Wydział Matematyki, Informatyki i Mechaniki UW

Sparsity

lecture notes

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1 Preliminaries

In this lecture, we will talk about undirected, loopless, simple, finite graphs. The letters G, H, ... always denote graphs, $\mathcal{C}$, $\mathcal{D}$, ... always denote classes of graphs. By $\mathcal{G}raphs$ we denote the class of all graphs.

Definition 1.1. A graph G is a pair (V, E), where

- V is finite set of vertices,
- $E \subseteq \binom{V}{2}$ is a set of edges, that is unordered pairs of vertices.

We will denote the cardinality of V by $|G|$ and the cardinality of E by $\|G\|$. Loopless means there are no self-loops (that is $\{v, v\} \notin E$ for any v). Simple graphs cannot have multiple edges between a pair of nodes. Undirected graphs are, well, not directed.

1.1 Planar graphs

We will study the basic notions and techniques which will be developed further throughout the course, as exemplified in the case of planar graphs.

Definition 1.2. A graph $G = (V, E)$ is planar if it can be embedded into $\mathbb{R}^2$. An *embedding* of a graph into $\mathbb{R}^2$ is a pair of mappings (r_V, r_E), where

- $r_V : V \to \mathbb{R}^2$ is injective.
- $r_E : E \to C(I, \mathbb{R}^2)$ satisfies following conditions:
  - $C(I, \mathbb{R}^2)$ is set of continuous functions from the closed unit interval I to $\mathbb{R}^2$;
  - For every $e \in E$, $r_E(e) : I \to \mathbb{R}^2$ is injective;
  - If $e = \{u, v\}$ then $\{r_E(e)(0), r_E(e)(1)\} = \{r_V(u), r_V(v)\}$;
  - If $e, e' \in E$ are two distinct edges, then $r_E(e)((0, 1)) \cap r_E(e')((0, 1)) = \emptyset$.

In other words, a graph G is planar if one can draw it on a the plane in such a way that edges do not intersect, as depicted in the example on the right.

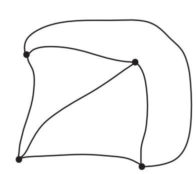


Figure 1: Example of a planar embedding of K_4

Remark 1.3. In the above definition of an embedding, we could replace $\mathbb{R}^2$ by any other topological space X. Then we can talk of graphs which *embed into* X. For instance, we will also talk about graphs which embed into the sphere, the torus, or a surface of genus g (a doughnut with g holes).

Remark 1.4. A more elegant (but equivalent) definition of an embedding is obtained by regarding a graph G as an abstract metric or topological space. This can be done by treating each edge of G as an open unit interval, and defining appropriately a distance function between the points of these intervals or the vertices of G. This defines a topological space M_G. More generally, a graph G is the same as a 1-dimensional *simplicial complex*, and any such complex defines a topological space M_G (called the *geometric realization*). Then, an embedding of G into a topological space X is just a continuous and injective mapping of M_G to X.

Fact 1.5. *A graph embeds into $\mathbb{R}^2$ if and only if it embeds into the two-dimensional sphere S^2.*

Proof. Observe that there is an embedding γ of $\mathbb{R}^2$ into S^2 whose image is S^2 with a point p removed. An example of such an embedding is the stereographic projection. Using this projection, any embedding of G into $\mathbb{R}^2$ can be converted into an embedding of G into S^2. Conversely, given an embedding of G into S^2, there is a point S^2 which is not in the image of the embedding (this follows from topology). Assume without loss of generality that this point is equal to p. Then, by composing with the inverse mapping γ^{-1}, the embedding of G into S^2 gets converted to an embedding of G into $\mathbb{R}^2$. $\square$

Remark 1.6. Any graph G embeds into $\mathbb{R}^3$. Indeed, it suffices to draw G on a plane in an arbitrary way – perhaps with intersecting edges – and then perturb the drawing slightly using the third dimension, so the edges no longer intersect.

Also, any graph G embeds into a surface of a sufficiently high genus g. It suffices to take any embedding of G into $\mathbb{R}^3$ and the draw sufficiently thin tubes along each edge, so that the edges are contained in the surface of these tubes. By merging these tubes at the vertices in a proper way, one gets a surface to which G embeds. The genus of this surface is bounded by the number of "basic cycles" in G.

Finally, observe that if G embeds into a surface S_g of genus g, then it also embeds into a surface of genus $g+1$. To see this, add a tiny handle to S_g, which does not touch the image of G. This gives a surface of genus $g+1$, into which G embeds.

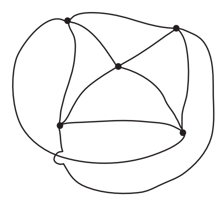


Figure 2: K_5, the smallest non-planar graph.

Definition 1.7 (Subgraph)**.** A graph $H = (V_H, E_H)$ is a subgraph of a graph $G = (V_G, E_G)$ (denoted $H \subseteq G$) if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

In other words, a subgraph of a graph G is a graph created by removing some vertices and/or edges from G.

Definition 1.8 (Induced subgraph)**.** Let $G = (V, E)$ and $S \subseteq V$. The subgraph of G induced by S is the graph $G[S] = (S, E \cap \binom{S}{2})$. If $S = V - T$, then $G[S]$ is also denoted by $G - T$. The induced subgraph is a subgraph obtained from G by removing some vertices. Of course, when a vertex is removed, all edges connected to it are also removed.

Fact 1.9. *A subgraph of a planar graph is also planar.*

We will now show probably the most famous theorem concerning planar graphs, Euler's theorem. First, we need to define what a face of an embedding is.

Definition 1.10. Let (r_V, r_E) be an embedding of G into $\mathbb{R}^2$. A *face* of the embedding is a (topologically) connected component of

$$\mathbb{R}^2 - \bigcup_{e \in E} \mathrm{Im}(r_E).$$

Now we can formulate Euler's theorem.

Theorem 1.11 (Euler)**.** *Let G be a connected planar graph with at least one edge. Then:*

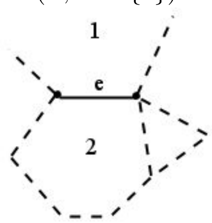
$$v - e + f = 2 \tag{1}$$

where $v = |G|$ *is the number of vertices,* $e = \|G\|$ *is the number of edges, and* f *is the number of faces in any embedding of* G*.*

Proof. The proof proceeds by induction on e.
Induction base:
G is a tree. Then $e = \|G\| = |G| - 1 = v - 1$ and there is only one face (formally, this requires a topological argument). Therefore

$$v - e + f = v - v + 1 + 1 = 2 \tag{2}$$

Induction step:
G is connected and is not a tree, therefore, there must be a cycle in G. Suppose that $E \ni e = \{u, v\}$ lies on a cycle and $G_e = (V, E - \{e\})$ is the graph obtained from G by removing e.



Intuitively, it is clear that removing e connects faces 1 and 2, therefore G_e has one edge and one face less than G. Equation (1) holds in G_e by inductive assumption, therefore it also holds in G.

What remains is to show that removing an edge from a cycle actually decreases the number of faces in G. This can be proved using the Jordan curve theorem but it is outside the scope of these notes. □

Euler's theorem is a powerful tool for proving properties of planar graphs. For example, we get the following.

Corollary 1.12. *Let* G *be planar. Then*

$$\frac{\|G\|}{|G|} < 3 \tag{3}$$

Definition 1.13. The value $\|G\|/|G|$ is called the *edge density* of the graph G.

Proof. First, observe that each edge belongs to at most 2 faces (unless G is). Furthermore, each face has at least 3 edges touching it. Therefore, $2e \geq 3f$. It follows that:

$$0 < 2 = v - e + f \leq v - e + \frac{2}{3}e = v - \frac{1}{3}e.$$

□

Definition 1.14. Let's denote $\Delta(G)$ by the maximal degree of the vertices in G and by $\delta(G)$ as the minimal degree. When G is clear form context, we will use simply Δ and δ. Additionally, we define $\mathbb{E}\text{deg}$ as average degree of vertex in G, so $\mathbb{E}\text{deg} = \frac{\sum_{v \in V} deg(v)}{|G|} = \frac{2\|G\|}{|G|}$.

Observe, that $\Delta \geq \mathbb{E}\text{deg} \geq \delta$. As for planar graphs, $\mathbb{E}\text{deg} < 6$, we get:

Corollary 1.15. *Each planar graph has a vertex of degree at most 5.*

Remark 1.16. This bound is tight. The edges of the regular icosahedron (the regular polyhedron with 20 faces, see Figure 3) form a graph which embeds into the sphere (the regular icosahedron is, topologically speaking, a sphere) and in which every vertex has degree 5.

Definition 1.17. We say that a graph G is *n-colorable* if there exists a coloring of the vertices using n colors, such that neighboring vertices get different colors.

Fact 1.18. *Planar graphs are 6-colorable.*

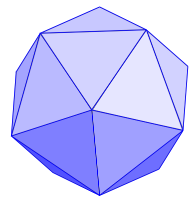


Figure 3: The regular icosahedron.

Proof. Induction on number of vertices, $|G|$.
Induction base:
$|G| \leq 6$. As there are at most 6 vertices, each can get a different color.
Induction step:
From 1.15, there exists a vertex v in G of degree at most 5. Let's remove this vertex. By the inductive assumption, the remaining graph can be colored using 6 colors. We apply this coloring to G and extend it to v by assigning to v any legal color. We can always do this, because v has at most 5 neighbours in G, therefore there is always one legal color available. □

Remark 1.19. It is not very difficult to improve the above construction to obtain a 5-coloring of any planar graph. The famous Four Color Theorem says that in fact any planar graph admits a 4-coloring.

1.2 Properties of k-degenerated graphs

Corollary 1.15 and the proof of Fact 1.18 lead us to the following notion.

Definition 1.20 (k-degeneration)**.** A graph G is k-degenerate if in any non-empty subgraph H of G contains a vertex of degree at most k (within H).

Examples:

- Trees are 1-degenerate.
- As a consequence of Corollary 1.15, planar graphs are 5-degenerate.

Fact 1.21. *A k-degenerate graph is $k+1$-colorable.*

Proof. The same as in the proof of Fact 1.18. □

Definition 1.22. The maximal average degree of graph G, denoted $\operatorname{mad}(G)$ is

$$\operatorname{mad}(G) = \max_{H \subseteq G} \mathbb{E}\deg(H) = \max_{H \subseteq G} \frac{2\|H\|}{|H|} \tag{4}$$

Maximal average degree and k-degeneration are closely related to each other.

Fact 1.23.
1. *Every graph G is $\lfloor \operatorname{mad}(G) \rfloor$-degenerate.*
2. *If G is k-degenerate then $\operatorname{mad}(G) < 2k$.*

Proof. For the first item, let H be a subgraph of G. Obviously $\delta_H \leq \mathbb{E}\deg(H) \leq \operatorname{mad}(G)$, because $\operatorname{mad}(G)$ is maximum over all subgraphs of G.

To prove the second item, suppose that G is k-degenerate. We will need following lemma:

Lemma 1.24. *For any graph G, there exists an (induced) subgraph $H \subseteq G$ s.t. $\delta_H > \frac{\|G\|}{|G|}$.*

From this lemma the upper bound for $\mathrm{mad}(G)$ follows easily: suppose that $\mathrm{mad}(G) \geq 2k$. Then there is a subgraph with $\frac{2\|H\|}{|H|} = \mathrm{mad}(G) \geq 2k$. Therefore $\frac{\|H\|}{|H|} \geq k$. By the lemma, there exists subgraph H' of H s.t. $\delta_{H'} > \frac{\|H\|}{|H|} \geq k$ which contradicts the assumption that G is k-degenerate.

It remains to prove Lemma 1.24. We will use a common and useful technique: iterative removal of vertices of small degree.

For a graph H, if there is a vertex v such that $deg_H(v) \leq \frac{\|H\|}{|H|}$, then by H' we denote the graph $H - \{v\}$ obtained by removing v. Observe that

$$\|H'\| \geq \|H\| - \frac{\|H\|}{|H|} = \frac{\|H\|}{|H|}(|H| - 1) = \frac{\|H\|}{|H|}|H'|, \tag{5}$$

so the operation $H \mapsto H'$ does not decrease the ratio $\frac{\|H\|}{|H|}$, i.e. the edge density.

We apply this operation iteratively to G until we obtain a graph H such that $\delta_H > \frac{\|H\|}{|H|}$:

$$G \supset G' \supset G'' \supset \cdots \supset H \tag{6}$$

In particular, $\delta_H > \frac{\|H\|}{|H|} \geq \frac{\|G\|}{|G|}$ which completes the proof. $\square$

We finish our study of graph degeneration with the following, useful characterization of k-degenerate graphs, in terms of orientations small indegree.

Definition 1.25. An *oriented graph* is a pair $\vec{G} = (V, \vec{E})$, where $\vec{E} \subseteq V \times V$ is a set of non-repeating pairs of elements of V (so there are no self-loops). We call elements of $\vec{E}$ *directed edges*. For a vertex $v \in V$, by $\mathrm{indeg}_{\vec{G}}(v)$ and $\mathrm{outdeg}_{\vec{G}}(v)$ we denote the *indegree* and *outdegree* of v in $\vec{G}$, i.e. the number of directed edges leading to (respectively, from) v.

We say that an oriented graph $\vec{G}$ is an *orientation* of a graph G, if they have the same vertex sets, and

- For each edge $\{u, v\}$ of G, either (u, v) or (v, u) is a directed edge of $\vec{G}$ (but not both); and
- conversely, for each directed edge (u, v) of $\vec{G}$, $\{u, v\}$ is an edge of $\vec{G}$.

Note that there are $2^{\|G\|}$ different orientations of G.

Theorem 1.26. *Let G be a graph. The following conditions are equivalent:*

1. *G is k-degenerate.*
2. *There is a linear ordering $\prec$ on V_G such that for each $v \in V_G$, the number of edges connected with v and $\prec$-greater than v is at most k, i.e.*

$$|\{u : v \prec u \text{ and } \{u, v\} \in E_G\}| \leq k.$$

3. *There is an acyclic orientation $\vec{G}$ of G such that each vertex has in-degree at most k.*

Proof. $1 \Rightarrow 2$**:**
We construct ordering $\prec$ as follows:

$$\begin{array}{lcl}
v_1 & - & \text{vertex of } G \text{ with degree at most } k \\
\curlywedge & & \\
v_2 & - & \text{vertex of } G - \{v_1\} \text{ with degree at most } k \\
\curlywedge & & \\
v_3 & - & \text{vertex of } G - \{v_1, v_2\} \text{ with degree at most } k \\
\curlywedge & & \\
\vdots & & \vdots \\
\curlywedge & & \\
v_n & - & \text{vertex of } G - \{v_1, \ldots, v_{n-1}\} \text{ with degree at most } k
\end{array}$$

Why is this construction correct? G is k-degenerate, so any subgraph of G has a vertex of degree at most k. The graphs $G_i = G - \{v_1, \ldots, v_i\}$ are induced subgraphs of G, therefore each v_i has at most k neighbours in G_i and all vertices greater than v_i with respect to $\prec$ belong to G_i so condition 2 holds.
$2 \Rightarrow 3$**:**
We direct G according to the ordering $\prec$:

$$(u, v) \text{ is an edge of } \vec{G} \iff \{u, v\} \in E \text{ and } u \succ v. \tag{7}$$

This ordering is acyclic (since $\prec$ is an ordering) and the in-degree is at most k.
$3 \Rightarrow 1$**:**
Let $\vec{G}$ be an orientation of G fulfilling the condition 3. Consider any nonempty subgraph H of G and orient it according to orientation of G, yielding $\vec{H}$. There exists a vertex v in $\vec{H}$ with $\mathrm{outdeg}_{\vec{H}}(v) = 0$. Otherwise, we would find a cycle in $\vec{H}$, which contradicts condition 3. Because $\mathrm{indeg}_{\vec{H}}(v) \leq k$, it follows that $deg_H(v) \leq k$. □

1.3 Algorithmic aspects

We will present a simple algorithm, which will be central in the remaining part of the course. The algorithm finds, for a given k-degenerate graph G, an acyclic orientation of indegree at most k, in linear time.

Proposition 1.27. *There is an algorithm for the following problem:*

> **Input***: A k-degenerate graph G, given by its adjacency list representation (i.e. a table associating with each vertex v the list of neighbors of v in G)*
>
> **Output***: An acyclic orientation $\vec{G}$ of outdegree at most k.*

whose running time is $O(k \cdot \|G\|)$.

Proof. The algorithm proceeds as follows. Let L_v denote the incidence list of a vertex v (i.e. the list of neighbors of v).

First, compute a list L of vertices of degrees at most k; initially, let $i = 1$. Then, repeat the following steps.

1. Remove a vertex v from the list L,
2. For every $w \in L_v$, remove v from the list L_w, and compute the new length of L_w; if this length is at most k, then add w to L,
3. Associate a number i with the vertex v, and increment i.

Terminate when L is empty.

The maintained invariant is that L is the set of vertices of degree at most k in the graph $G - S$, where S is the set of vertices to which a number has been associated. In the end, every vertex has a distinct number associated to it. It follows from the proof of Theorem 1.26 that this induces a linear ordering $\prec$ which satisfies the second condition of the theorem. From this – similarly, but dually as in Theorem 1.26 – we can easily compute an acyclic orientation of outdegree at most k: for each vertex v of G, remove from its incidence list those vertices u, such that the number associated with u is smaller than the number associated with v. The resulting structure is the list representation of a graph $\vec{G}$ which satisfies the required condition. □

Applications There are two common data structures for representing graphs:

- the adjacency matrix, whose size is $O(|G|^2)$, and checking adjacency can be done in time $O(1)$
- the adjacency list representation, whose size is $O(\|G\|)$, but checking adjacency is in time $O(\|G\|)$.

For k-degenerate graphs there is a data structure which combines the benefits of both representations: it is the representation computed by the algorithm in Proposition 1.27. Indeed, given this data structure, checking for adjacency of u and v can be done in time $O(k)$: test whether v is on the adjacency list of u and vice versa.

1.4 Fixed-parameter tractability

Fix an input alphabet A, and an alphabet for *parameters*, B.

Definition 1.28 (FPT)**.** A decision problem $L \subseteq (A^+, B^+)$:

> **Input**: P, $w \in A^+$ (called the *input*) $p \in B^+$ (called the *parameter*)
>
> **Output**: Does (w, p) belong to L?

is called *Fixed Parameter Tractable* if there is an algorithm deciding $(w, p) \in L$ in time $f(|p|) \cdot |w|^d$ for some computable function f and for some constant d. We say that L is *fixed-parameter linear* if $d = 1$.

Fact 1.29. *Fix a number k. The following problem is decidable in time $O(\|G\|)$:*

> **Input***: A k-degenerate graph G, given by its list representation, and a number n*
>
> **Decide***: Does G contain a clique of size n?*

More precisely, the following problem is fixed-parameter linear:

> **Input***: A k-degenerate graph G, given by its list representation, and a number n, where k, n are the parameters*
>
> **Decide***: Does G contain a clique of size n?*

One of the guiding themes of this course is to provide FPT algorithms for problems involving sparse graphs. These algorithms will be obtained as generalizations of the above algorithm.

