Fast bisimulation-checking for normed context-free processes

BASICS summer school, 19-23.08.2013

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joint work with Wojciech Czerwiński and Sibylle Froeschle

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- W. Czerwiński, *Partially-commutative context-free graphs*, PhD Thesis, University of Warsaw, 2012.

- W. Czerwiński, S. Froeschle, S. Lasota, *Partially-commutative context-free processes: expressibility and tractability,* Information and Computation, 2011.

- W. Czerwiński, S. Lasota, *Fast equivalence-checking for normed context-free processes,* FSTTCS 2010.

- W. Czerwiński, S. Froeschle, S. Lasota, *Partially-commutative context-free processes,* CONCUR 2009.

Outline

- Background
    - Context-free graphs and commutative context-free graphs
    - Bisimulation equivalence problem
    - Norm
    - History of the problem
    - Unique decomposition
    - Naive algorithm
- Efficient algorithm for BPA and BPP
    - Outline of the algorithm
    - Refinement
    - Efficient computation of refinement for BPA
    - Time-cost analysis
    - Partially-commutative context-free graphs

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Context-free grammars (CFG)

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$$\Sigma = \{a, b\} \qquad V = \{B, S\}$$

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stateless pushdown automata
without ε-transitions

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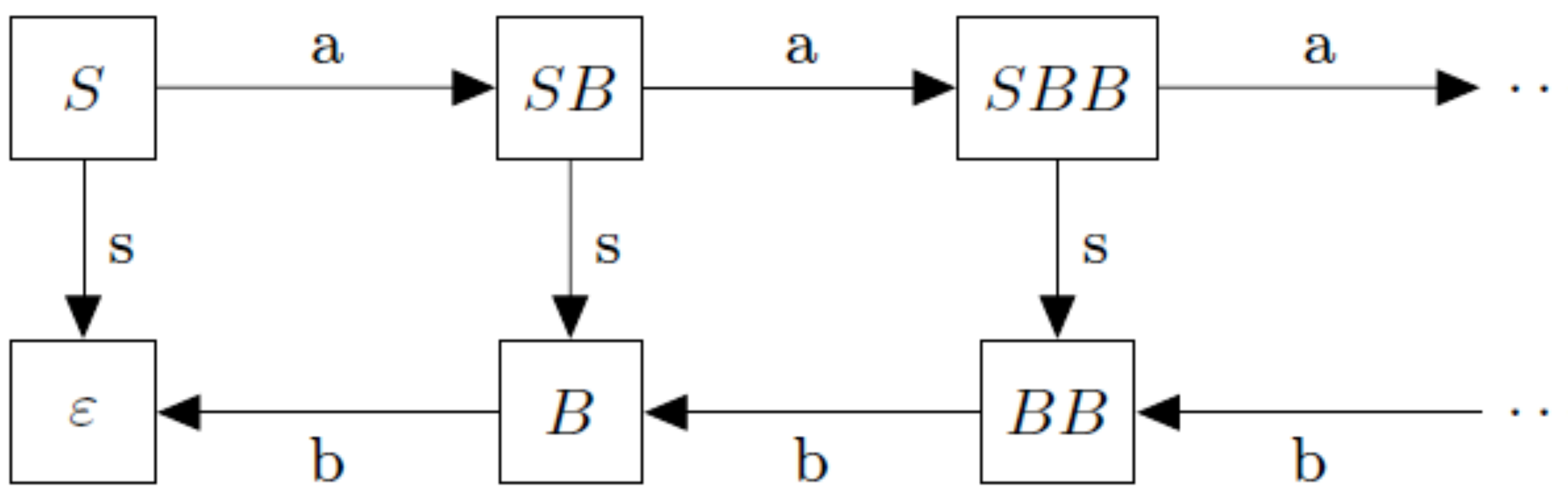

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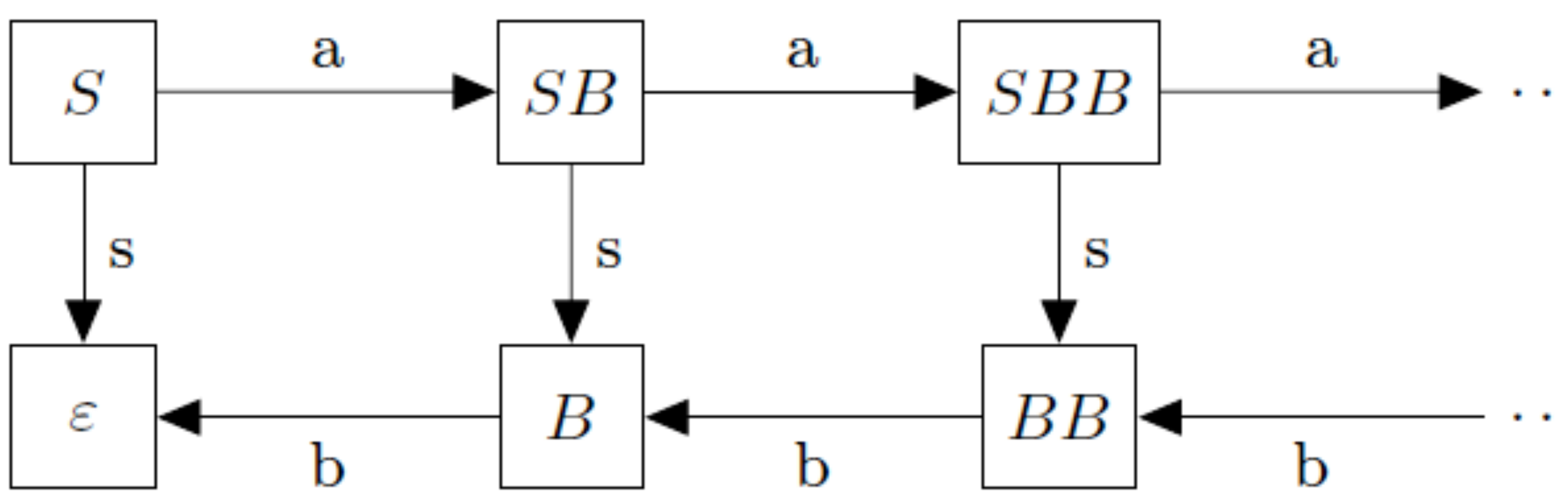

the simplest model of recursive programs

Deterministic context-free graphs

Simple grammars

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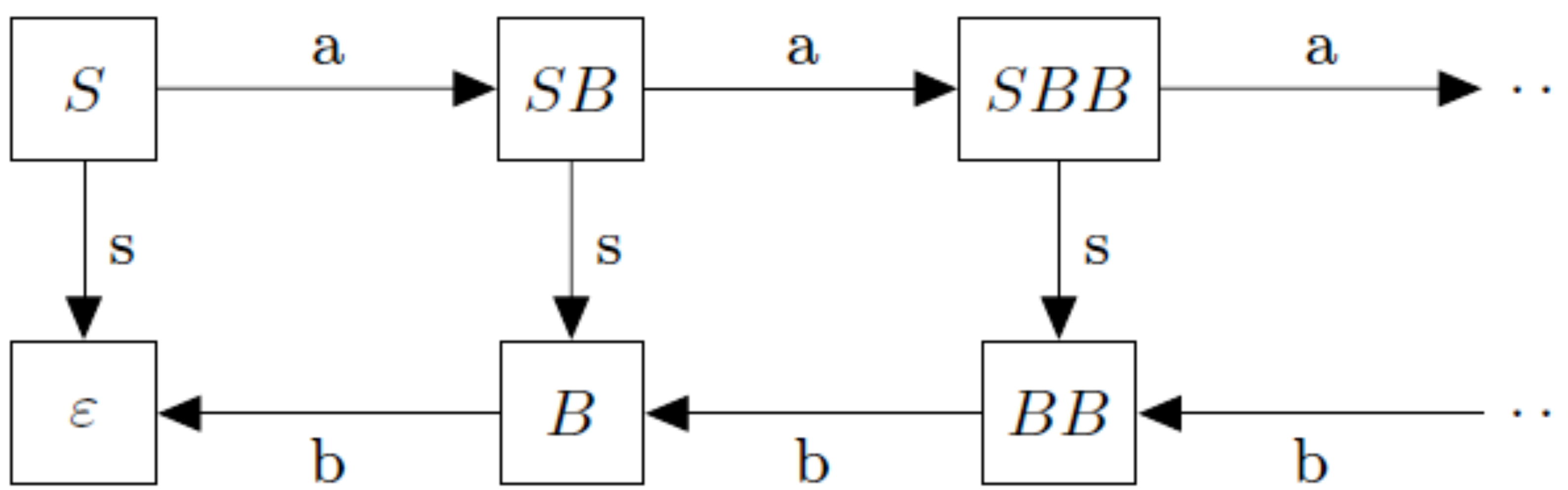

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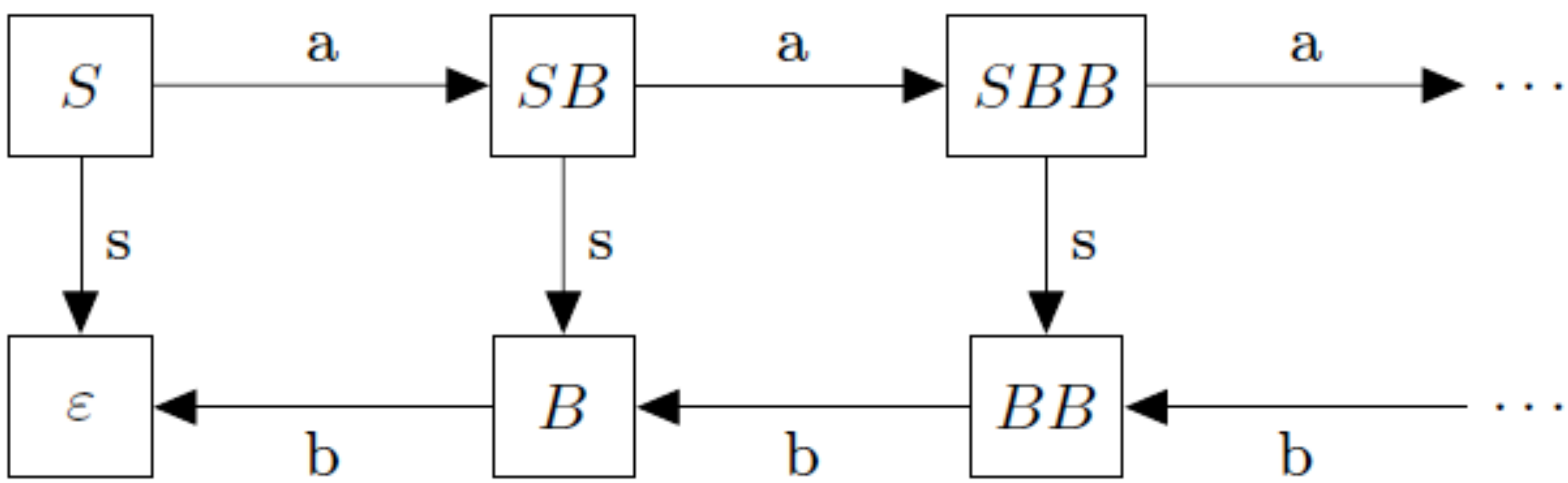

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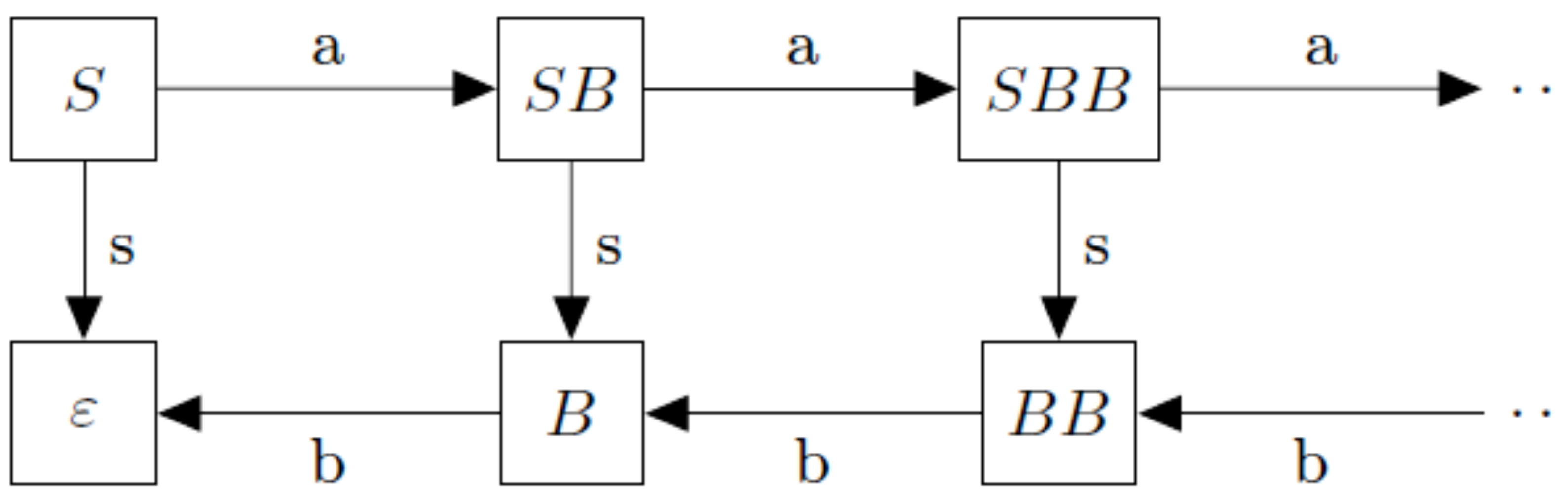

determinism: for every alphabet letter *a*, every process has at most one *a*-successor

Commutative context-free graphs

Basic Parallel Processes (BPP)

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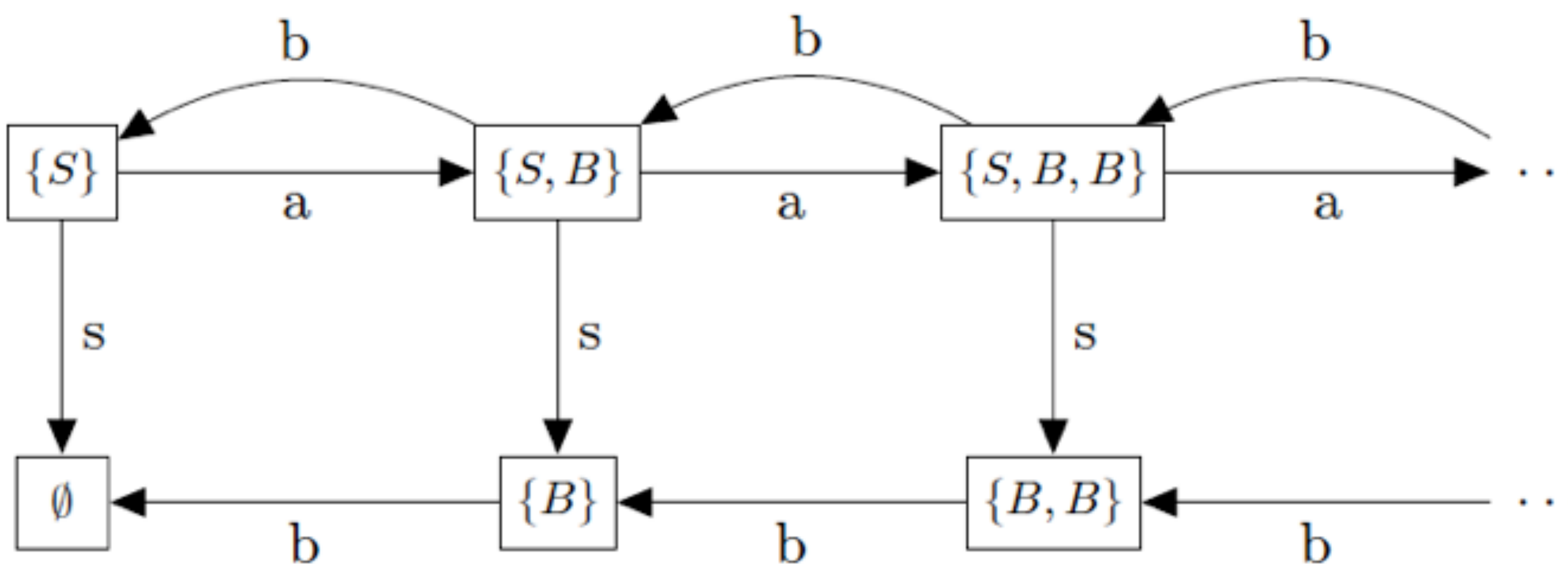

