Computation with atoms

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University of Warsaw

joint work with
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LSV, ENS Cachan, 2014.07.08

Outline

- Sets with atoms
- Models of computation in sets with atoms
- Are sets with atoms useful?

Sets with atoms

Sets with atoms

sets with urelements
permutation models
[Fraenkel, Mostowski '30ies]

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nominal sets
[Gabbay, Pitts '99]

named sets
[Pistore, Montanari '97]

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nominal G-sets
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sets with symmetry

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hereditarily finitely-supported sets

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Fraenkel-Mostowski sets

sets with atoms

Atoms allow us to model:

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- data values

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Atoms

Atoms

Atoms are a fixed logical structure

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atoms	atom automorphisms

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Atoms are a parameter in the following

Sets with atoms

Sets with atoms

Classical sets are built using ∅ and { }

e.g. {∅, {∅}, {{∅}}, {∅,{∅}}}

Sets with atoms

Classical sets are built using $\varnothing$ and { }

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Sets with atoms are built using $\varnothing$ and { } and atoms

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Examples: • ∅

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legality depends on atom automorphisms

Legal sets with atoms

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Extend atom automorphisms π to all sets element-wise,
e.g. $\pi(\{a, b, c\}) = \{\pi(a), \pi(b), \pi(c)\}$

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Sets supported by ∅ are called equivariant

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Examples:

- $\varnothing$
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support = atoms that you use in order to "define" a set

Examples:

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Sets with atoms

Sets with atoms

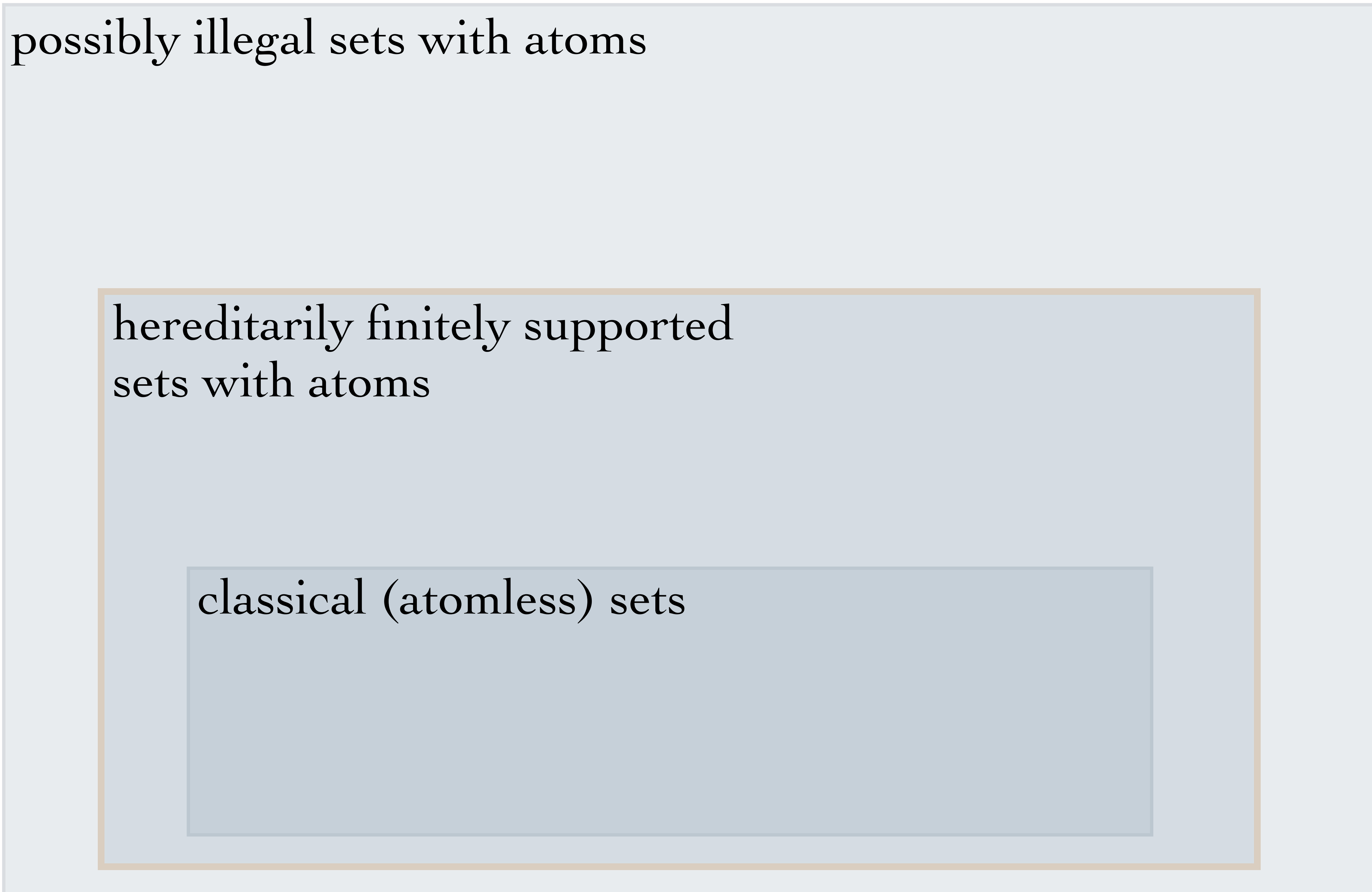
classical (atomless) sets

Sets with atoms

possibly illegal sets with atoms

classical (atomless) sets

Sets with atoms



relax finiteness to...

relax finiteness to...

...finiteness up to atom automorphism

Orbit-finite sets

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x, y are in the same orbit if $\pi(x) = y$ for an atom automorphism π

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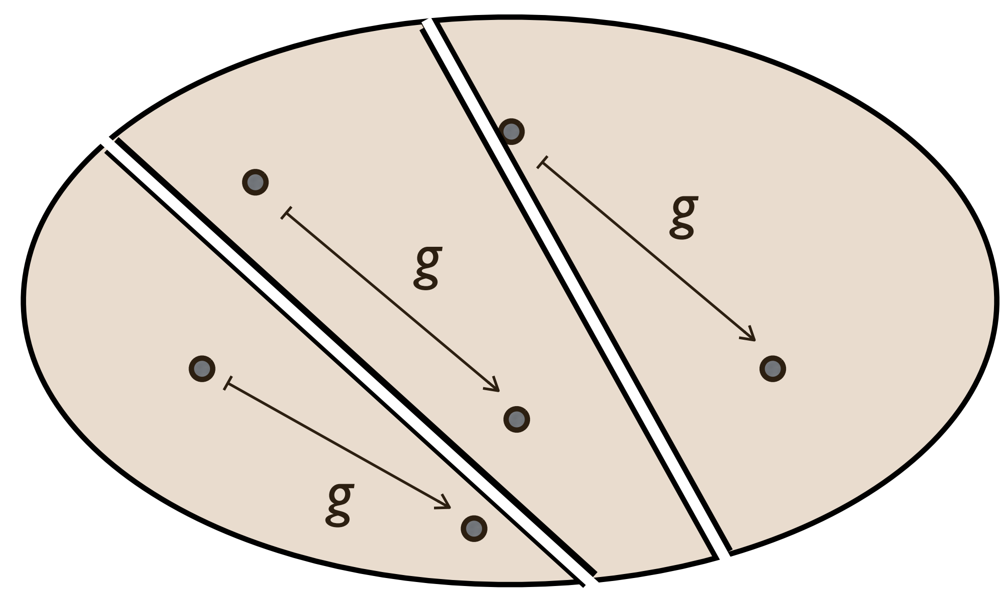

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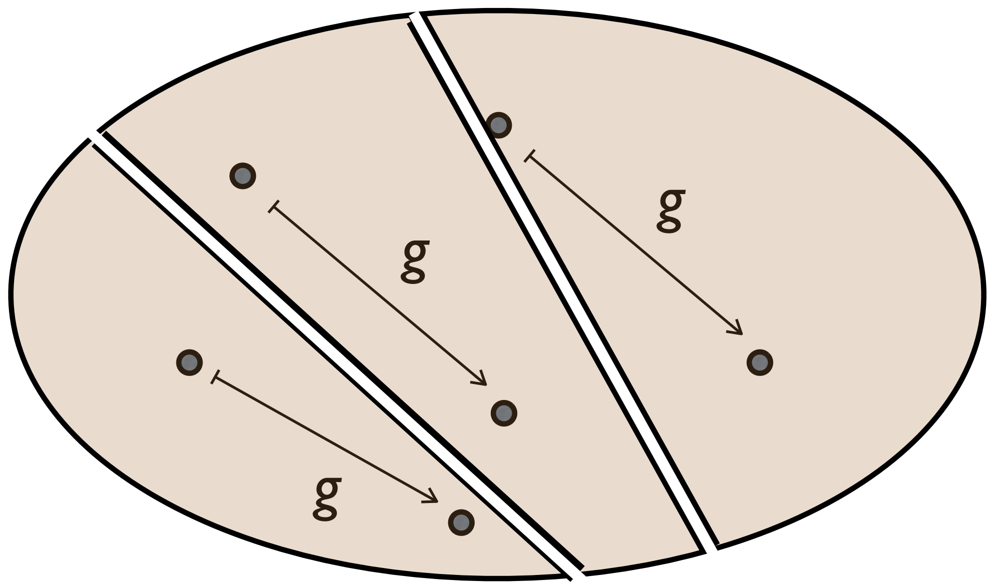

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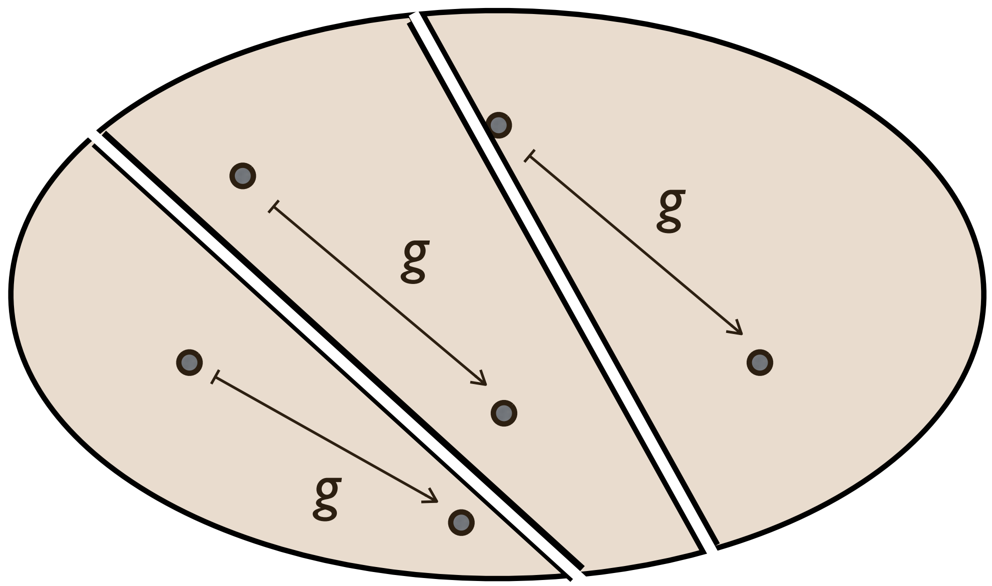

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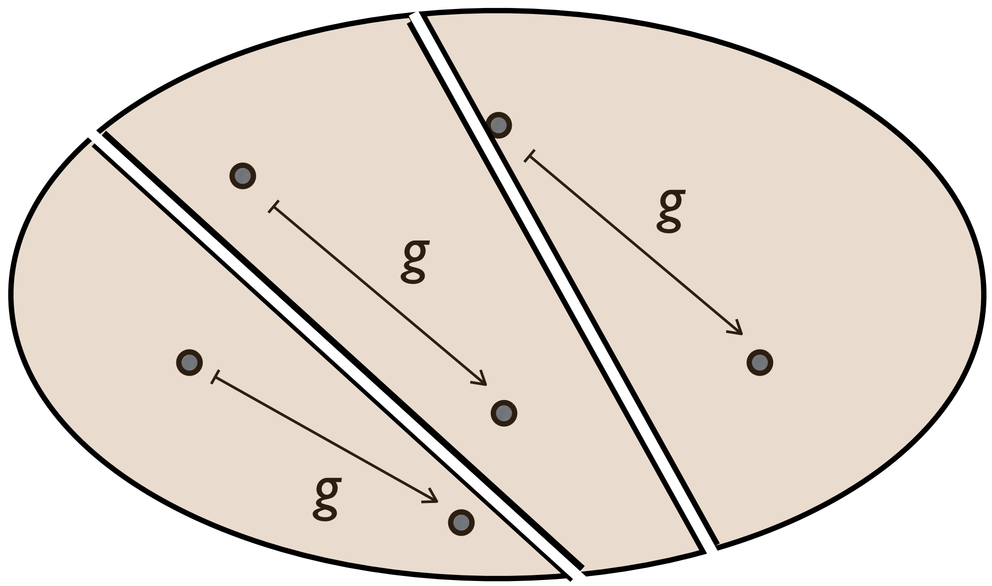