Remark 1.30. A widely believed complexity-theoretical assumption is that the following problem is not decidable in time $O(f(n) \cdot |G|^d)$, for any fixed number d:

> **Input**: A graph G, given by its list representation, and a number n
>
> **Decide**: Does G contain a clique of size n?

Excercise 1.1. Show that the following problem is not decidable in time $O(\|G\| \cdot \|H\|)$:

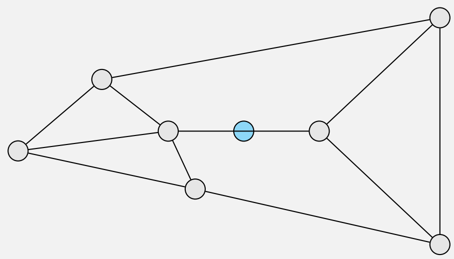
> **Input**: A k-degenerate graph G, given by its list representation, and a graph H
>
> **Decide**: Does G contain a subgraph isomorphic to H?

2 Minors

2.1 Definitions

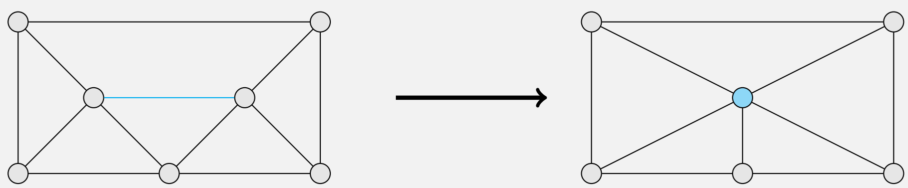
Definition 2.1 (subdivision)**.** A *subdivision* of G is obtained by inserting new vertices on edges.

Example:



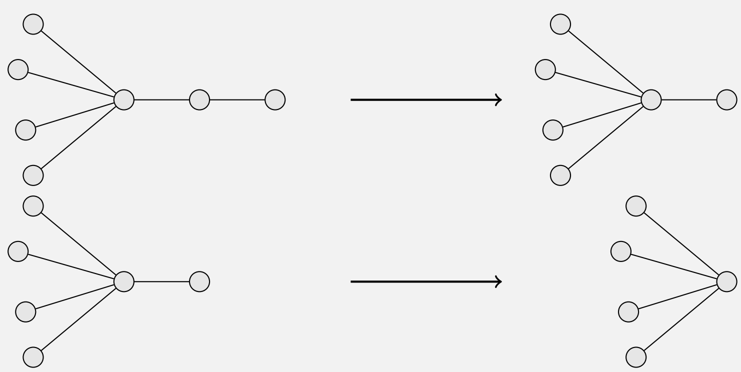
Definition 2.2 (contraction)**.** A *contraction* of an edge $e \in G$ i obtained by replacing the edge by a vertex.

Example:



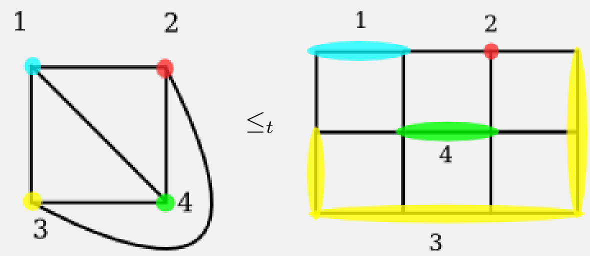
Definition 2.3 (topological contraction)**.** A contraction of $\{u, v\}$ is *topological* if either u or v has $deg \leq 2$ (almost inverse of subdivision).

Example:



Definition 2.4 (minor)**.** Let H, G be two graphs. We say that H is a *minor* of G ($H \leq_m G$) if H can be obtained from a subgraph of G by a series of contractions. We say that H is a *topological minor* of G (written $H \leq_t G$) if the contractions can be made topological.

Example:



Fact 2.5. *Suppose that H is obtained from G by any series of operations of the following types:*

1. *Edge removal,*
2. *Vertex removal,*
3. *Edge contraction.*

Then H is a minor of G. If the edge contractions are topological, then H is a topological minor of G.

Corollary 2.6. $\leq_m$ *and* $\leq_t$ *are partial orders.*

There is yet another, useful characterization of minors.

Fact 2.7. *Let H and G be graphs. Then H is a minor of G if and only if there is a family $(P_v)_{v \in V_H}$ of subgraphs of G such that:*

- *each graph P_v is connected,*
- *P_v and P_w are vertex-disjoint for $v \neq w$*
- *for each edge $\{u, v\} \in E_H$ there is an edge between P_u and P_v in G.*

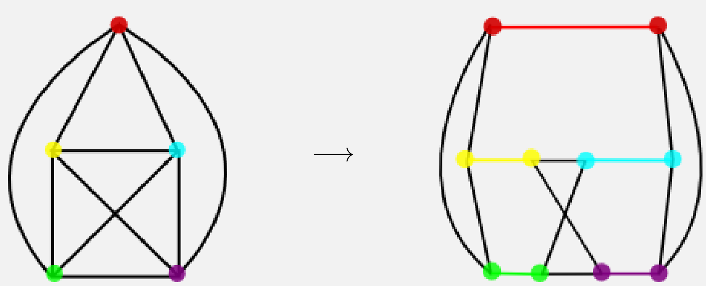
We call the family $(P_v)_{v \in V_H}$ as above an *H-decomposition* of G.

2.2 Minors vs. topological minors

The following fact is obvious.

Fact 2.8. *If H is a topological minor of G then H is a minor of G.*

Is the other implication true? Consider K_5, and replace every vertex of degree four by a pair of connect vertices of degree 3:



Clearly, $K_5 \leq_m G$. We claim that $K_5 \not\leq_t G$. Indeed, this is a consequence of the following.

Fact 2.9. *If $H \leq_t G$ then $\Delta(H) \leq \Delta(G)$.*

It is not difficult to prove the following.

Fact 2.10. *Let G, H be graphs, and suppose that $\Delta(H) \leq 3$. Then, $H \leq_m G$ if and only if $H \leq_t G$.*

Corollary 2.11. *$\leq_m$ and $\leq_t$ are equivalent on graphs with $\Delta \leq 3$.*

2.3 Excluding minors

Definition 2.12 (minor-closed)**.** A class $\mathcal{C}$ is *minor-closed* if for any $G \in \mathcal{C}$ and any graph H if $H \leq_m G$ then $H \in \mathcal{C}$. We say that $\mathcal{C}$ is *proper* if $\mathcal{C} \neq \mathcal{G}raphs$. When speaking of minor-closed classes, we implicitly assume that they are proper. We say that a class $\mathcal{C}$ *excludes a minor* if it is contained in a proper minor-closed class of graphs. In other words, there exists a graph G which is not a minor of any graph from $\mathcal{C}$.

We define analogous notions for topological minors: we may say that a class is *topologically-minor-closed*, or that it *excludes a topological minor.*

Lemma 2.13. *Planar graphs are minor-closed. They are also closed under topological minors.*

Intuitively, it is clear that contracting an edge of a planar graph yields a planar graph. A formal proof requires a topological argument.

Theorem 2.14 (Kuratowski 30)**.** *The following conditions are equivalent for a graph G*
1) G is planar
2) $K_{3,3} \not\leq_t G$ and $K_5 \not\leq_t G$

Proof. $1 \to 2$. We show that $K_{3,3}$ is not planar, by showing that it does not embed to the sphere. Observe that $K_{3,3}$ is a cycle $x_1, x_2, \ldots, x_6, x_1$ of length 6, together with the three diagonals: $\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}$.

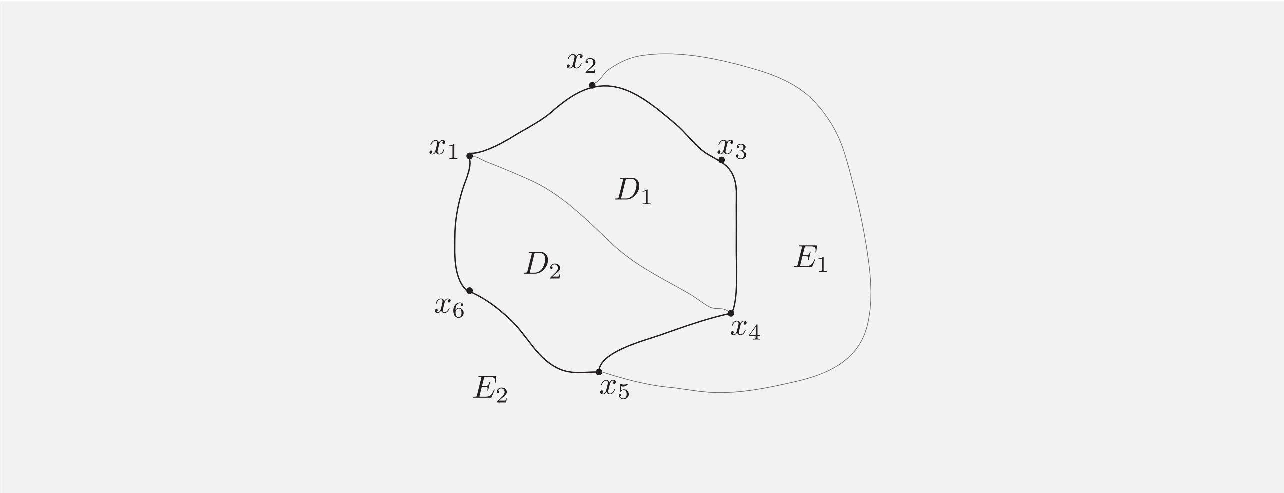
In any embedding of $K_{3,3}$ to the sphere, the image Γ of this cycle splits the sphere into two faces (by the Jordan curve theorem); let these faces be denoted D and E (see image below).

Now, we need to draw one of the three diagonals, say $\{x_1, x_4\}$. The corresponding curve is a connected topological space, so it must be contained in a connected component of $\mathbb{R}^2 - \Gamma$, so it is either contained in D or in E. Without loss of generality, assume that the curve joining x_1 and x_4 is contained in D. This curve splits (again, using the Jordan curve theorem) D into two faces. Let D_1 be the face adjacent to x_1, x_2, x_3, x_4, and let D_2 be the face adjacent to x_1, x_4, x_5, x_6.

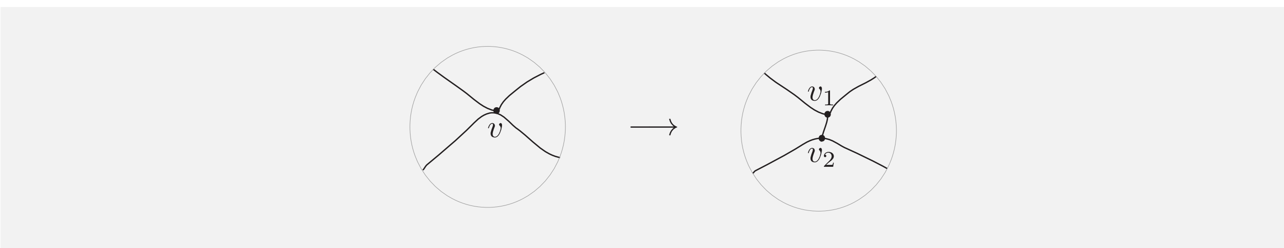
Now we need to draw the diagonal $\{x_2, x_5\}$. The vertex x_2 has two faces adjacent to it: D_1 and E, and the vertex x_5 has two faces adjacent to it: D_2 and E. A curve joining x_2 and x_5 can be only contained in one connected component, so it needs to be contained in E.

This curve splits E into two faces, call them E_1 and E_2. Assume that E_1 is the face which is adjacent to x_3, then E_2 is the face which is adjacent to x_6.

Now, x_3 is adjacent to the faces D_1 and E_1, whereas x_6 is adjacent to the faces E_2 and D_2. Therefore, there does not exist a curve joining x_3 with x_6, which does not intersect the previously drawn curves. This proves that $K_{3,3}$ is not planar.



The fact that K_5 is not planar is a simple consequence. Indeed, assume that K_5 is planar. Split one of the vertices v of K_5 into two vertices v_1, v_2 of degree 3 each, connected to each other (see image below). This yields a graph G, which is also planar. This graph contains $K_{3,3}$ as a subgraph – a contradiction, since $K_{3,3}$ is not planar.



This finishes the proof of the implication $1 \to 2$, since planar graphs are closed under topological minors (indeed, under minors in general). The implication $2 \to 1$ is more involved... □

There is a very similar theorem due to Wagner; the difference is that it speaks about minors rather than topological minors. In a sense, Kuratowski's theorem is stronger than Wagner's theorem, since forbidding topological minors is a weaker requirement than forbidding minors.

Theorem 2.15 (Wagner 37)**.** *The following conditions are equivalent for a graph G*
1) G is planar
2) $K_{3,3} \not\leq_m G$ and $K_5 \not\leq_m G$

Proof. Neither $K_{3,3}$ nor K_5 are planar, and planar graphs are closed under minors. This proves the implication from 1 to 2.

The other implication follows from Kuratowski's theorem: indeed, if a graph G does not contain $K_{3,3}$ nor K_5 as a minor, then it does not contain either of them as a topological minor, so it is planar. □

Remark 2.16. It is also not difficult to deduce Kuratowski's theorem from Wagner's theorem.

Wagner's conjecture (proved by Robertson and Seymour in a series of papers from 1983-2004).
If $\mathcal{C}$ is any minor-closed class of graphs then there exist finitely many graphs G_1, ..., G_k (*forbidden minors*) such that the following are equivalent for any graph G
1) $G \in \mathcal{C}$
2) $G_1 \not\leq_m G, G_2 \not\leq_m G, \ldots,$ and $G_k \not\leq_m G$.

Definition 2.17 (w.q.o.)**.** A partial order $\leq$ over a set X is called a *well quasi-order* (*w.q.o.*) if for any sequence x_1, x_2, $x_3, \ldots$ of elements of X there exist indices $i < j$ such that $x_i \leq x_j$.

For example, the linear order over $\mathbb{N}$ is a w.q.o. A more involved example is the subword order over the set of finite words over a finite alphabet A: v is a subword of w if v can be obtained from w by arbitrarily removing letters.

Fact 2.18. *Let $\leq$ be a partial order over a set X. The following conditions are equivalent.*

1. *$\leq$ is a w.q.o.*
2. *For any sequence x_1, x_2, $x_3, \ldots$ there exists an infinite subsequence $x_{n_1}, x_{n_2}, \ldots$ such that*

$$x_{n_1} \leq x_{n_2} \leq \ldots.$$

3. *For any downward-closed subset C of X, there exists a finite set of "forbidden elements", $\{x_1, x_2, \ldots, x_n\}$, such that for all $x \in X$,*

$$x \in C \text{ if and only if } x_1 \nleq x, x_2 \nleq x, \ldots, \text{ and } x_n \nleq x,$$

The following theorem, due to Robertson and Seymour, is considered one of the deepest results of graph theory. It says that Wagner's conjecture holds.

Theorem (The Graph Minor Theorem) 2.18.1. *$\leq_m$ is a w.q.o. on graphs.*

Below is a table describing the forbidden minors for some minor-closed classes of graphs.

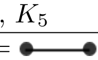
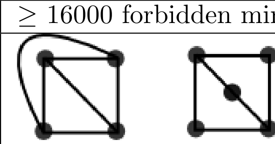
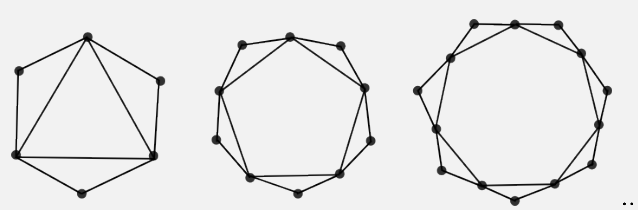
minor-closed class	**forbidden minors**
planar graphs	$K_{3,3}$, K_5
edgeless graphs	$K_2 =$ 
forests	K_3
toric graphs	$\geq$ 16000 forbidden minors
outer-planar graphs	
graphs which embed into the surface of genus g	some very long list

Table 1: minor-closed graphs and their forbidden minors

The corresponding result is false for $\leq_t$. The fact that $\leq_t$ is not a w.q.o. is witnessed by the sequence of graphs depicted below. This shows that topological minors and minors sometimes behave very differently.



2.4 Algorithmic aspects

In order to check if $H \leq_m G$, it suffices to guess a subgraph of G and a sequence of contractions which lead to a graph isomorphic to H (the isomorphism can be also guessed). This yields the following.

Fact 2.19. *The following decision problem (MINOR)*

> **Input***: Graphs H, G*
>
> **Output***: Is $H \leq_m G$*

is decidable in NP.

Wagner's theorem therefore implies the following.

Corollary 2.20. *Planarity is in co-NP.*

The work of Robertson and Seymour in fact implies the following result.

Theorem 2.21. *MINOR is FPT where the parameter is $|H|$ – it can be decided in time $f(|H|) \cdot |G|^3$*

Remark 2.22. The known bound for the function $f(|H|)$ is at least exponential in $|H|$. Note that this is unavoidable, assuming that Hamiltonicity cannot be decided in subexponential time: testing whether G has a Hamiltonian cycle amounts to testing whether the n-cycle is a minor of G, where $n = |G|$.

Theorem 2.21 implies that planarity, as well as embeddability into a fixed surface of genus g, can be decided in polynomial time. This result, however, can be proved by simpler means (without the Robertson-Seymour theorem), and giving a better bound:

Theorem 2.23. *There exists a linear time algorithm for the following problem:*

> **Input***: A graph G.*
>
> **Output***: "Yes" if G is planar; if G is not planar, describe how $K_{3,3}$ or K_5 is a minor of G.*

Remark 2.24. The theorem generalizes to surfaces of a fixed genus g.

2.5 Density bounds for minor-closed classes

Minor-closed classes of graphs are sparse. This is implied by the following.

Theorem 2.25. *There is a function f such that for any number h and graph G such that $K_h \not\leq_m G$,*

$$\frac{\|G\|}{|G|} \leq f(h).$$

Remark 2.26. The above theorem is not difficult to prove. The optimal bound is $f(h) = \gamma \cdot h \cdot \sqrt{log(h)}$, for some constant γ; this is a theorem proved independently by Kostochka and Thomason; it is substantially more difficult to prove than the above theorem. A result analogous to Theorem 2.25 holds for topological minors. Note that the result for topological minors is more general than the result for minors, since excluding a minor implies excluding a topological minor.

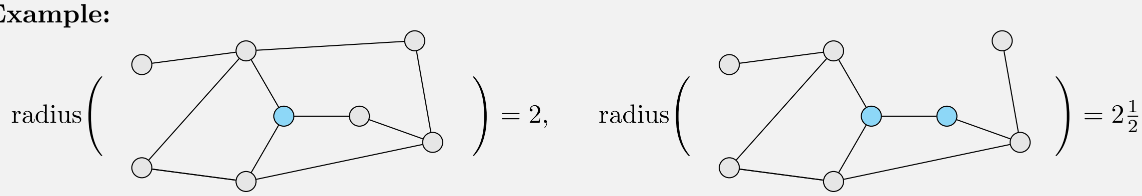
Corollary 2.27. *Let $\mathcal{C}$ be a proper (topological) minor-closed class. Then there exists a constant C such that every graph $G \in \mathcal{C}$ has edge density bounded by C. In particular, the graphs in $\mathcal{C}$ are k-degenerate for some constant k (since $\mathcal{C}$ is closed under taking subgraphs).*

3 Shallow minors

In this section we introduce *shallow minors*, which are simply minors obtained with bags of small radius.