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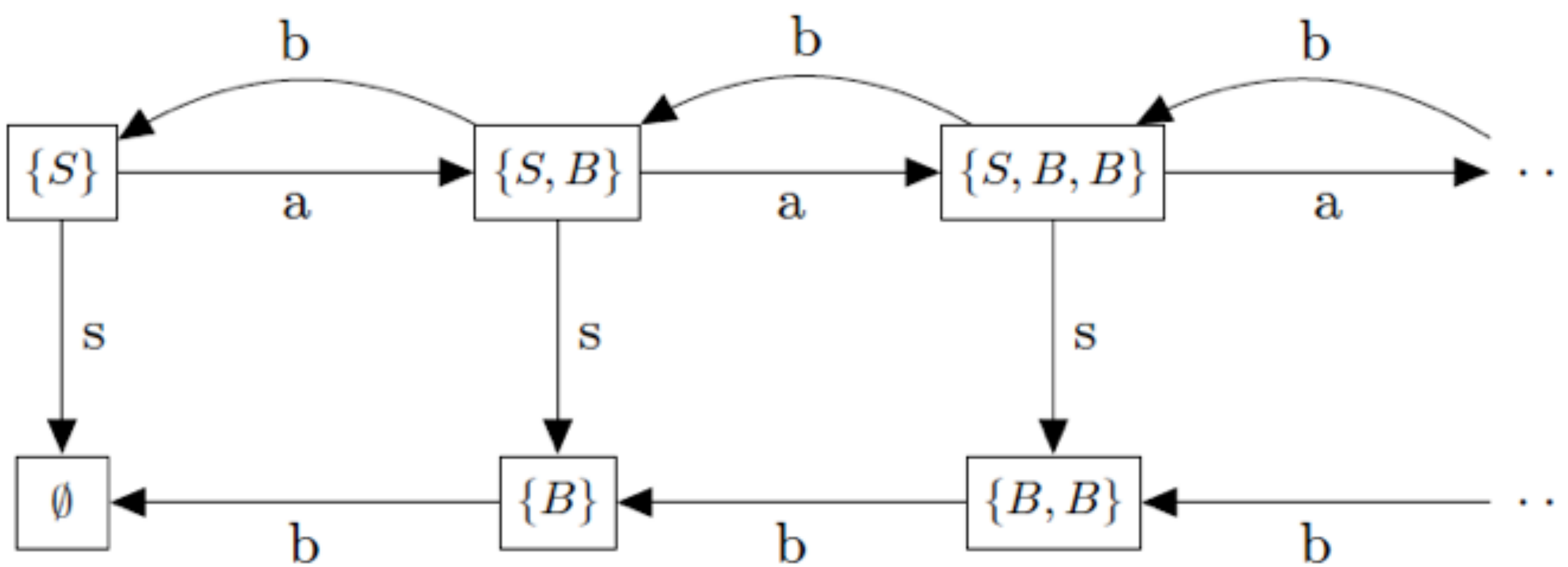
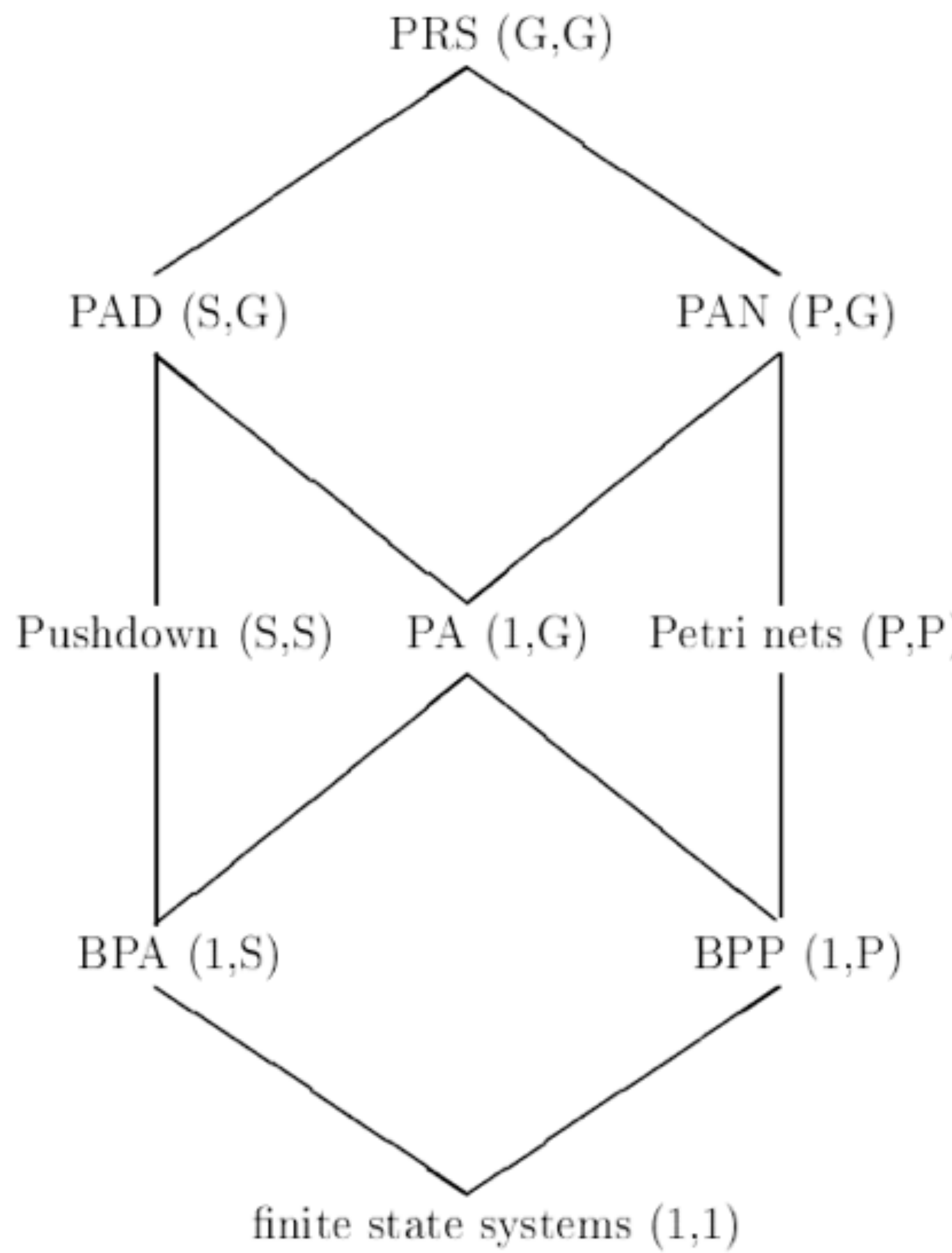


the simplest model of parallel programs

PRS hierarchy



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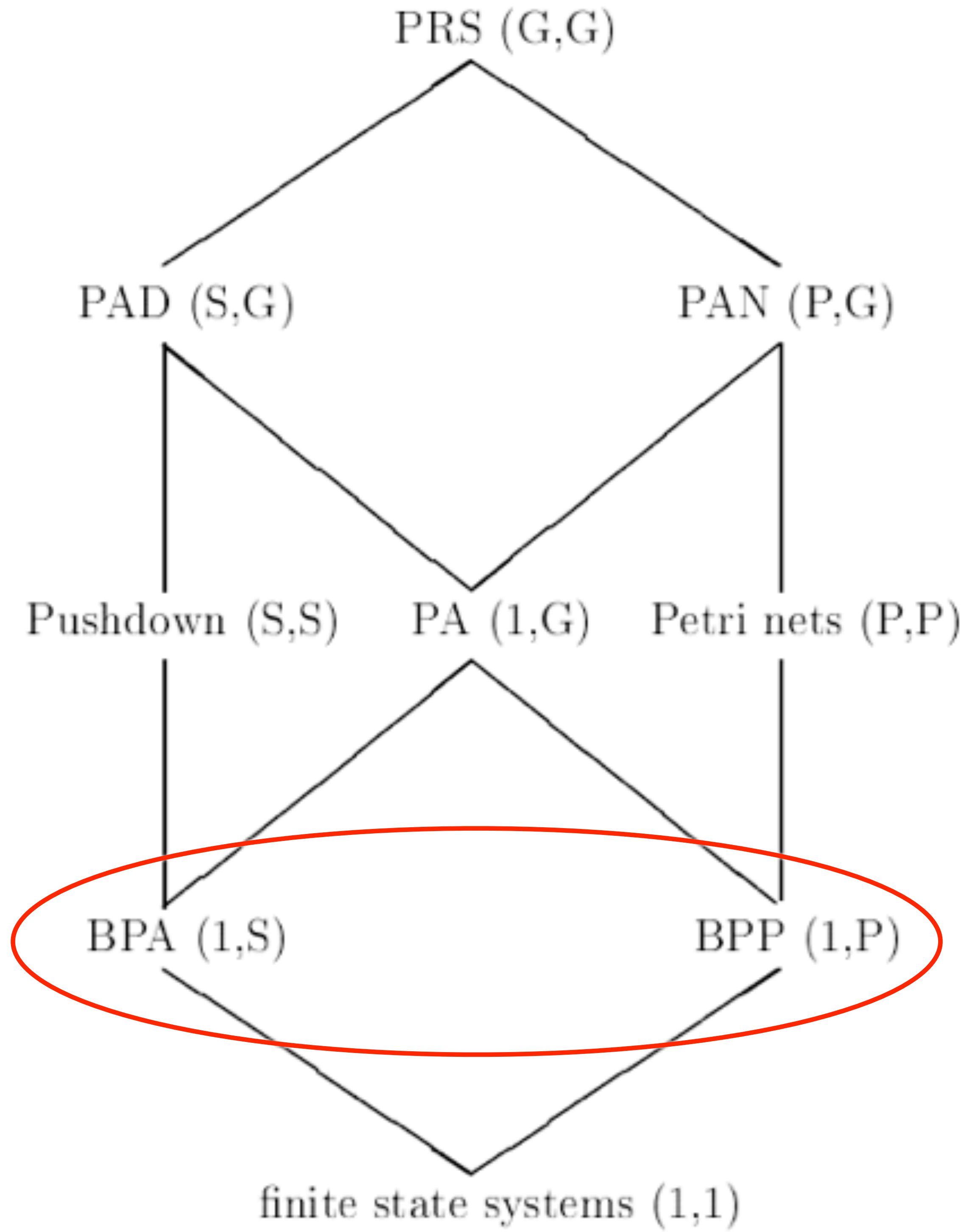

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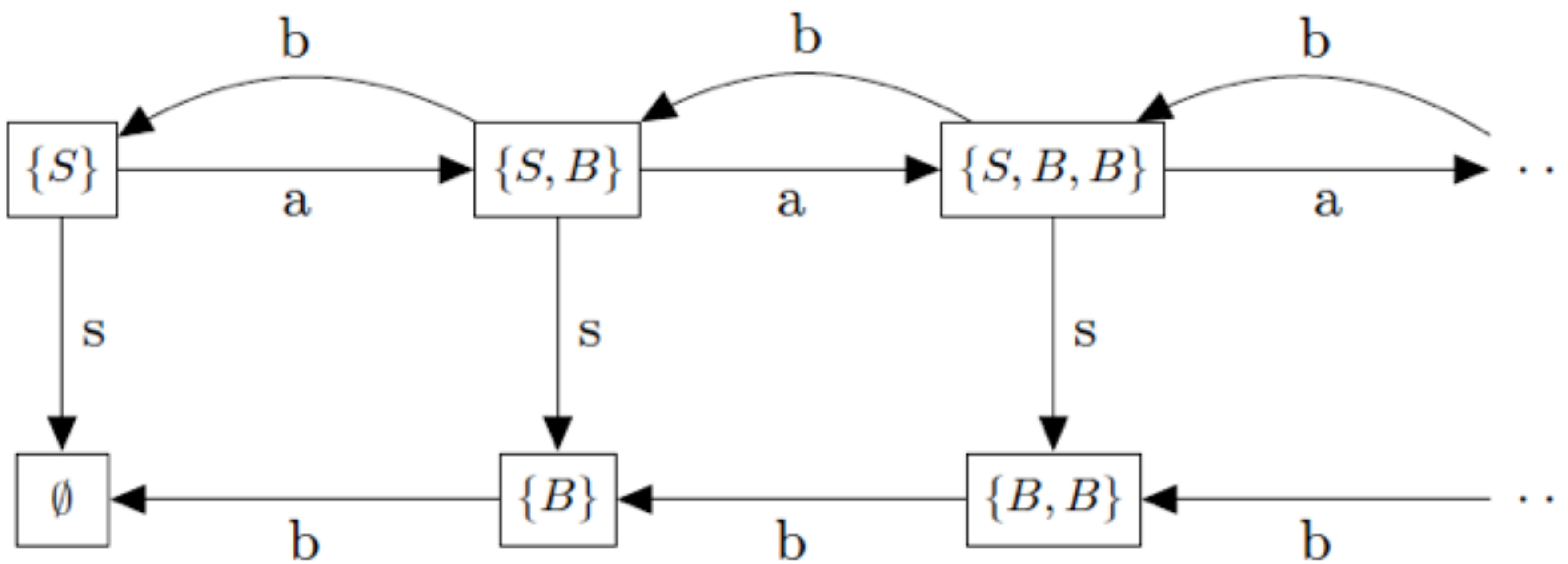
Bisimulation Game



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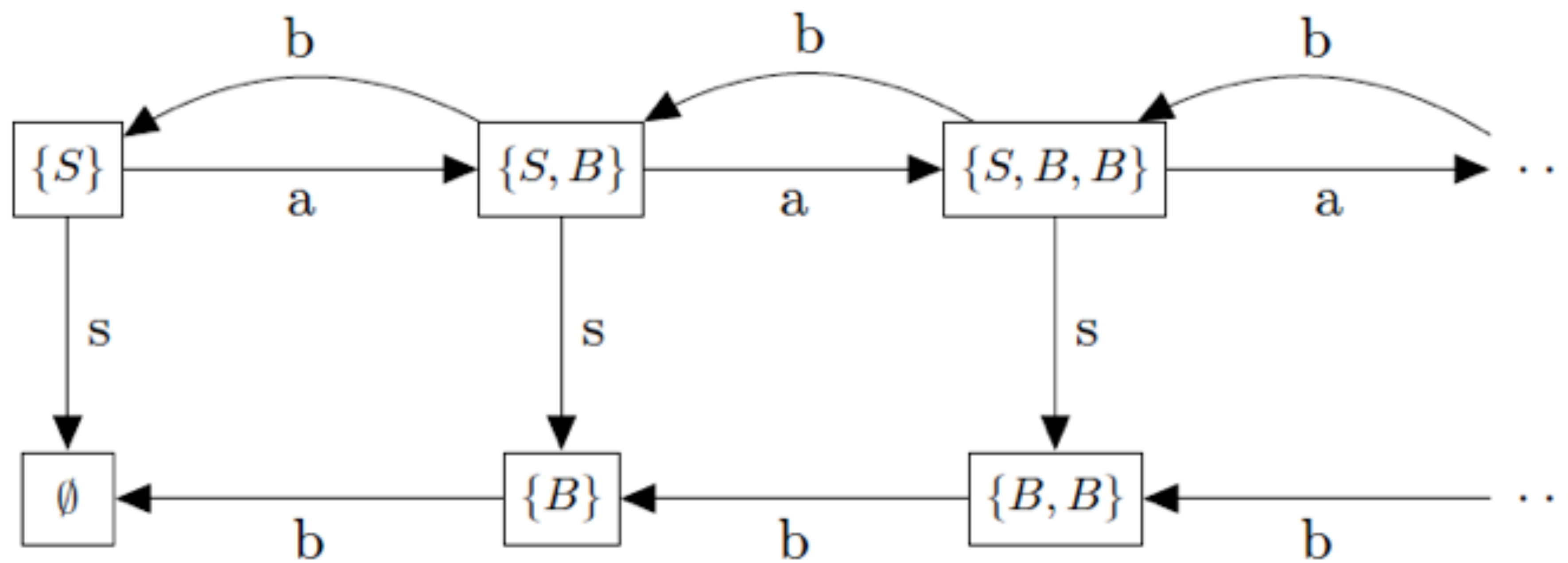

Two-player game (between Spoiler and Duplicator) proceeding in rounds

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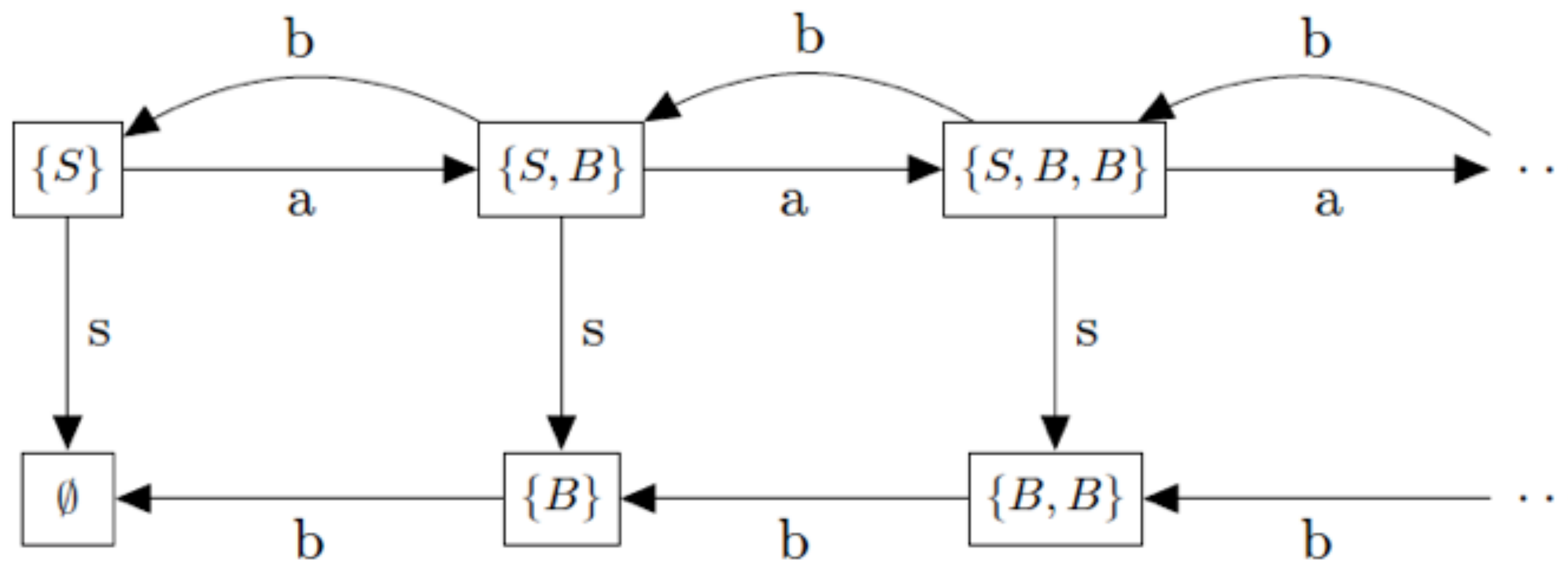