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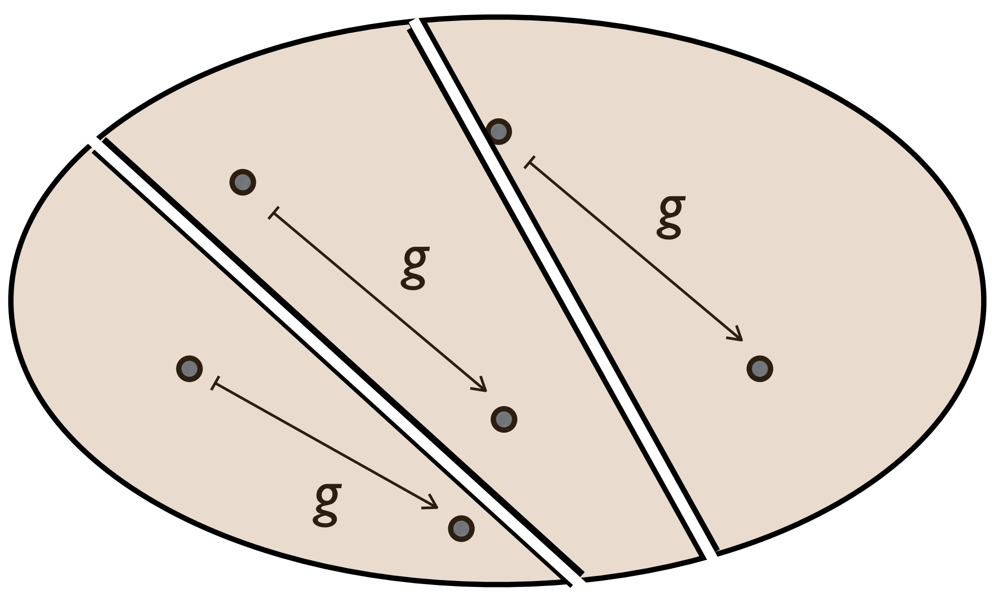

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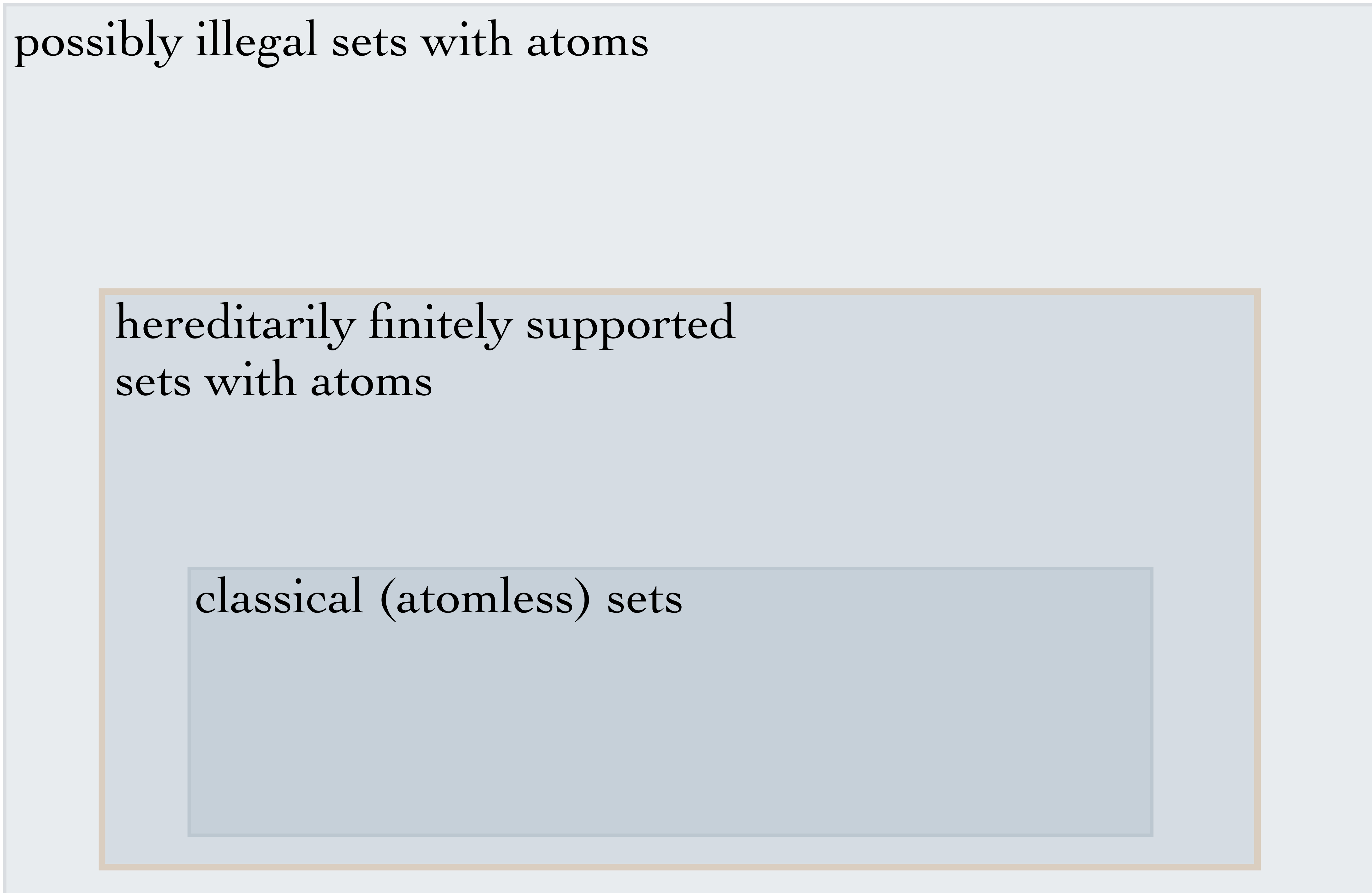
Orbit-finite sets with atoms



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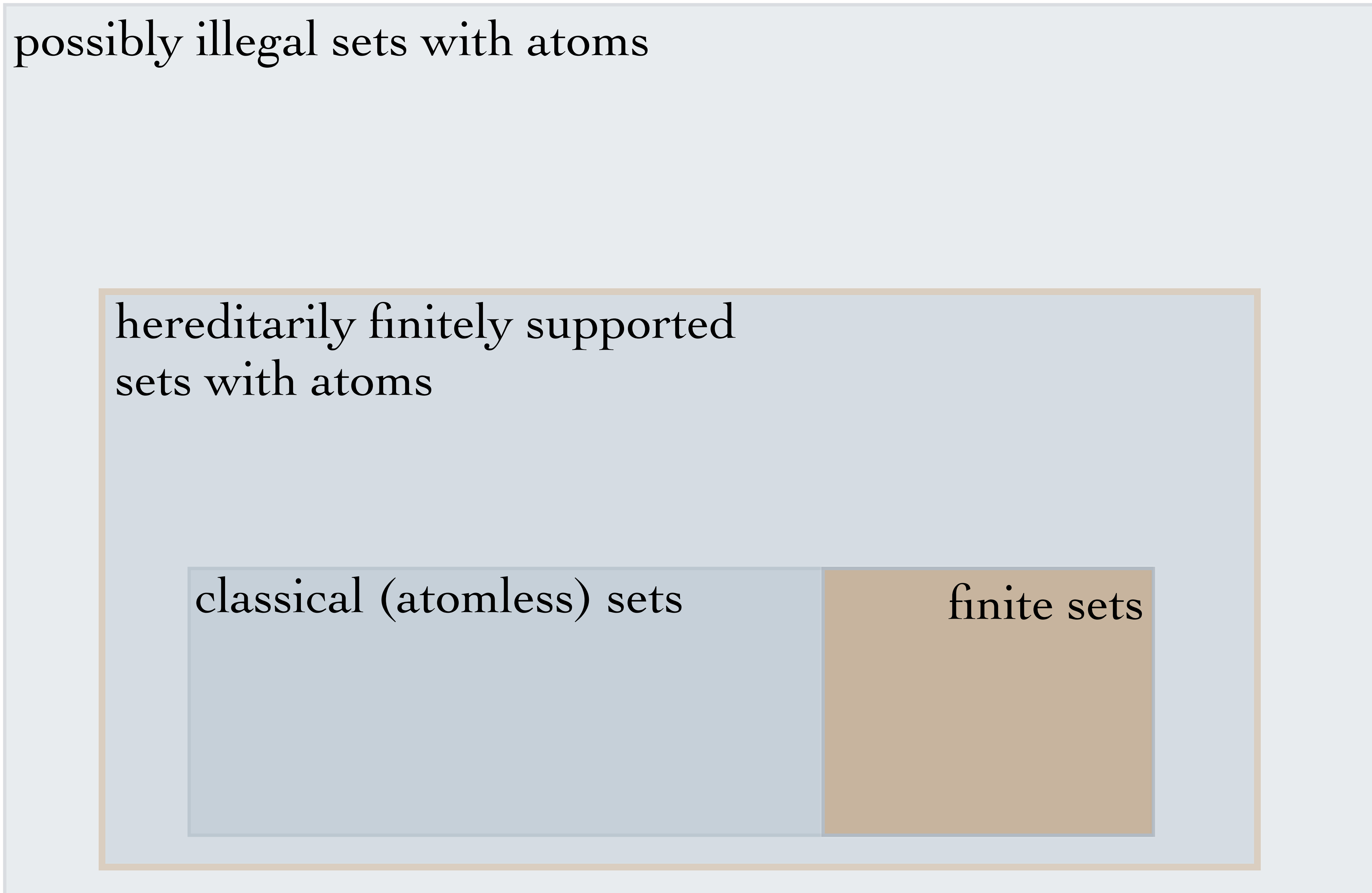

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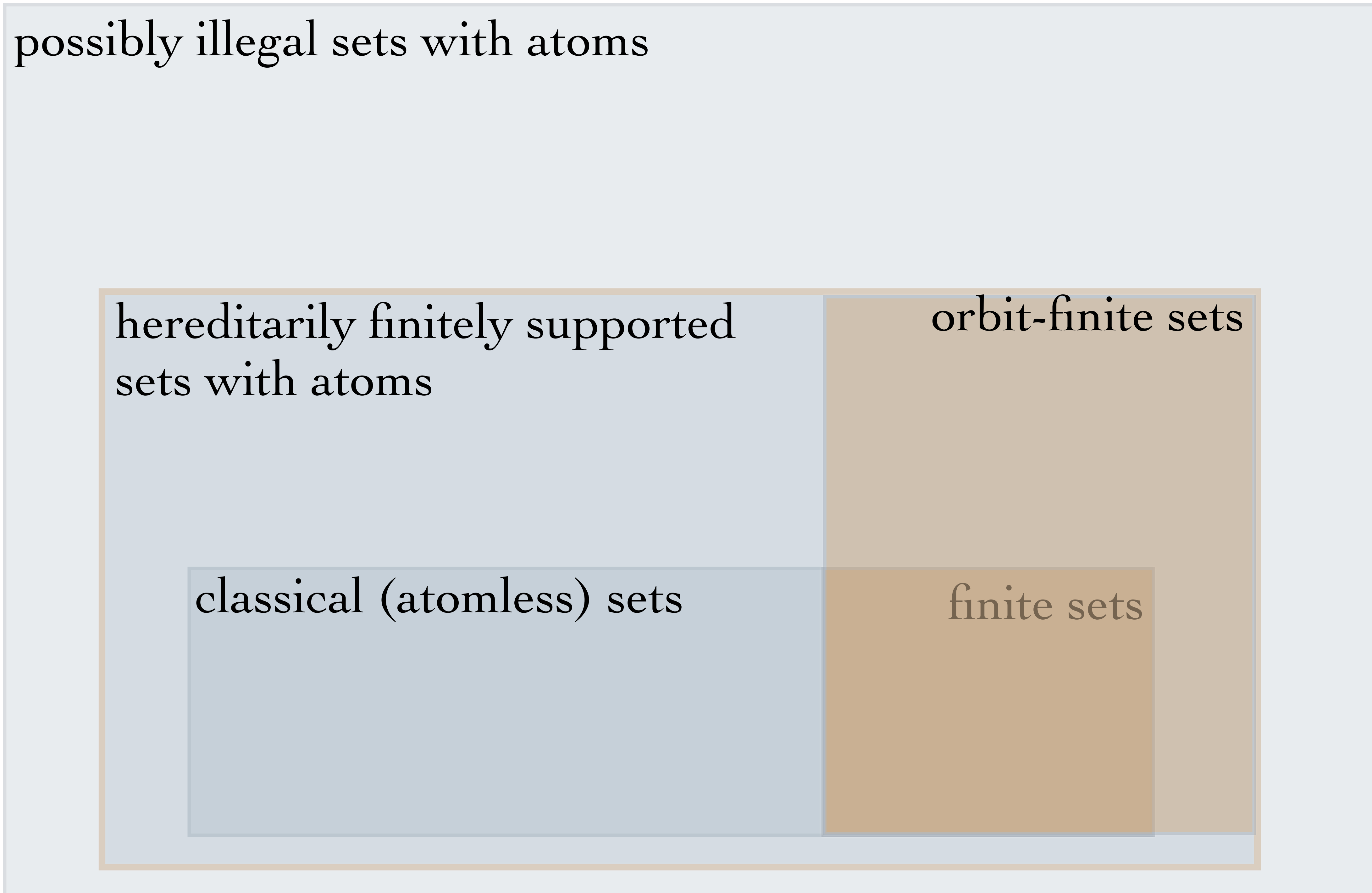