Definition 3.1 (radius)**.** The *radius* of a graph G is the smallest d such that for some vertex $v \in G$ we have $\forall_{w \in G}\ d_G(v, w) \leq d$. Sometimes we add that $\mathrm{radius}(G) \leq d + \frac{1}{2}$ if there is an edge $\{u, v\}$ of G such that $\forall_{w \in G}\ d_G(u, w) \leq d \vee d_G(v, w) \leq d$.

Example:



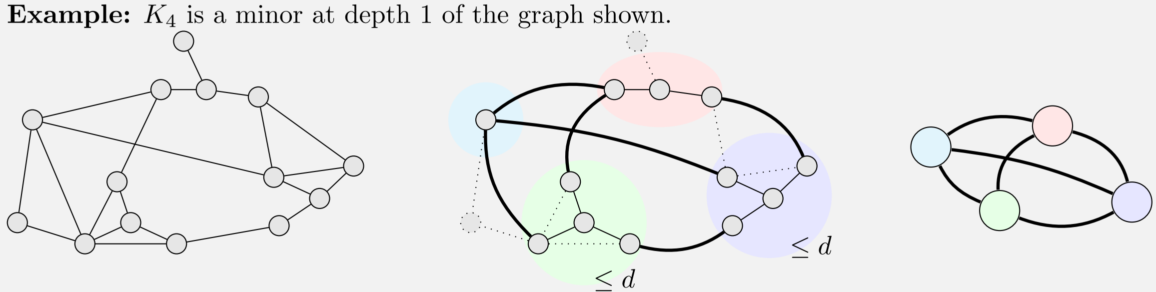
Observe that a graph of radius d has a spanning tree of radius d (a tree of shortest paths). A related notion, the diameter $\mathrm{diam}(G) = \max_{u,v \in G} d_G(u, v)$, is equivalent up to a factor of 2 ($\mathrm{radius}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{radius}(G)$; ironically diam = radius for cycles C_n).

Definition 3.2 (shallow minor)**.** *H is a shallow minor of G at depth d (or simply a minor at low depth) if there are subgraphs $(P_v)_{v \in V_H}$ of G such that:*

- each graph P_v is connected, for $v \in V_H$
- P_v and P_w are vertex-disjoint, for $v \neq w$
- for each edge $\{u, v\} \in E_H$ there is an edge between P_u and P_v in G,
- each P_v has radius at most d.

Definition 3.3 (shallow topological minor)**.** *H is a shallow topological minor of G at depth d if G contains (as a subgraph) a $\leq 2d$-subdivision of H (i.e. each edge is subdivided into a path of radius at most d).*

Example: K_4 is a minor at depth 1 of the graph shown.



We write $G\nabla d$ for the set of minors of G at depth d and $G\tilde{\nabla} d$ for shallow topological minors. Observe that $G\nabla 0$ is simply the set of subgraphs of G, while $G\nabla|G| = G\nabla\infty$ is the set of all minors of G. For a class of graphs $\mathcal{C}$, we write

$$\mathcal{C}\nabla d \stackrel{\text{def}}{=} \bigcup_{G \in \mathcal{C}} G\nabla d,$$
$$\mathcal{C}\tilde{\nabla} d \stackrel{\text{def}}{=} \bigcup_{G \in \mathcal{C}} G\tilde{\nabla} d.$$

The following inclusions hold for any class $\mathcal{C}$:

$$\begin{array}{ccccccccc}
\mathcal{C} & \subseteq & \mathcal{C}\nabla 0 & \subseteq & \mathcal{C}\nabla 1 & \subseteq & \mathcal{C}\nabla 2 & \subseteq & \dots \\
\| & & \| & & \cup\!\mid & & \cup\!\mid & & \\
\mathcal{C} & \subseteq & \mathcal{C}\tilde{\nabla} 0 & \subseteq & \mathcal{C}\tilde{\nabla} 1 & \subseteq & \mathcal{C}\tilde{\nabla} 2 & \subseteq & \dots
\end{array}$$

Definition 3.4 (resolution)**.** The *resolution* of $\mathcal{C}$ is the sequence of classes $\mathcal{C}\nabla 0$, $\mathcal{C}\nabla 1$, $\mathcal{C}\nabla 2$, ... Similarly, the *topological resolution* of $\mathcal{C}$ is the sequence of classes $\mathcal{C}\tilde{\nabla} 0$, $\mathcal{C}\tilde{\nabla} 1$, $\mathcal{C}\tilde{\nabla} 2$, ...

4 Nowhere-density

In this section we define a robust notion of sparseness that generalizes minor-closed classes (e.g. planar graphs, graphs of bounded genus, graphs excluding K_n as a minor), classes of bounded degree, and other classes arising in algorithmics and logic that intuitively seem “sparse”.

Definition 4.1 (nowhere-density)**.** We say that a class of graphs $\mathcal{C}$ is *nowhere-dense* if for each number d, $\mathcal{C}\nabla d \neq \mathcal{G}raphs$.

Examples:

- Let $\mathcal{P}$ be the class of planar graphs (or any other proper minor-closed class). Then $\mathcal{P}\nabla d = \mathcal{P} \neq \mathcal{G}raphs$, so $\mathcal{P}$ is nowhere-dense.
- Let Δ_3 be the class of graphs of degree ≤ 3. For each d the class $\Delta_3\nabla d$ is a class of bounded degree (an easy inductive argument shows that $\Delta_3\nabla d = \Delta_{3\cdot 2^d}$), therefore Δ_3 is nowhere-dense, even though $\Delta_3\nabla\infty = \bigcup_d \Delta_3\nabla d = \mathcal{G}raphs$.

Definition 4.2 (logarithmic density)**.** The *logarithmic density* of a graph G is defined as $\ell\mathrm{dens}(G) = \frac{\log \|G\|}{\log |G|}$ (if the graph has no edges define $\ell\mathrm{dens}(G) = -\infty$).

Examples:

- For any graph with edges $1 \leq \|G\| < |G|^2$, thus $0 = \frac{\log 1}{\log |G|} \leq \ell\mathrm{dens}(G) < \frac{\log(|G|^2)}{\log |G|} = \frac{2\log |G|}{\log |G|} = 2$.
- For any planar graph G we have $\|G\| < 3|G|$, thus $\ell\mathrm{dens}(G) < \frac{\log 3|G|}{\log |G|} = 1 + \frac{\log 3}{\log |G|}$.
- For cliques $\ell\mathrm{dens}(K_n) = \frac{\log \frac{n(n-1)}{2}}{\log n} = \frac{\log n^2 + \log \frac{n-1}{n} + \log \frac{1}{2}}{\log n} = 2 + \frac{\log(1-\frac{1}{n})-1}{\log n} \xrightarrow{n\to\infty} 2$.
- For complete bipartite graphs $\|K_{n,n}\| \simeq \frac{1}{2}\|K_{2n}\|$, so $\ell\mathrm{dens}(K_{n,n})$ is only smaller by approximately $\frac{\log 2}{\log n}$. Therefore we still have $\ell\mathrm{dens}(K_{n,n}) \xrightarrow{n\to\infty} 2$.

We extend any graph parameter $f : \mathcal{G}raphs \to \mathbb{R} \cup \{-\infty, \infty\}$ to a parameter of classes

$$f(\mathcal{C}) \stackrel{\text{def}}{=} \sup_{G\in\mathcal{C}} f(G)$$

For example, the best possible bound on the chromatic number of planar graphs is $\chi(\mathcal{P}) = 4$. We also write

$$\bar{f}(\mathcal{C}) \stackrel{\text{def}}{=} \limsup_{G \in \mathcal{C}} f(G) \stackrel{\text{def}}{=} \lim_{n\to\infty} \sup_{\substack{G\in C \\ |G|>n}} f(G)$$

Observe that $\lim\limits_{n\to\infty}$ could be replaced by $\inf\limits_{n\in\mathbb{N}}$, so this is always defined.

Finally for any class $\mathcal{C}$ the two resolutions $\mathcal{C}^{\nabla}$ and $\mathcal{C}^{\tilde{\nabla}}$ define the following two values:

$$\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \stackrel{\text{def}}{=} \lim_{d\to\infty} \limsup_{G\in\mathcal{C}\nabla d} \ell\text{dens}(G) \qquad \overline{\ell\text{dens}}(\mathcal{C}^{\tilde{\nabla}}) \stackrel{\text{def}}{=} \lim_{d\to\infty} \limsup_{G\in\mathcal{C}\tilde{\nabla} d} \ell\text{dens}(G).$$

Remark 4.3. Observe that $\lim\limits_{d\to\infty}$ could be replaced by $\sup\limits_{d\in\mathbb{N}}$ (we take more and more graphs into account), so $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla})$ and $\overline{\ell\text{dens}}(\mathcal{C}^{\tilde{\nabla}})$ are defined for any class of graphs.

Lemma 4.4. *For any class of graphs $\mathcal{C}$, $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \in \{-\infty, 0\} \cup [1, 2]$.*

Proof.

- If a class $\mathcal{C}$ contains only finitely many graphs which have edges, then $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla})$ is $-\infty$. Otherwise, after taking minors at depth d and disregarding finitely many graphs, the class will still have graphs with edges, so $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \geq 0$.
- If all graphs in $\mathcal{C}$ have a number of edges bounded by c, as we disregard all graphs with less than n vertices, the logarithmic density will tend to $\frac{\log c}{\log n} \to 0$. Otherwise there are graphs with arbitrarily many edges, so $C\nabla 0$ contains arbitrarly large subgraphs that eliminate isolated vertices – these subgraphs have $\ell\text{dens}(G) \geq 1$, so $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \geq 1$.
- $\ell\text{dens}(G) \leq 2$, so $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \leq 2$.

$\square$

Surprisingly, no class takes a value strictly between 1 and 2, that is $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \in \{-\infty, 0, 1, 2\}$, which is the statement of the **trichotomy theorem** (named so because $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla})$ can take only three different values, if we merge the degenerate cases of 0 and $-\infty$). Also, the strong notion of sparseness this implies coincides with nowhere-density. We prove both statements in one general theorem after giving some examples.

Examples:

- For planar graphs $\mathcal{P}\nabla d = \mathcal{P}$, so $\overline{\ell\text{dens}}(\mathcal{P}^{\nabla}) = \limsup_{G\in\mathcal{P}} \ell\text{dens}(G)$. But $\ell\text{dens}(G) < 1 + \frac{\log 3}{\log |G|}$, so as we disregard small graphs, the supremum of $\ell\text{dens}(G)$ will tend to 1.
- Similarly for classes of degree bounded by c. At every step d of the resolution we have $\ell\text{dens}(G) < 1 + \frac{\log c^d}{\log |G|} \to 1$.
- If a class $\mathcal{C}$ contains arbitrarily large cliques $(K_{n_i})_{i\in\mathbb{N}}$ in some step of the resolution, so will every step after it. Thus $\overline{\ell\text{dens}}(\mathcal{C}^{\nabla}) \geq \limsup\limits_{i\in\mathbb{N}} \ell\text{dens}(K_{n_i}) = \lim\limits_{n\to\infty} \ell\text{dens}(K_n) = 2$.
- The same holds for complete bipartite graphs, because $\lim\limits_{n\to\infty} \ell\text{dens}(K_{n,n}) = 2$.

Theorem 4.5. *Let $\mathcal{C}$ be any class of graphs. The following conditions are equivalent to nowhere-density:*

1) $\forall_d \mathcal{C}\nabla d \neq \mathcal{G}raphs$ *$\tilde{1}$)* $\forall_d \mathcal{C}\tilde{\nabla} d \neq \mathcal{G}raphs$ *(no step of the resolution contains all graphs)*

2) $\forall_d \exists_{n\in\mathbb{N}} K_n \notin \mathcal{C}\nabla d$ *$\tilde{2}$)* $\forall_d \exists_{n\in\mathbb{N}} K_n \notin \mathcal{C}\tilde{\nabla} d$ *(no step of the resolution contains all cliques)*

3) $\overline{\ell dens}(\mathcal{C}^{\nabla}) \leq 1$ *$\tilde{3}$)* $\overline{\ell dens}(\mathcal{C}^{\tilde{\nabla}}) \leq 1$

4) $\overline{\ell dens}(\mathcal{C}^{\nabla}) < 2$ *$\tilde{4}$)* $\overline{\ell dens}(\mathcal{C}^{\tilde{\nabla}}) < 2$

The first item is the definition of a nowhere-dense class.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \tilde{1} \arrow[d] \\
2 \arrow[u, bend left] \arrow[r] \arrow[d, bend left, "5.2"] & \tilde{2} \arrow[u, bend right] \arrow[l, bend right, "5.3"'] \arrow[d, bend right, "5.1"'] \\
3 \arrow[r] \arrow[d] & \tilde{3} \arrow[d] \\
4 \arrow[r] \arrow[uu, bend left] & \tilde{4} \arrow[uu, bend right]
\end{tikzcd}$$

Implications $1 \to \tilde{1}$, $2 \to \tilde{2}$, $3 \to \tilde{3}$, $4 \to \tilde{4}$ are trivial, because $\mathcal{C}\tilde{\nabla} d$ is smaller than $\mathcal{C}\nabla d$.
Implications $2 \to 1$, $\tilde{2} \to \tilde{1}$, $3 \to 4$, $\tilde{3} \to \tilde{4}$ are trivial, because they're just stronger statements.
The implications $1 \to 2$ and $\tilde{1} \to \tilde{2}$ are also easy, since excluding some graph with n nodes as a (topological) minor at depth d implies excluding the clique K_n as a (topological) minor at depth d.

Implications $4 \to 2$, $\tilde{4} \to \tilde{2}$ follow from the contrapositive: if $\mathcal{C}\nabla d$ contains all cliques for some number d (and all larger ones), then $\overline{\ell \text{dens}}(\mathcal{C}^{\nabla}) = 2$ (see third example above).

Therefore, to show the equivalence of the non-tilde items (1,2,3,4), it remains to prove the implication $2 \to 3$. This is done in Section 5.2. Similarly, to show the equivalence of the tilde items $\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}$, it suffices to prove the implication $\tilde{2} \to \tilde{3}$; we do this in Section 5.1. Finally, to show the equivalence of the non-tilde items with the tilde items, it suffices to show the implication $\tilde{2} \to 2$; we do this in Section 5.3.

5 Proof of Theorem 4.5

Lemma 5.1. *Let G be a graph and $\varepsilon > \frac{1}{|G|}$. Then there exists a subgraph $H \subseteq G$ such that*

(a) $\delta(H) \geq (1-\varepsilon)\frac{||G||}{|G|}$,

(b) $||H|| > \varepsilon ||G||$.

Proof. Let us iteratively remove an arbitrary vertex with degree less than $\delta := (1-\varepsilon)\frac{||G||}{|G|}$ (with accidental edges); if such a vertex does not exist then we do nothing. One gets a sequence of graphs

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_{|G|} = H.$$

Graph H either satisfies (a) or is just an empty graph. But in each step we remove at most δ edges, so

$$||H|| \geq ||G|| - |G|\delta = \varepsilon ||G|| > 0.$$

So (b) holds and, as H is a non-empty graph, (a) holds too. $\square$

5.1 Proof of the implication $\tilde{2} \to \tilde{3}$

We prove by contraposition. Assume that $\overline{\ell\text{dens}}(\mathcal{C}^{\tilde{\nabla}}) > 1$. That implies that there exists an $\epsilon > 0$ and a d_0 that for all n there exists a graph G such that

$$\begin{aligned} \frac{||G||}{|G|} &\geq |G|^{\epsilon} \\ G &\in \mathcal{C}\tilde{\nabla} d_0 \\ |G| &> n \end{aligned}$$

We want to show that there is a d that for all n $K_n \in \mathcal{C}\tilde{\nabla} d$. We fix $\epsilon > 0, d_0$ such as above.

Proposition 5.2. *Let $\epsilon > 0$. There exist $n_0 \in \mathbb{N},\ c \in \mathbb{N},\ \mu > 0$ such that for all graphs G*

$$\begin{cases} \delta(G) & \geq |G|^{\epsilon} \\ |G| & \geq n_0 \end{cases} \qquad \Longrightarrow \qquad K_{|G|^{\mu}} \in G\tilde{\nabla} c.$$

The implication $\tilde{2} \to \tilde{3}$ follows for d that satisfies $2d + 1 = (2c+1)(2d_0+1)$, $G\tilde{\nabla} d \subseteq G\tilde{\nabla} c\tilde{\nabla} d_0$

Lemma 5.3. *Fix $\alpha = 5$. For $0 < \epsilon < 1$ there exist $n_0 \in \mathbb{N}$ such that for every graph G, if*

$$\begin{aligned} n \overset{def}{=} |G| &\geq n_0 \\ \delta(G) &\geq |G|^{\epsilon}, \end{aligned}$$

then

either $\quad 1) K_{n^{\epsilon^{\alpha}}} \in G\tilde{\nabla} 1$

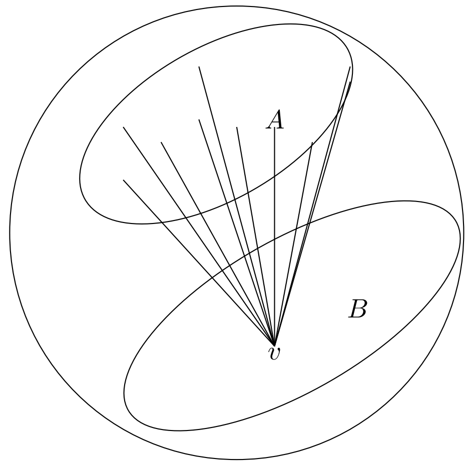
or $\quad 2) \exists H \in G\tilde{\nabla} 1,\ \delta(H) \geq \frac{1}{8}(\frac{1}{4}|H|)^{\epsilon \frac{1-\epsilon^{\alpha}}{1-\epsilon-\epsilon^{\alpha}}} \geq |H| \geq \frac{1}{2} n^{\epsilon - \epsilon^{\alpha}}$

Proof. The proof proceeds in two steps.

Step 1 Find $A, B \subseteq V_G$ such that

$$A \cap B = \emptyset \tag{5.1}$$
$$|B| \geq \frac{1}{2} n \tag{5.2}$$
$$n^{1-\epsilon+\epsilon^{\alpha}} \leq |A| \leq 4n^{1-\epsilon+\epsilon^{\alpha}} \tag{5.3}$$
$$\forall v \in B \qquad deg_A(v) \geq n^{\epsilon^{\alpha}} \tag{5.4}$$

where $deg_A(v) \overset{def}{=}$ number of neighbors of v in A.