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Configurations = pairs of processes $(\alpha, \beta) \in V^* \times V^*$

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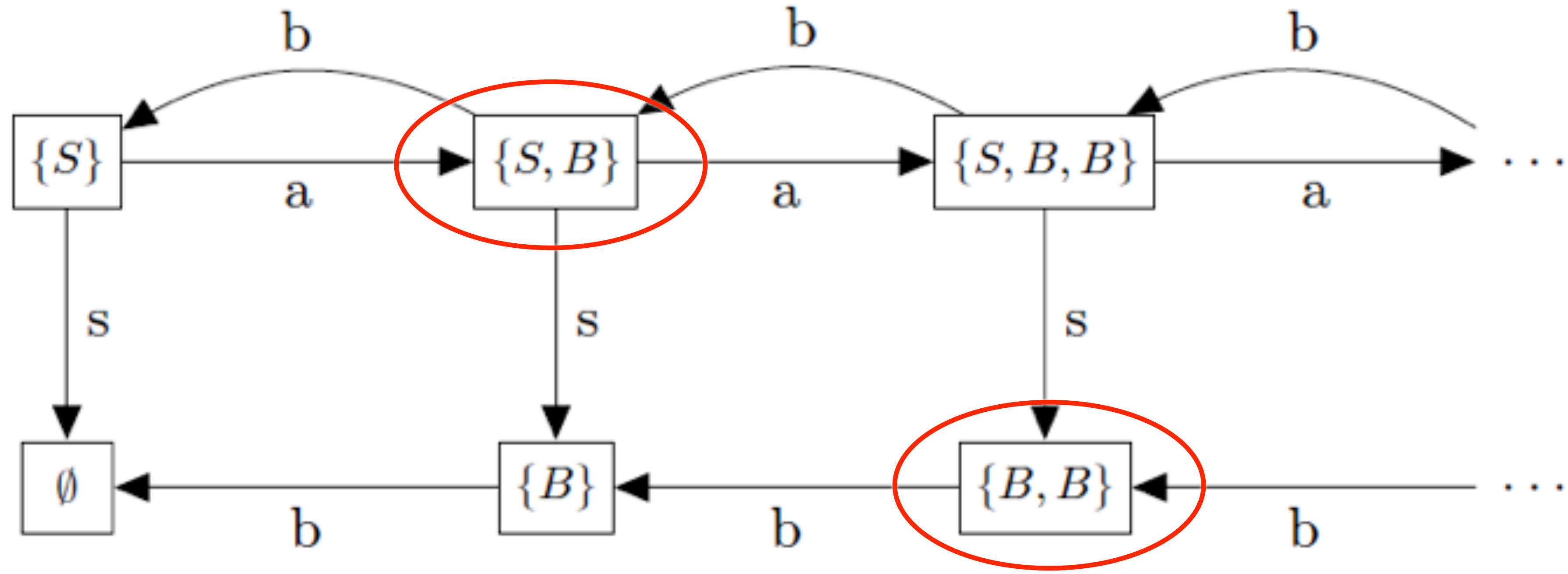

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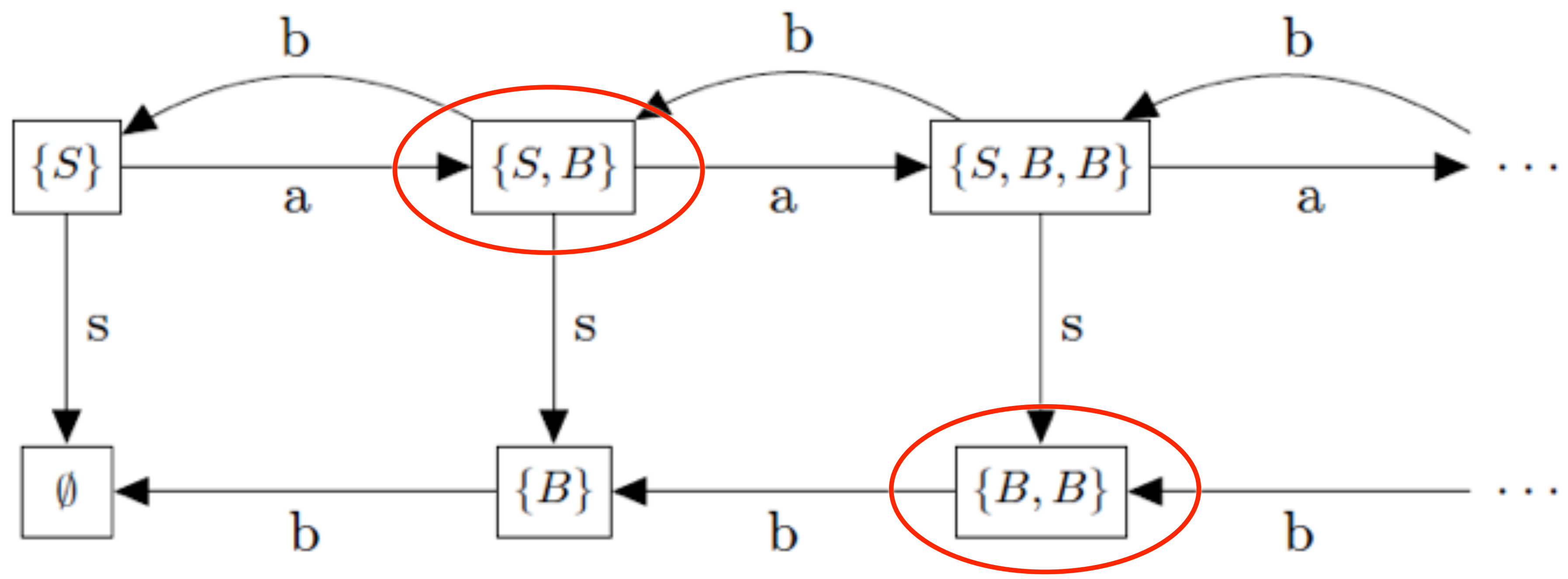
Spoiler's move



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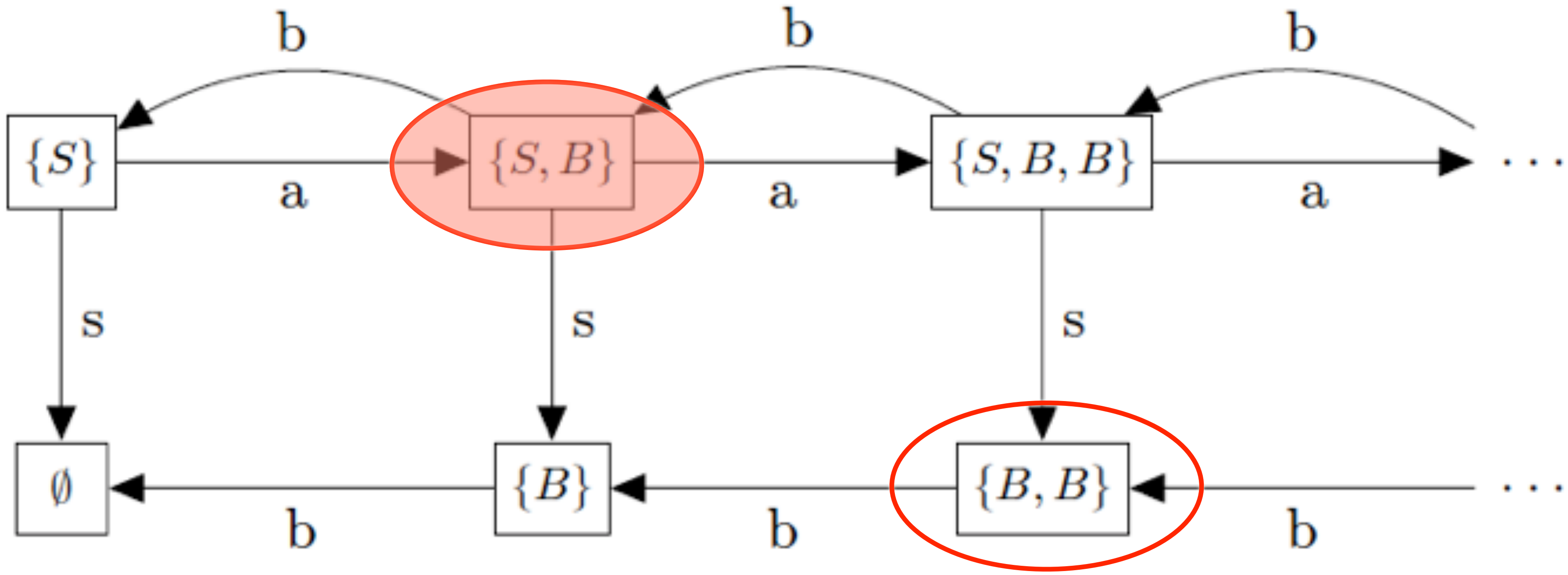

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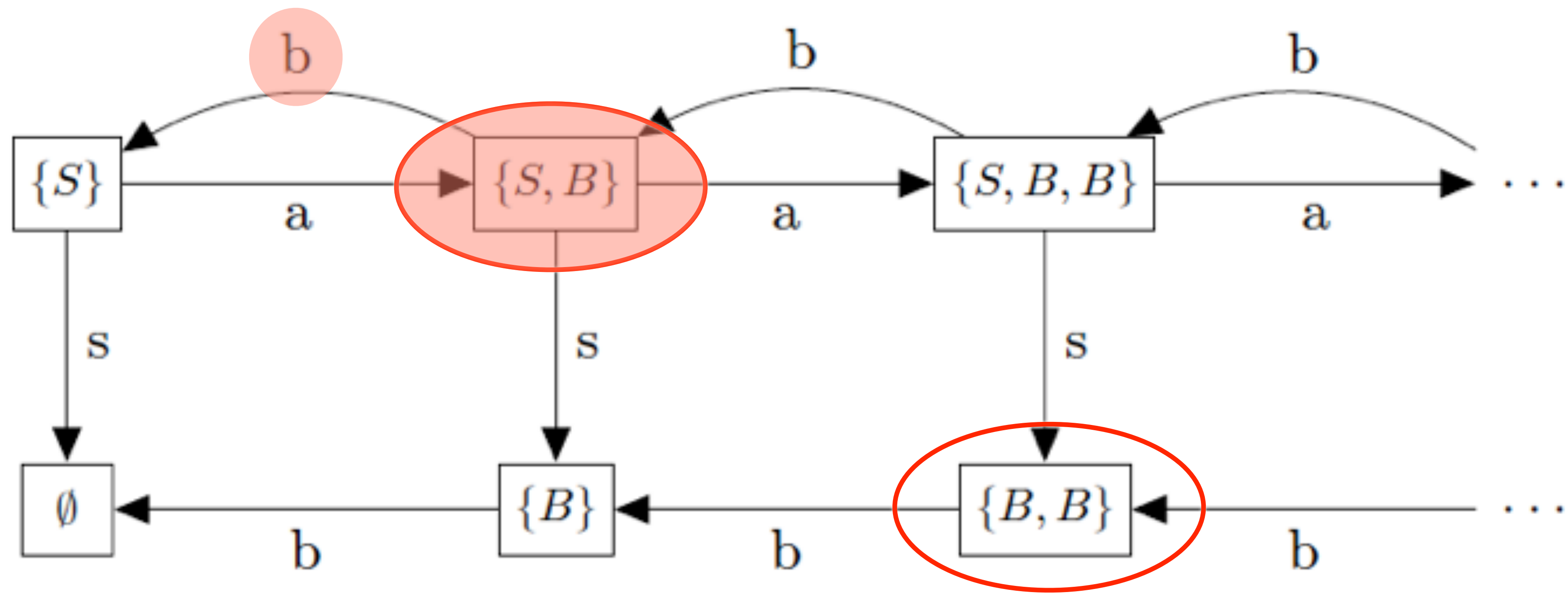

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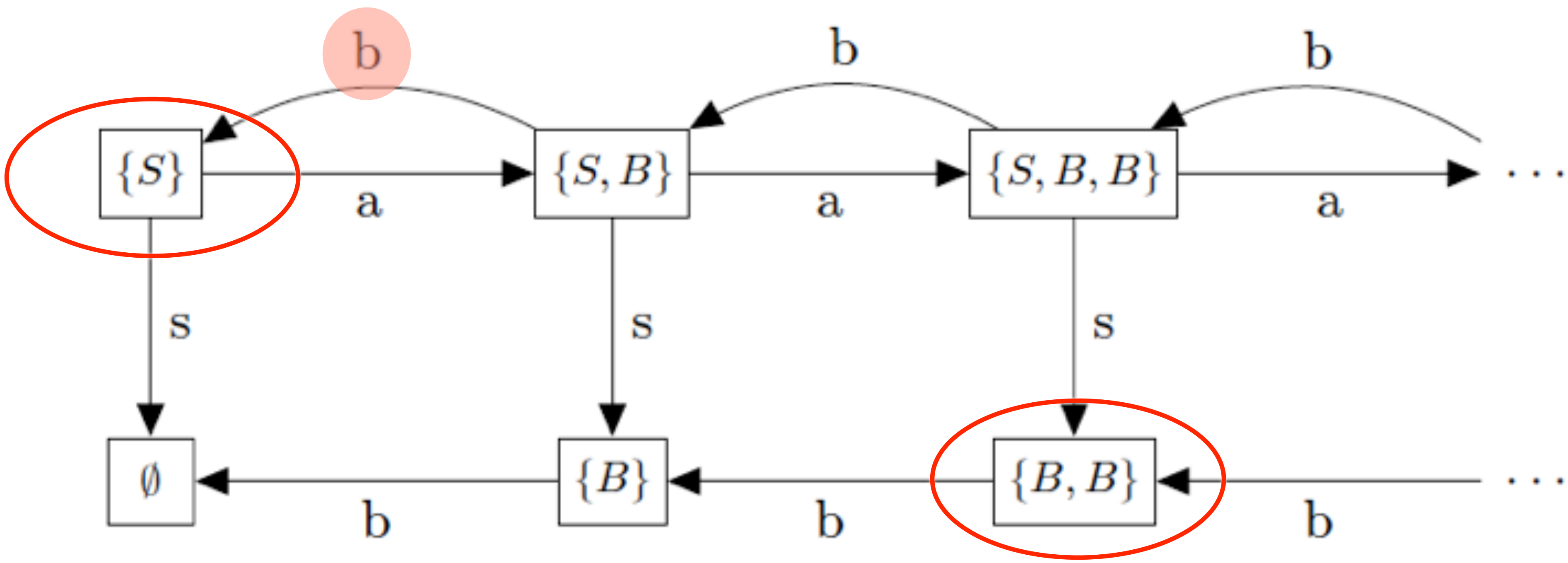

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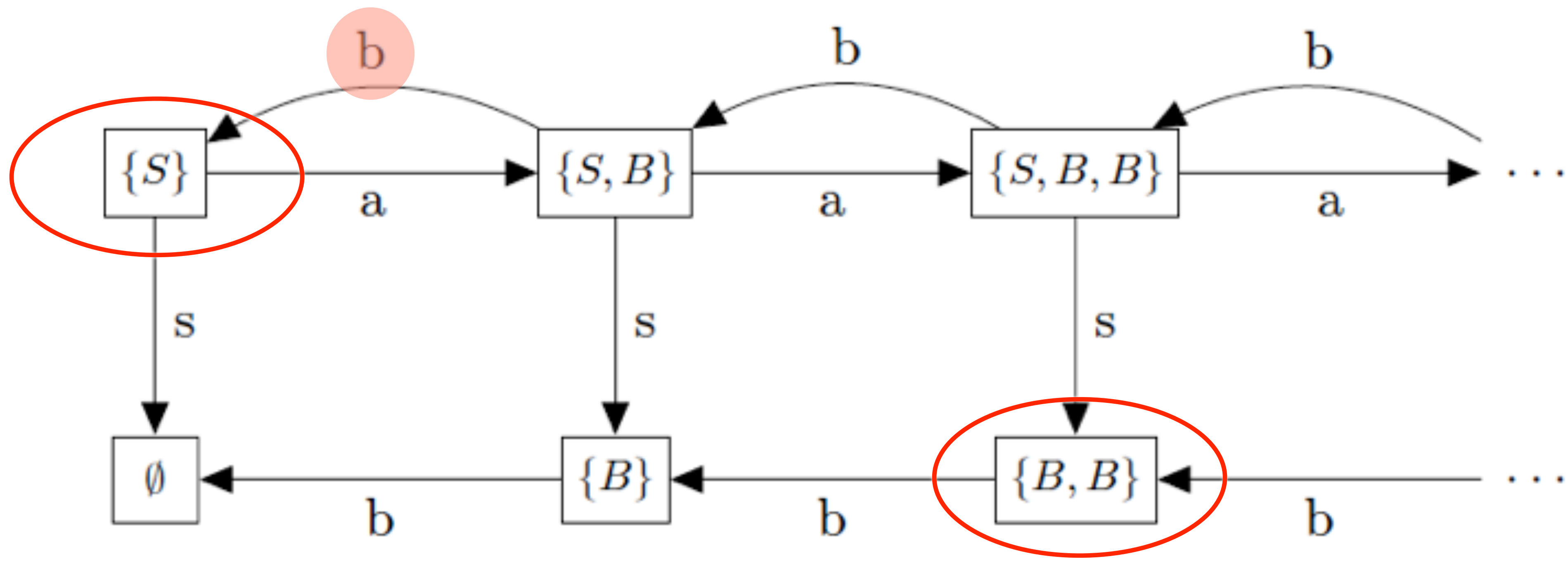