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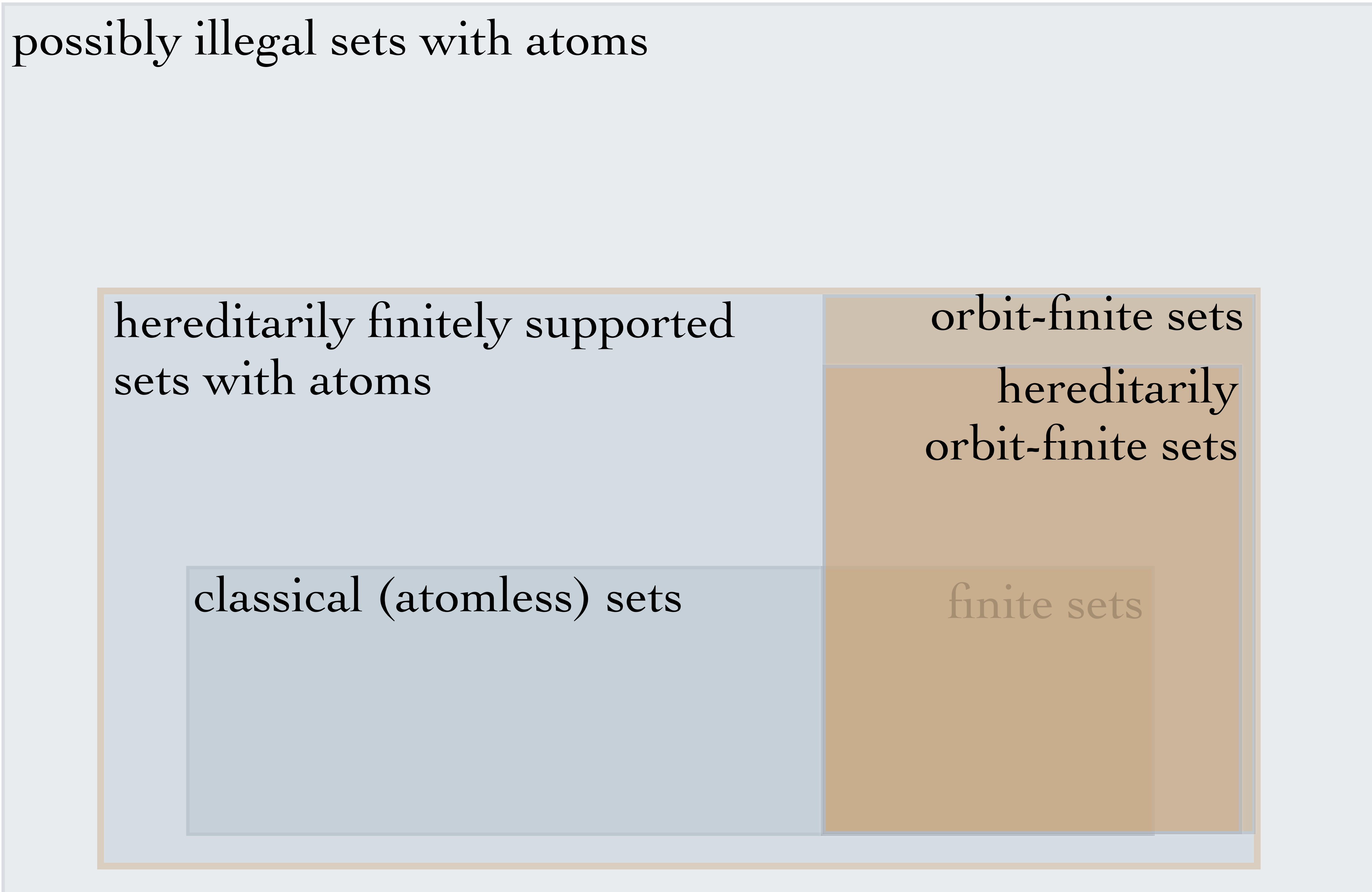

We've relaxed finiteness to orbit-finiteness.

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Are orbit-finite sets finitely representable?

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When atoms are homogeneous and relational,

legal subsets of atoms^n = quantifier-free definable subsets of atoms^n

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$x_1 < x_2 < 7$ supported by $\{7\}$

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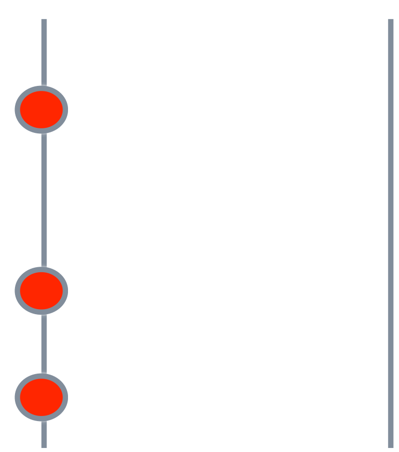

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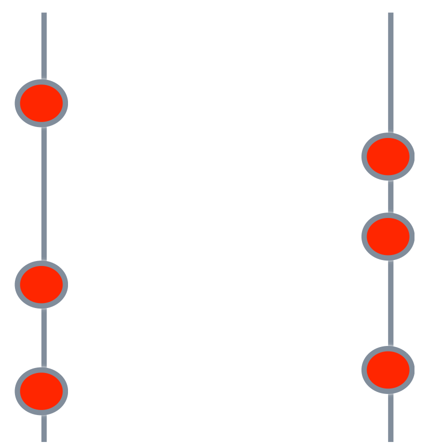

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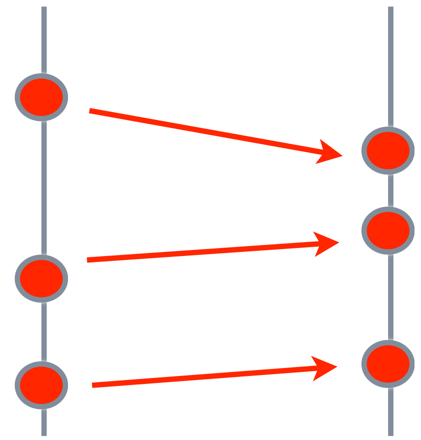

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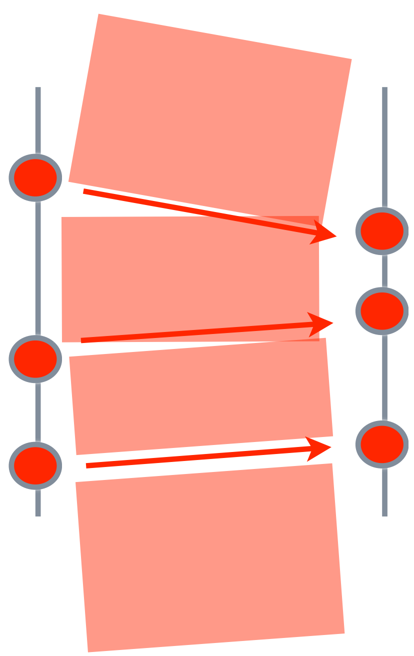

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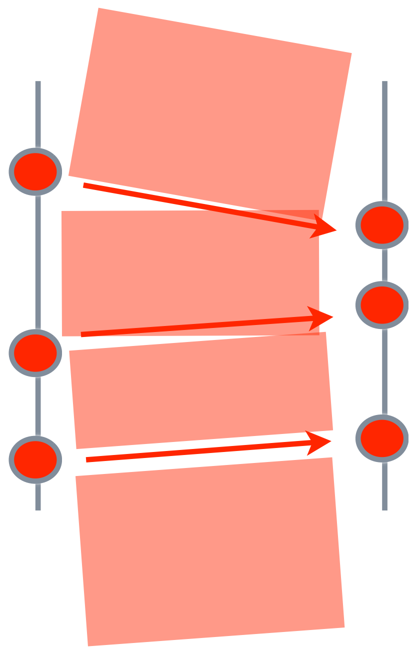
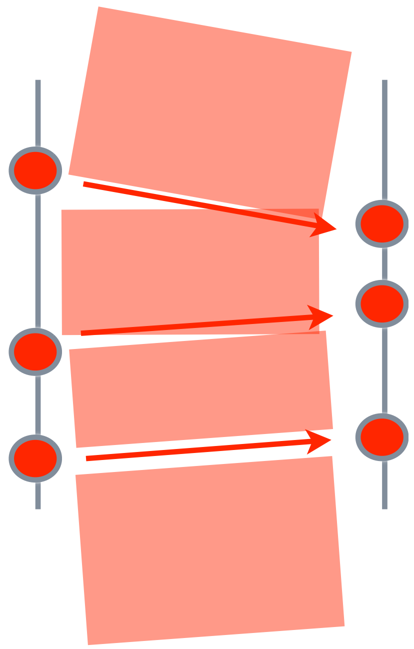


a homogeneous structure is uniquely determined by its finite induced substructures

Homogeneous atoms



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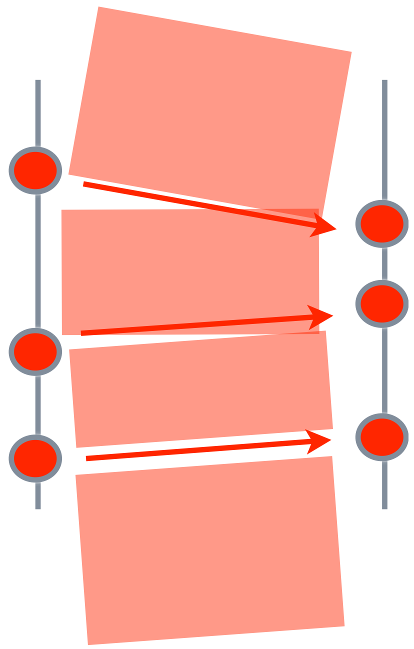

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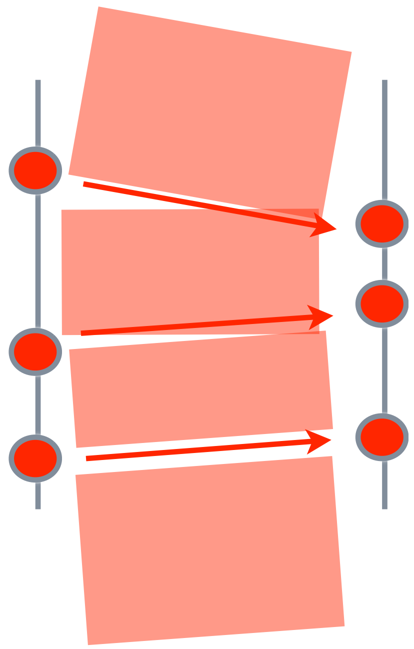

equality atoms (N, =)
total order atoms (Q, <)

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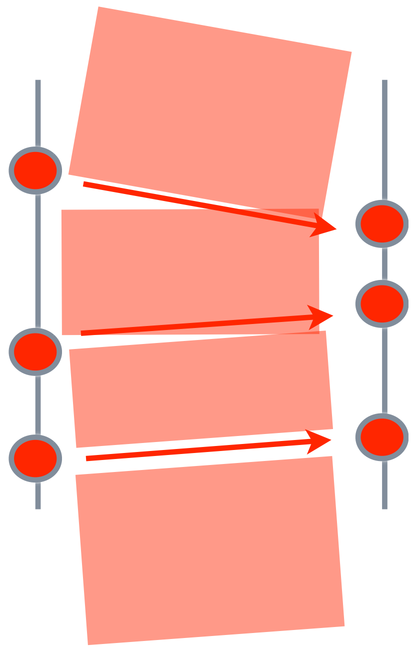

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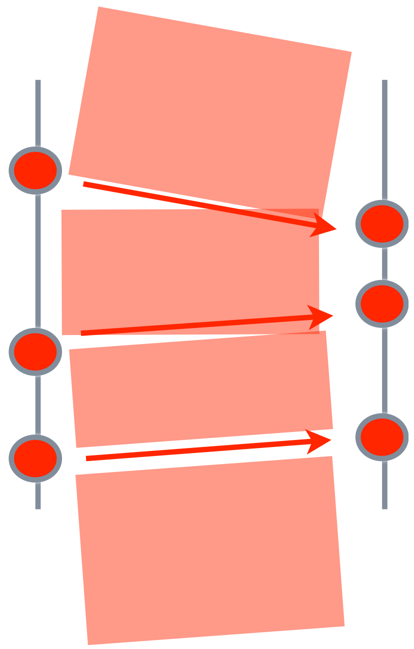

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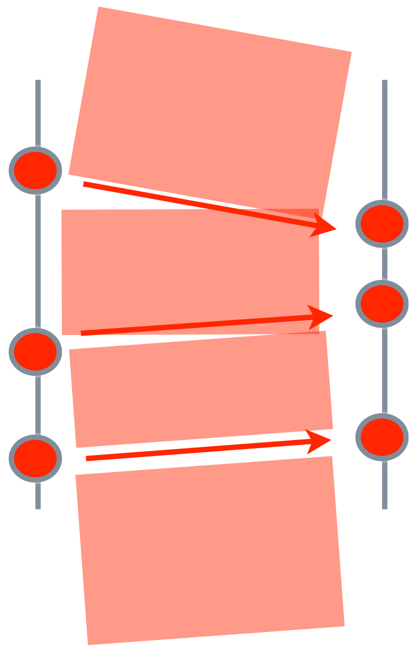

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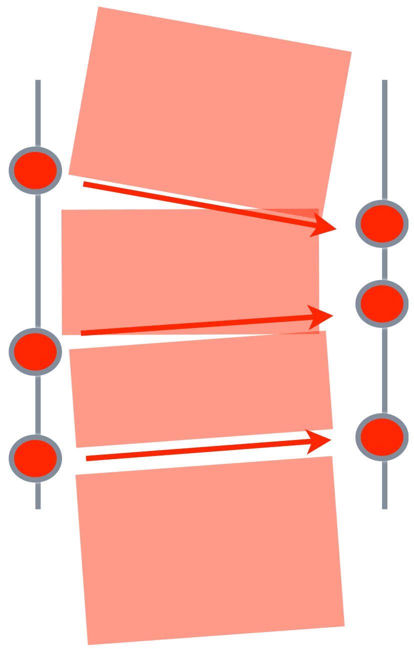
total order atoms (Q, <)
~~integer atoms (Z, <)~~
(Q, <, +1)