We construct A by randomy choosing vertices from V_G with probability

$$p = 2n^{-\epsilon+\epsilon^\alpha}$$

The expected size of A is:

$$\mathbb{E}|A| = p \cdot n = 2n^{1-\epsilon+\epsilon^\alpha}$$

Given the Chernoff's inequality we can control the size of A. That is for sufficiently large n we have:

$$\mathbb{P}(|A| > 2pn) < \frac{1}{4}$$
$$\mathbb{P}(|A| < \frac{1}{2}pn) > \frac{1}{4}$$

so with probabily at least $\frac{1}{2}$ the size of A does not exceed the desired size:

$$\mathbb{P}(|A| \notin [n^{-\epsilon+\epsilon^\alpha}\ ,\ 4n^{-\epsilon+\epsilon^\alpha}]) < \frac{1}{2} \tag{5.5}$$

Choose any $v \in V_G$. Then we calculate estimations for A-neighbors of v:

$$deg_A(v) \geq n^\epsilon$$
$$\mathbb{E}[deg_A(v)] \geq n^\epsilon p = 2n^{\epsilon^\alpha}$$
$$\mathbb{P}(deg_A(v)\ <\ n^{\epsilon^\alpha}) < \frac{1}{5}$$

for sufficiently large n.
Lets define

$$B' = \{v :\ deg_A(v) \geq n^{\epsilon^\alpha}\}$$

The expected value of number of vertices not in B' is:

$$\mathbb{E}|V_G - B'| = \mathbb{E}\sum_v [deg_A(v) < n^{\epsilon^\alpha}] = \sum_v \mathbb{E}[deg_A(v) < n^{\epsilon^\alpha}] < \sum_v \frac{1}{5} = \frac{n}{5}$$

Using Markov's inequality we obtain:

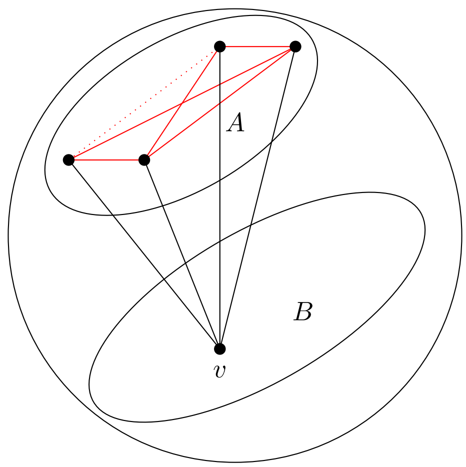
$$\mathbb{P}(|V_G - B'| \geq \frac{2}{5}n) \cdot \frac{2}{5}n \leq \mathbb{E}|V_G - B'| = \frac{n}{5}$$
$$\Rightarrow \qquad \mathbb{P}(|V_G - B'| \geq \frac{2}{5}n) < \frac{1}{2} \tag{5.6}$$

From inequalities (7.5) and (7.6) it follows that with positive probability A is such that

$$\begin{aligned} n^{1-\epsilon+\epsilon^\alpha} \leq |A| &\leq 4n^{1-\epsilon+\epsilon^\alpha} \\ \frac{3}{5}n &\geq |B'| \end{aligned}$$

Let $B = B' - A$. Then $|B| \geq \frac{1}{2}n$ and the sets A and B satisfy the required conditions (7.1) – (7.4).

Step 2 We construct a graph G' with a vertex set A and edges corresponding to the elements of B. Initially, G' has no edges. We add new edges to G' as follows. We begin by choosing an element of B in some arbitrary way.



If $N_G(v) \cap A$ form a clique in G' we terminate the construction of G'. Otherwise we add a missing edge associated with v and move on to the next arbitrarily chosen vertex of B. We terminate the construction if we have used all vertices of B.

Either we have found a clique of size n^{ϵ^α} (because of (7.4))

$$K_{n^{\epsilon^\alpha}} \in G\tilde{\nabla}1$$

or we have used all vertices of B and thereby constructed $G' \in G\tilde{\nabla}1$ such that $||G'|| = |B|$ and $|G'| = |A|$. It follows, that

$$\frac{||G'||}{|G|} = \frac{|B|}{|A|} \geq \frac{\frac{1}{2}n}{4n^{1-\epsilon+\epsilon^\alpha}} = \frac{1}{8}n^{\epsilon-\epsilon^\alpha}$$

We can now choose a subgraph $H \subseteq G'$ such that:

$$4n^{1-\epsilon+\epsilon^\alpha} \geq |A| \geq |H| \geq \frac{1}{2}|G'| = \frac{1}{2}|A|,$$

and solve for n:

$$n \geq (\frac{1}{4}|H|)^{\frac{1}{1-\epsilon+\epsilon^\alpha}},\ \delta(H) \geq \frac{1}{8}n^{\epsilon-\epsilon^\alpha} \geq \frac{1}{8}(\frac{1}{4}|H|)^{\frac{\epsilon-\epsilon^\alpha}{1-\epsilon+\epsilon^\alpha}} = \frac{1}{8}(\frac{1}{4}|H|)^{\epsilon+\epsilon^2\frac{1-\epsilon^{\alpha-1}-\epsilon^{\alpha-2}}{1-\epsilon+\epsilon^\alpha}}$$

$\delta(H) > H$ is impossible and

$$\begin{aligned} \delta(H) &\geq \frac{1}{8}n^{\epsilon-\epsilon^\alpha} \\ |H| &\leq 4n^{1-\epsilon+\epsilon^\alpha} \end{aligned}$$

That means if

$$\frac{1}{8}n^{\epsilon-\epsilon^\alpha} \quad > \quad 4n^{1-\epsilon+\epsilon^\alpha},$$

then the second possibility does not hold. Hence the following corollary: □

Corollary 5.4. *If $\frac{1}{32} > n^{1-2\epsilon+2\epsilon^\alpha}$ then the second possibility does not hold. This is the case when $1-2\epsilon+2\epsilon^\alpha$ is negative and n is large enough. For $\alpha = 5$, $\quad 1-2\epsilon+2\epsilon^\alpha \quad$ is negative for all $\quad 0,56 < \epsilon < \frac{3}{4}$.*

To prove Proposition 5.2 we apply the lemma iteratively. At each step we either obtain a clique or a subraph with larger minimal degree. In the first case we are done and the proposition is proved. Note that the second case can occur only finitely many (depending on ϵ) times in our iteration. This is because in each step we increase the minimal degree of the obtained graph at least by a constant. Assuming that we began with

$$\begin{cases} \epsilon_0 > 0 \\ \delta(H) \geq |H|^{\epsilon_0} \end{cases}$$

we then obtain the following sequence

$$\epsilon_0 \quad \longrightarrow \quad \epsilon_1 = \epsilon_0 + \epsilon_0^2 \cdot c \quad \longrightarrow \quad \epsilon_2 = \epsilon_1 + \epsilon_1^2 \cdot c \quad > \quad \epsilon_0 + 2\epsilon_0^2 \cdot c \quad \longrightarrow \quad \ldots \quad \longrightarrow \quad \epsilon_n \quad > \quad \epsilon_0 + n\epsilon_0^2 \cdot c$$

We see that after finitely many steps we obtain an ϵ that is greater than 0.56 and using the corollary above we find the desired clique.

5.2 Proof of the implication $2 \to 3$

The proof follows closely the proof of the implication $\tilde{2} \to \tilde{3}$.
We prove by contraposition. Assume that $\overline{\ell\text{dens}}(\mathcal{C}^\nabla) > 1$. That implies that there exists an $\epsilon > 0$ and a d_0 that for all n there exists a graph G such that

$$\begin{aligned} \frac{||G||}{|G|} \;&\geq\; |G|^\epsilon \\ G \;&\in\; \mathcal{C}\nabla d_0 \\ |G| \;&>\; n \end{aligned}$$

We want to show that there is a d that for all n $K_n \in \mathcal{C}\nabla d$. Notice that in proposition 5.2 if $K_{|G|^\mu} \in G\tilde{\nabla} c$ than also $K_{|G|^\mu} \in G\nabla c$. Thus the implication $2 \to 3$ follows from proposition 5.2 for d that satisfies $2d+1 = (2c+1)(2d_0+1)$, $G\nabla d_0 \nabla c \subseteq G\nabla d$.

5.3 Proof of the implication $\tilde{2} \to 2$

We will use the following notation:

$$\omega(G) \stackrel{\text{def}}{=} \max\{n : K_n \subseteq G\}$$

$$\omega(\mathcal{C}) = \sup_{G\in\mathcal{C}} \omega(G).$$

Using this notation, we can rewrite the implication $\tilde{2} \to 2$ in the following way:

$$\forall_d \omega(\mathcal{C}\tilde{\nabla} d) < \infty \qquad \Rightarrow \qquad \forall_d \omega(\mathcal{C}\nabla d) < \infty.$$

Proposition 5.5. *Let G be a graph and $d \in \mathbb{N}$. Then,*

$$\omega(G\tilde{\nabla} d) \leq \omega(G\nabla d) \leq 2\omega(G\tilde{\nabla}(3d+1))^{d+1}.$$

From this we deduce an analogous result, when G is replaced by a class of graphs $\mathcal{C}$. The implication $\tilde{2} \to 2$ follows from the second inequality: if $\omega(\mathcal{C}\tilde{\nabla}d)$ is finite for any d then so is $2\omega(\mathcal{C}\tilde{\nabla}(3d+1))^d$ and in consequence $\omega(\mathcal{C}\nabla d) < \infty$ for any d.

Proof. The first inequality follows from the inclusion $G\tilde{\nabla}d \subseteq G\nabla d$. We therefore focuns on the second inequality.

Let $\omega(G\nabla d) = p$. G contains K_p as a minor at depth d. Let $\{w_1, \ldots, w_p\}$ be the set of the trees of radius at most d (we will call them „bags") corresponding to the nodes of this clique. We will also add the edges connecting bag i to $p-1$ other bags to the tree w_i. That way each tree has exactly $p-1$ leafs and depth at most $d+1$ (maximum distance from a selected node, which we will call the „centre of the bag", to the leafs).

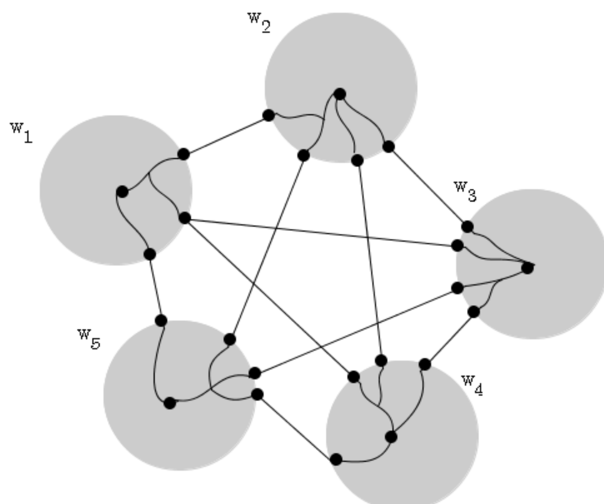


Figure 4: Example of the initial minor K_p (for $p = 5$).

We construct q sets of bags: $M_1, \ldots, M_q$. The construction proceeds in steps:

In each step i we have a working set of bags W_i. Initially $W_1 = \{w_1, \ldots, w_p\}$. In step i we select any bag $w_i \in W_i$. Let's direct w_i from leafs to the centre of the bag. There exists a node $v_i \in w_i$ with $\operatorname{indeg}(v_i)$ at least $(|W_i| - 1)^{\frac{1}{d+1}}$. Suppose q is chosen so that:

$$\operatorname{indeg}(v_i) \geq (|W_i| - 1)^{\frac{1}{d+1}} \geq q - 1. \tag{5.7}$$

We can then define M_i as a set containing w_i and $q-1$ other bags from W_i such that the paths from v_i to those bags are disjoint:

We then put $W_{i+1} = W_i - M_i$. After q iterations we get q sets: $M_1, \ldots, M_q$.
Because every two bags are connected by an edge, it follows that K_q is a topological minor of G at depth at most $3d+1$.

We will now find q such that condition (5.7) is fulfilled. After $q-1$ iterations we have $q-1$ sets M_i of size q, so:

$$|W_i| \geq p - (q-1)q \qquad \text{for } i \leq q$$

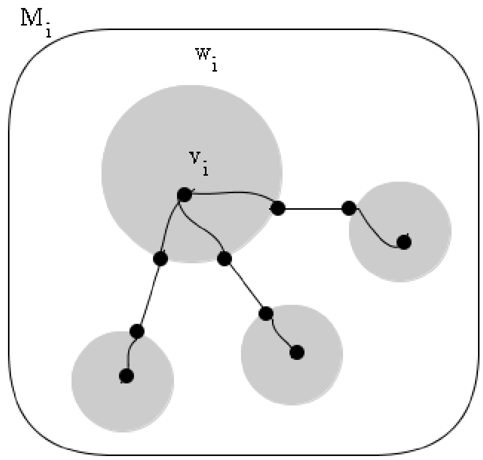


Figure 5: Construction of set M_i.

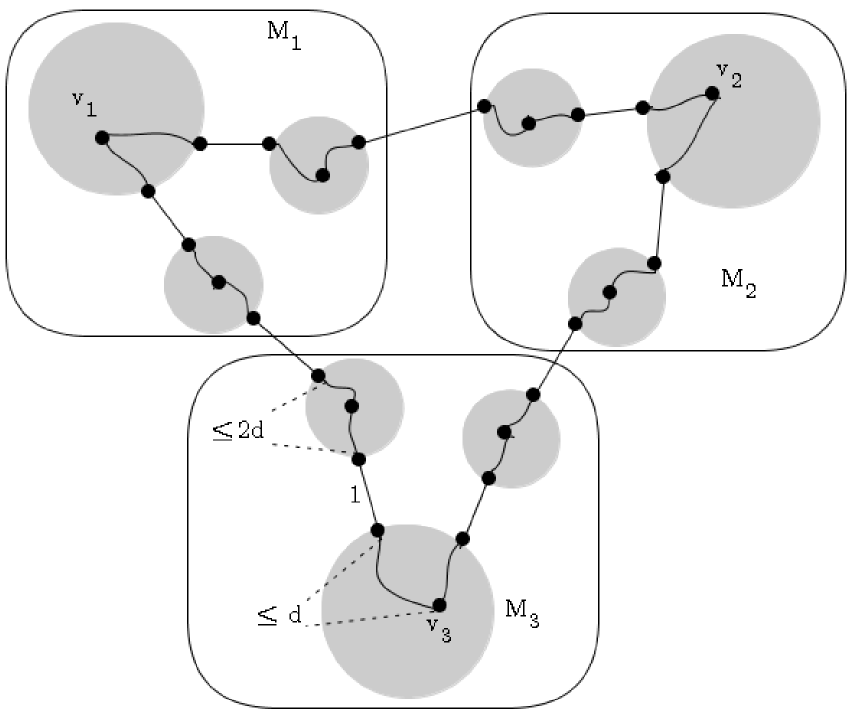


Figure 6: Example of the constructed topological minor K_q (for $q = 3$).

For (5.7) to be fulfilled it is enough that:

$$(p - (q-1)q - 1)^{\frac{1}{d+1}} \geq q - 1$$

$$p - 1 \geq (q-1)^{d+1} + (q-1)q,$$

from which we can define q as:

$$q = \left\lfloor \left(\frac{p}{2}\right)^{\frac{1}{d+1}} \right\rfloor.$$

$K_q \in G\tilde{\nabla}(3d+1)$ so:

$$\omega(G\tilde{\nabla}(3d+1)) \geq q = \left\lfloor \left(\frac{p}{2}\right)^{\frac{1}{d+1}} \right\rfloor,$$

from which we conclude that:

$$\omega(G\nabla d) = p \leq 2\omega(G\tilde{\nabla}(3d+1))^{d+1}.$$

$\square$

6 Decompositions

In this section, we will study two notions of a graph decomposition. The idea of decompositions is to decompose a graph into simple parts which form a simple structure. In tree decompositions, the parts form a tree, and each part is of bounded size.

Definition 6.1 (tree decomposition)**.** Let G be a graph. A *tree decomposition* of G is a pair:

- a forest $T = (B, E_T)$, called the *decomposition tree*, and
- a family of sets $\{B_b\}_{b \in B}$ (the *bags* of the decomposition) such that:
  1. Each bag B_b is a set of vertices of G;
  2. The family $\{B_b\}_{b \in B}$ is a covering of G – both vertexwise and edgewise – meaning that each vertex of G belongs to some bag, and for each edge e of G, there is some bag which contains both endpoints of e;
  3. For each $v \in V_G$ the set $\{b : v \in B_b\}$ is connected in T.

We will be interested in finding tree decompositions of small width.

Remark 6.2. The decomposition tree is a forest, which might be not connected, i.e. not be a tree. However, one can always turn the decomposition tree T into a connected forest T', by adding an extra root node b_0, connected with one vertex in each component of T, and defining the corresponding bag B_{b_0} to be the empty set. This operation does not increase the width of the decomposition.

Definition 6.3 (decomposition width)**.** The *width* of a tree decomposition is $\max_{b \in B}(|B_b| - 1)$.

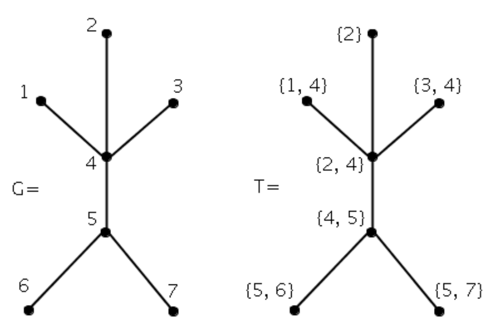


Figure 7: Example of a tree decomposition of width 1.

Definition 6.4 (treewidth)**.** The *treewidth* of a graph G (denoted $\mathrm{tw}(G)$) is defined as the minimum width of any tree decomposition of G.

Fact 6.5. *If $G \subseteq H$ then* $\mathrm{tw}(G) \leq \mathrm{tw}(H)$.

Fact 6.6. $\mathrm{tw}(G \,\dot{\cup}\, H) = \max(\mathrm{tw}(G), \mathrm{tw}(H))$.

Lemma 6.7. *Let G be a graph. If G is connected, then every decomposition tree of G is connected. If G is disconnected, then every decomposition tree of G whose bags are nonempty is disconnected.*

Proof. For one implication, we show that a path in G induces a path in T. For the other implication, we show that a path in T induces a path in G. $\square$

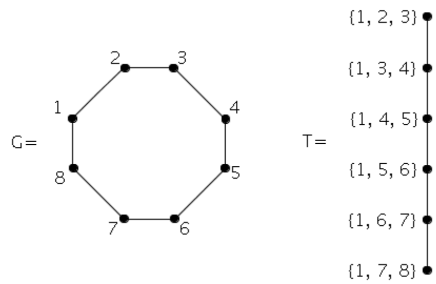


Figure 8: Example of a tree decomposition of width 2.

6.1 Separators

Definition 6.8 (separator)**.** Let G be a connected graph. A set of vertices $S \subseteq V_G$ is a *separator* if the graph $G - S$ has at least two connected components.

