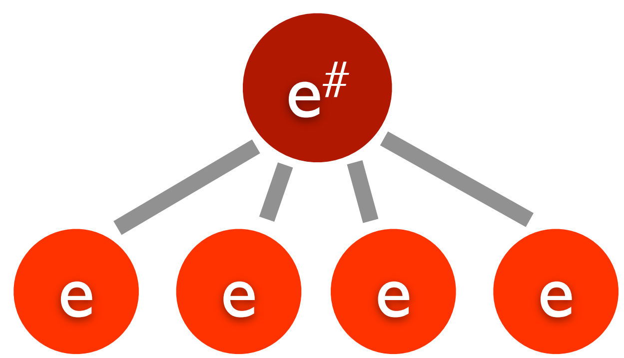

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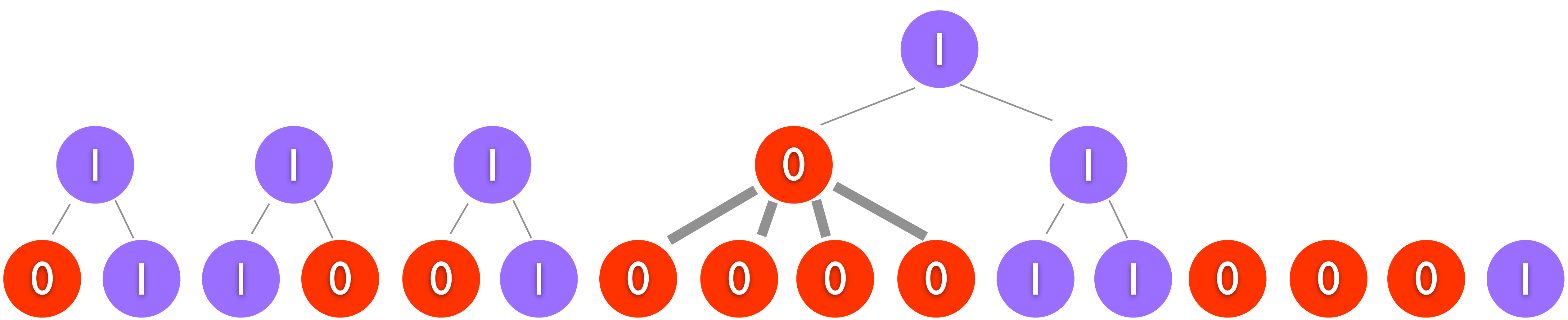

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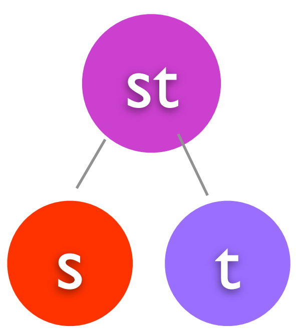

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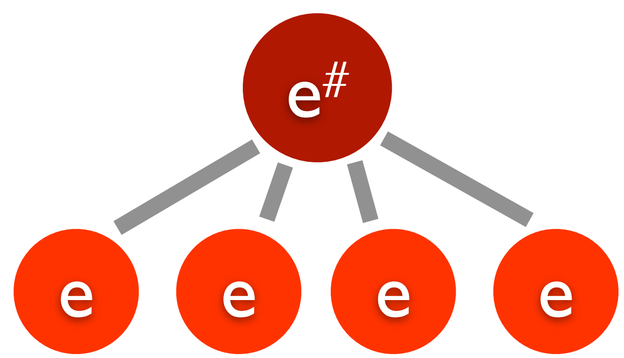

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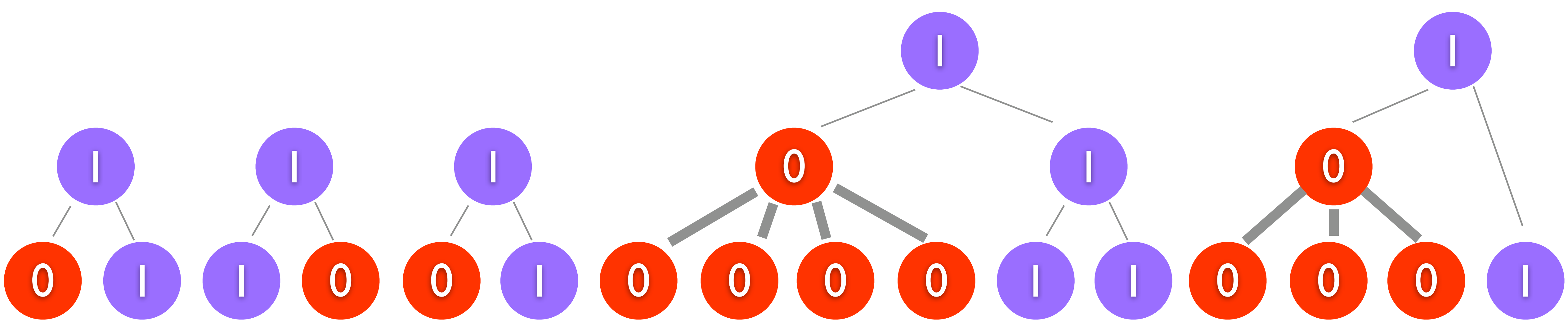