Homogeneous atoms



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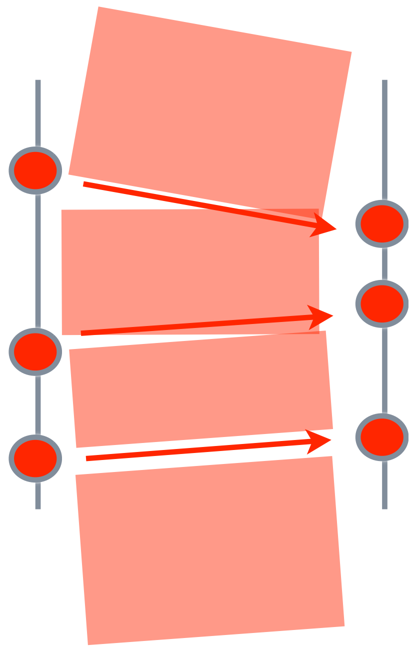
Homogeneous atoms



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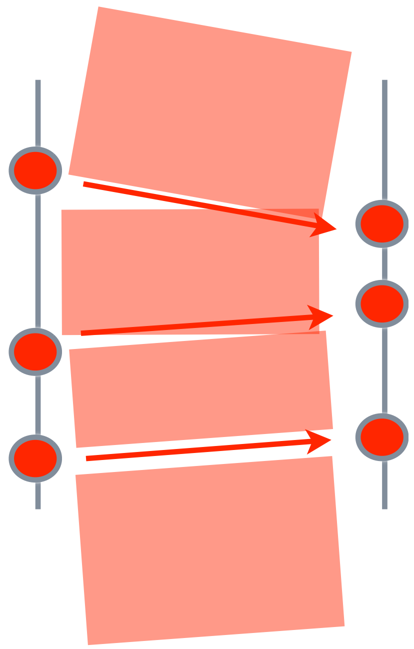

equality atoms (N, =)
total order atoms (Q, <)
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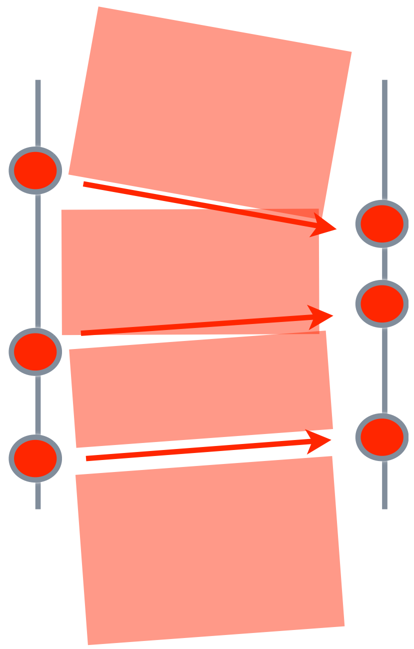
~~(Q, <, +1)~~
∅
universal (random) graph

Homogeneous atoms



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total order atoms (Q, <)
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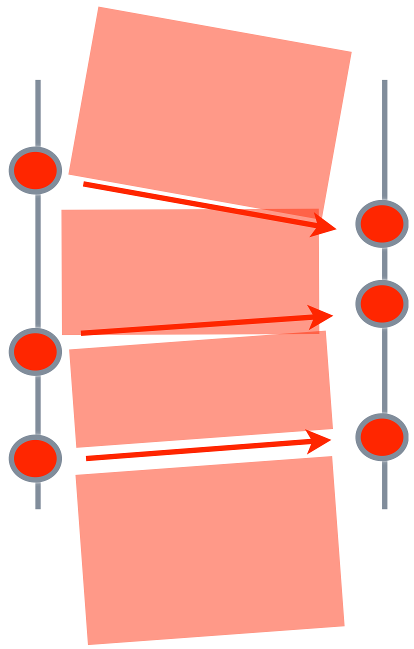
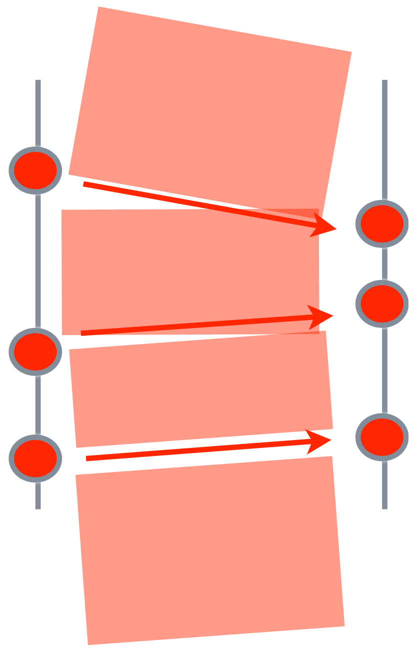
universal (random) graph
universal partial order

Homogeneous atoms



equality atoms $(\mathbb{N}, =)$
total order atoms $(\mathbb{Q}, <)$
~~integer atoms $(\mathbb{Z}, <)$~~
~~$(\mathbb{Q}, <, +1)$~~
$\emptyset$
universal (random) graph
universal partial order
universal equivalence relation

Homogeneous atoms



equality atoms (N, =)
total order atoms (Q, <)
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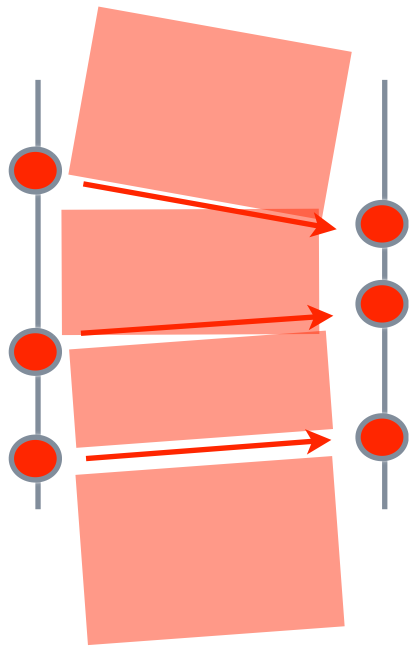
universal (random) graph
universal partial order
universal equivalence relation
universal tournament

Homogeneous atoms



equality atoms (N, =)
total order atoms (Q, <)
~~integer atoms (Z, <)~~
~~(Q, <, +1)~~
∅
universal (random) graph
universal partial order
universal equivalence relation
universal tournament
...

Atoms are assumed in the sequel to be oligomorphic and effective.

Outline

- Sets with atoms

- Models of computation in sets with atoms

- Are sets with atoms useful?

Automata

Automata

- alphabet A

Automata

- alphabet A

atoms $\times$ (a finite set)

Automata

atoms $\times$ (a finite set)

- alphabet A
- states Q

Automata

- alphabet A
- states Q

atoms × (a finite set)

$\text{atoms}^n \times (\text{a finite set})$

Automata

- alphabet A
- states Q
- $\delta \subseteq Q \times A \times Q$

Automata

- alphabet A
- states Q
- $\delta \subseteq Q \times A \times Q$
- $I, F \subseteq Q$

Automata

- alphabet A
- states Q
- $\delta \subseteq Q \times A \times Q$
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} orbit-finite sets instead of finite ones

Automata

- alphabet A
- states Q
- $\delta \subseteq Q \times A \times Q$
- I, F $\subseteq$ Q

} orbit-finite sets instead of finite ones

Deterministic automata:

- $\delta : Q \times A \to Q$

input alphabet: atoms

language: "exactly two different atoms appear"

states:

transitions:

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions:

initial state:

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input alphabet: atoms

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number of registers may vary from one orbit to another

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number of registers may vary from one orbit to another

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta((), a) =$	(a)	$a \in \text{atoms}$

if in state () atom a is read, goto state (a)

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

number of registers may vary from one orbit to another

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \rightarrow Q$

$\delta((), a) =$	(a)	$a \in$ atoms
$\delta((a), b) =$	(ab)	$a \neq b$

if in state (a), an atom b ≠ a is read, goto state (ab)

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

number of registers may vary from one orbit to another

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta((), a) =$	(a)	$a \in$ atoms
$\delta((a), b) =$	(ab)	$a \neq b$
$\delta((a), b) =$	(a)	$a = b$

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

number of registers may vary from one orbit to another

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta((), a) =$	(a)	$a \in$ atoms
$\delta((a), b) =$	(ab)	$a \neq b$
$\delta((a), b) =$	(a)	$a = b$
$\delta((ab), c) =$	reject	$c \neq a, b$

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

number of registers may vary from one orbit to another

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta((), a) =$	(a)	$a \in$ atoms
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$\delta((ab), c) =$	reject	$c \neq a, b$

initial state: $()$

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

number of registers may vary from one orbit to another

states: $Q = \text{atoms}^{\leq 2} \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta((), a) =$	(a)	$a \in$ atoms
$\delta((a), b) =$	(ab)	$a \neq b$
$\delta((a), b) =$	(a)	$a = b$
$\delta((ab), c) =$	reject	$c \neq a, b$

initial state: ()

accepting states: atoms^2

input alphabet: atoms

language: "exactly two different atoms appear"

states:

transitions:

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions:

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

registers are not necessarily ordered

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions:

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

registers are not necessarily ordered

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta(\emptyset, a) = \{a\}$ $a \in$ atoms

if in state $\emptyset$ atom a is read, goto state {a}

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

registers are not necessarily ordered

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta(\emptyset, a) = \{a\}$ $\quad a \in \text{atoms}$

$\delta(\{a\}, b) = \{a, b\}$ $\quad a, b \in \text{atoms}$

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

registers are not necessarily ordered

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta(\emptyset, a) = \{a\}$ $\quad a \in$ atoms

$\delta(\{a\}, b) = \{a, b\}$ $\quad a, b \in$ atoms

$\delta(\{a, b\}, c) =$ reject $\quad c \neq a, b$

if in state {a, b}, atom c ≠ a, b is read, reject

initial state:

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

registers are not necessarily ordered

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

transitions: $\delta : Q \times A \to Q$

$\delta(\varnothing, a) =$	$\{a\}$	$a \in$ atoms
$\delta(\{a\}, b) =$	$\{a, b\}$	$a, b \in$ atoms
$\delta(\{a, b\}, c) =$	reject	$c \neq a, b$

initial state: $\varnothing$

accepting states:

input alphabet: atoms

language: "exactly two different atoms appear"

registers are not necessarily ordered

states: $Q = \mathcal{P}_{\leq 2}(\text{atoms}) \cup \{\text{reject}\}$

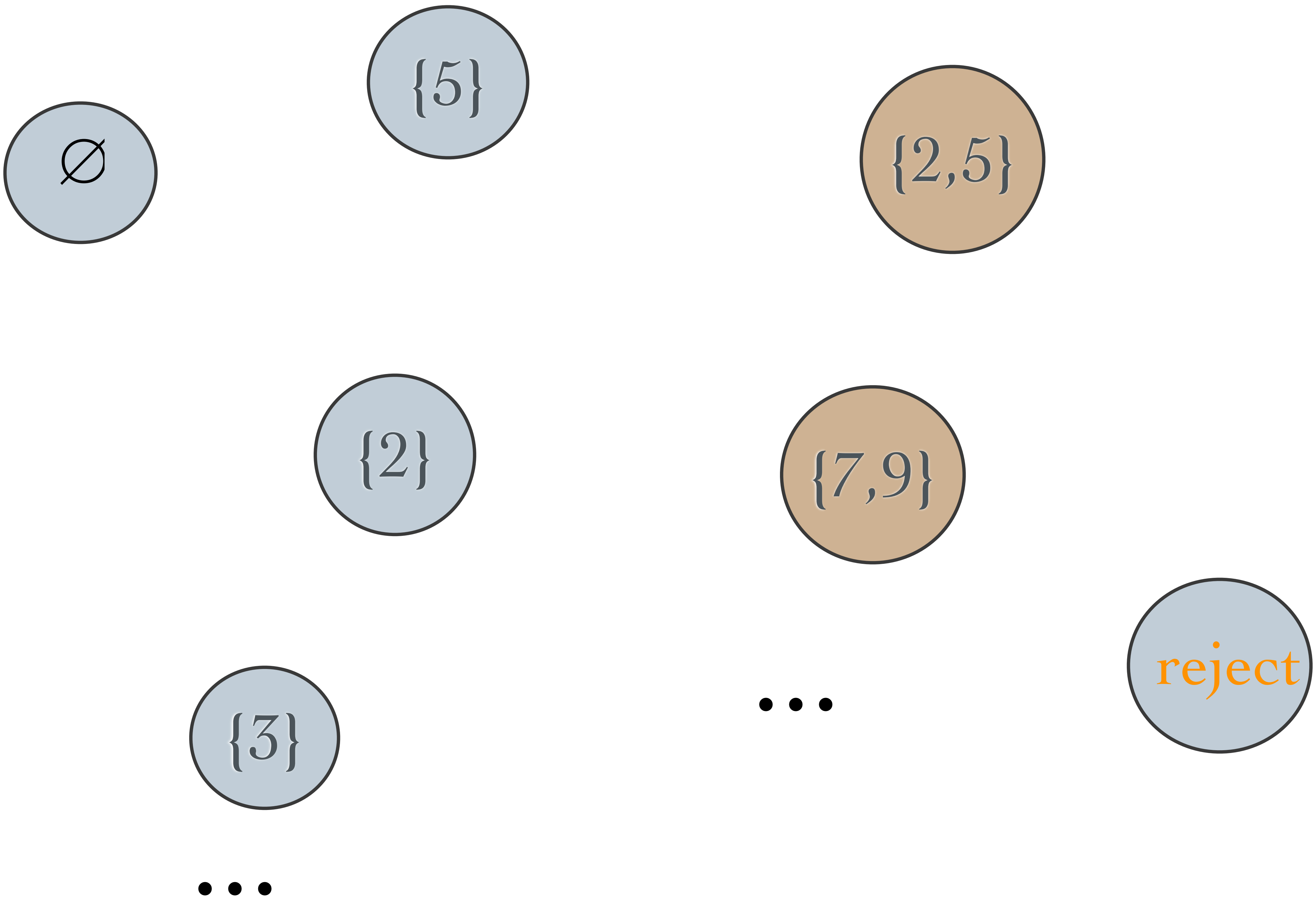
transitions: $\delta : Q \times A \to Q$