Proposition 6.9. *Let G be a connected graph such that $\operatorname{tw}(G) \leq n$. Then either $|G| \leq n+1$ or G has a separator S such that $|S| \leq n$.*

Lemma 6.10. *Let G be a graph with a given decomposition tree T. Let $\{b, c\}$ be an edge of T, and let T' be obtained from T by removing the edge $\{b, c\}$. Let G' be the graph $G - (B_b \cap B_c)$. Then T' is a decomposition tree for G', where the bags of T' are obtained by restricting the bags of T to G'.*

Proof. It is clear that the bags of T' form a covering of G'. It remains to prove that for any vertex v of G', the set of vertices in T' whose corresponding bags contain v is connected.

Fix a vertex v of G. Let S be the set of vertices in T whose corresponding bags contain v. Since T is a tree decomposition, the set S is connected in T.

Clearly, S is exactly the set of vertices in T' whose corresponding bags contain v. It remains to prove that S is connected in T'. Suppose that S is not connected in T'. Since S is connected in T and is not connected in T', it follows that S must contain both endpoints of the edge $\{b, c\}$ (this is because T is a tree, T' is obtained from T only by removing the edge $\{b, c\}$). But the bags of T' corresponding to the endpoints b and c have empty intersection. In particular, v cannot belong to both these bags. Therefore, b or c does not belong to S, a contradiction. □

Lemma 6.11. *Let T be a tree decomposition of G, such that no bag of the decomposition is contained in another bag. If $\{b, c\}$ is an edge of T, then $B_b \cap B_c$ is a separator in G.*

Proof. Let G' be the graph $G-(B_b \cap B_c)$. This graph has a decomposition tree T' as described in the lemma. This tree T' is not connected, as it is obtained from the tree T by removing one edge $\{b, c\}$. Moreover, the tree T' does not contain an empty bag – indeed, this would imply that the tree T has a bag which is contained in G_b and G_c.

Fact 6.7 implies that G' is disconnected. This proves that $B_b \cap B_c$ is a separator. □

Proof of Proposition 6.9. Take a decomposition tree T of G, whose width is at most n. We may assume that no bag of the decomposition is contained in another bag – otherwise, we can remove the smaller bag.

If $|G| > n+1$ then there are two nodes b and c which are connected by an edge in the decomposition tree T. The previous lemma then implies that $B_b \cap B_c$ is a separator in G. As both B_b and B_c have at most $n+1$ elements, and neither is contained in the other, the size of $B_b \cap B_c$ is at most n. □

Corollary 6.12. *Cycles have treewidth* 2.

Proof. Let C be a cycle. We have seen in the examples C has treewidth at most 2. Since C does not have a separator of size 1, it follows that C must have treewidth at least 2. □

Corollary 6.13. *Let G be a graph. Then G has treewidth at most 1 if and only if G is a forest.*

Proof. We have seen in the examples that a forest has treewidth at most 1. Conversely, if G has treewith at most 1, then it cannot contain a cycle, since cycles have treewidth 2. Therefore, G is a forest. $\square$

Definition 6.14. Let G be a graph, W be a set of its vertices, and α be a real number such that $0 < \alpha < 1$. A set S of vertices of G is called an *α-separator* for W if the following conditions hold:

- Some two vertices of W are in distinct components of $G - S$, and
- No component of G has more than $\alpha|W|$ vertices of W.

The following proposition is a strengthening of Proposition **??**

Proposition 6.15. *Let G be a connected graph of treewidth at most n, and let W be a set of at least $n+2$ vertices of G. Then there is a $\frac{1}{2}$-separator for W.*

Proof. Let T be a tree decomposition for G, of width at most n. Choose a root r for T. We call a subgraph of G *big* if it contains more than $\frac{1}{2}|W|$ vertices of W.

If B_v is a $\frac{1}{2}$-separator for W, we are done. Otherwise, some component D of $G - B_v$ is big; let w be the corresponding child of v.

If no component of $G - B_v$ is big, then we are done – the required separator is B_v.

Otherwise, one of the components of $G - B_v$ is big. We argue that this must be a component corresponding to a child of w, and not to v, the parent of w.

Indeed, the component C of $G - B_v$ which intersects B_v cannot be big: by Lemma 6.11, $B_v \cap B_w$ splits G into two disjoint sets, one of which contains C, and the other contains D. It is impossible that both C and D are big.

Therefore, C is not big, so there must be a child of w whose corresponding component is big.

We repeat the argument, with w being the root instead of v. If B_w is a $\frac{1}{2}$-separator for W, we are done. Otherwise, some component of $G - B_w$ is big, and this component must correspond to a child of w, and so on. Finally, we must find a vertex w such that B_w is a $\frac{1}{2}$-separator for W. $\square$

Fact 6.16. *Show that an $n \times n$ grid (with n^2 vertices) has treewidth at least n.*

Proof. Exercise. $\square$

6.2 Treewidth and minors

Fact 6.17. *If $G \leq_m H$, then $\mathrm{tw}(G) \leq \mathrm{tw}(H)$.*

Corollary 6.18. *For every n, the class of graphs of treewidth at most n is minor-closed.*

The following theorem is a special case of the Graph Minor Theorem (Theorem 2.18.1).

Theorem 6.19. *For every n, the minor relation is a w.q.o. over the class of graphs of treewidth at most n.*

Remark 6.20. For $n = 1$, this result is known as Kruskal's tree theorem. Its proof is not very difficult. For $n > 1$, the proof follows similar lines. The above theorem is one of the ingredients of the Graph Minor Theorem. Another, more difficult ingredient is the following result.

Theorem 6.21. *Let $\mathcal{C}$ be a class of graphs. The following conditions are equivalent:*

1. *$\mathcal{C}\nabla\infty$ contains all planar graphs.*
2. *$\mathcal{C}\nabla\infty$ contains all square grids,*
3. *$\mathrm{tw}(\mathcal{C}) = \infty$,*

Proof. The implication $1 \to 2$ is obvious. The implication $2 \to 3$ follows from Fact 6.16. The implication $3 \to 1$ is difficult. It can be restated as follows:

> For every $n \in \mathbb{N}$ there is a number $k_n \in \mathbb{N}$ such that $\mathrm{tw}(G) > k_n$ implies that G contains the $n \times n$-grid as a minor.

This result is a theorem of Robertson and Seymour. □

6.3 Algorithmic aspects of computing the treewidth

Theorem 6.22. *The following problem* $(*)$ *is NP-complete:*

- ***Input:*** *A graph* G *and a number* k
- ***Output:*** *Does* G *have treewidth at most* k*?*

However, we will se that the problem $(*)$ is FPT, if k is taken as the parameter:

Theorem 6.23 (Bodlaender)**.** *There is an algorithm for* $(*)$ *running in time* $O(|G| \cdot 2^{poly(k)})$*, which moreover constructs a corresponding tree decomposition (if it exists).*

Proof. We will only prove a weaker version of this theorem, where the running time is $O(|G|^2 \cdot 9^k)$. The idea is to recursively find good separators. For the proof, see e.g. this introductory text by Jiří Fiala: `iti.mff.cuni.cz/series/2003/132.ps`, Section 4.1. □

6.4 Treedepth

A *rooted forest* is a disjoint union of rooted trees. Let F be a rooted forest. For two nodes v, w of F, we say that v is an *ancestor* of w, and write $v \leq w$, if v are in the same connected component and v lies on the path connecting w with the root. By $<$ we denote the irreflexive relation corresponding to $<$.

The *height* of F is the maximal length of a chain with respect to $<$, i.e. the maximal number of vertices on a root-to-leaf path in F.

By F^+ we denote the graph $(V_F, \{\{v, w\} : v < w \text{ in } F\})$; we call F^+ the *closure* of F.

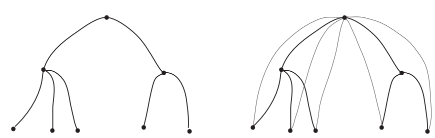


Figure 9: A forest and its closure.

Definition 6.24. Let G be a graph. The *treedepth* of G, denoted $\mathrm{td}(G)$, is the minimal height of a forest F such that G embeds into F^+.

Example 6.25. *If* G *is a star (i.e. a tree of depth* 2*), then* $\mathrm{td}(G) = 2$*. If* G *is a discrete set, then* $\mathrm{td}(G) = 1$*. The treedepth of a path with* n *nodes is* $O(\log(n))$*. The treedepth of the* n*-clique is* n*.*

The following fact is obtained easily by induction on $\mathrm{td}(G)$.

Fact 6.26. *Let* G *be a graph. Then:*

$$\mathrm{td}(G) = \begin{cases} 1 & \text{if } |G| = 0 \\ \max\{\mathrm{td}(H) : H \text{ is a connected component of } G\} & \text{if } G \text{ is disconnected} \\ 1 + \min\{\mathrm{td}(G - \{v\}) : v \in V_G\} & \text{if } G \text{ is connected} \end{cases}$$

Corollary 6.27. *Let P_n denote the path with n vertices. Then* $\mathrm{td}(P_n) = \lceil \log_2(n+1) \rceil$.

Proposition 6.28. *If G has no path of order greater than n, then* $\mathrm{td}(G) \leq n$.

Proof. Without loss of generality, assume that G is connected. Run a DFS on G. By the assumption on G, the depth of the search tree is at most n. The graph G embeds into the closure of this search tree. $\square$

7 Bounded Expansion

Definition 7.1 (Greatest reduced average degree)**.** Let $\mathcal{C}$ be a class of graphs and let $d \in \mathbb{N}$ be a number . The *greatest reduced average degree* at depth d of a graph $G \in \mathcal{C}$ is

$$\nabla_d(G) = \max_{H \in G \nabla d} \frac{\|H\|}{|H|}.$$

Definition 7.2 (Bounded expansion)**.** A class of graphs $\mathcal{C}$ has *bounded expansion* if $\nabla_d(\mathcal{C}) < \infty$ for all $d \in \mathbb{N}$, which means:

$$\max_{H \in \mathcal{C} \nabla d} \frac{\|H\|}{|H|} < f(d).$$

Fact 7.3. *Let a class of graphs $\mathcal{C}$ exclude a minor, from the fact that $\mathcal{C}$ has bounded density follows:*

$$\nabla_d(\mathcal{C}) < const.$$

Fact 7.4. *If $\mathcal{C}$ is nowhere dense, then $\nabla_d(G) \leq |G|^{o(1)}$.*

Proof. From theorem 4.5 we know that nowhere density is equivalent to the following condition:

$$\lim_{d \to \infty} \limsup_{H \in \mathcal{C} \nabla d} \frac{\log \|H\|}{\log |H|} \leq 1.$$

By a simple reformulation of above statement we get:

$$\forall d \forall \epsilon \exists n \qquad \begin{matrix} G \in \mathcal{C} \nabla d \\ |G| > n \end{matrix} \quad \text{such that } \frac{\log \|G\|}{\log |G|} \leq 1 + \epsilon.$$

We can now remove logarithms from inequality, and receive a bound on graph density:

$$\frac{\|G\|}{|G|} < |G|^{\epsilon}.$$

□

Theorem 7.5 (Dvorak at al.; Grohe and Kreutzer)**.** *Let $\mathcal{C}$ be a class of bounded expansion graphs. Then the following problem:*

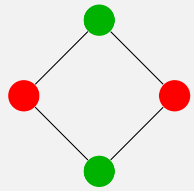
- *Input: A formula ϕ of First Order logic (treated as a parameter) and a graph G.*
- *Decide: Does $G \models \phi$ hold?*

is in FPT. More precisely, there exists an algorithm working in a time $f(|\phi|) \cdot |G|$.

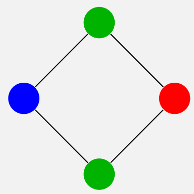
Proof. We will prove this theorem later. □

8 Coloring

Definition 8.1 (p-tree-width coloring)**.** Let G be a graph and p be a number. A coloring $c : V_g \to C$ is a p-tree-width coloring, if for any subgraph $H \subseteq G$ colored with $i \leq p$ colors, $\mathrm{tw}(H) \leq i - 1$.

Example 8.2. 

Any subgraph using exactly one color has tree width 0. This is a 1-tree-width coloring.

Example 8.3. 

This is a 2-tree-width coloring.

Definition 8.4 (p-tree-width chromatic number)**.** Let G be a graph. By the *p-tree-width chromatic number* of G we call the minimal number of colors needed in a p-tree-width coloring of G and denote it $\chi_p^{\mathrm{tw}}(G)$.

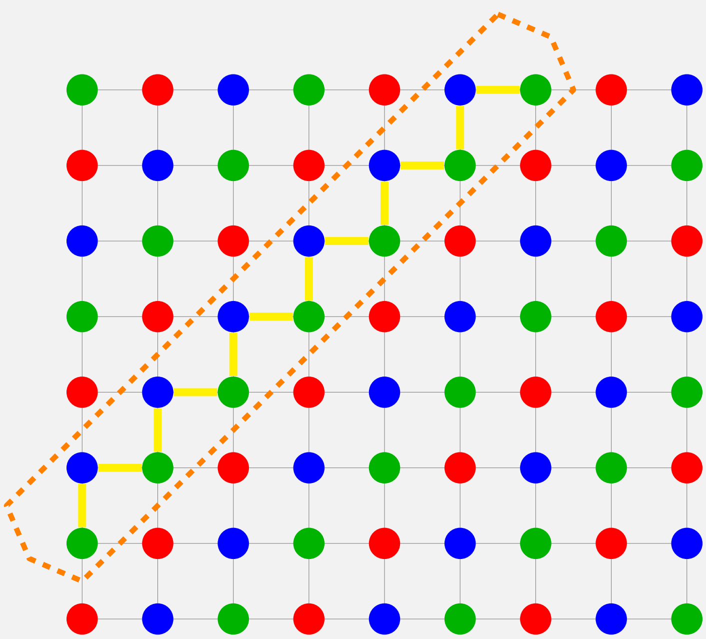
Fact 8.5. *Let G be a graph. Then:*

$$\chi_2^{\mathrm{tw}}(G) \leq n$$

iff G has a proper coloring with n colors, such that every cycle has more than 3 colors.

Fact 8.6. *For any graph G,*

$$\chi_1^{\mathrm{tw}}(G) = \chi(G).$$

Example 8.7. 

Any graph induced by 2 colors is a sum of paths.

$$\chi_2^{\mathrm{tw}}(G) = 3$$

Theorem 8.8. *Let $\mathcal{C}$ be a class of graphs which excludes a minor. Then for all $p \in \mathbb{N}$*

$$\chi_p^{\mathrm{tw}}(\mathcal{C}) < \infty.$$

Proof. We will prove a more general theorem later. ☐

Definition 8.9 (p-tree depth coloring)**.** Let G be a graph and p be a number. A coloring $c : V_G \to C$ is a p-tree-depth coloring, if for any subgraph $H \subseteq G$ colored with $i \leq p$ colors, $\mathrm{td}(H) \leq i$.

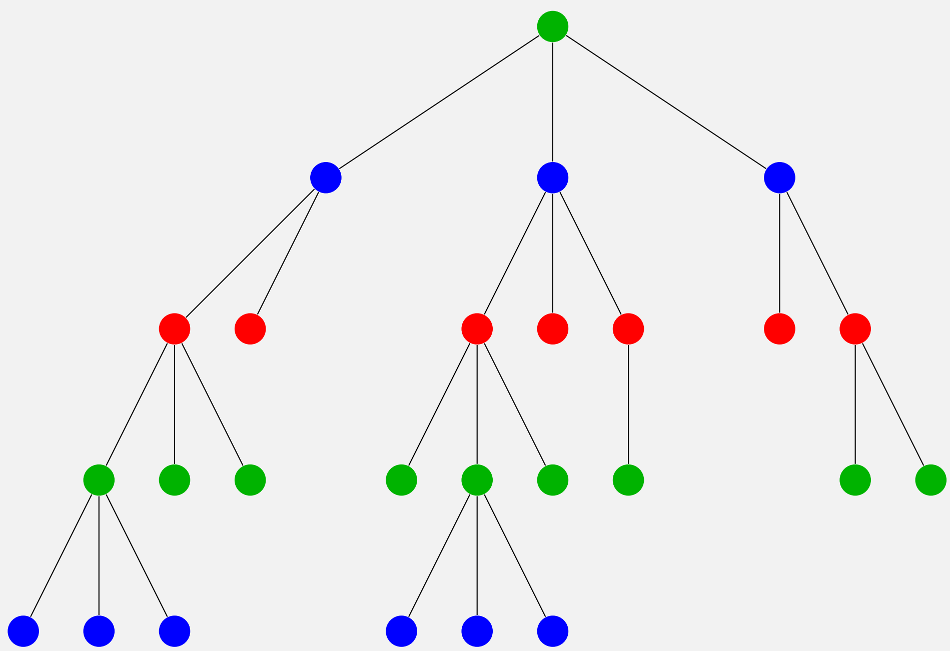
Definition 8.10 (p-centered coloring)**.** Let G be a graph and p be a number. A coloring $c : V_G \to C$ is a p-centered coloring, if for any connected subgraph $H \subseteq G$ one of the following conditions holds:

- H has a vertex v_i with a unique color. More precisely, $\exists i : (v_i) = c(v_j) \iff i = j$
- H uses at least p colors.

Fact 8.11. *For any graph G,*

$$\chi_1^{\mathrm{td}}(G) = \chi_1^{\mathrm{tw}}(G) = \chi(G).$$

Example 8.12.



$$\chi_p^{cent}(T) = p$$

Definition 8.13 ($f_\_$)**.** By $f_\_$ we denote a family of graph parameters, $f_\_ = (f_1, f_2, f_3, ...)$, where $f_n : \mathcal{G}raphs \to \mathbb{R} \cup \{-\infty, +\infty\}$. If $f_\_, g_\_$ are two such families, we write $f_\_ \preceq g_\_$ when there exist increasing functions s, t, such that $f_n(G) \leq t(g_{s(n)}(G))$ for any graph G and $n \in \mathbb{N}$.

Theorem 8.14. *The following inequalities holds:*

$$\begin{tikzcd}
\chi^{\mathrm{tw}}_{-} \arrow[r, phantom, "\preceq"] \arrow[dr, phantom, "\preceq" sloped] & \chi^{\mathrm{td}}_{-} \arrow[r, phantom, "\succeq"] \arrow[d, phantom, "\preceq" sloped] & \nabla_{-} \arrow[dl, phantom, "\preceq" sloped] \\
& \chi^{cent}_{-} &
\end{tikzcd}$$

Proof.

1. We will show that $\chi_p^{tw}(G) \leq \chi_p^{td}(G)$ Consider a p-tree-depth coloring of a graph G with n colors. This coloring is also a p-tree-width coloring.
2. $\chi_{-}^{td} \preceq \chi_{-}^{cent}$ Let G has a $p+1$-centered coloring with n colors. We will show that this is also a p-tree-depth coloring of G. From this follows $\chi_p^{td}(G) \leq \chi_{p+1}^{cent}(G)$. First we are going to show that if $H \subseteq G$ uses $i \leq p$ colors, then $\mathrm{td}(H) \leq i$. We proceed by induction on i:
   - For $i = 1$ it is indeed correct.
   - Without loss of generality we can assume that H is connected. Then either:
     - H uses more than $p+1$ colors, which is a contradiction with the assumptions.
     - H has a vertex v with an unique color.