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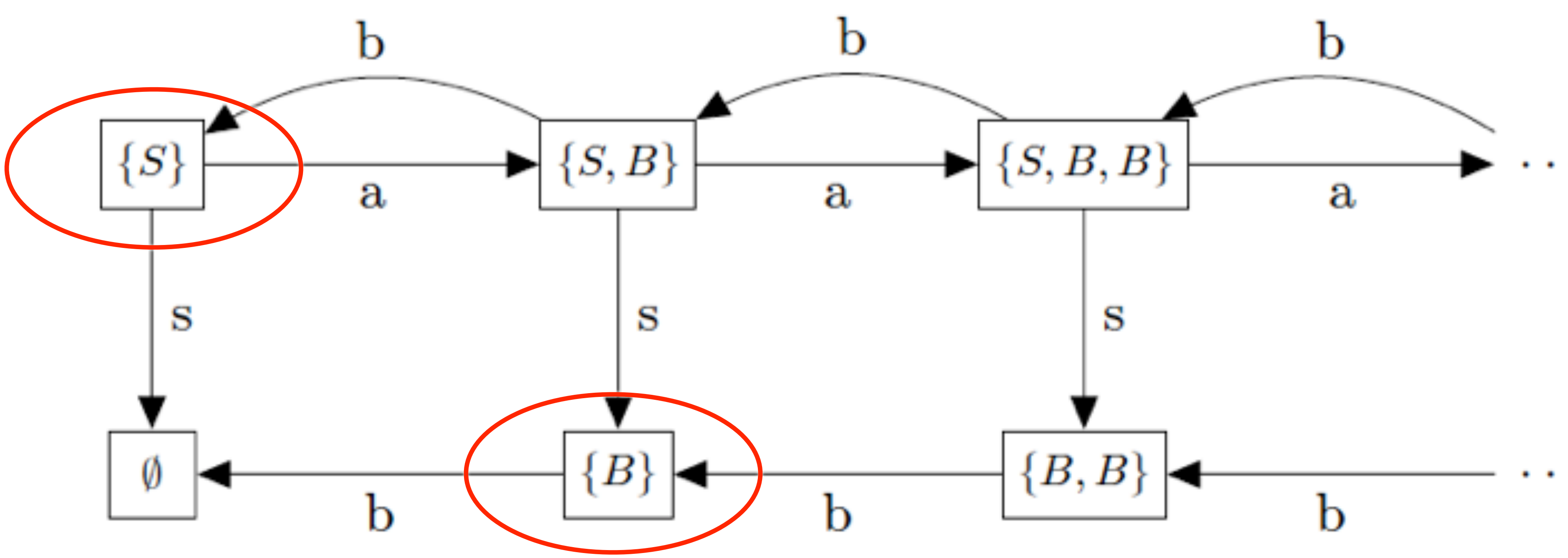

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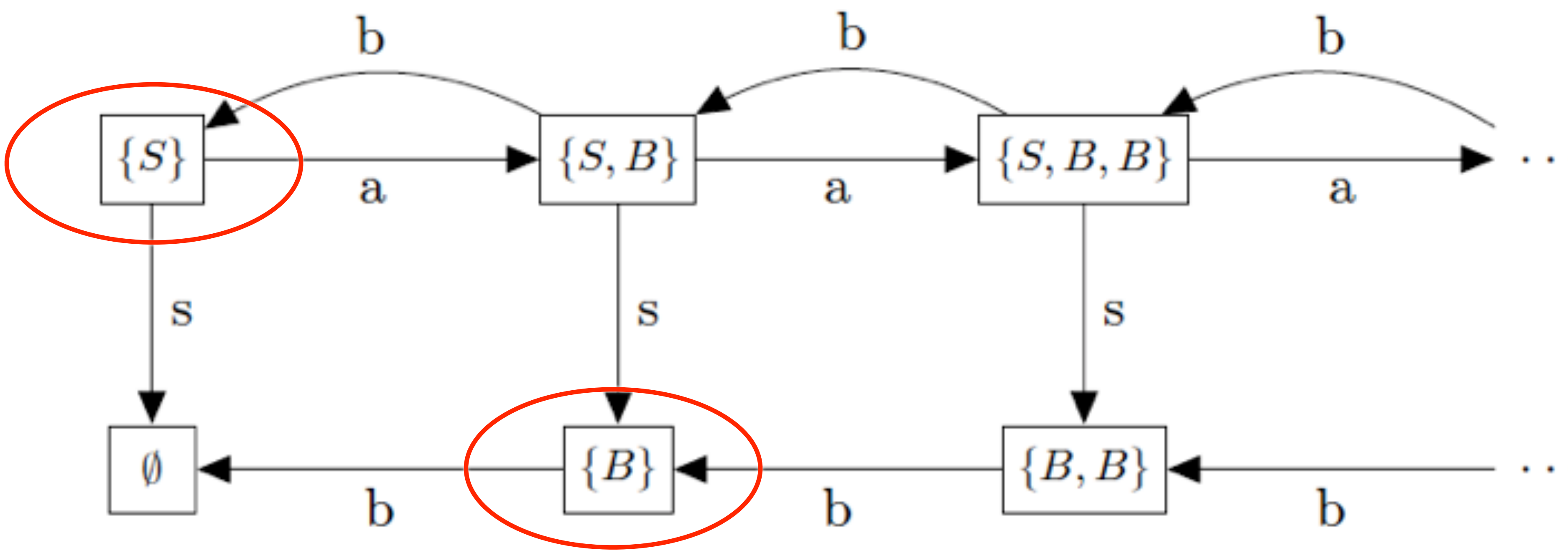

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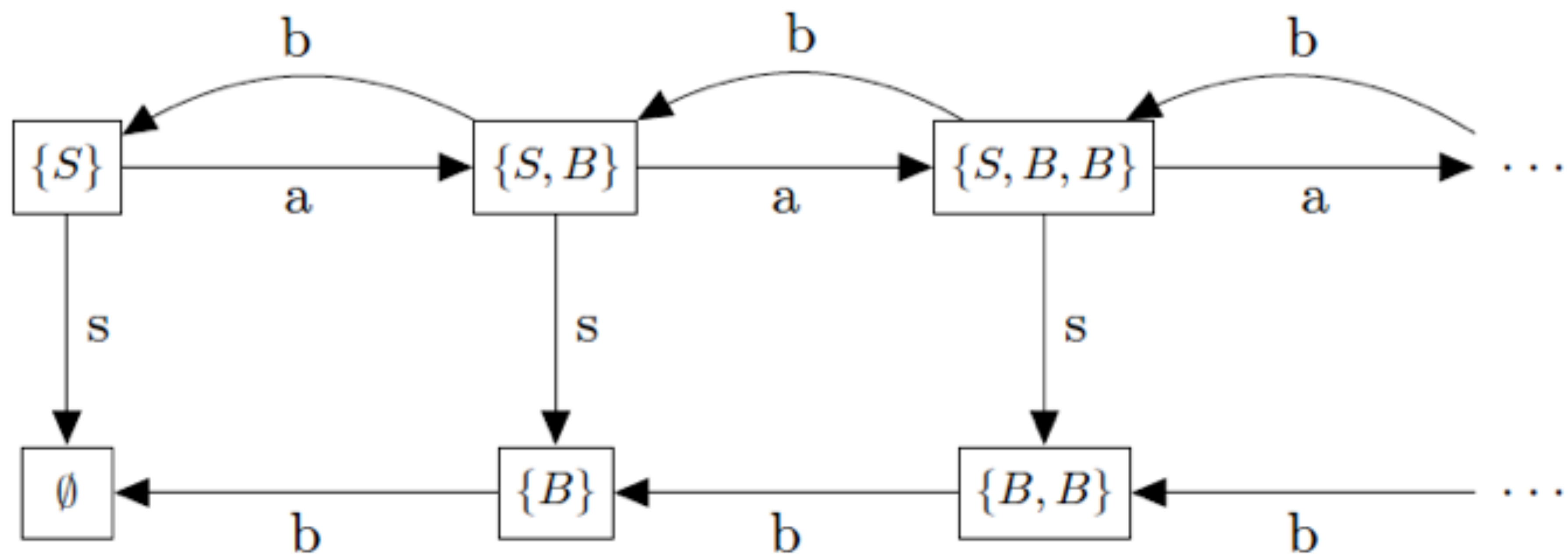

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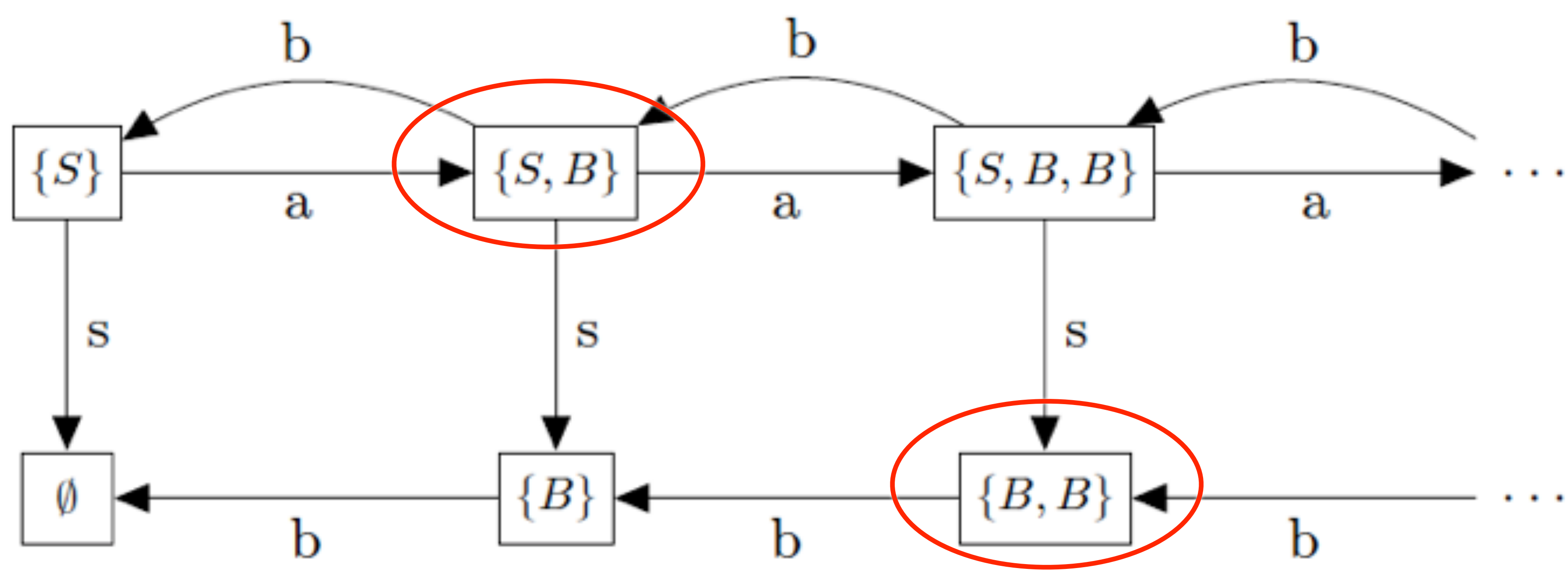

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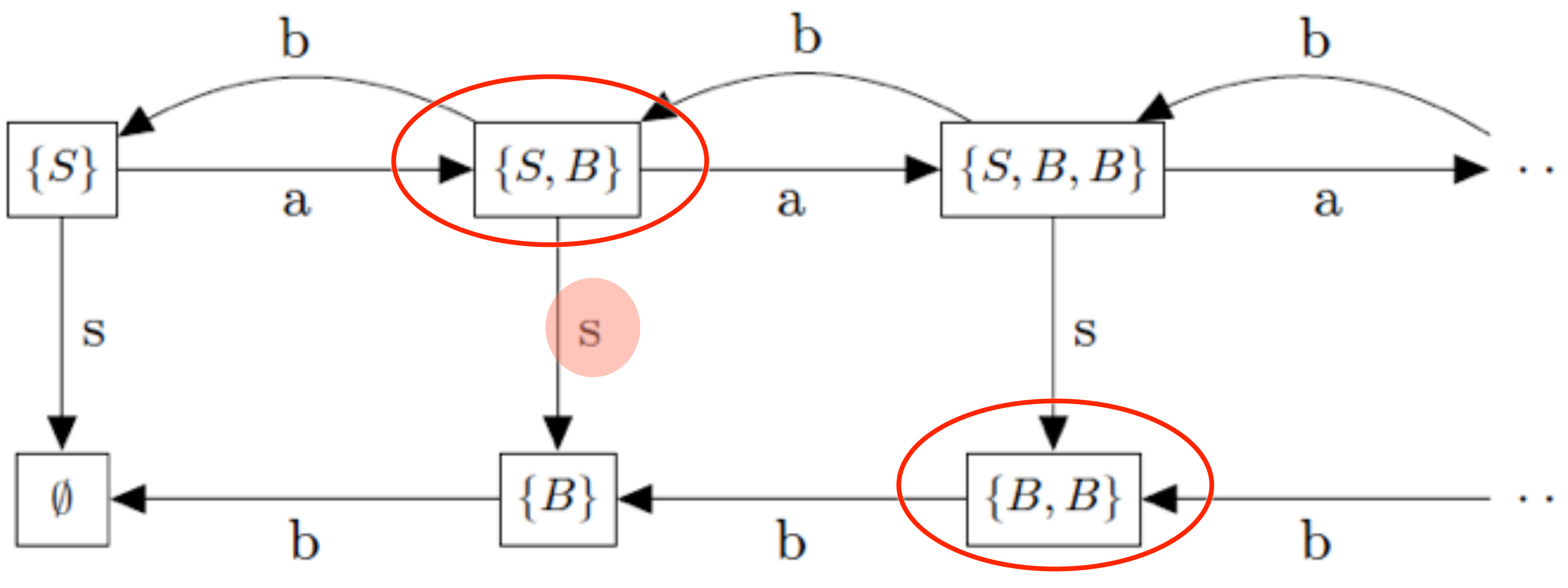

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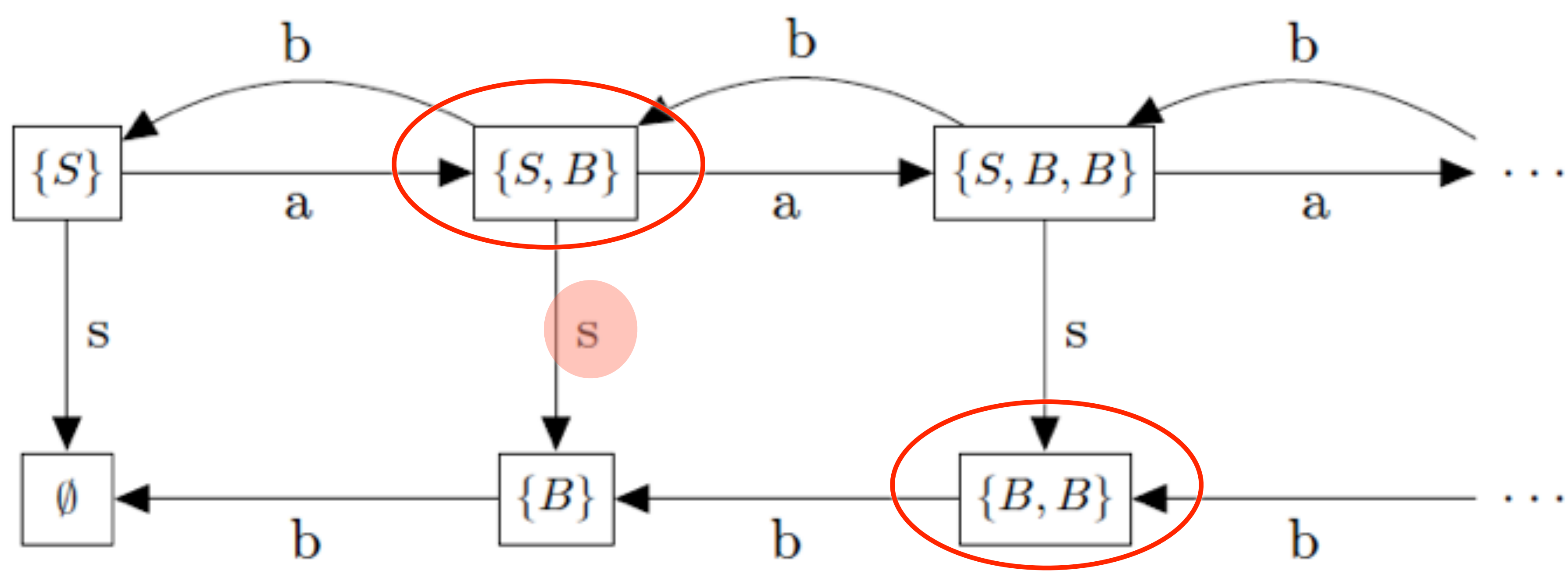

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$\alpha \sim \beta$ iff Duplicator has a winning strategy from (α, β)