$\delta(\emptyset, a) =$	$\{a\}$	$a \in$ atoms
$\delta(\{a\}, b) =$	$\{a, b\}$	$a, b \in$ atoms
$\delta(\{a, b\}, c) =$	reject	$c \neq a, b$

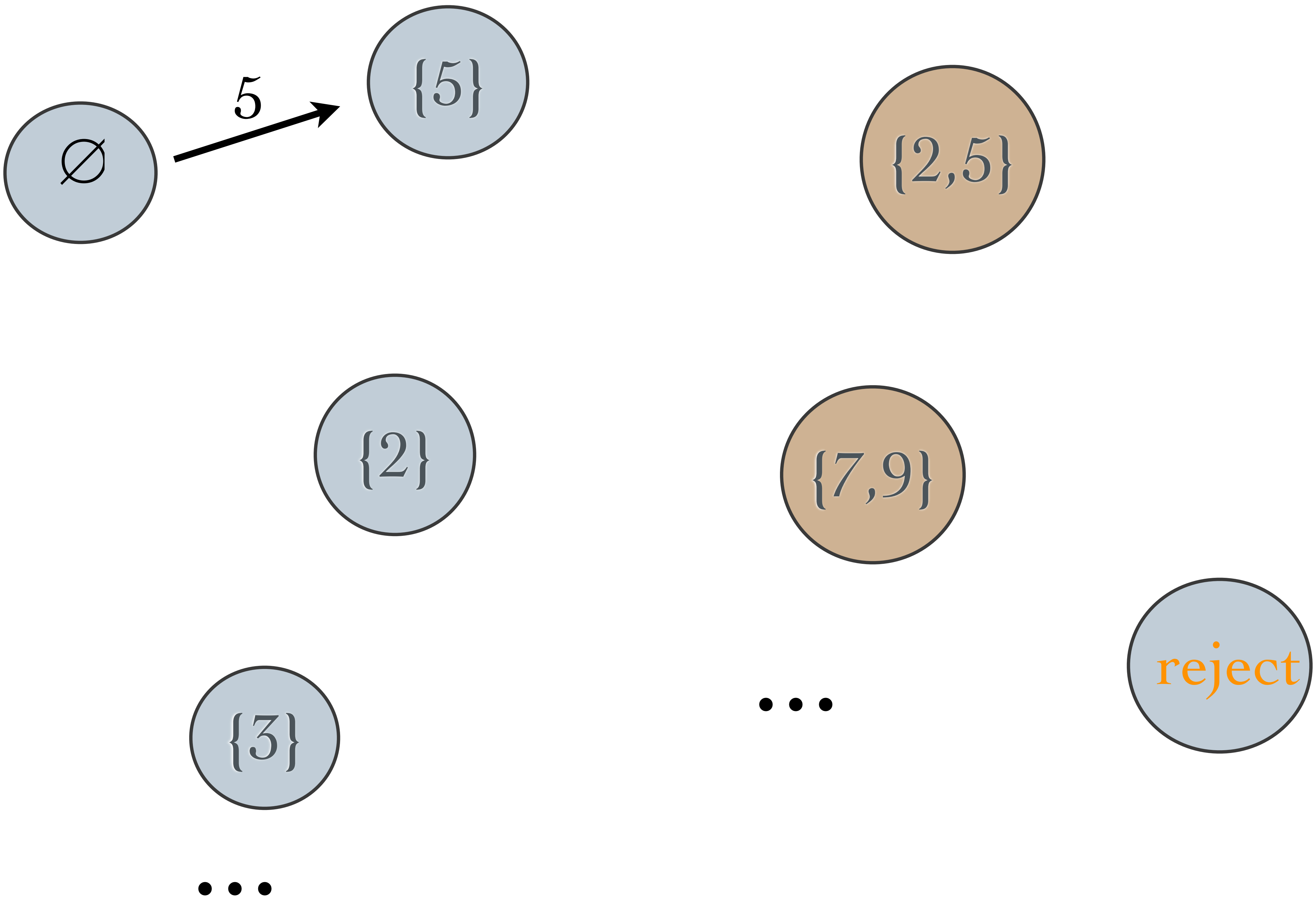
initial state: $\emptyset$

accepting states: $\mathcal{P}_2(\text{atoms})$

"exactly two different atoms appear"



"exactly two different atoms appear"



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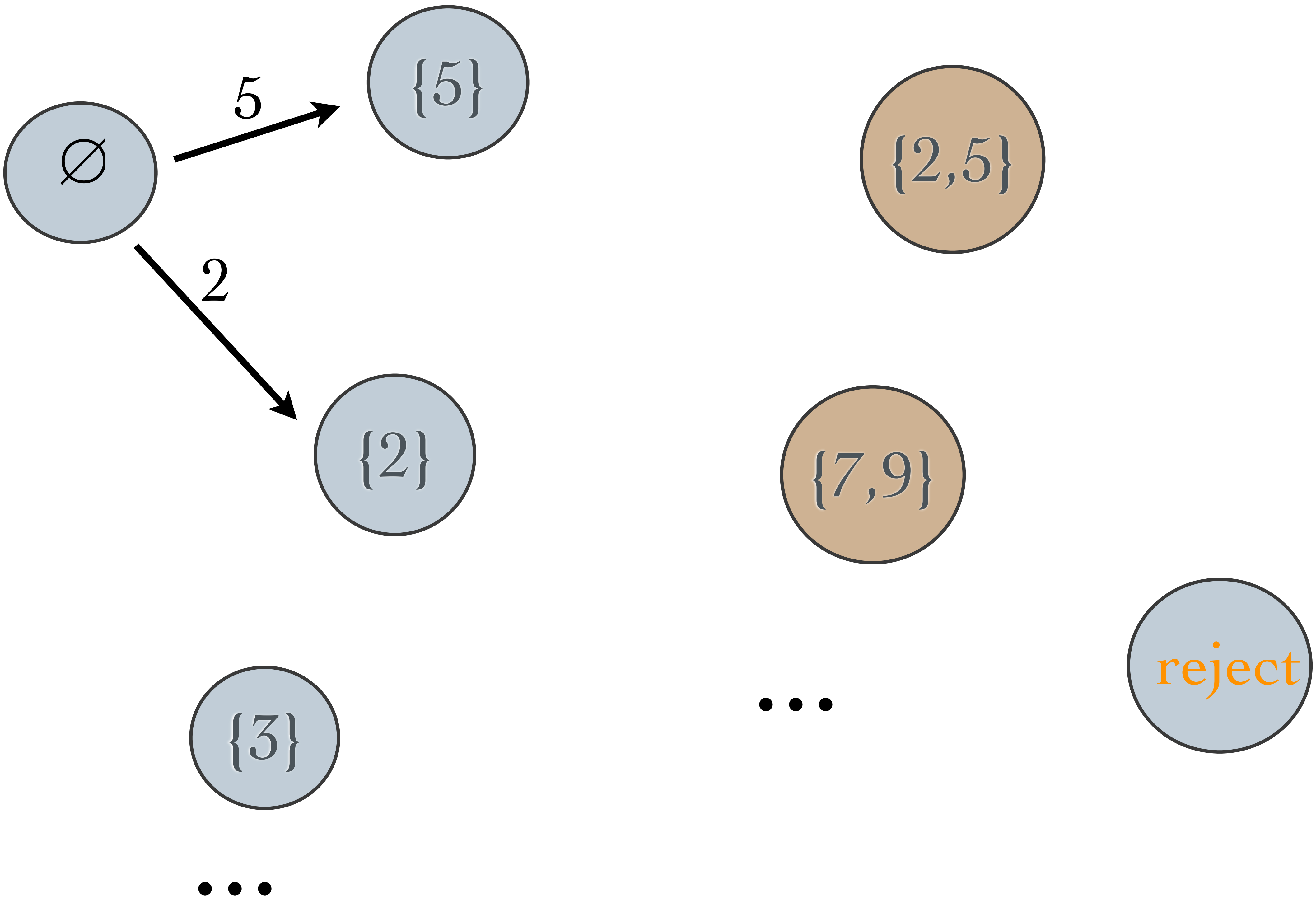

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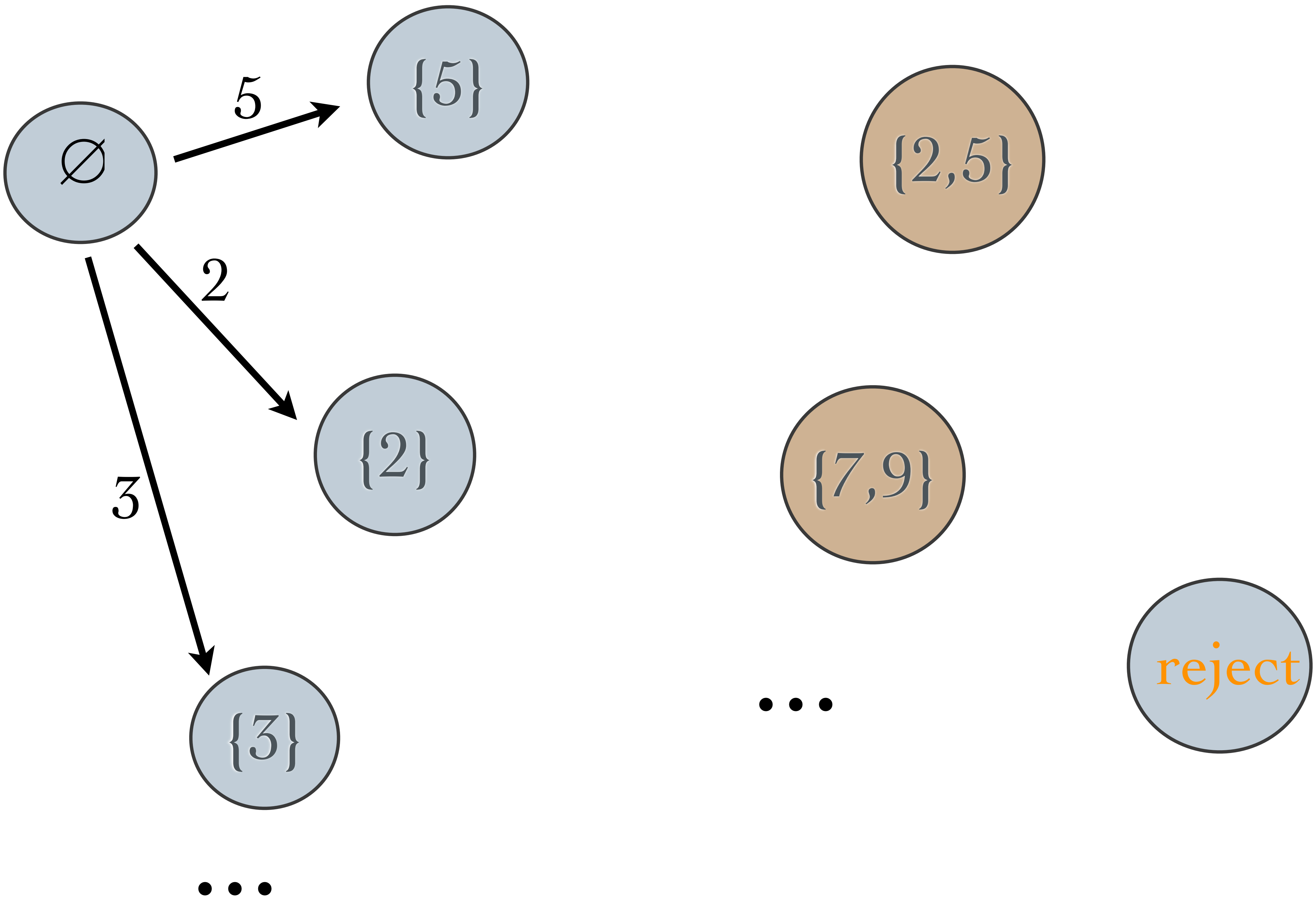