   Then $H - \{v\}$ uses $i-1$ colors. By inductive assumption, we now find a tree decomposition with a depth $i-1$ and add v as a root, while keeping the remaining edges.
3. To be continued.

□

Corollary 8.15. *Let $\mathcal{C}$ be a class of graphs. Following conditions are equivalent:*

- *$\mathcal{C}$ has bounded expansion.*
- *$\chi_p^{tw}(\mathcal{C}) < \infty$ for all $p \in \mathbb{N}$.*
- *$\chi_p^{td}(\mathcal{C}) < \infty$ for all $p \in \mathbb{N}$.*
- *$\chi_p^{cent}(\mathcal{C}) < \infty$ for all $p \in \mathbb{N}$.*

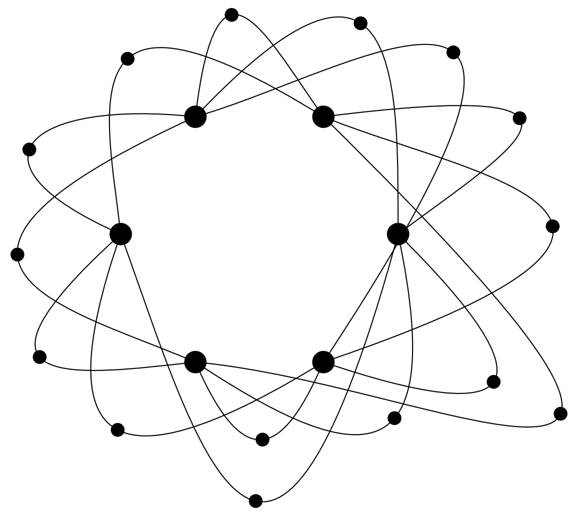


Figure 10: K_6 with subdivided edges.

9 $\chi_{-}^{\mathrm{td}} \preceq \nabla_{-}$

We now show that $\chi_{-}^{\mathrm{td}} \preceq \nabla_{-}$.

Definition 9.1. A *forest resolution* of a graph G is a sequence $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \ldots$, such that each graph is obtained from the previous one by applying the following rules, whenever possible: TODO: list of rules.

Note that all rules are applied simultaneously.

Our intermediate goal is to show that after performing one step of the resolution, we can, in some sense, bound the growth of the in-degree. Observe that a simplest approach does not work, that is there does not exist a function f, such that $\mathrm{maxdeg}(\vec{G}_{i+1}) \leq f(\mathrm{maxdeg}(\vec{G}_i))$. A possible example is shown in Fig. 10. Note that the maximum degree grows up to $\Theta(n)$, no matter how the edges are directed.

However, in the example above $\nabla(G)$ is very big. It turns out that if this is not the case, we can derive the desired bound. Namely, we can show that for some function f

$$\mathrm{maxdeg}(\vec{G}_{i+1}) \leq f(\mathrm{maxdeg}(\vec{G}_i), \nabla(\vec{G}_i))$$

Lemma 9.2. *There exists a function f such that for any oriented graph $\vec{G}$ and its arbitrary forest augumenation $\vec{G}'$ we have*

$$\nabla_r(G') \leq f(r, \mathrm{maxindeg}(\vec{G}), \nabla_{2r+1/2}(\vec{G})).$$

(G' denotes graph $\vec{G}'$ after dropping the orientation.)

Proof. We first show that $G' \in (G \bullet K_{d+1})\nabla 1$, where $d = \mathrm{maxindeg}(\vec{G})$. Observe that $G \bullet K_{d+1}$ can be constructed in two steps. First, for each $v \in V(G)$ we add a clique C_v of size $d+1$. Then for every directed edge $uv \in E(\vec{G})$, we add directed edges from every vertex in C_u to every vertex in C_v.

We now construct a subgraph $H \subseteq (G \bullet K_{d+1})$. Inside each clique C_v fix arbitrary vertex to be a *sink vertex* (denoted by $s(C_v)$) of this clique. Now for each edge vw of $\vec{G}$ we add a single edge to H. This edge will connect the sink vertex $s(C_v)$ with some non-sink vertex of C_w. We choose the end vertex inside C_w in such a way, that every vertex in H has at least one incoming edge. This can be done, as there are $d = \mathrm{maxindeg}\,\vec{G}$ vertices different from the sink in every clique.

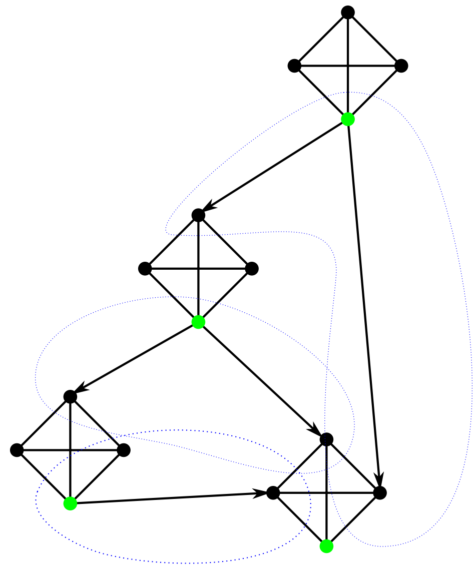


Figure 11: Graph H. Sets of edges that are contracted to one vertex are marked with dotted lines.

We obtain a graph H depicted in Fig. 11. Clearly, $H \subseteq (G \bullet K_{d+1})$. H is a collection of disjoint *stars*: every edge starts in some sink and every vertex has at most one in-edge.

In order to show that $G' \subseteq (G \bullet K_{d+1})\nabla 1$, we construct a minor H' of $G \bullet K_{d+1}$, and show that G' is a subgraph of H'. To obtain H' we contract all edges from H. Since H consists of stars, H' is a minor of H at depth 1 and thus also $H' \in (G \bullet K_{d+1})\nabla 1$. It remains to show that $G' \subseteq H'$.

There exists a natural correspondence between vertices of G and vertices of H'. Namely, each vertex v corresponds to a vertex created by contracting out-edges of $s(C_v)$ (although it is a slight abuse of notation, we will call this vertex $s(C_v)$). We now have to show that all edges of G' can be traced in H'. There are three types of edges in G':

1. Edges belonging to $\vec{G}$. Consider an edge $uv \in \vec{G}$. In $(G \bullet K_{d+1}$ there is an edge between $s(C_u)$ to $s(C_v)$. Since these vertices are not contracted to one vertex, the same edge exists in H'.

2. Edges added by the "transitive" rule. Assume that $uv, vw \in \vec{G}$. Then in H there is an edge from u to a vertex $v' \in C_v$. Moreover, in $G \bullet K_{d+1}$ there is an edge from v' to $s(C_w)$. After the contraction, the first of these edges gets contracted, so there is an edge between $s(C_u)$ and $s(C_v)$.

3. Edges added by the "fraternal" rule. If $uv, wv \in \vec{G}$ then in H one can go from $s(C_u)$ to $s(C_v)$ by following first a directed edge, then an edge inside a clique and then going backwards along a directed edge. Similarly as in case 1, after the contraction the corresponding vertices are connected with an edge.

We conclude that $G' \in (G \bullet K_{d+1})\nabla 1$. It follows easily that $\nabla_r(G') \leq \nabla_r((G \bullet K_n)\nabla 1)$.

We now use the following property:

$$\nabla_r(K\nabla 1) = \max_{H \in K\nabla 1 \nabla r} \frac{||H||}{|H|} \leq \nabla_{2r+1/2}(K)$$

to derive

$$
\begin{aligned}
\nabla_r((G \bullet K_n)\nabla 1) &\le \nabla_{2r+1/2}(G \bullet K_{d+1}) \\
&\le f(\tilde{\nabla}_{g(2r+1/2)}(G \bullet K_{d+1})) \\
&\le F(\tilde{\nabla}_{g(2r+1/2)}(G), d, r) \\
&\le F(\nabla_{g(2r+1/2)}(G), d, r)
\end{aligned}
$$

for some functions f, g and F, which do not depend on G or d. This completes the proof. □

Corollary 9.3. *There exists a family of functions $R_i(X, Y)$, such that for every graph $\vec{G}$ there exists an augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \ldots$ in which*

$$\text{maxindeg}(\vec{G}_i) \le R_i(\text{maxindeg}(\vec{G}), \nabla_{2^{i-1}-1}(G)).$$

Proof. We use induction on i to prove the following claim. There exists an augumentation such that for every r and i we have:

$$\text{maxindeg}(\vec{G}_i) \le R_i(\text{maxindeg}(\vec{G}), \nabla_{2^{i-1}-1}(G))$$

and

$$\nabla_r(G_i) \le S_i(r, \text{maxindeg}(\vec{G}), \nabla_{2^{i-1}(r+1)-1}(G)).$$

For $i = 1$, we have to show that

$$\text{maxindeg}(\vec{G}_1) \le R_1(\text{maxindeg}(\vec{G}), \nabla_0(G)))$$

and

$$\nabla_r(G_1) \le S_1(r, \text{maxindeg}(\vec{G}), \nabla_r(G)),$$

which follows immediately, since $G_1 = G$.

Now, the induction step. We first use Lemma 9.2 and then the induction hyphothesis twice:

$$
\begin{aligned}
\nabla_r(G_{i+1}) &\le f(r, \text{maxindeg}(G_i), \nabla_{2r+1/2}(\vec{G}_i)) \\
&\le f(r, R_i(\text{maxindeg}(G), \nabla_{2^{i-1}-1}G), S_i(r, \text{maxindeg}(\vec{G}), \nabla_{2^{i-1}(2r+1/2+1)-1})(G)) \\
&\le S_{i+1}(r, \text{maxindeg}(\vec{G}), \nabla_{2^i(r+1)-1}(G)).
\end{aligned}
$$

This gives us a bound on $\nabla_r(G_{i+1})$.

In order to bound maxindeg$(\vec{G}_{i+1})$, consider a resolution $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \ldots \subseteq \vec{G}_i$, which satisfies both claims. Our goal is to find such an augmentation $\vec{G}_{i+1}$ of $\vec{G}_i$, that we can bound the increase in the maximum in degree.

There are at most d^2 "transitive" edges entering any vertex of $\vec{G}_{i+1}$, where $d = \text{maxindeg}(\vec{G}_i)$. Let us now deal with fraternal edges, added by the second augmentation rule. Denote by H the graph consisting of fraternal edges. From Lemma 9.2 we have that $\nabla_0(H) \le f(0, d, \nabla_{1/2}(\vec{G}_i))$. Hence we can direct H, so that maxindeg$(\vec{H})$ is bounded as follows

$$
\begin{aligned}
\text{maxindeg}(H) &\le f(0, d, \nabla_{1/2}(\vec{G}_i)) \\
&\le f(0, d, S_i(1/2, \text{maxindeg}(\vec{G}), \nabla_{2^{i-1}(1/2+1)-1}(G)) \\
&\le f(0, d, S_i(1/2, \text{maxindeg}(\vec{G}), \nabla_{2^{i-1}(1/2+1)-1}(G)) \\
&\le f(0, d, S_i(1/2, \text{maxindeg}(\vec{G}), \nabla_{2^i-1}(G))
\end{aligned}
$$

Using this orientation, we have

$$\text{maxindeg}(\vec{G}_{i+1}) \le d + d^2 + f(0, d, S_i(1/2, \text{maxindeg}(\vec{G}), \nabla_{2^i-1}(G)))$$

Now, since $d = \text{maxindeg}(\vec{G}_i) \le R_i(\text{maxindeg}(\vec{G}), \nabla_{2^{i-1}-1}(G))$ we can bound $\text{maxindeg}\, \vec{G}_{i+1}$ by a function $R_{i+1}(\text{maxindeg}(\vec{G}), \nabla_{2^{i-1}-1}(G))$.

This completes the proof.

□

Remark. It is possible to make R_i polynomial functions.

Lemma 9.2 allows us to bound the in-degree in a forest augmentation of the graph. Our next goal is to show that when we perform the augmentation of G and the in-degree does not grow too much, after a small number of steps the augmented graph contains a forest order containing G.

Lemma 9.4. *Fix $p \in \mathbb{N}$. Consider any graph G and its resolution $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \ldots$.*
Then, for $T(d) = 1 + (d-1)(2 + \lceil \log_2 p \rceil)$, where $d = td(G)$, either

- *$\vec{G}_T$ contains an acyclicly oriented clique of size p, or*
- *$\vec{G}_T$ contains a forest order $\vec{F}$, such that $G \subseteq \vec{F}$.*

Proof. The proof proceeds by induction on d. For $d = 1$, it is easy (G consists solely of isolated vertices).

Now, an induction step. Consider G, such that $td(G) = d+1$. W.l.o.g let us assume that G is connected. Hence, there exists a vertex s, such that $td(G - \{s\}) = d$. Then $G - \{s\}$ is a collection of connected graphs H^i.

$$\vec{H}^i_t := \vec{G}_t[V_{H^i}]$$

Moreover, $\vec{H}^i_1 \subseteq \vec{H}^i_2 \subseteq \ldots$ is a super-resolution of H^i. We now apply the induction hyphothesis to the resolution of H^i. Consider a moment T. If any graph contains a directed clique of size p, we are done. Thus, let us now assume the opposite, that is the other case holds for all $\vec{H}^i_T$. Thus, each $\vec{H}^i_T$ contains a forest order $\vec{F}_i$ containing H^i. We now have to show that after a small number of steps, the entire graph will contain a forest order containing G.

Consider a set of vertices $D_i = \{v \in H^i | \exists_w v \ge w \to s\}$ ($\to$ means there is an edge in H^i_T). Hence, in the next step of resolution, all vertices of D_i get connected to s and in the following, we get a clique on vertices of $D = \bigcup D_i$. If $|D| \ge p$, we are done, we get a clique and it can be made an undirected one (TODO: how?). Otherwise, the total size of D_i's is less than p.

We pick a hamiltonian path in $\vec{G}_{T+2}$ than goes through all vertices in D. This gives us an ordering of vertices inside D_i's, in particular of the vertices, whose descendants are outside D_i. □

The following lemma is a consequence of Lemma 9.4.

Lemma 9.5. *Let $p \in \mathbb{N}$. Consider a graph G together with its resolution*

$$\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \dots$$

For $S = S(p) = 1 + (p-1)(2 + \lceil \log_2 p \rceil)$ at least one of the following holds:

(a) $\vec{G}_S$ contains an acyclic oriented clique of size p, or

(b) $\operatorname{td}(G) \leq p - 1$ and $\vec{G}_S$ contains a forest order $\vec{F}$ such that $G \subseteq F$.

Proof. (1) If $\operatorname{td}(G) \leq p-1$ then by Lemma 9.4 (a) or (b) holds for $\vec{G}_T$ where $T = 1 + (\operatorname{td}(G) - 1)(2 + \lceil \log_2 p \rceil) < S$, hence (a) or (b) also holds for $\vec{G}_S$.

(2) Suppose $\operatorname{td}(G) = p$. We show that G contains an acyclic oriented clique of size p. Assume it does not. By Lemma 9.4, $\vec{G}_S$ contains a forest order $\vec{F}$ such that $G \subseteq F$. Then F has depth at least p, because $\operatorname{td}(G) = p$. Therefore $\vec{F}$ contains a path of length p. Since $\vec{F}$ is a forest order, each two vertices on this path are joined by exactly one edge in $\vec{F}$, whose orientation is consistent with the orientation of the path. Hence the vertices on the path induce an acyclic oriented clique of size p in $\vec{F} \subseteq \vec{G}_S$. Contradiction.

(3) Suppose $\operatorname{td}(G) > p$. Then there exists $H \subseteq G$ such that $\operatorname{td}(H) = d$. It is clear from the definition of a resolution that there exists a resolution r of H

$$\vec{H} = \vec{H}_1 \subseteq \vec{H}_2 \subseteq \dots$$

such that $\vec{H}_i \subseteq \vec{G}_i$ for $i \in \mathbb{N}_+$. Indeed, by simply restricting the resolution of G to H and removing edges unnecessarily added to $\vec{H}$ in the resolution of G, we obtain a desired resolution of H. By applying (2) to H with the resolution r, we conclude that $\vec{H}_S \subseteq \vec{G}_S$ contains an acyclic oriented clique of size p.

□

Corollary 9.6. *For a graph G and its resolution $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \dots$ we have:*

$$\chi_p^{\text{cent}}(G) \leq 2\,\operatorname{maxindeg}(\vec{G}_{S(p)}) + 1$$

where S is the function from Lemma 9.5.

Proof. Let $D = \operatorname{maxindeg}(\vec{G}_{S(p)})$. First, we show that $G_{S(p)}$ has a proper coloring using at most $2D+1$ colors.[1] We have:

$$\operatorname{mad}(G_{S(p)}) = \max_{H \subseteq G_{S(p)}} \frac{2\|H\|}{|H|} = \max_{H \subseteq G_{S(p)}} \frac{2\sum_{v \in \vec{H}} \operatorname{indeg}(v)}{|H|} \leq \max_{H \subseteq G_{S(p)}} 2\,\operatorname{maxindeg}(\vec{H}) = 2\,\operatorname{maxindeg}(\vec{G}_{S(p)})$$

This implies that $G_{S(p)}$ is $2D$-degenerate, so it may be colored with $2D+1$ colors (remove a vertex v of degree $2D$ and inductively color the rest of the graph, then color v with a color not used by any of its $2D$ neighbors).

So let $c : V_G \to \mathbb{N}$ be a coloring of $G_{S(p)}$ with $2D+1$ colors. Obviously, c is then also a coloring of G, since $G \subseteq G_{S(p)}$ and $V_G = V_{G_{S(p)}}$. We will show that c is a p-centered coloring of G.

Let $H \subseteq G$ be connected. As in the proof of Lemma 9.5, there exists a resolution $\vec{H} = \vec{H}_1 \subseteq \vec{H}_2 \subseteq \dots$ such that $\vec{H}_i \subseteq \vec{G}_i$ for $i \in \mathbb{N}_+$. Let $S = S(p)$. If $\vec{H}_S \subseteq \vec{G}_S$ contains an acyclic oriented clique of size p, then $c_{\restriction V_H}$ uses at least p colors. Otherwise, by Lemma 9.5, $\vec{H}_S$ contains a forest order $\vec{F}$ such that $H \subseteq F$. Because H is connected, $\vec{F}$ has exactly one root u, which is connected with every other vertex in $\vec{F} \subseteq \vec{G}_S$. Thus u must have a unique color in $\vec{F}$, and hence also in $H \subseteq F$. □

[1] For a directed graph $\vec{G}$ we denote by G the undirected graph obtained from $\vec{G}$ by forgetting edge orientation and removing duplicate edges.

By Corollary 9.3, there exist families of functions $R_i(x, y)$, $\tilde{R}_i(x, y)$ and $f_i(x, y)$ ($i = 1, 2, \ldots$) such that for any graph G there exists a resolution of G

$$\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \ldots$$

such that

$$\begin{aligned} \operatorname{maxindeg}(\vec{G}_i) &\leq R_i(\operatorname{maxindeg}(\vec{G}), \nabla_{2^{i+1}-1}(G)) \\ &\leq \tilde{R}_i(\nabla_0(G), \nabla_{2^{i+1}-1}(G)) \\ &\leq f_i(\nabla_{2^{i+1}-1}(G)) \end{aligned}$$

The resolution of G for which the above inequalities hold is called the *canonical* resolution, and $\vec{G}_i$ in this resolution is denoted by $\rho_i(G)$.