Bisimulation equivalence

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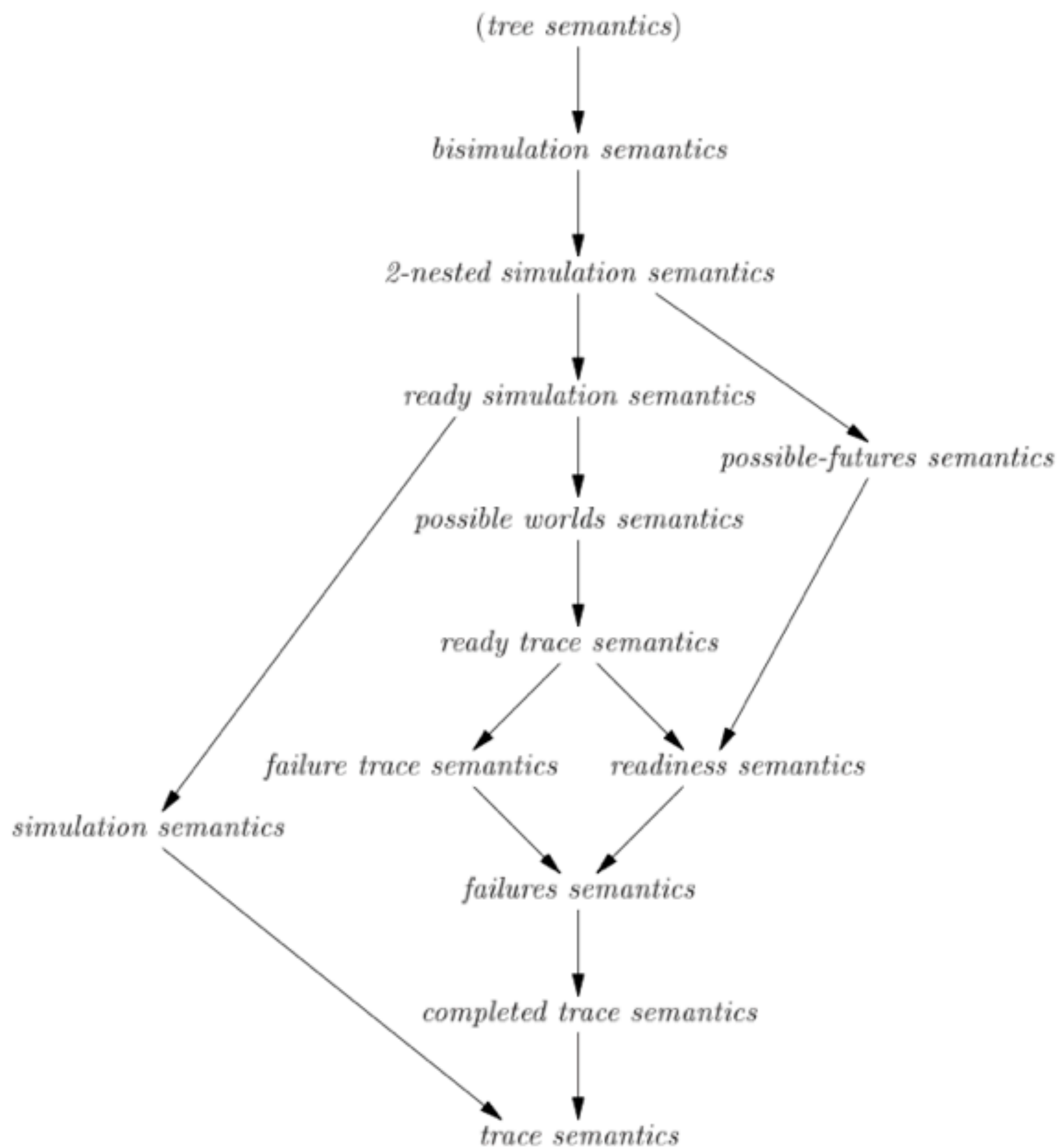
Bisimulation equivalence is the greatest bisimulation.

Other equivalences



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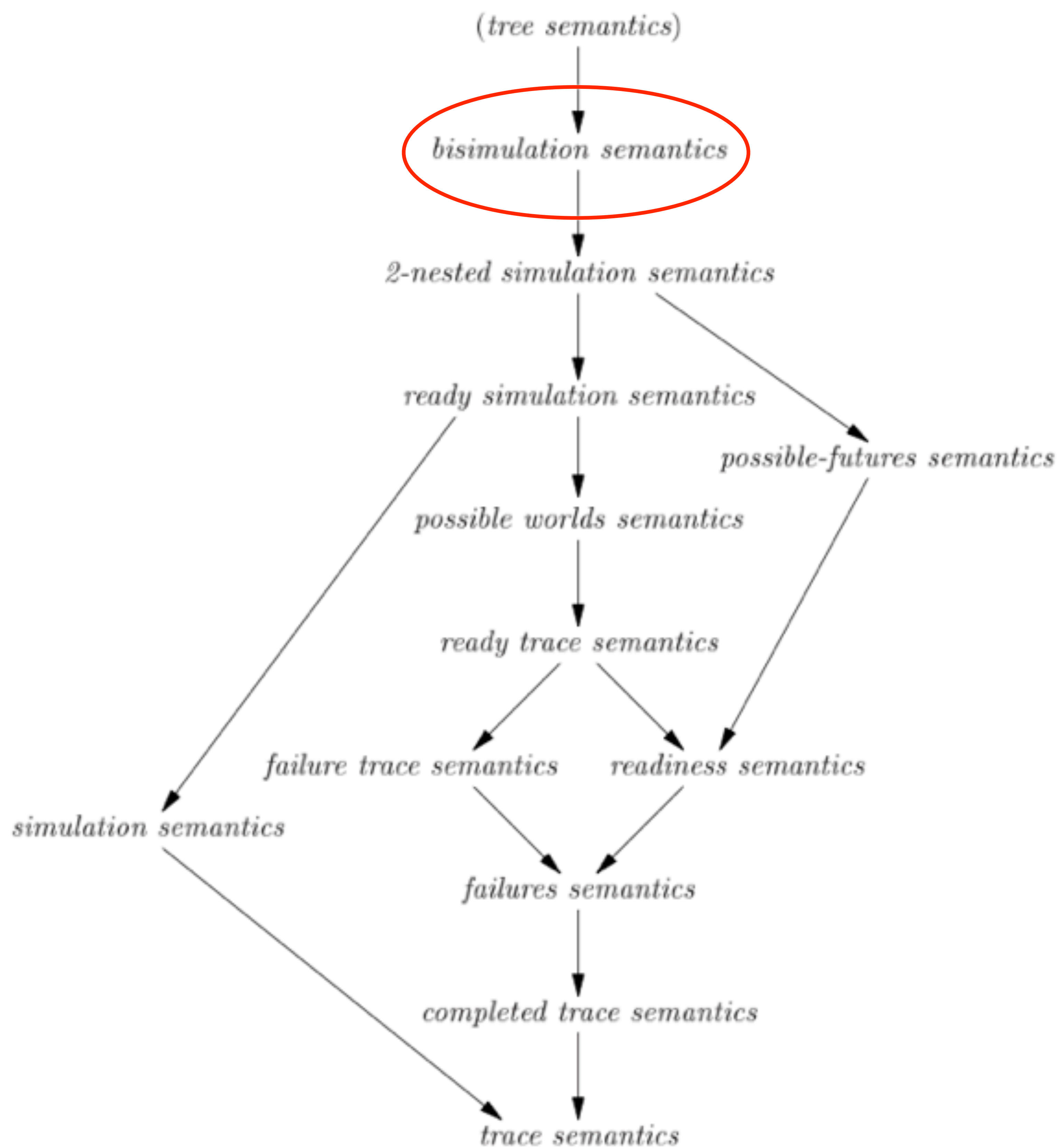

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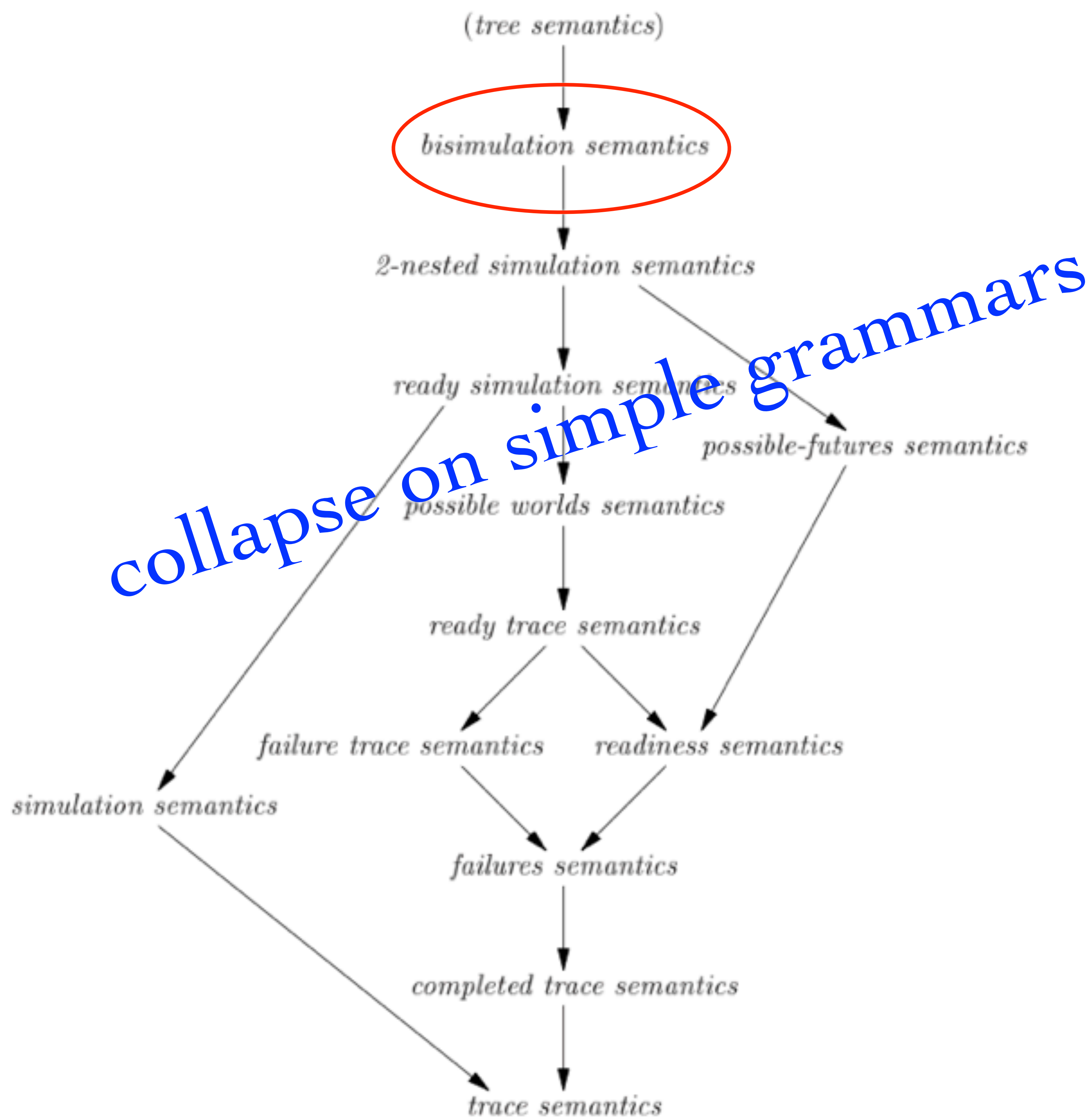

Decision problem

Given a plain/commutative CFG and two processes

$$\alpha, \beta \in V^*$$

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Given a plain/commutative CFG and two processes

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decide whether they are equivalent

$$\alpha \sim \beta \quad ?$$

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Norm

Norm of a variable X is the length of the shortest path to ε

$$X \xrightarrow{a_1} \ldots \xrightarrow{a_1} \varepsilon$$

(if such path exists)

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the empty process

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Example:

$$\begin{array}{cccccccccccc}
X_1 & \xrightarrow{a} & \varepsilon & X_2 & \xrightarrow{a} & X_1 & X_2 & \xrightarrow{b} & X_1X_1 & X_3 & \xrightarrow{a} & X_2 \\
X_4 & \xrightarrow{a} & X_3X_2 & X_4 & \xrightarrow{b} & X_4 & X_5 & \xrightarrow{a} & X_4 & X_5 & \xrightarrow{a} & X_3X_3
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$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

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Norms can be easily computed

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$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

Norms can be easily computed

$$X \xrightarrow{a} Y\,Z \qquad |X| = |Y| + |Z| + 1$$

Norm

A variable X is normed if X has finite norm.

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if X generates a word.

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if X generates a word.

A grammar is normed if every its variable is normed.

Norm

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Normedness simplifies the bisimulation equivalence problem but does not simplify the language equivalence problem

Outline

- Background
  - Context-free graphs and commutative context-free graphs
  - Bisimulation equivalence problem
  - Norm
  - **History of the problem**
  - Unique decomposition
  - Naive algorithm
- Efficient algorithm for BPA and BPP
  - Outline of the algorithm
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Complexity of bisimulation equivalence

	normed	unnormed
BPA	P-complete	EXPTIME-hard in 2-EXPTIME
BPP	P-complete	PSPACE-complete

Complexity of bisimulation equivalence

even reachability is NP-complete

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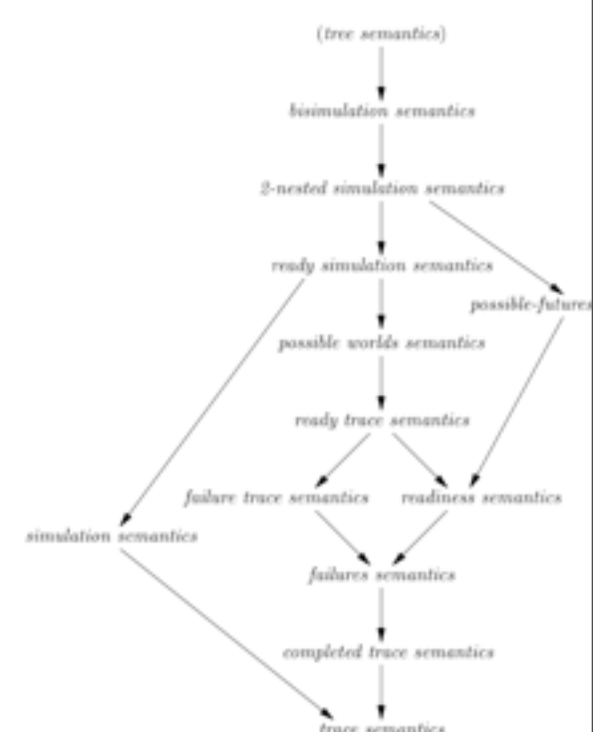

essentially all other equivalences are undecidable

Complexity of bisimulation equivalence

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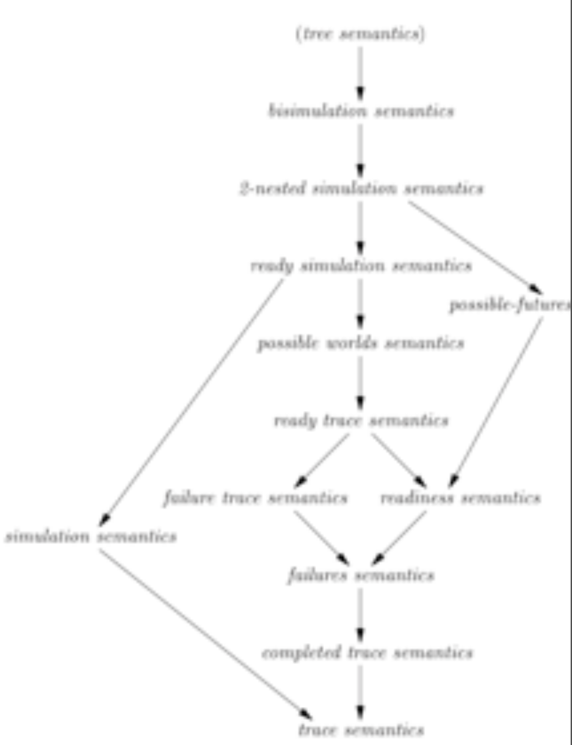

This motivates searching for efficient polynomial-time algorithms, aiming at understanding better the structure of bisimulation over BPA and BPP

History (upper bound)

	normed	unnormed
language equiv. simple grammars	•n^7 [Bastien, Czyżowicz, Frączak, Rytter 2005] •n^6 [L., Rytter 2006] **•n^3 [Czerwiński's PhD thesis 2012]**	•2-EXPTIME [Korenjak, Hopcroft 1966] •EXPTIME [Caucal 1990]
~ BPA	•decidable [Beaten, Bergstra, Klop 1987] •EXPTIME [Huettel, Stirling 1991] •NP^NP [Huynh, Tian 1994] •P [Hirshfeld, Jerrum, Moller 1996] •n^8 [L., Rytter 2006] **•n^5 [Czerwiński, L. 2010]** **•n^4 [Czerwiński's PhD thesis 2012]**	•decidable [Christensen, Hirshfeld, Stirling 1995] • 2-EXPTIME [Burkart, Caucal, Steffen 1995] •2-EXPTIME [Jancar 2013]
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History (upper bound)

- bisimulation equivalence in presence of silent steps (weak bisimulation, branching bisimulation)

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- state-extended BPA and BPP (pushdown systems, Petri nets)

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- finiteness up to bisimulation
- BPA vs. BPP
- ...

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Congruence

$$\alpha \;\sim\; \alpha' \;\; \text{and} \;\; \beta \;\sim\; \beta' \implies \alpha\,\beta \;\sim\; \alpha'\,\beta'$$

Fact: Bisimulation equivalence is a congruence, both over BPA and BPP.