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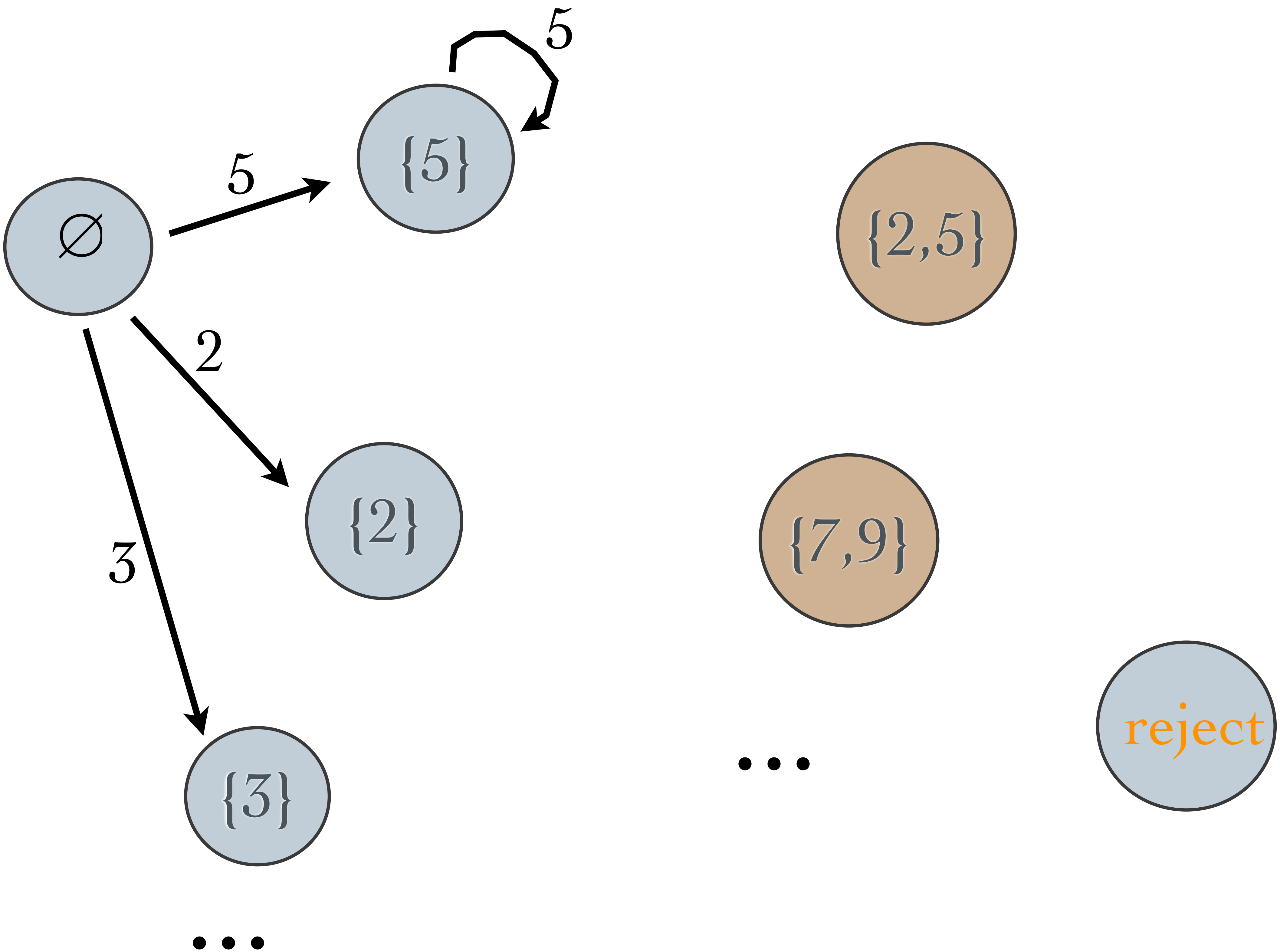
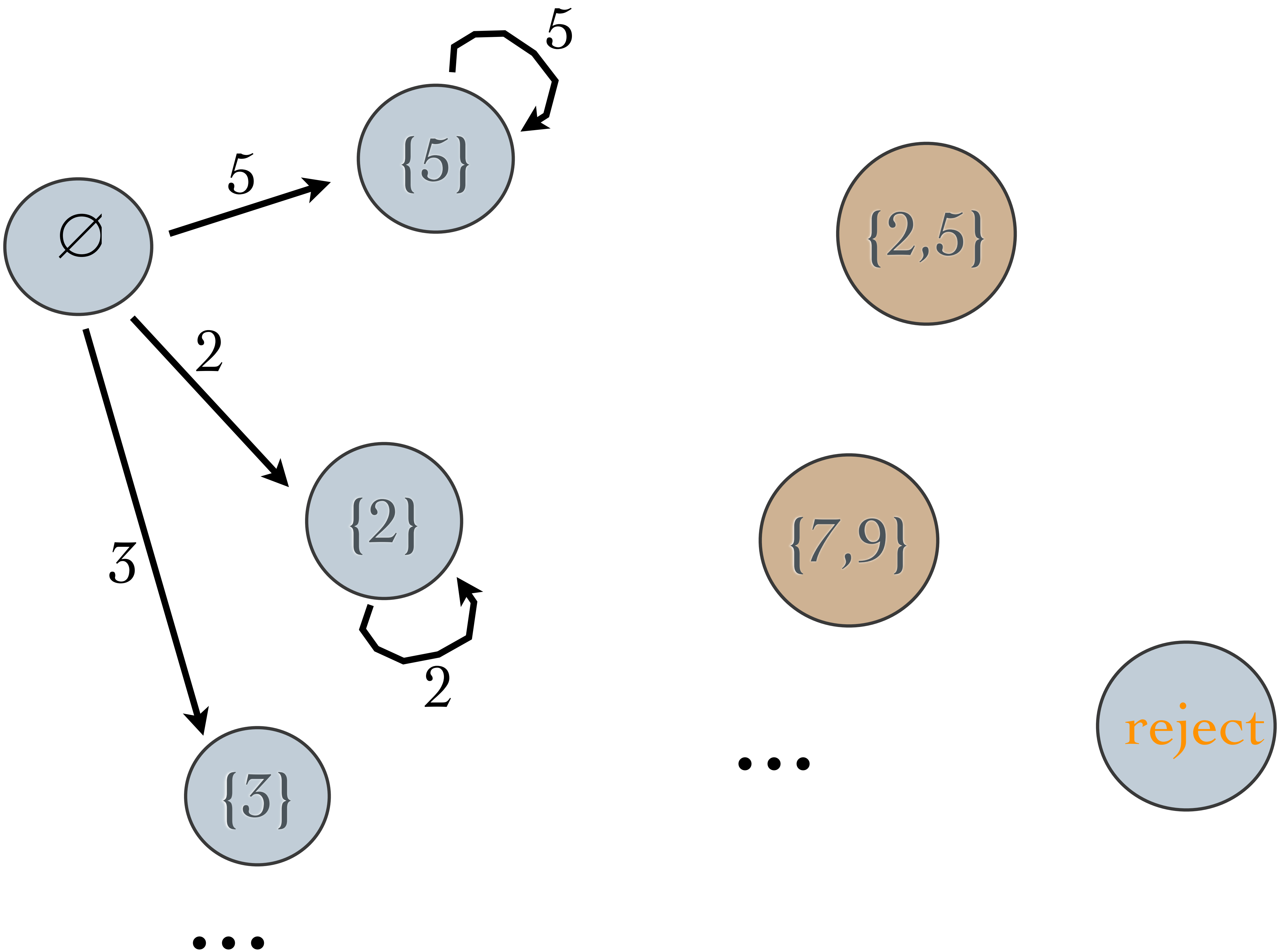


"exactly two different atoms appear"



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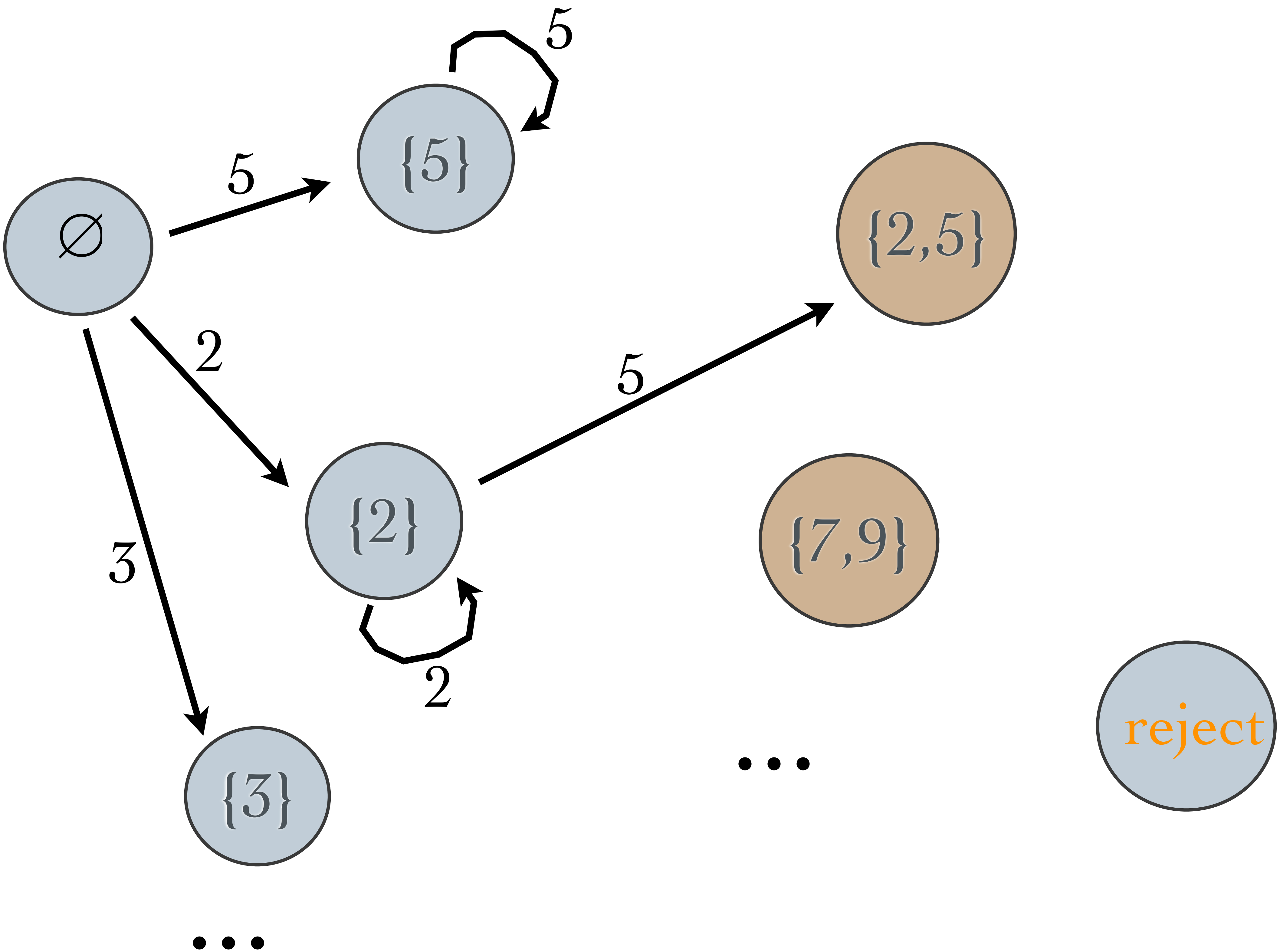

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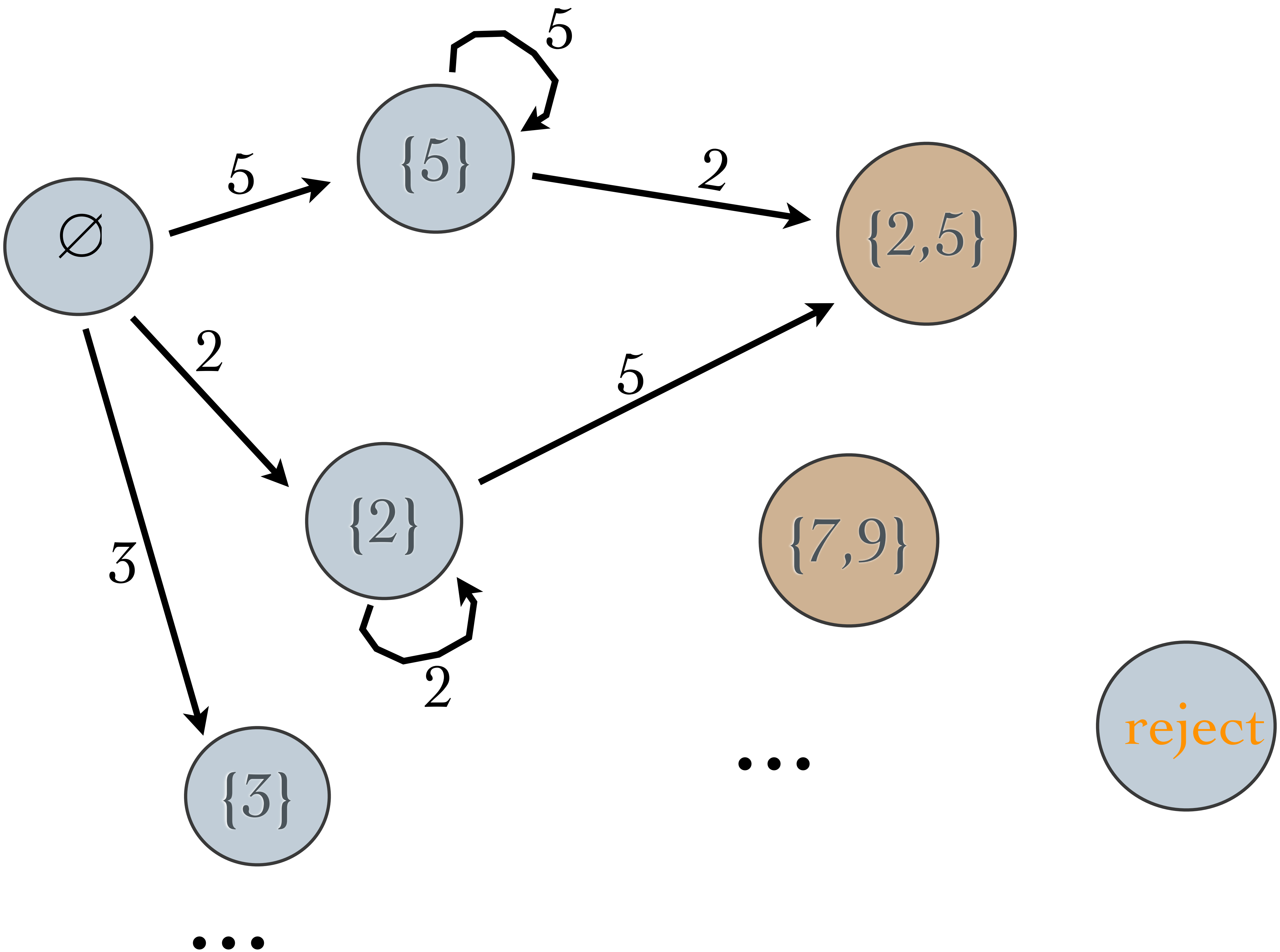