Corollary 9.7.

$$\chi_p^{\text{cent}}(G) \leq 2 \operatorname{maxindeg}(\rho_{S(p)}(G)) + 1 \leq f_{S(p)}(\nabla_{g(p)}(G))$$

for S and f as above, and an appropriate function g.

Conclusion

$$\begin{array}{ccccc} \chi_{\_}^{\text{tw}} & \preceq & \chi_{\_}^{\text{td}} & \succeq & \nabla_{\_} \\ & \rotatebox{-135}{$\succeq$} & \rotatebox{-90}{$\preceq$} & & \rotatebox{90}{$\preceq$} \\ & & \chi_{\_}^{\text{cent}} & \preceq & \operatorname{maxindeg}(\rho_{\_}) \end{array}$$

10 Algorithmic aspects

Theorem 10.1. *Let $p \in \mathbb{N}$. There exists an FPL algorithm which given a graph G computes:*

(1) canonical resolution of G up to the p-th augmentation:

$$\vec{G} = \vec{G}_1 \subseteq \ldots \subseteq \vec{G}_p$$

($\nabla_{2^{p+1}-1}(G)$ is a parameter),

(2) p-centered coloring of G ($\nabla_{g(p)}(G)$ is a parameter, with g as in the previous section),

(3) p-tree-depth coloring of G ($\nabla_{h_1(p)}(G)$ is a parameter, for some function h_1 independent of G).

(4) p-tree-width coloring of G ($\nabla_{h_2(p)}(G)$ is a parameter, for some function h_2 independent of G).

Proof. We assume that the representation of G is by adjacency lists, i.e. G is given as an array mapping vertices to lists of their neighbors.

(1) Obviously, $\operatorname{mad}(G) = 2\nabla_0(G) \leq 2\nabla_{2^{p+1}-1}(G)$, and the last value is a parameter. The graph G is $\lfloor \operatorname{mad}(G) \rfloor$-degenerate (see Section 1.2), so by Theorem 1.26 there exists an orientation $\vec{G}$ of G with $\operatorname{maxindeg}(\vec{G}) \leq \operatorname{mad}(G)$. This orientation may be found by the method presented in the proof of Theorem 1.26. We show that this method may implemented in time $O(\operatorname{mad}(G)|G|)$. First, we compute the degree of every vertex of G and store the degrees in an array D mapping vertices to their degrees. This may be done in time $O(|G| + \|G\|) = O(\operatorname{mad}(G)|G|)$. We maintain a list L of vertices with degree $\leq \operatorname{mad}(G)$. The main algorithm works as follows.

1. Remove a vertex v from L.

2. Decrease the degrees of neighbours of v, adding them to L if their degree drops to $\leq \text{mad}(G)$, unless they were in L before.
3. Inductively orient $G - \{v\}$.
4. Orient all edges from neighbours of v in $G - \{v\}$ to v.

The cumulative time spent on step 2 is $O(\|G\|) = O(\text{mad}(G)|G|)$. Note that there is no need to update the adjacency lists of the neighbours of v in step 3 (the adjacency lists in recursive calls will contain additional vertices, which we may simply ignore). Therefore, the whole algorithm works in time $O(\text{mad}(G)|G|)$.

Thus G may be oriented to $\vec{G} = \vec{G}_1$ with low maxindeg. Note that a directed graph may be represented by an array of adjacency lists, each of length $\leq$ maxindeg – simply store in an adjacency list for a vertex v only those verties w for which there is an edge $w \to v$. This enables, for a given vertex v, to list in time $O(\text{maxindeg})$ all vertices w such that $w \to v$.

Assuming $\text{maxindeg}(\vec{G}_i) \leq f_i(\nabla_{2^{i+1}-1}(G))$, it now suffices to show how to compute from $\vec{G}_i$ a graph $\vec{G}_{i+1}$ such that $\text{maxindeg}(\vec{G}_{i+1}) \leq R_{i+1}(\text{maxindeg}(\vec{G}_i), \nabla_{2^{i+2}-1}(G))$, in time linear in $|G|$, where the functions f_i and R_i do not depend on G. Let $D_i = \text{maxindeg}(\vec{G}_i)$. We form two graphs.

1. The graph $\vec{A}$ formed from $\vec{G}_i$ by adding all transitive edges (see the definition of augmentation). This graph may be computed in $O(D_i^2|G|)$ (for each vertex w consider every path of length two ending in w). Note that the in-degree of each vertex may grow by at most D_i^2.
2. The undirected graph B formed from $\vec{G}_i$ by adding fraternal edges and dropping the orientation. By Lemma 9.2 we have
$$\text{mad}(B) = 2\nabla_0(G) \leq f(D_i, \nabla_{1/2}(G_i))$$
for some f not depending on G. Let B' be B with only the fraternal edges. Then $\text{mad}(B') \leq \text{mad}(B)$ is bounded, so we may orient B' in linear time, obtaining $\vec{B}'$. Adding to $\vec{B}'$ the directed edges from $\vec{G}_i$ we obtain a graph $\vec{B}$, whose maximum in-degree is still bounded by a function of $\text{maxindeg}(G_i)$ and $\nabla_{2^{i+2}-1}(G)$.

Now we join the two graphs $\vec{A}$ and $\vec{B}$ into one directed graph $\vec{G}_{i+1}$. This graph still has maxindeg bounded by a function of $\text{maxindeg}(G_i)$ and $\nabla_{2^{i+2}-1}(G)$.

(2) Using (1) we find the $S(p)$-th augmentation, where S is as in Corollary 9.6. We know that $\text{maxindeg}(\vec{G}_{S(p)}) \leq f(\nabla_{g(p)}(G))$ for some function f not depending on G. Let $D = f(\nabla_{g(p)}(G))$. This implies that $\vec{G}_{S(p)}$ is $2D$-degenerate, so it may be colored with $2D + 1$ colors in linear time (see the proof of Corollary 9.6). The proof of Corollary 9.6 then implies that this coloring is a p-centered coloring.

(3) Follows from (2) and the fact that every $(p + 1)$-centered coloring is also a p-tree-depth coloring (see Theorem 8.14).

(4) Follows from (3) and the fact that every p-tree-depth coloring is a p-tree-width coloring (see Theorem 8.14).

□

Definition 10.2. We say that $\mathcal{C}$ has *effectively bounded expansion* if
$$\forall_p \nabla_p(\mathcal{C}) \leq f_{\mathcal{C}}(p)$$
for a computable function $f_{\mathcal{C}}$.

Theorem 10.3. *Let $\mathcal{C}$ be a class with effectively bounded expansion. Then the following problem is FPL.*

Input: *Graph $G \in \mathcal{C}$ and a formula φ (a parameter) of the form:*

$$\varphi \equiv \exists X.(|X| \leq p) \vee (G[X] \models \psi)$$

where $p \in \mathbb{N}$ and ψ is an MSO formula.

Output: $G \models \varphi$

Proof. The algorithm is as follows.

1. Find a $\leq p$-tree-depth coloring $c : V_G \to C$ of G, where

   $$|C| \leq \chi_p^{\mathrm{td}}(G) \leq h(f_{\mathcal{C}}(g(p)))$$

   for some functions h, g not depending on G. For $D \in \binom{C}{p}$ we denote $G[D] = G[c^{-1}(D)]$. We have $\mathrm{td}(G[D]) \leq p$.

2. Repeat the following for each $D \in \binom{C}{p}$.

   - Run DFS on $G[D]$.
   - This gives a spanning tree of $G[D]$ of depth $\leq 2^p$ (the tree-depth of a path of length m is exactly $\lceil \log_2 m \rceil$, so if $G[D]$ contains a path of length $> 2^p$, then $\mathrm{td}(G[D]) > p$).
   - This gives a tree decomposition of $G[D]$ of width $\leq 2^p + 1$.
   - We apply Courcelle's theorem to check $G[D] \models \varphi$ in linear time.

3. If for some $D \in \binom{C}{p}$ we obtain $G[D] \models \varphi$, then output **YES**, otherwise output **NO**.

If the algorithm outputs **YES**, then $G[D] \models \varphi$ for some $D \in \binom{C}{p}$. Hence $G \models \varphi$. Conversely, if $G \models \varphi$ then there exists a subset of vertices X such that $|X| \leq p$ and $G[X] \models \psi$. The set X uses at most p colors, so $G[X] \subseteq G[D]$ for some $D \in \binom{C}{p}$. Thus $G[D] \models \varphi$. This shows that the algorithm is correct.

Now we analyze the running time. The first step takes $f_1(p)|G|$ time, for some function f_1 not depending on G. Each of the $\binom{|C|}{p}$ iterations in the second step takes time $f_2(\varphi)|G|$, where f_2 does not depend on G. Because $|C|$ is bounded by a function of p, we conclude that the algorithm is FPL. $\square$

11 FO formulas

Definition 11.1. A signature Σ over a formal language is a list

$$\Sigma \quad = \quad (R_1, R_2, ..., R_k, f_1, f_2, ..., f_l, ar)$$

where R_i are relation symbols or predicates, f_j are function symbols and ar is the *arity* function defined

$$\Sigma \ni \sigma \qquad \xrightarrow{ar} \qquad arity(\sigma) \in \mathbb{N}$$

Example 11.2. *A signature for a class of oriented, coloured graphs is*

$$\Sigma \quad = \quad (E, C_1, C_2, ..., C_l)$$

where E is a binary relation defining edges and C_i are unary relations defining colourings ie. $C_i(v)$ means that edge v has colour i.

For each relation symbol $R_i \in \Sigma$ there exists a unique arity relation $R_i^{\mathbb{A}}$, where $\mathbb{A}$ is a structure over Σ. For each function symbol $f_j \in \Sigma$ there exists a unique arity function $f_i^{\mathbb{A}}$

$$f_i^{\mathbb{A}} : \mathbb{A}^n \quad \longrightarrow \quad \mathbb{A}$$

Example 11.3. *A structure over*

$$\Sigma \quad = \quad (E, C_1, ..., C_k)$$

is an oriented graph (loopless) where each edge can have multiple colours.

FO formulas (over a signature Σ) can use symbols of Σ and $\vee, \wedge, \neg, \forall, \exists, =$ as in

$$\varphi(v) = \forall w \quad E(v, w) \ \vee \ E(w, v)$$

. Let $\mathbb{A}$ be a structure over Σ, then we denote

$$\begin{aligned} \varphi(\overline{x}) \quad &- \quad \textit{formula over } \Sigma \textit{ with free variables } \overline{x} = (x_1, ..., x_k) \\ \varphi(\mathbb{A}^k) \quad &= \quad \{\overline{v} \in \mathbb{A}^k \qquad \mathbb{A} \models \varphi[\overline{x} \to \overline{v}]\} \end{aligned}$$

If φ is a sentence, then $k = 0$ and $\varphi(\mathbb{A}^0) \ \in \ \{\emptyset, \{\epsilon\}\}$, where $/epsilon$ is an empty tuple.

Theorem 11.4. ($Dvo\check{r}\grave{a}k$, $Kr\grave{a}l$, $Thomas$ $-$ 2010, $Grohe$, $Krentzer$ $-$ 2011)
Let $\mathcal{C}$ be a class of graphs with effectively bounded expansion. Then the following problem is FPL: ($|G| \cdot f(\varphi)$, f computable)

Input: *Sentence $\varphi \in$ FO (parameter) and a graph $G \in \mathcal{C}$*

Output: *Whether $G \models \varphi$*

Example 11.5. *Independent set of size k*

$$\exists v_1 \ \exists v_2, \ ... \ , \exists v_k \bigwedge_{\substack{1 \leq i, \ j \leq k \\ i \neq j}} \neg \ E(v_i, v_j) \ \wedge \ v_i = v_j$$

Example 11.6. *Dominating set of size k*

$$\exists v_1 \ \exists v_2, \ ... \ , \exists v_k \ \forall w \bigvee_{1 \leq i \leq k} E(v_i, w)$$

12 Colored pointer structures

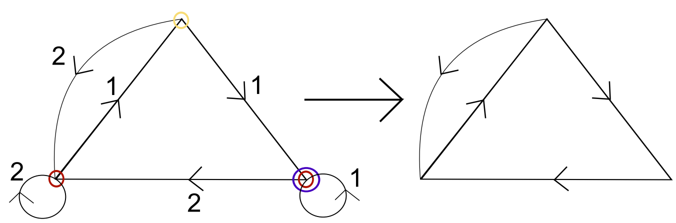
Fix $d \in \mathbb{N}$. We define a *Colored d-pointer structure* over a signature Σ as

$$(C_1, ..., C_k, f_1, ..., f_d)$$

where C_i are unary relations with any k and f_j are the pointes with fixed d.

To simplify notation we will use $v \xrightarrow{i} w$ istead of $f_i(v) = w$.

Example 12.1. *2-pointer structure*



For a pointer structure H, by Graph(H) we denote the underlying undirected graph, defined by:

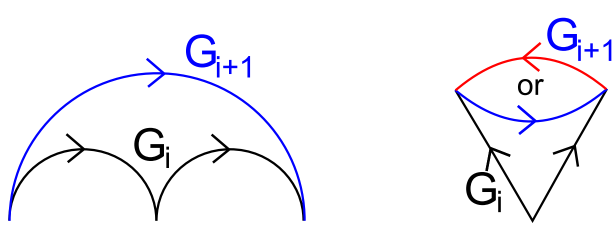
$$\text{Graph}(H) = (supp\ H, \bigcup_{i=1}^{d} graph(f)\ -\ diagonal)$$

Now we are going to reverse the arrows with regard to the last lecture.

Notation. For an oriented graph $\overrightarrow{G}$ we define

$$\overrightarrow{G_i} \stackrel{def}{=} Rev(\overrightarrow{\varphi_i}(Rev(\overrightarrow{G})))$$

where Rev denotes reversing of the arrows. We obtain $\overrightarrow{G_{i+1}}$ from $\overrightarrow{G_i}$ as follows:



Let D be a class of oriented graphs. We say that D has efectivelly bounded resolution if there is a computable sequence $d_1, d_2, ...$ such that for all n and for all graphs $\overrightarrow{G} \in D \quad deg\overrightarrow{G_n} \leq d_n$.

Fact 12.2. *From previous lectures it follows that the class of unoriented graphs $\mathcal{C}$ has a bounded expansion iff the is a class of oriented graphs D with bounded resolution such that $\mathcal{C} = Undirect(D)$.*

Definition 12.3. Assume that D has bounded resolution. We define a family of pointer structures.

$$D_n = \{H : H \text{ is a } d_n\text{-pointer structure such that } Graph(H) \in \overrightarrow{G_n} \text{ for some } \overrightarrow{G} \in D\}$$

(Proof based on Kazana, Segoufin, 2013)

Theorem 12.4. *Let D be a class of oriented graphs with bounded resolution. Then the following problem is FPL:*

> **Input**: *Formula φ and an indicator structure H*
>
> **Output**: *Whether $H \models \varphi$*

Main step is the following proposition:

Proposition 12.5. *Let $p \in \mathbb{N}$ and $\psi(\overline{x}, y)$ be a d_p-pointer quantifierless formula. Then there exists $q \in \mathbb{N}$ and a d_q-pointer quantifierless formula $\varphi(\overline{x})$ such that for any structure $H \in D_p$ there is a d_q-pointer structure $H' \supseteq H$ such that*

$$(\exists y\ \psi)(H) = \varphi(H')$$

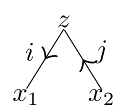
Moreover q and φ are computable from p and ψ and H' is computable from H in time $O(|H|)$.

Example 12.6. *Lets consider the following formula:*

$\varphi(x_1, x_2) = d(x_1, x_2) \leq 2 \ \Leftrightarrow$
$E(x_1, x_2) \ \vee \ E(x_2, x_1) \ \vee \ (\exists z\ E(z, x_1) \wedge E(z, x_2)) \ \vee \ (\exists z\ E(z, x_1) \wedge E(x_2, z)) \ \vee \ (\exists z\ E(x_1, z) \wedge E(x_2, z))$

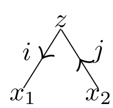
We now act as follows

Step 1: d_1-pointer formula

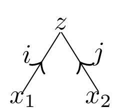
$(x_1 = x_2) \ \vee \ \bigvee_{i=1}^{d_1} x_1 \xrightarrow{i} x_2 \ \vee \ \bigvee_{i,j} \exists z$ 

$\vee \ \ldots$

Step 2: Simplification

$\exists z$ 

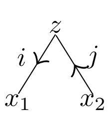
$\longmapsto x_1 = f_i(f_j(x_2))$

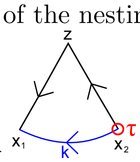
$\exists z$ 

$\longmapsto f_i(x_1) = f_j(x_2)$

we are left with the case of "forks".

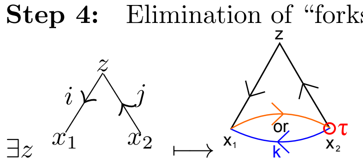
Step 3: Reduction of the nesting depth to 1.



in $H_1 \longmapsto$ 

in H_2 $\quad x_1 = f_k(x_2) \wedge \tau(x_2)$. τ means having colour τ in the augumentation. $\tau(v) \Leftrightarrow (f_i(f_j(v)) = f_k(v))$.

Step 4: Elimination of "forks":



$$x_2 = f_k(x_1) \wedge \tau(x_1), \qquad \tau(x_1) \Leftrightarrow \exists z f_i(z) = x_1 \qquad f_j(z) = f_k(f_i(z))$$

We obtained a formula ψ which is quantifierless and has a nesting depth of 1 (later referred to as a simple formula). This means that for any structure H in linear time we can compute a structure H' such that $\varphi(H) = \psi(H')$.

We say that a formula ϕ is compatible with signature H if the signature of ϕ is contained in H.

Fix a class $\mathcal{D}$ of directed graphs, a sequence d_n and families $\mathcal{D}_1, \mathcal{D}_2, \ldots$ of pointer structures, where $\mathcal{D}_n$ is a familiy of d_n-pointer structures H such that $Graph(H) = \vec{G}_n$ for some $G \in \mathcal{D}$.

Definition 12.7. Let ϕ be a d_p-pointer structure and ψ a d_q-pointer structure, where $p \leq q$. We say that ϕ and ψ are equivalent modulo augumentations if for every $H \in \mathcal{D}_p$ that is compatible with ϕ there exists $H' \in \mathcal{D}_q$ computable in $O(f(\phi)|H|)$ time, such that

- $\phi(H) = \psi(H')$
- H' is a $q - p$-augumentation of H.

Proposition 12.8. *Let $p \in \mathbb{N}$ and $\psi(\bar{x}, y)$ be a simple d_p-pointer formula. Then, there exists $q \in \mathbb{N}$ and $\phi(\bar{x})$ – a simple d_q-pointer formula, such that ϕ and $\exists_y \psi$ are equivalent modulo augumentations.*

Moreover, q and ϕ are computable from p and ψ.