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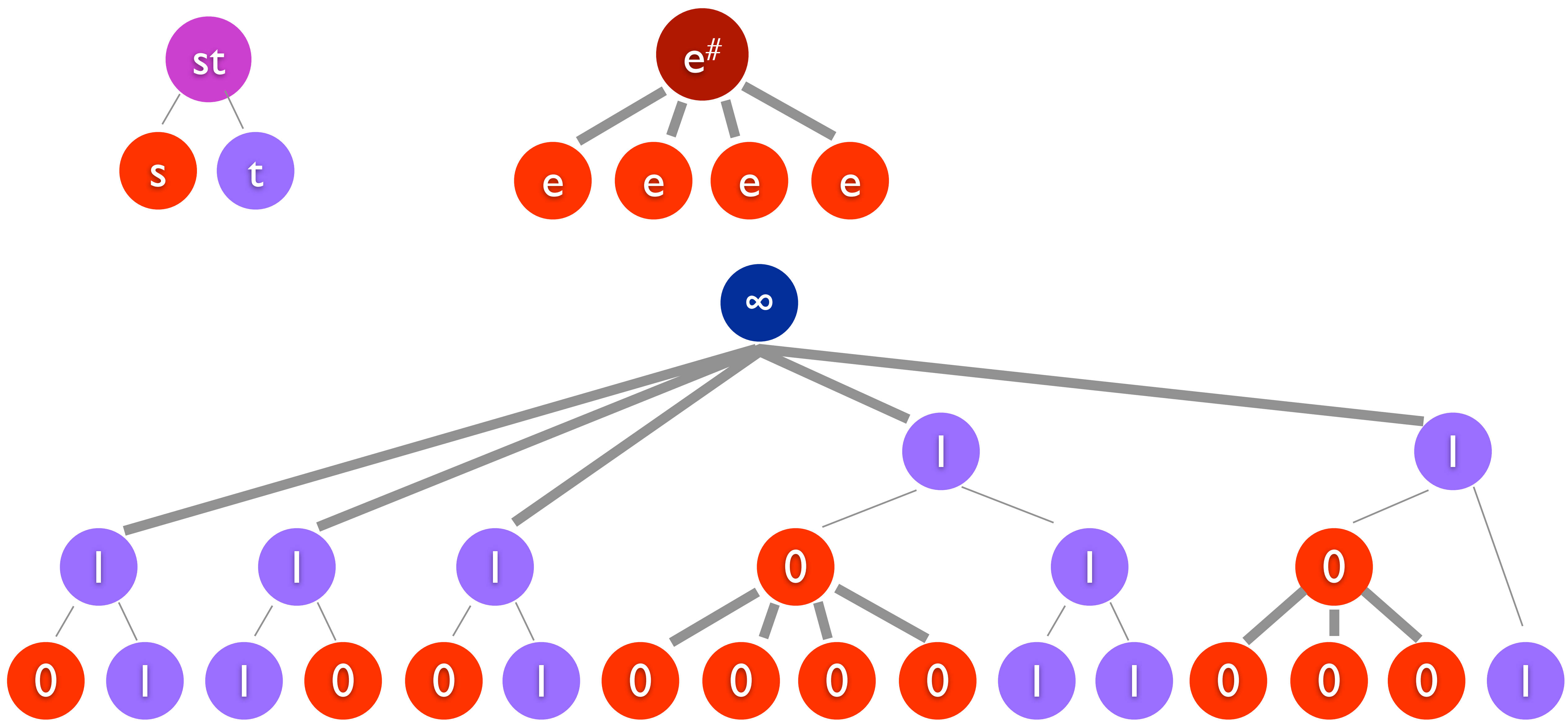

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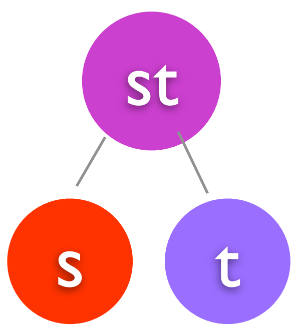

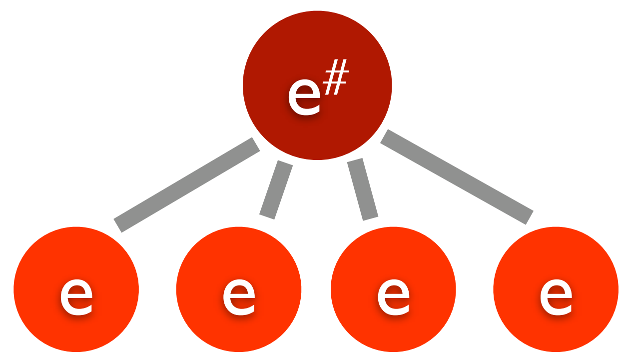

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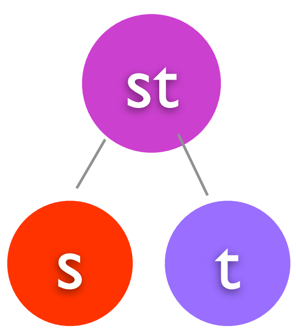

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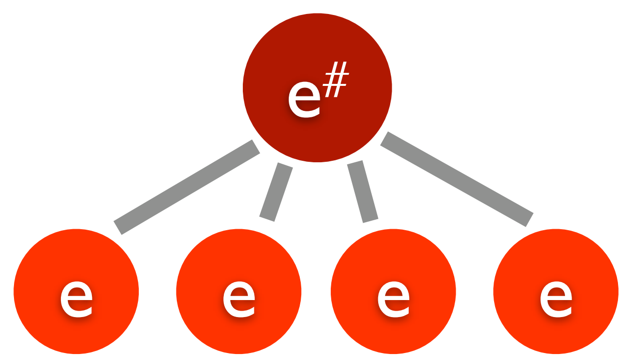

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Theorem. For any finite stabilization semigroup S and word $w \in S^+$ there exists a factorization tree over w of height $\leq 9|S|^2$.

Thank you for your attention!