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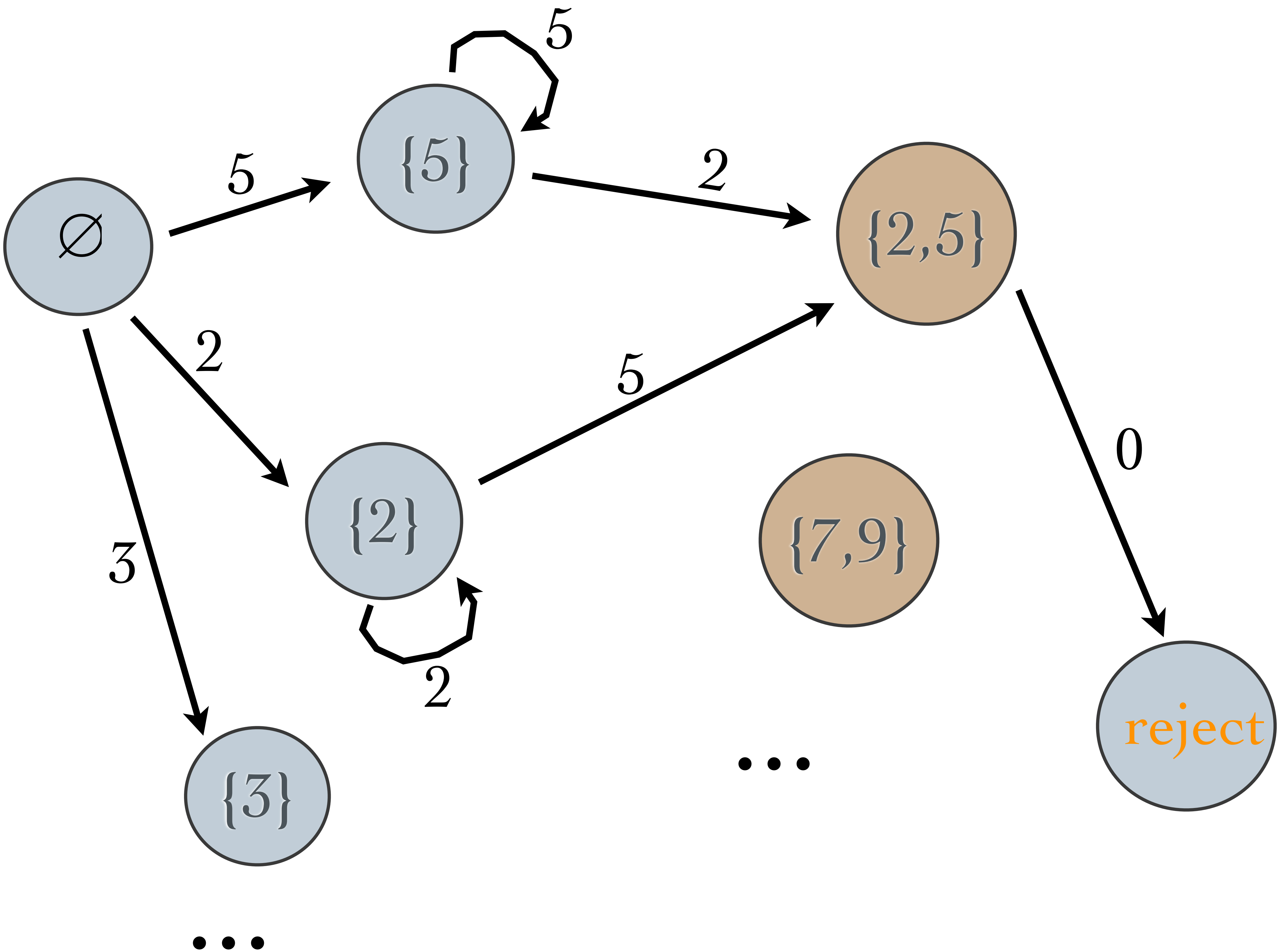

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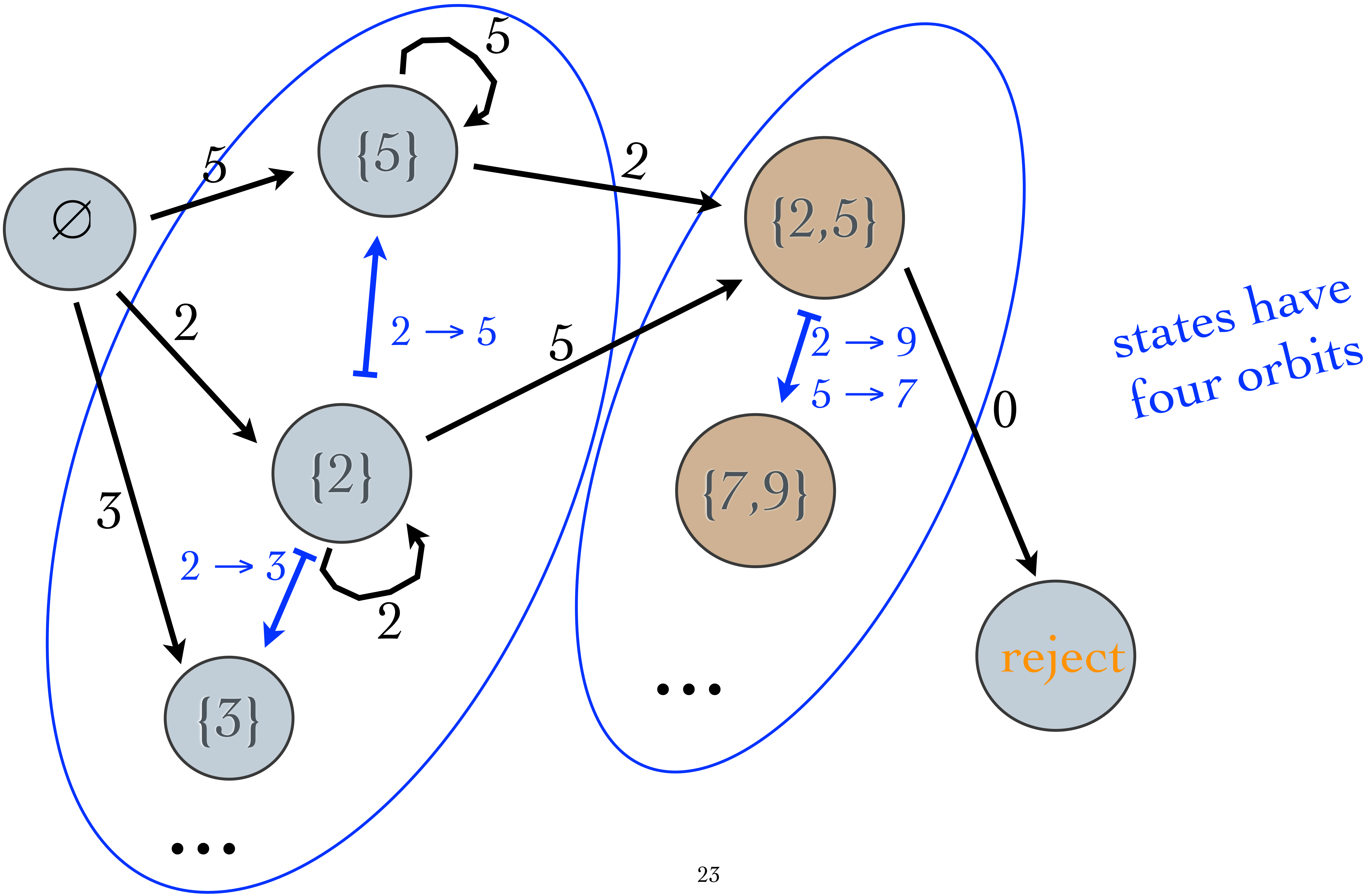

Slight generalization of register automata:

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- number of registers may vary from one orbit to another

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- alphabet letters may contain more than one atom

Slight generalization of register automata:

- number of registers may vary from one orbit to another
- registers are not necessarily ordered
- alphabet letters may contain more than one atom

this is not a design decision,
but a property of orbit-finite sets

Expressive power

register automata
with equality tests
x = y

=

automata with
equality atoms (N, =)
over alphabet
atoms × (a finite set)

Expressive power

register automata
with equality tests
$x = y$

=

automata with
equality atoms ($\mathbb{N}$, =)
over alphabet
atoms × (a finite set)

register automata
with inequality tests
$x \leq y$

=

automata with
total order atoms ($\mathbb{Q}$, ≤)
over alphabet
atoms × (a finite set)

Minimization

register automata
with equality tests
x = y

=

automata with
equality atoms (N, =)
over alphabet
atoms × (a finite set)

Minimization

deterministic
register automata
with equality tests
$x = y$

=

deterministic
automata with
equality atoms ($\mathbb{N}$, =)
over alphabet
atoms × (a finite set)

Minimization

deterministic
register automata
with equality tests
x = y

=

deterministic
automata with
equality atoms (N, =)
over alphabet
atoms × (a finite set)

do not minimize

Minimization

deterministic
register automata
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=

deterministic
automata with
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over alphabet
atoms × (a finite set)

do not minimize

do minimize

Myhill-Nerode Theorem

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Theorem:

L is recognized by a deterministic automaton

iff

the set of L-equivalence classes is orbit-finite

Myhill-Nerode Theorem

Theorem:

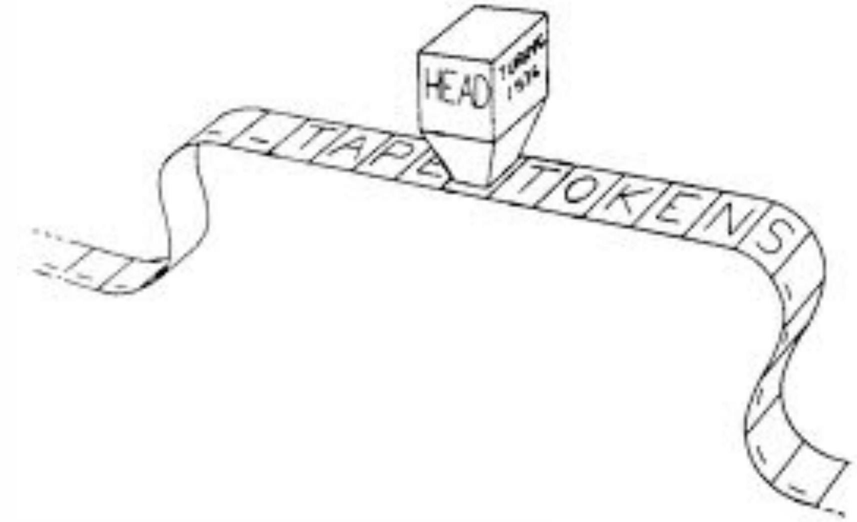
L is recognized by a deterministic automaton

iff

the set of L-equivalence classes is orbit-finite

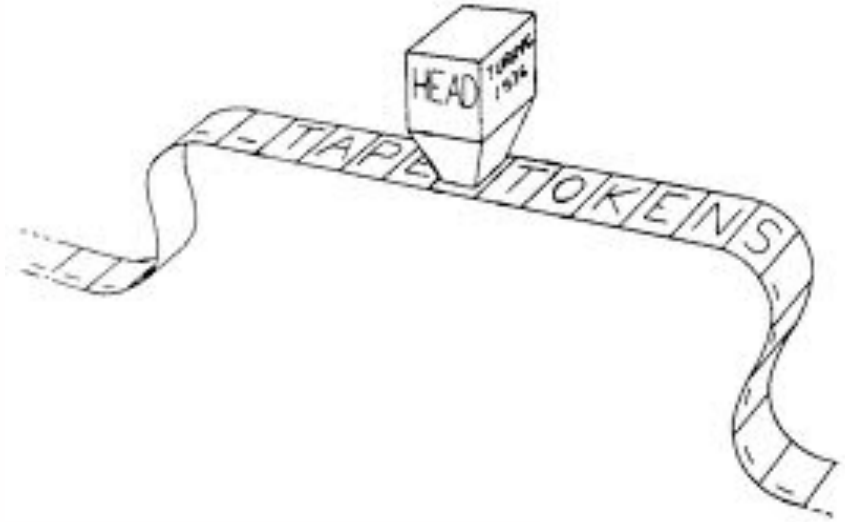
The equivalence classes are states of the minimal automaton for L

Turing machines



- tape alphabet A
- states Q
- subset $\delta \subseteq Q \times A \times Q \times A \times \{\leftarrow, \rightarrow, \downarrow\}$
- subsets $I, F \subseteq Q$