From the above preposition, we will derive:

Theorem 12.9. *Let $\mathcal{D}$ be a class with bounded augumentation and let $\phi(\bar{x})$ be a d_1-pointer formula. Then we can compute $q \in \mathbb{N}$ and a q-pointer simple formula $\psi(\bar{x})$, such that ψ and ϕ are equivalent modulo augumentations.*

Before we give the proof of the Theorem, we need the following fact:

Lemma 12.10. *Let ϕ be an arbitrary d_p-pointer formula. Then, there exists a formula ψ such that*

- *ψ has the same quantifier depth as ϕ.*
- *ψ has no nested functions.*
- *ψ is a d_q-pointer formula for $q =$??.*
- *ϕ and ψ are equivalent modulo augumentations.*

Proof. TODO ☐

We can now prove the Theorem.

Proof. The proof proceeds on the induction on the construction of ϕ.

- If ϕ has no quantifiers then we can simply use Lemma 12.10.
- If $\phi = \phi_1 \vee \phi_2$ then we apply the induction hyphothesis to ϕ_1 and ϕ_2. We set $\psi := \psi_1 \vee \psi_2$. This is equialent to ϕ, since $\phi(H) = \phi(H_1) \cup \phi(H_2)$. There exists H' which can be colored so that $\phi_1(H) = \psi(H'_C)$ (and similarly for ϕ_2). Then $\phi(H) = \psi(H'_{C \cup D})$.
- If $\phi = \neg\phi'$, we do a similar (and a bit simpler) thing.

- If $\phi = \exists_y \phi'(\dots, y)$, we apply the induction hyphothesis to ϕ' and then use Proposition 12.8.

□

Corollary 12.11. *Let ϕ be a d_1-pointer structure. For a given d_1-pointer structure H, after preprocessing in $O(|H|)$ time we can answer to questions "Does $\bar{x} \in \phi(H)$?" in constant time.*

Proof. We compute ψ and graph H' (in $O(|H|)$). It now suffices to check if $\bar{x} \in \psi(H')$, since the size of ψ is constant. □

Lemma 12.12. *Let $H \in \mathcal{D}_p$ and $u \in H$. Let $q = p + d_p$ and let H' be a d_p-augumentation of H.*
Then H' contains a linear order on the the successors of u.

Proof. After one augumentation, we get a directed clique on the successors of u. It is a well-known fact that a directed clique contains a Hamiltonian cycle. Thus, after $\lceil \log_2 d_p \rceil$ more augumentations, we obtain a linear order on the successors. □

13 Independence

Definition 13.1. Set $A \subseteq V_G$ is *d-independent* if

$$\forall_{v,w \in A,\ v \neq w} d(v,w) > d.$$

Definition 13.2. Set $A \subseteq V_G$ is *d-scattered* if

$$\forall_{v,w \in A,\ v \neq w} N_d(v) \cap N_d(w) = \emptyset.$$

Fact 13.3. *A is d-scattered $\iff$ A is $2d$-independent.*

Let $\mathcal{C}$ be a class of graphs and let $s : \mathbb{N} \to \mathbb{N}$ be a function.

Definition 13.4 (quasi-wide class)**.** Class $\mathcal{C}$ is *quasi-wide with margin s* if for all $r, m \geq 0$ there exists an $N > 0$ such that:

$$\begin{cases} G \in \mathcal{C} \\ |G| \geq N \end{cases} \implies \begin{cases} \exists_{S \subseteq V_G} |S| \leq s(r) \\ G - S \quad \text{contains an } r\text{-scattered set of size at least } m \end{cases}$$

Examples:

- $K_{1,k}$ graphs (stars) are quasi-wide with margin $s(r) = 1$ ($N(r,m) = m+1$).
- Graphs with degree at most d are quasi-wide with margin $s(r) = 0$.
- Trees are quasi-wide with margin $s(r) = 1$.
- Graphs excluding H as a minor are quasi-wide with margin $s(r) = |H|$.

Definition 13.5 (uniformly quasi-wide class)**.** Class $\mathcal{C}$ is *uniformly quasi-wide with margin s* if for all $r, m \geq 0$ there exists an $N = N_{\mathcal{C}}(r,m) > 0$ such that:

$$\begin{cases} G \in \mathcal{C} \\ w \subseteq V_G \\ |w| \geq N \end{cases} \implies \begin{cases} \exists_{S \subseteq V_G} |S| \leq s(r) \\ w \quad \text{contains an } r\text{-scattered subset of } G - S \text{ of size at least } m \end{cases}$$

Theorem 13.6. *Let $\mathcal{C}$ be a class of graphs that is closed under taking induced subgraphs. The following conditions are equivalent:*

1. *$\mathcal{C}$ is quasi-wide,*
2. *$\mathcal{C}$ is uniformly quasi-wide,*
3. *$\mathcal{C}$ is nowhere-dense.*

We will only proove the implication $2 \Rightarrow 3$. The implication follows easily from the following:

Lemma 13.7. *Let $\mathcal{C}$ be a uniformly quasi-wide class and let $h > N_{\mathcal{C}}(r+1, s(r+1)+1)$, then $K_h \notin G \nabla r$.*

Proof. We prove by contraposition. Assume that $K_h \in G \nabla r$ where $G \in \mathcal{C}$.
Let $w = \{u_1, u_2, \ldots, u_h\}$ where u_i are vertices of G such that $u_i \in B_i \subseteq N_r(u_i)$, where B_i are the sets of vertices we contract to obtain K_h.
By the definition of quasi-wideness, there exists a set $S \subseteq V_G$ with $|S| \leq s(r+1)$ such that w a subset A, where A is an $(r+1)$-scattered set in $G - S$ and $|A| \geq s(r+1) + 2$.
The sets $\{B_i\}$ are disjoint, so there exist $1 \leq i < j \leq h$ such that $u_i, u_j \in A$ and $S \cap B_i = \emptyset = S \cap B_j$. There is an edge between some vertex in B_i and some vertex in B_j so $N_{r+1}(u_i) \cap N_{r+1}(u_j) \neq \emptyset$, which contradicts with the assumption that A is $(r+1)$-scattered in $G - S$. $\square$

We will now focus on an FPT algorithm for the problem of finding a dominating set on nowhere dense classes of graphs.

Definition 13.8. We say that a set $X \subseteq V_G$ *d-dominates* a set $W \subseteq V_G$ if

$$\forall_{w \in W} \exists_{x \in X} d(w, x) \leq d.$$

Theorem 13.9. *Let $\mathcal{C}$ be a nowhere-dense class of graphs. Then the following problem:*

> **Input***: Graph $G \in \mathcal{C}$, a set $W \subseteq V_G$*
>
> **Parameters***: k, d*
>
> **Output***: A set X which d-dominates W such that $|X| \leq k$*

can be solved in time $f_{\mathcal{C}}(k, d) \cdot |G|^3$ (it is FPT).

To prove this theorem we will use the following lemmas:

Lemma 13.10. *Given a graph $G \in \mathcal{C}$, a set $W \subseteq V_G$ and parameters r, m we can compute in time $O(|G|^2)$ the sets:*
$S \subseteq V_G$, such that $|S| \leq f_{\mathcal{C}}(r) - 2$ where $f_{\mathcal{C}}(r) = \min\{k : K_k \notin \mathcal{C}\nabla r\}$ and $A \subseteq W$, $|A| \geq m$, such that A is r-scattered in $G - S$ (that is - we can compute the sets form the definition of uniform quasi-wideness).

Lemma 13.11. *The following problem is FPT for all graphs:*

> **Input***: Graph $G \in \mathcal{C}$, a set $W \subseteq V_G$ of size w*
>
> **Parameters***: k, d, w*
>
> **Output***: A set X of size k which d-dominates W.*

Proof. We give the algorithm:
For every parition of W into k sets $\{W_1, W_2, \ldots, W_k\}$ we check (in $O(|G|^2)$ time) whether $\bigcap_{w \in W_i} N_d(w) \neq \emptyset$ for every i (that is, if for each set W_i there is a vertex that d-dominates W_i). ☐

Lemma 13.12. *The following problem can be solved in $f(k, d) \cdot |G|^2$ time:*

> **Input***: Graph $G \in \mathcal{C}$, a set $W \subseteq V_G$ such that $|W| > N(d, (k+2)(d+1)^s)$ where $s = f_{\mathcal{C}}(d) - 2$*
>
> **Output***: A vertex $w \in W$ such that for any set $X \subseteq V_G$ with $|X| \leq k$, X d-dominates W iff X d-dominates $W - \{w\}$.*

Proof. By lemma 13.10 we can compute sets S and A such that $|A| = m \geq (k+2)(d+1)^s$, $|S| \leq s$ and A is d-scattered in $G - S$.
For each $a \in A$, we compute the distance vector $v_a = (\mathrm{dist}|_{d+1}(a, s))_{s \in S}$. There are at most $(d+1)^s$ distinct distance vectors.
Since $|A| \geq (k+2)(d+1)^s$, there are at least $k+2$ distinct vertices $a_1, \ldots, a_{k+2} \in A$ such that $v_{a_i} = v_{a_j}$ for $1 \leq i \leq j \leq k+2$. We will show that $w = a_1$ satisfies the lemma.
The implication from left to right is obvious. Now suppose X d-dominates $W - \{a_1\}$. The sets $N_d^{G-S}(a_i)$ for $1 < i \leq k+2$ are disjoint, and since there are $k+1$ of them, then at least one $(N_d^{G-S}(a_j))$ does not contain any element of X. But since X dominates $W - \{a_1\}$ there is a path of length at most d from some $x \in X$ to a_j, which must go through an element of S.
But $v_{a_1} = v_{a_j}$, so there is also a path of length at most d from x to a_1, thus X d-dominates W. ☐

We can now give the algorithm which prooves the Theorem 13.9:

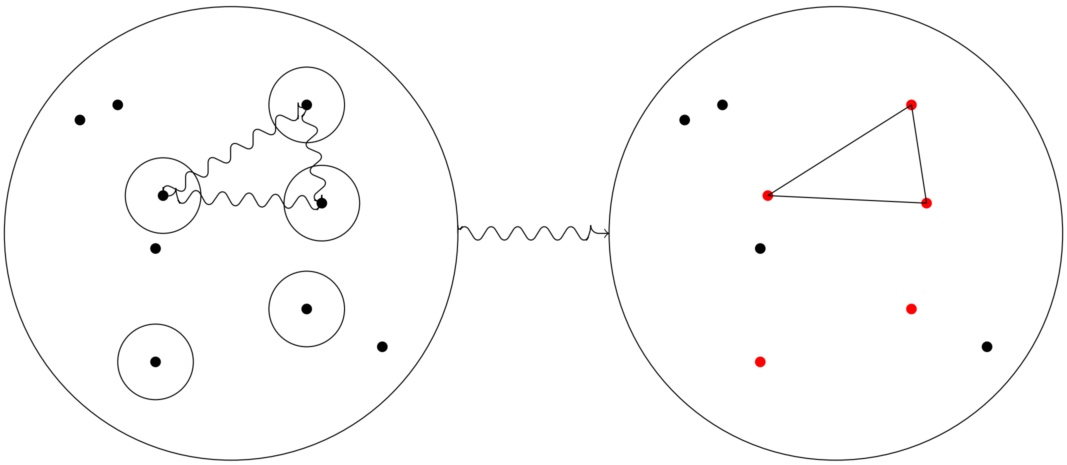
Proof. Using lemma 13.12 we iteratively remove elements from set W, as long as $|W| > N(d, (k+2)(d+1)^s)$.
Once $|W| \leq N$ we use lemma 13.11 to find the d-dominating set. ☐

Let us recall the following

Theorem 13.13 (Ramsey)**.** *There exists a function $R\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all G sattisfying $|G| \geq R(p,q)$ graph G either contains a p-clique or q-independent set.*

Now we are going to prove the implication (1) $\Rightarrow$ (2) in Theorem 13.6.

Lemma 13.14. *Let G be a graph and A be its $2r$-independent subset satisfying $|A| \geq R(c,n)$. Then either $K_c \in G\nabla r$ or A contains a $(2r+1)$–independent subset of size n.*



Proof. Let us merge each r-neighbourhood of some point in A. This transforms graph G into some graph $G' \in G\nabla r$ and a subset A into some $A' \subseteq G'$. Then either

- A' contains some big clique hence $K_c \in G\nabla r$ or
- A' contains some big 1-independent set and then A contains a big $(2r+1)$–independent set.

□

Lemma 13.15. *There exists a function $\theta\colon \mathbb{N}^4 \to \mathbb{N}$ such that if G is a graph and A its $(2r+1)$–independents subset satisfying $|A| \geq \theta(m,a,b,s)$ then at least one of the following satisfies:*

- *$K_a \in G\nabla r\nabla 2$,*
- *$K_{s+1,b} \in G\nabla r$,*
- *There exists a subgraph $S \subseteq G$ such that $|S| \leq s$ and $A\backslash S$ contains a $(2r+2)$-independent subset of size at least m.*[2]

Proof. Let us merge each r-neighbourhood of some point in A. Let us define G' like in Lemma 13.14. Then at least one of the following conditions is satisfied:

- G' contains 1-partition of K_a,
- G' contains $K_{s+1,b}$,
- it is possible to remove at most s vertex from A such that new graph (denoted by A') contain 2-nowhere-dense subset of size at least m.

□

[2]in this statement we treat $A\backslash S$ as a subset of $G\backslash S$

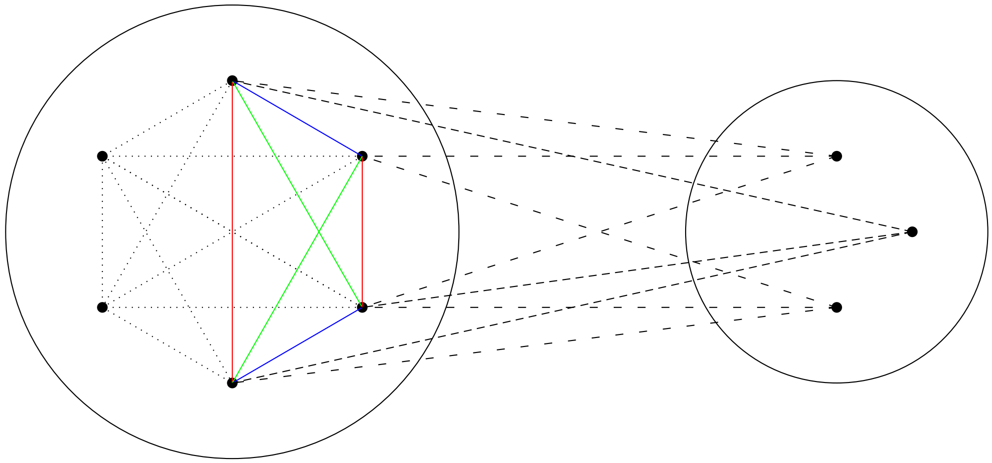


Figure 12: Example $n = 3$, $A = 6$

Lemma 13.16. *Let $G = (A \cup B, E)$ be a bipartite graph, $|A| \geq R(p, q, n)$. Then at least one of the following conditions is satisfied:*

(i) *A contains 2-independent set of size p,*

(ii) *A contains a main vertexes of 1-partition of clique K_q*

(iii) *B contains a vertex with degree at least n.*

Proof. Let us suppose that $\deg v \leq n$ for all $v \in B$. Let us color $\binom{A}{2}$ using $\leq \binom{n}{2}$ colors (plus one extra color "transparent") such that each $N(v)$ does not contain two edges with the same color.

By Ramsey's Theorem we have a monochromatic K_q or a transparent K_p. If a clique is not transparent we have 1-partition of K_q.

Otherwise we have a transparent K_p. But no vertex in B coincide with two vertexes in K_p. Hence K_p is a 2-independent set. $\square$

Lemma 13.17. *Let $G = (A \cup B, E)$ be a bipartite graph, $|A| \geq \theta(m, a, b, s)$. Then at least one of the following conditions is satisfied:*

(i) *There exists $A' \subseteq A$ and $B' \subseteq B$, $|A| = m$, $|B| = s$ such that each vertex from $B - B'$ has at most one neighbour in A' (Figure 13),*

(ii) *A contains a main vertexes of 1-partition of K_a,*

(iii) *B contains $s + 1$ vertexes of $K_{s+1,b}$.*

Proof. We prove it by induction with respect to s. For $s = 0$ it is just the Lemma 13.16.
Inductive step $s \to s + 1$. We apply the same lemma for $n = \theta(p, q, n, s)$. Then (i) and (ii) in Lemma 13.16 and this one implies respectively. The only case left is that B contains a vertex (denoted by v) with degree at least n.

Let $A_0 = N(v)$ and $B_0 = N(A_0) \backslash \{v\}$. Then ,by induction applied to the graph $(A_0 \cup B_0, E \cap A_0 \times B_0)$, one obtain sets A_0' and B_0'. It is easy to verify that sets $A' = A_0'$ and $B' := B_0' \cup \{v\}$ satisfies (i). $\square$

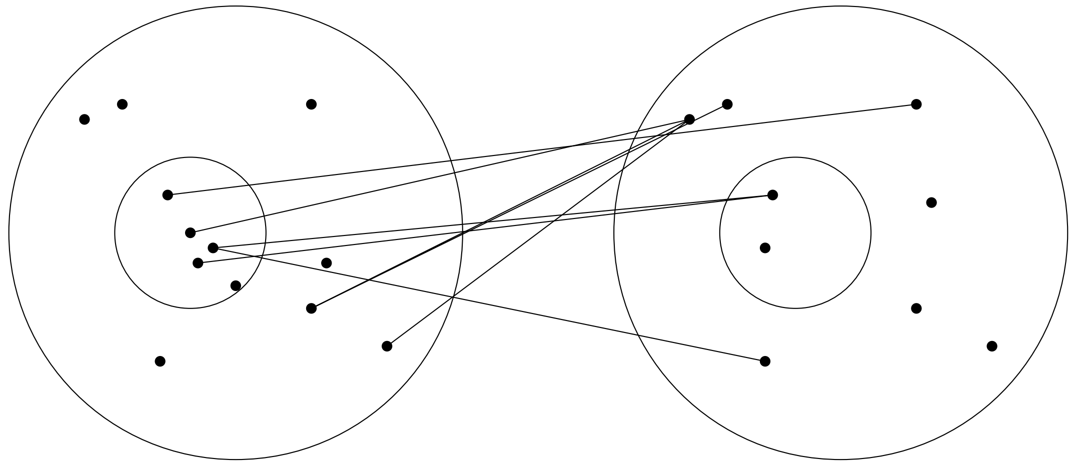


Figure 13: Case (i)

Corollary 13.18. *Let G be a graph. $A \subseteq G$ be a independent subgraph with at least $\theta(m, a, b, s)$ vertexes. Then at least one of the following conditions is satisfied:*

- *G contain a subset Δ of size at most s, disjoint with A, such that A as a subset of $G \backslash \Delta$ has a 2-independent set of size n,*
- *G contains a 1-partition of K_a,*
- *G contains $K_{s+1,b}$.*