From now on we restrict ourselves to

normed

BPA and BPP

Unique decomposition
(both for BPA and BPP)

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order variables according to norm: $|X_1| \leq |X_2| \leq \ldots \leq |X_n|$

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Def: A variable is decomposable if it admits a non-trivial decomposition; otherwise it is prime.

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Thm: Bisimulation equivalence has unique decomposition property, both over BPA and BPP.

Unique decomposition

Example:

$$\begin{array}{llllllllllll}
X_1 & \xrightarrow{a} & \varepsilon & X_2 & \xrightarrow{a} & X_1 & X_2 & \xrightarrow{b} & X_1X_1 & X_3 & \xrightarrow{a} & X_2 \\
X_4 & \xrightarrow{a} & X_3X_2 & X_4 & \xrightarrow{b} & X_4 & X_5 & \xrightarrow{a} & X_4 & X_5 & \xrightarrow{a} & X_3X_3
\end{array}$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

Unique decomposition

Example:

is it BPA or BPP?

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

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prime variables

decomposable variables

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prime variables

X_1

$X_2 \not\sim X_1\, X_1$

decomposable variables

$X_3 \sim X_1\, X_2$

$\mathrm{dec}(X_3)$

Unique decomposition

is it BPA or BPP?

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prime variables

X_1

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prime variables

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Algebraic perspective

unique decomposition property says that

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quotient of V^* by bisimulation equivalence $\sim$ is a free semigroup (resp. free commutative semigroup).

Cancellation

$$\alpha\gamma \sim \beta\gamma \implies \alpha \sim \beta$$

Cancellation

$$\alpha\gamma \ \sim \ \beta\gamma \quad \Longrightarrow \quad \alpha \ \sim \ \beta$$

Fact: unique decomposition $\quad \Longrightarrow \quad$ cancellation

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Proof: (both for BPA and BPP)

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extend dec to all processes:

$$\mathrm{dec}(\alpha\,\beta) = \mathrm{dec}(\alpha)\,\mathrm{dec}(\beta)$$

Cancellation

$$\alpha\,\gamma \;\sim\; \beta\,\gamma \quad\Longrightarrow\quad \alpha \;\sim\; \beta$$

Fact: unique decomposition $\Longrightarrow$ cancellation

Proof: (both for BPA and BPP)

extend dec to all processes:

$$\mathrm{dec}(\alpha\,\beta) \;=\; \mathrm{dec}(\alpha)\,\mathrm{dec}(\beta)$$

$$\alpha \sim \beta \quad\Longleftrightarrow\quad \mathrm{dec}(\alpha) = \mathrm{dec}(\beta)$$

Cancellation

$$\alpha\gamma \sim \beta\gamma \implies \alpha \sim \beta$$

Fact: unique decomposition $\implies$ cancellation

Proof: (both for BPA and BPP)

extend dec to all processes:

$$\mathrm{dec}(\alpha\,\beta) = \mathrm{dec}(\alpha)\,\mathrm{dec}(\beta)$$

$$\alpha \sim \beta \iff \mathrm{dec}(\alpha) = \mathrm{dec}(\beta)$$

and observe:

$$\mathrm{dec}(\alpha)\,\mathrm{dec}(\gamma) = \mathrm{dec}(\beta)\,\mathrm{dec}(\gamma) \implies \mathrm{dec}(\alpha) = \mathrm{dec}(\beta)$$

BPA and BPP differ

proofs of unique decomposition

$$\begin{array}{c} \text{unique decomposition} \\ \text{BPA} \Uparrow \quad \Downarrow \text{BPP} \\ \text{cancellation} \end{array}$$

[Hirshfeld, Jerrum, Moller 1996]

Base

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Base = prime variables + decompositions of other variables

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Primes and decompositions may be chosen arbitrarily

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$\mathrm{dec}_B(X_3)$

$$B \quad = \quad (\{X_1, X_2, X_4, X_5\}, \quad \{X_3 = X_1\, X_2\})$$

Base

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\begin{array}{llllllllllll}
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\end{array}
$$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\mathrm{dec}_B(X_3)$

$$B = (\{X_1, X_2, X_4, X_5\}, \ \{X_3 = X_1\, X_2\})$$

$$B = (\{X_1, X_4\}, \ \{X_2 = X_1, X_3 = X_1\, X_1, \ X_5 = X_4\})$$

It is sufficient to restrict to

norm-preserving

bases

$$|X| = |\mathrm{dec}_B(X)|$$

It is sufficient to restrict to norm-preserving bases

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Example:

$$\begin{array}{llllllllllll}
X_1 & \xrightarrow{a} & \varepsilon & X_2 & \xrightarrow{a} & X_1 & X_2 & \xrightarrow{b} & X_1 X_1 & X_3 & \xrightarrow{a} & X_2 \\
X_4 & \xrightarrow{a} & X_3 X_2 & X_4 & \xrightarrow{b} & X_4 & X_5 & \xrightarrow{a} & X_4 & X_5 & \xrightarrow{a} & X_3 X_3
\end{array}$$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\mathrm{dec}_B(X_3)$

$$B \;=\; (\{X_1, X_2, X_4, X_5\}, \;\; \{X_3 = X_1\, X_2\})$$

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extend dec_B to all processes :

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Fact: A congruence has unique decomposition property
iff
it is represented by a base

Outline

- Background
  - Context-free graphs and commutative context-free graphs
  - Bisimulation equivalence problem
  - Norm
  - History of the problem
  - Unique decomposition
  - **Naive algorithm**
- Efficient algorithm for BPA and BPP
  - Outline of the algorithm
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  - Efficient computation of refinement for BPA
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  - Partially-commutative context-free graphs

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Given a plain/commutative CFG and two processes

$$\alpha, \beta \in V^*$$

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For solving the bisimulation equivalence problem, it is sufficient to compute a bisimulation base containing the decomposition $X = Y$ (if any such exists)

Naive algorithm

Naive algorithm

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the size of B is at most exponential, as it has to be norm-preserving

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in BPA, we need a compressed representation of decompositions

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$$B \;\subseteq\; \exp(\equiv_B) \quad\Longrightarrow\quad \equiv_B \subseteq\; \exp(\equiv_B)$$

[Caucal 1990]

Compressed representation of strings

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- straight-line programs

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- acyclic morphisms

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Example:

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- exponential compression
- efficient manipulation without decompression

Operations on compressed strings

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$$A \longrightarrow B\,C$$

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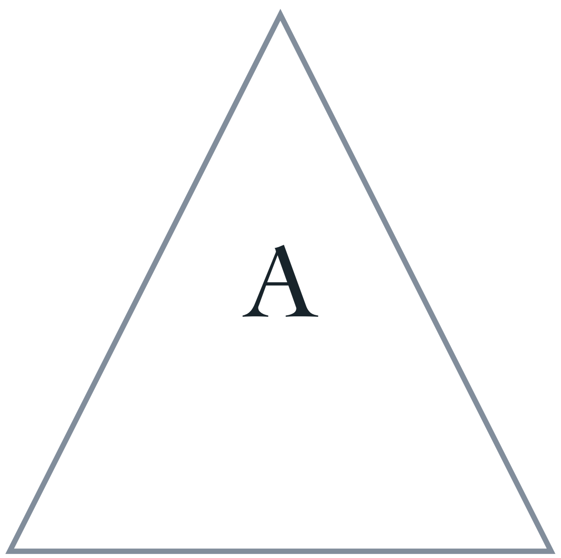

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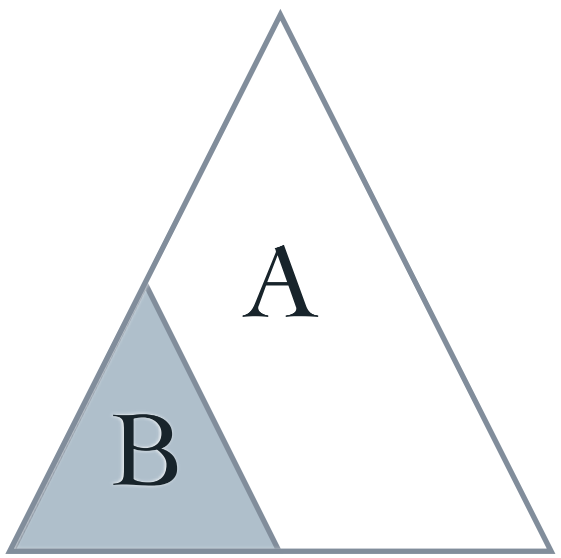

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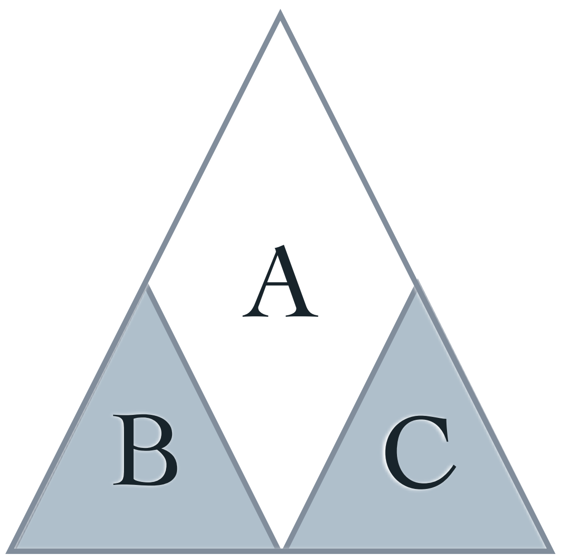

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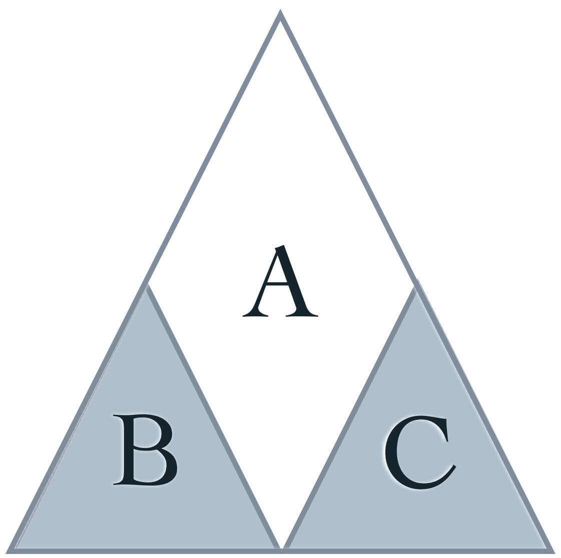

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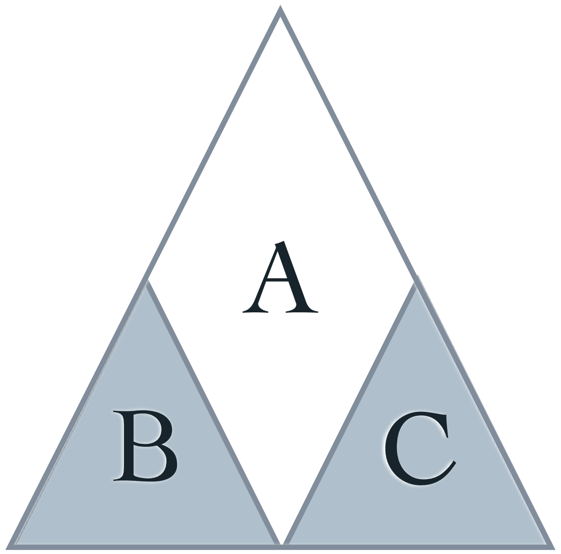

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equality test A = X ?

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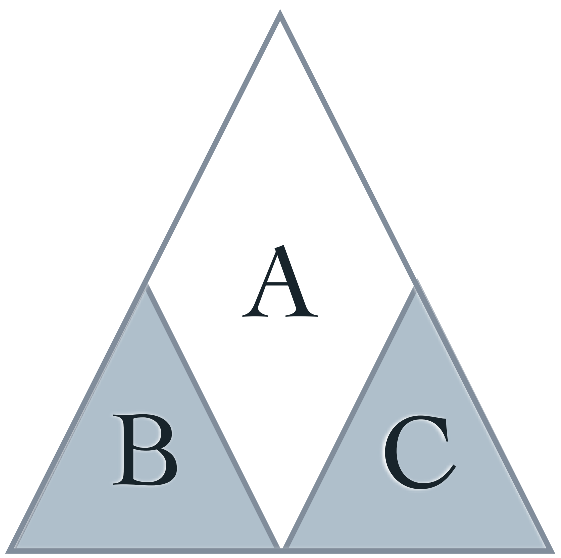

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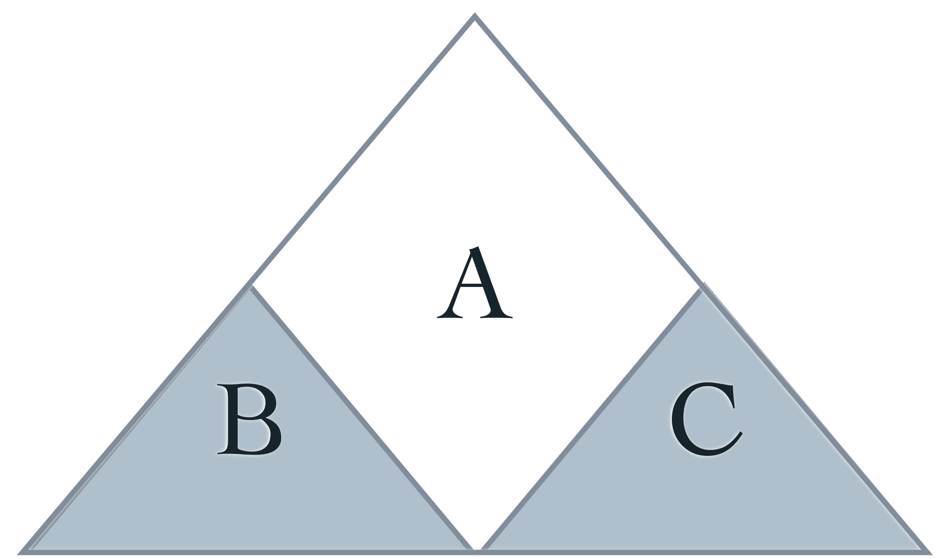

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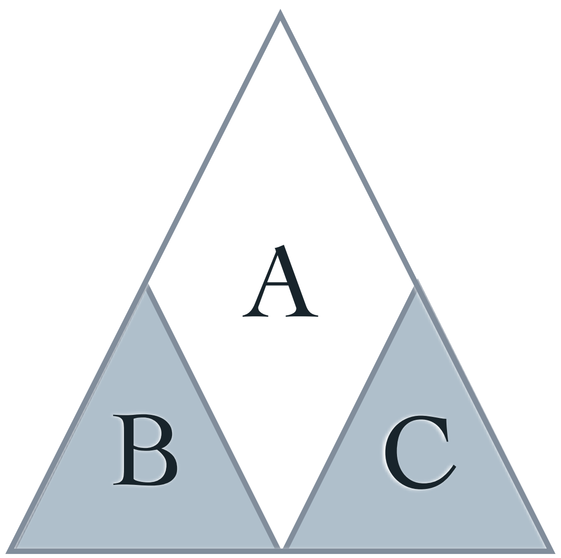