Turing machines



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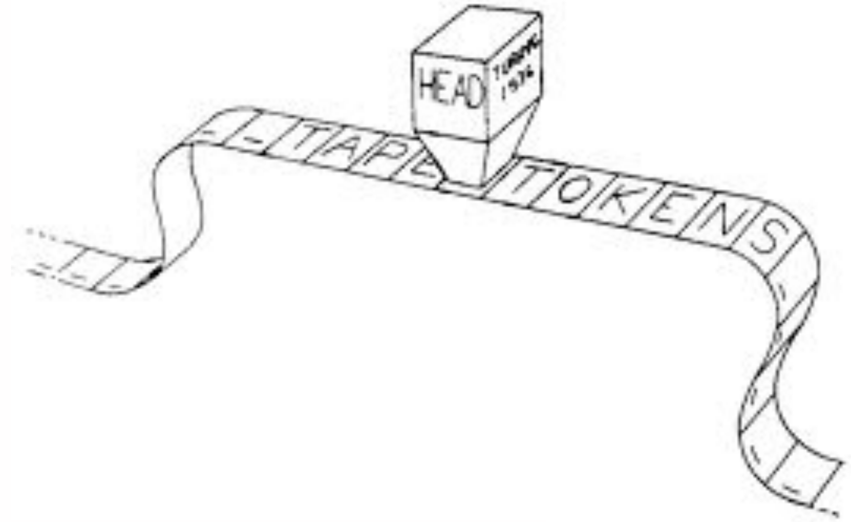
} orbit-finite sets instead of finite ones

Turing machines



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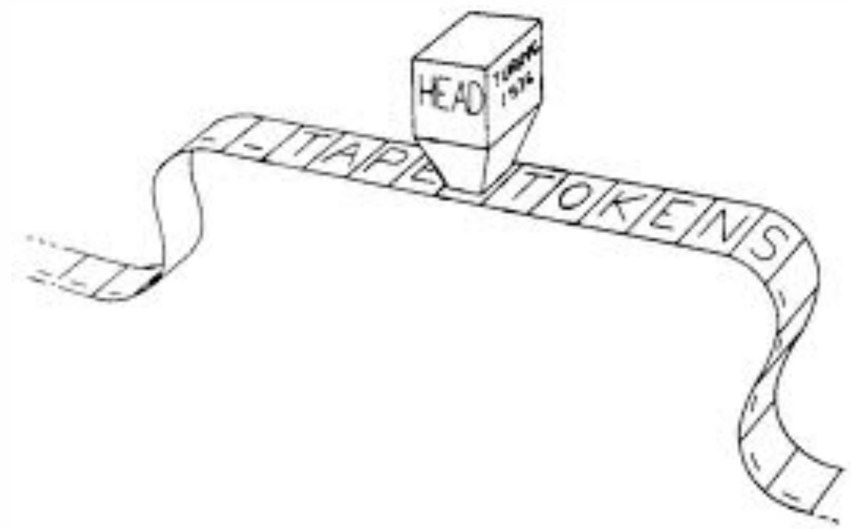
} orbit-finite sets instead of finite ones

Configurations = $A^* \times Q \times A^*$

Turing machines



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Configurations = $A^* \times Q \times A^*$

Deterministic machines:

- $\delta : Q \times A \rightarrow Q \times A \times \{\leftarrow, \rightarrow, \downarrow\}$

input alphabet: atoms

language:

tape alphabet:

states:

transitions:

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \dots a_n \;:\; a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet:

states:

transitions:

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms $\cup$ $\{\perp\}$

states:

transitions:

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms ∪ {⊥}

states: Q = atoms ∪ {start, accept, ret}

transitions:

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms ∪ {⊥}

states: Q = <u>atoms</u> ∪ {start, accept, ret}

transitions: δ : Q × A → Q × A × {←,→,↓}

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \dots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: $A = \text{atoms} \cup \{\perp\}$

states: $Q = \underline{\text{atoms}} \cup \{\text{start}, \text{accept}, \text{ret}\}$

transitions: $\delta : Q \times A \to Q \times A \times \{\leftarrow, \rightarrow, \downarrow\}$

$\delta(\text{start}, a) = (\underline{a}, \perp, \rightarrow)$ $\qquad a \in \text{atoms}$

if in state start atom a is read from tape, goto state <u>a</u>, write ⊥ on tape, and move right

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms ∪ {⊥}

states: Q = <u>atoms</u> ∪ {start, accept, ret}

transitions: δ : Q × A → Q × A × {←,→,↓}

δ(start, a) = (<u>a</u>, ⊥, →) a ∈ atoms

δ(<u>a</u>, b) = (<u>a</u>, b, →) a ≠ b, a, b ∈ atoms

if in state <u>a</u> atom b ≠ a is read from tape, stay in state <u>a</u>, write b on tape, and move right

input alphabet: atoms

language: "no atom appears twice":

$$\{a_1 a_2 \ldots a_n \;:\; a_i \neq a_j \text{ when } i \neq j\}$$

tape alphabet: A = atoms ∪ {⊥}

states: Q = atoms ∪ {start, accept, ret}

transitions: $\delta : Q \times A \to Q \times A \times \{\leftarrow, \rightarrow, \downarrow\}$

$\delta(\text{start}, a) = (\underline{a}, \bot, \rightarrow)$ a ∈ atoms

$\delta(\underline{a}, b) = (\underline{a}, b, \rightarrow)$ a ≠ b, a, b ∈ atoms

$\delta(\underline{a}, B) = (\text{ret}, B, \leftarrow)$ a ∈ atoms

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms ∪ {⊥}

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δ(start, a) =	(<u>a</u>, ⊥, →)	a ∈ atoms
δ(<u>a</u>, b) =	(<u>a</u>, b, →)	a ≠ b, a, b ∈ atoms
δ(<u>a</u>, B) =	(ret, B, ←)	a ∈ atoms
δ(ret, a) =	(ret, a, ←)	a ∈ atoms

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms ∪ {⊥}

states: Q = <u>atoms</u> ∪ {start, accept, ret}

transitions: $\delta : Q \times A \to Q \times A \times \{\leftarrow, \rightarrow, \downarrow\}$

δ(start, a) = (<u>a</u>, ⊥, →) a ∈ atoms

δ(<u>a</u>, b) = (<u>a</u>, b, →) a ≠ b, a, b ∈ atoms

δ(<u>a</u>, B) = (ret, B, ←) a ∈ atoms

δ(ret, a) = (ret, a, ←) a ∈ atoms

δ(ret, ⊥) = (start, ⊥, →)

input alphabet: atoms

language: "no atom appears twice":

$\{a_1 a_2 \ldots a_n \ : \ a_i \neq a_j \text{ when } i \neq j\}$

tape alphabet: A = atoms ∪ {⊥}

states: Q = <u>atoms</u> ∪ {start, accept, ret}

transitions: δ : Q × A → Q × A × {←,→,↓}

δ(start, a) = (<u>a</u>, ⊥, →) a ∈ atoms

δ(<u>a</u>, b) = (<u>a</u>, b, →) a ≠ b, a, b ∈ atoms

δ(<u>a</u>, B) = (ret, B, ←) a ∈ atoms

δ(ret, a) = (ret, a, ←) a ∈ atoms

δ(ret, ⊥) = (start, ⊥, →)

δ(start, B) = (accept, B, →)

Pushdown automata

- alphabet A
- states Q
- stack alphabet S
- $\delta \subseteq Q \times (A \cup \{\varepsilon\}) \times S \times Q \times S^*$
- $I, F \subseteq Q$

Pushdown automata

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- stack alphabet S
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Deterministic pushdown automata: ...

Theorem:

Pushdown automata = prefix-rewriting

Context-free grammars

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- terminal symbols $A \subseteq S$
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Context-free grammars = pushdown automata

Petri nets

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- places P

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Configurations = finite multisets of places $M_{fin}(P)$

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general Petri nets	elementary nets
data Petri nets	general Petri nets

places: atoms × (a finite set)

Outline

- Sets with atoms
- Models of computation in sets with atoms
- Are sets with atoms useful?

usefulness of sets with atoms in infinite-state verification

usefulness of sets with atoms in infinite-state verification

- orbit-finite abstractions

usefulness of sets with atoms in infinite-state verification

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- clarification and unification of known methods

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- relationships with other fields

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Corollary: Emptiness of pushdown automata is decidable

Potential application to orbit-infinite abstractions in analysis of recursive program.

clarification and unification

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Theorem: reachability is decidable for alternating automata with one register/clock

[Ouaknine, Worrel '05]
[L., Walukiewicz '05]
[Demri, Lazic '09]

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Idea: regions = orbits

solver of already solved problems

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- coverabily of timed/data Petri nets
  [Lazic, Newcomb, Ouaknine, Roscoe, Worrell '12]
  [Abdulla, Nylen '01]
- c. s. reachability of timed/data lossy FIFO automata
  [Abdulla, Atig, Cederberg '12]
- emptiness of timed/data pushdown systems
  [Abdulla, Atig, Stenman '12]
  [Dubov, Kaminski '09]
- relating timed and data variants
  [Figueira, Hofman, L.'10]
  [Bonnet, Finkel, Haddad, Rosa-Velardo '10]

solver of previously unsolved problems

solver of previously unsolved problems

- minimization of deterministic register automata
  [Bojańczyk, Klin, L. '11]
- machine-independent characterization of det. timed languages
  [Bojańczyk, L. '12]
- verification of database-driven systems
  [Bojańczyk, Segoufin, Toruńczyk '13]

generator of new problems

generator of new problems

- decidability of reachability for Petri nets
- decidability of equivalence of deterministic pushdown automata
- ...

relationships with other fields

relationships with other fields

- CSP theory
- descriptive complexity

} [Klin, L., Ochremiak, Toruńczyk '14]

- model-theory
  - homogenizability
  - automorphisms with bounded color classes
- finite permutation groups
- ...

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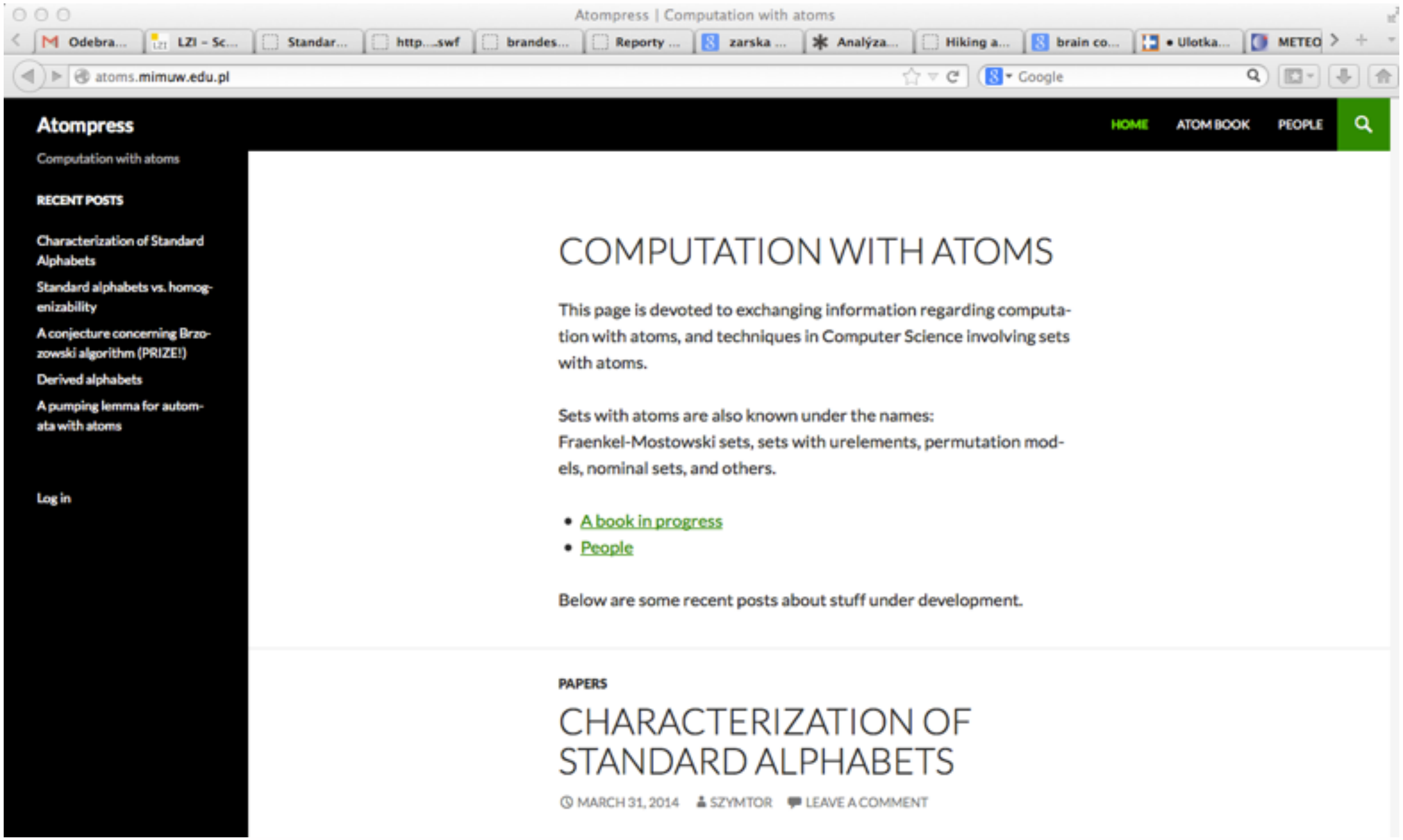

thank you!