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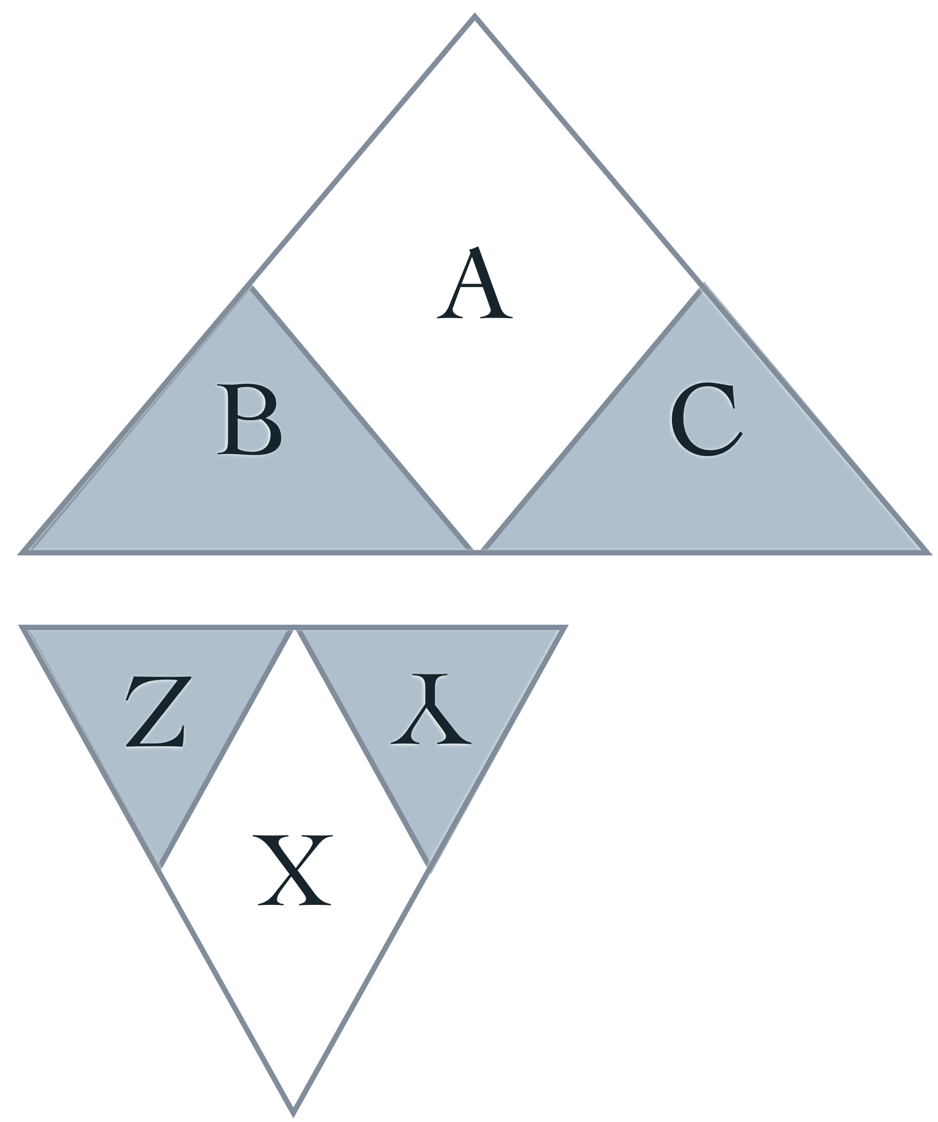
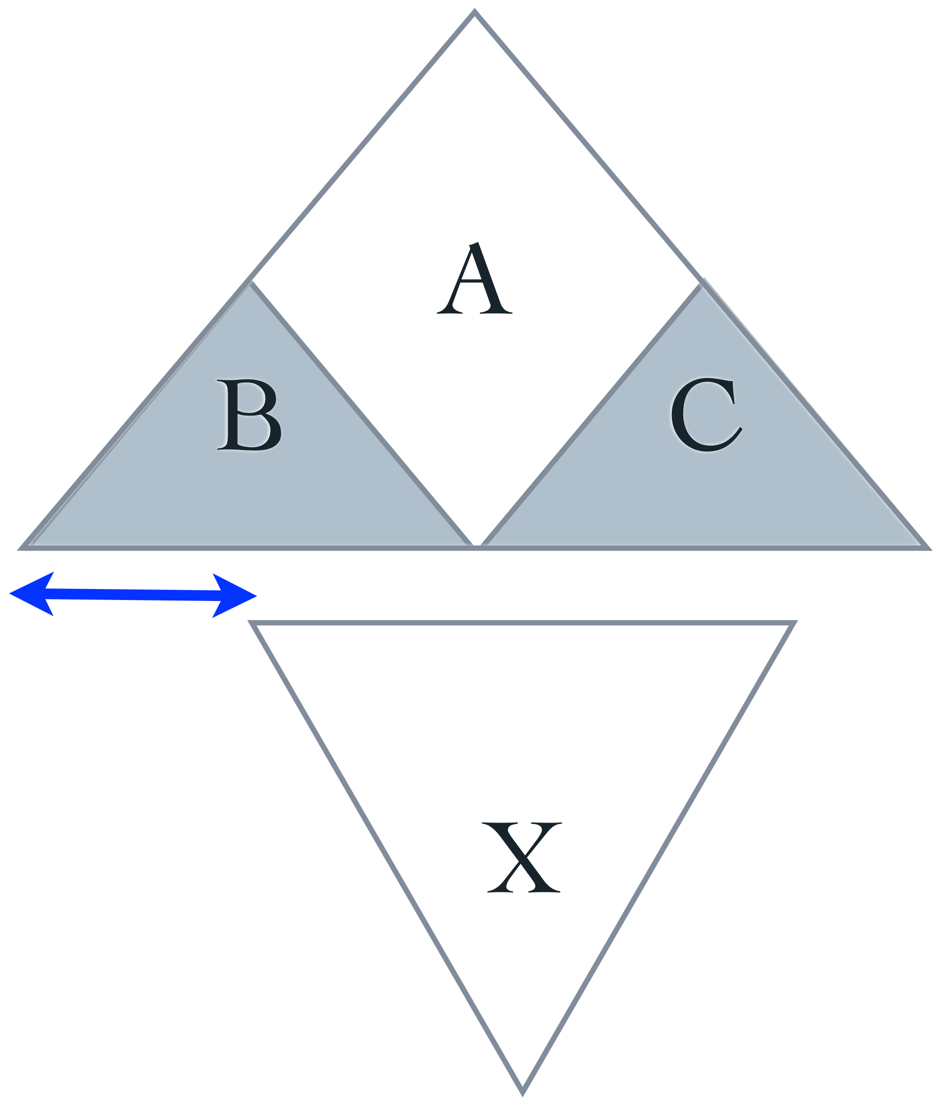


Operations on compressed strings

occurence test



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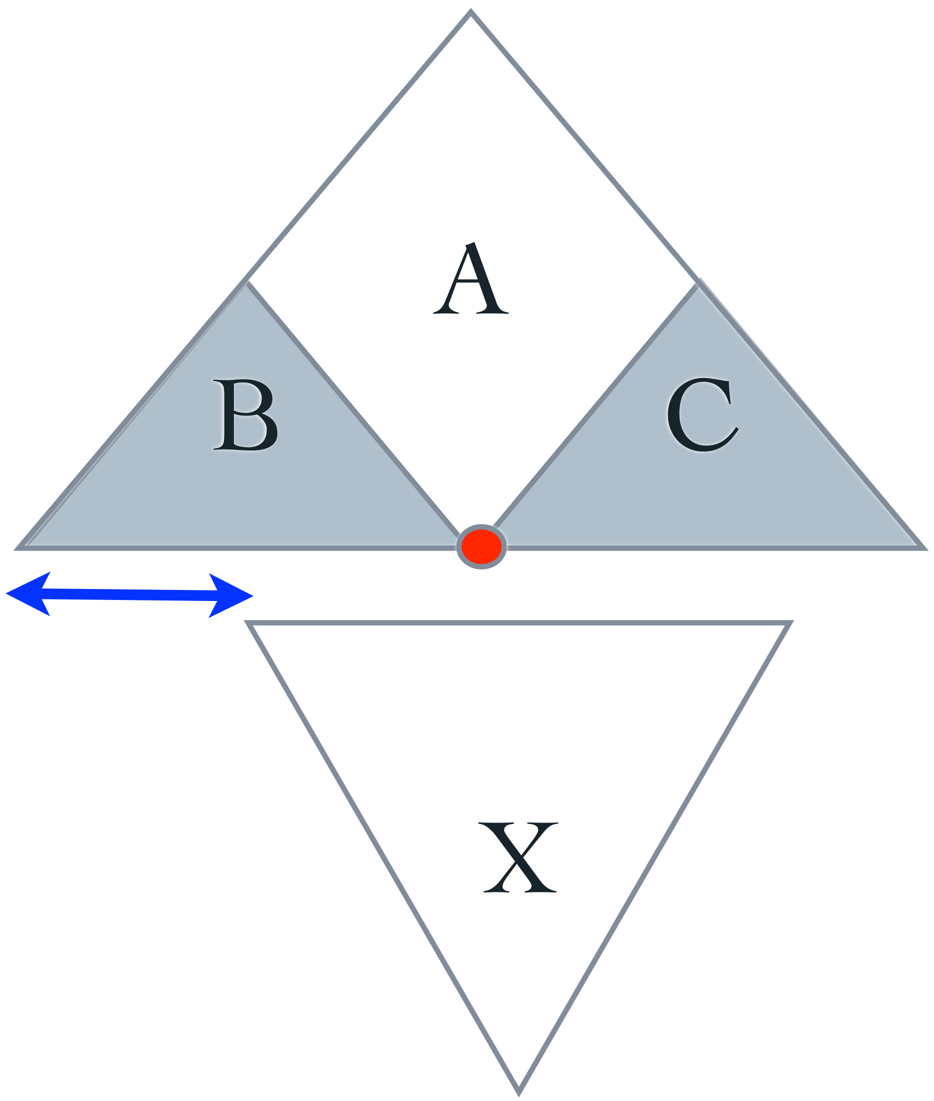

Operations on compressed strings

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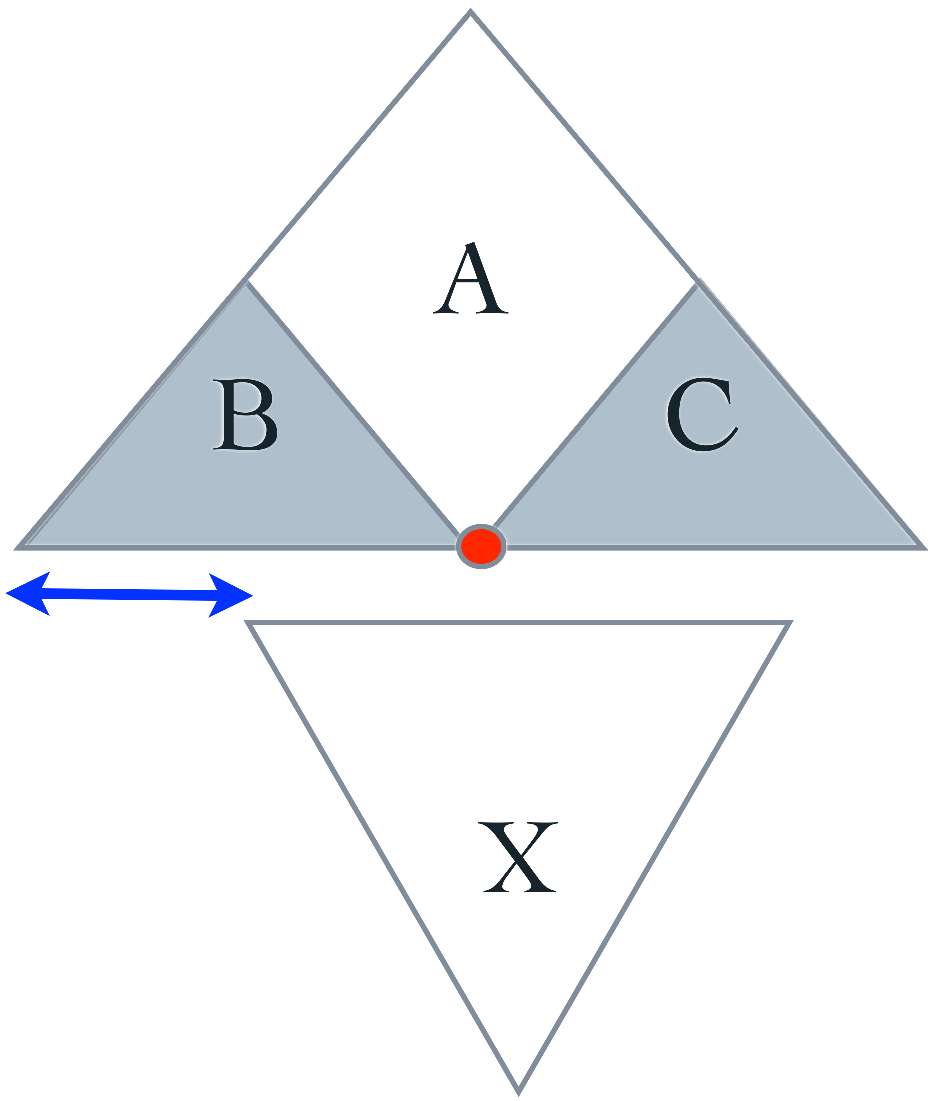

Fact:

The set of occurrences of X in A touching the cutting point is an arithmetic progression

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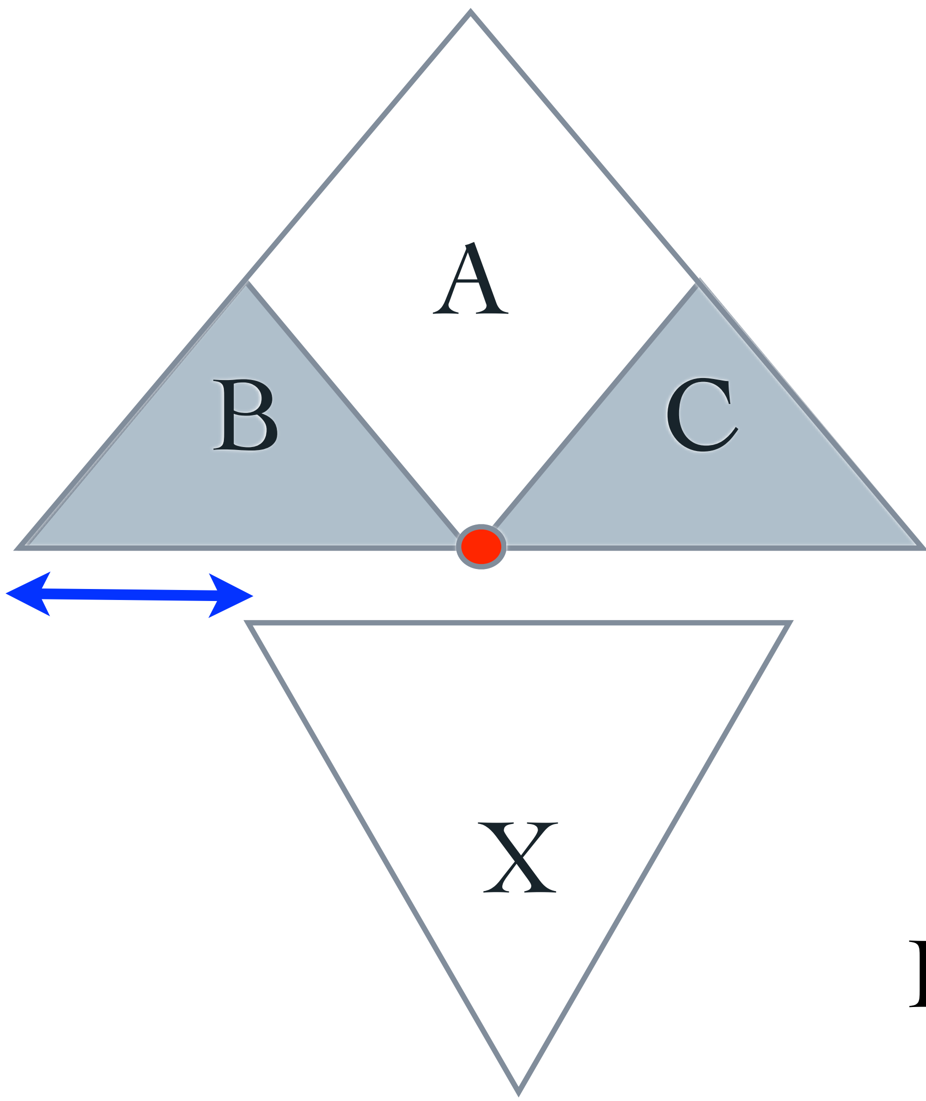

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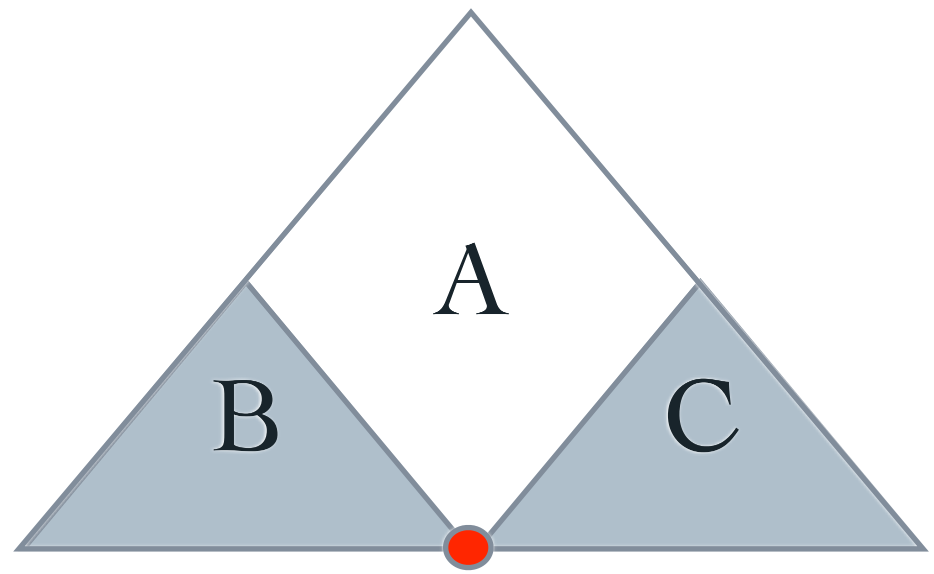
How to compute the occurrence table?

Computing occurrence table



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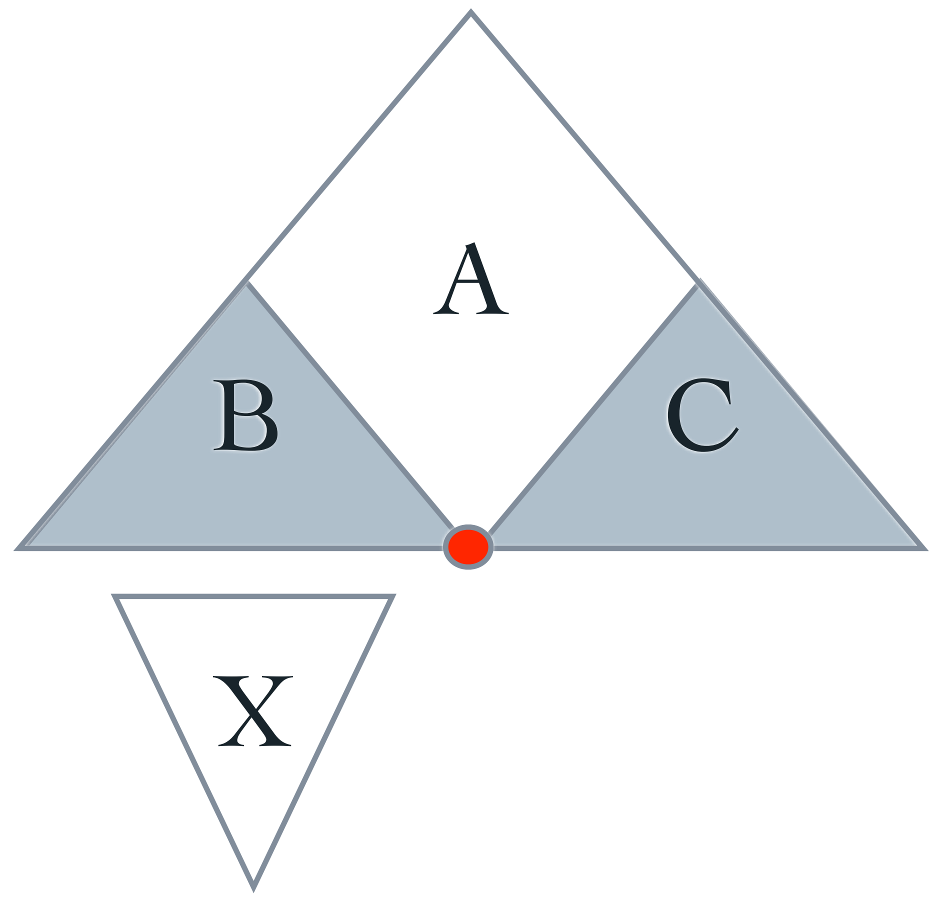

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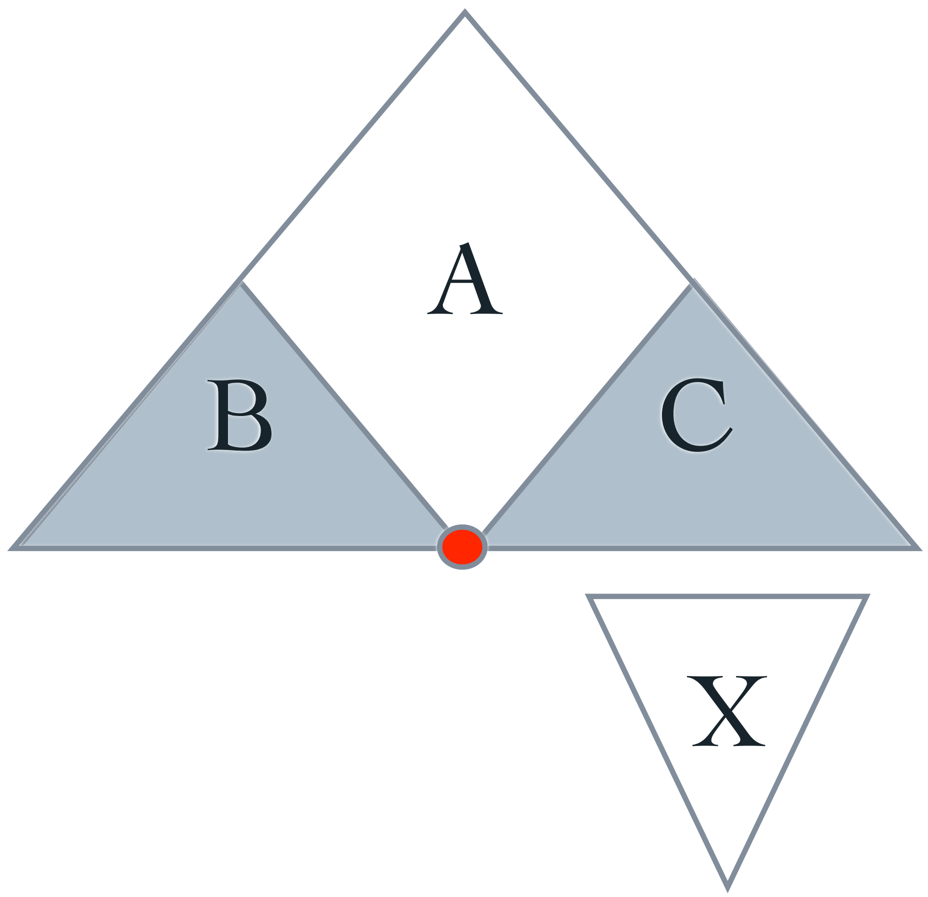

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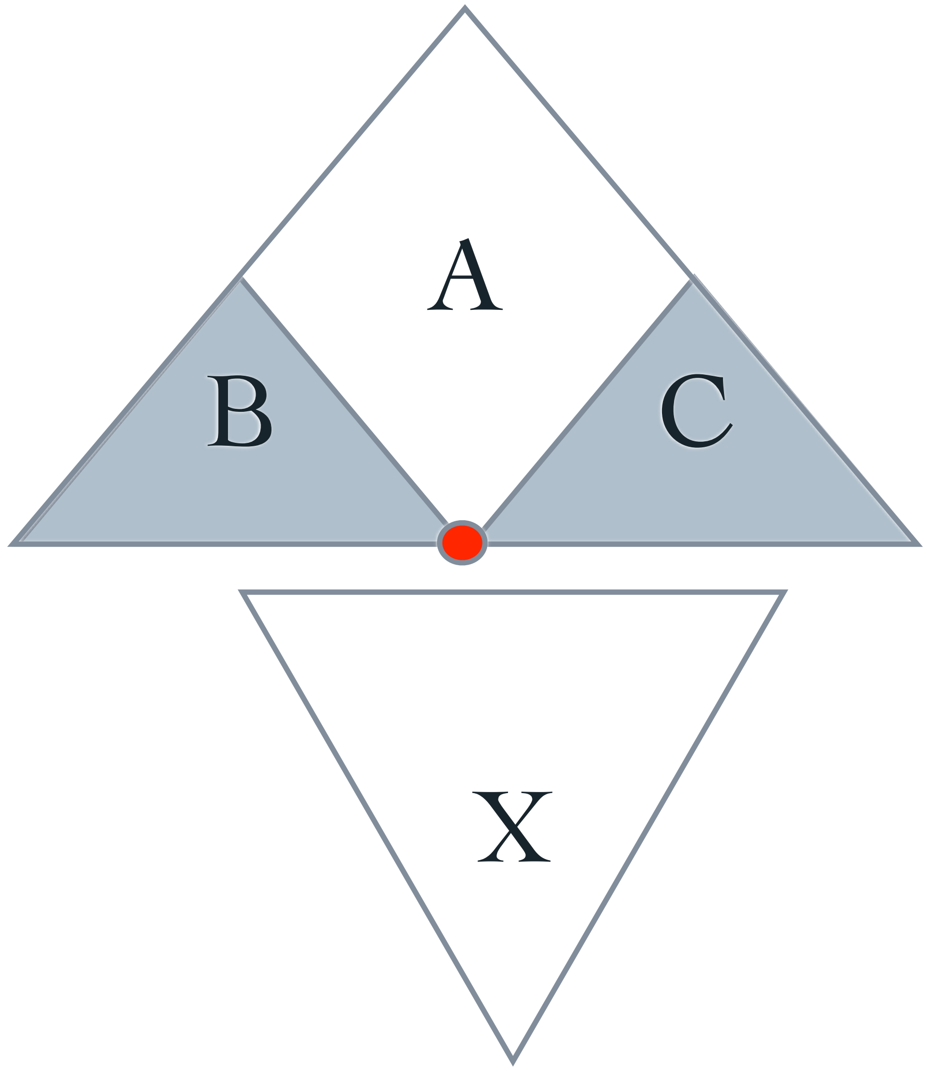

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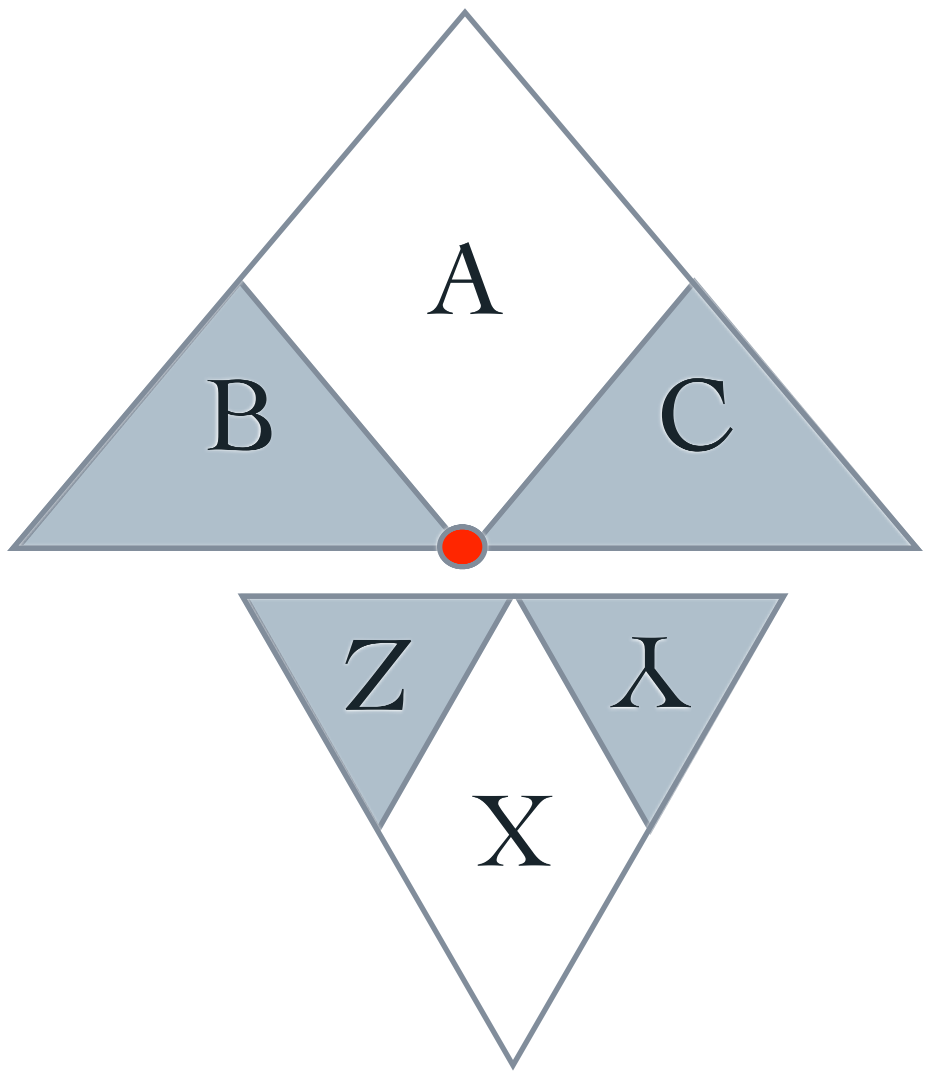

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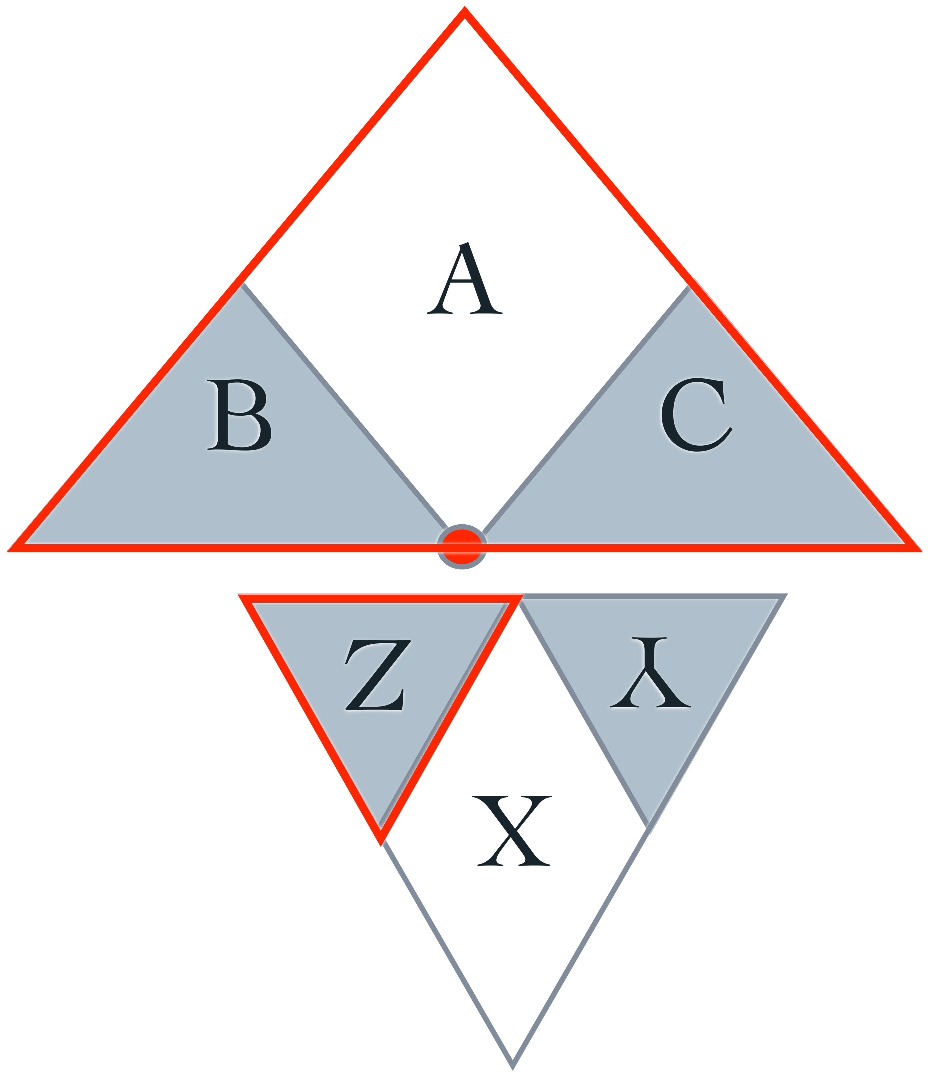

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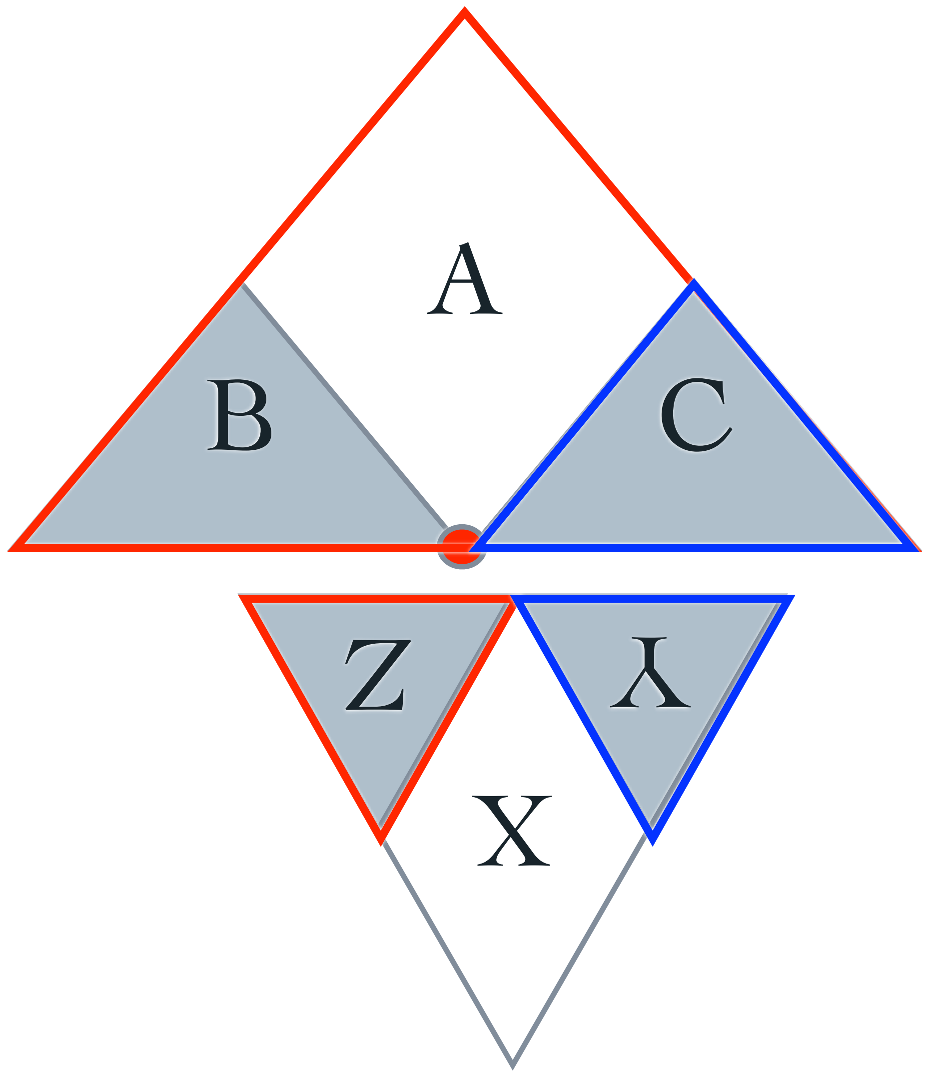

BPA and BPP differ
polynomial-time algorithms

A sequence of over-approximating congruences:

$$\equiv_{B_0} \supseteq \equiv_{B_1} \supseteq \equiv_{B_2} \supseteq \ldots \supseteq \equiv_{B_n} = \sim$$

yielded by an efficient refinement step:

$$B_i \mapsto \mathrm{ref}(B_i) = B_{i+1}$$

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- application of BPP approximation scheme to BPA
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- efficient operations on compressed strings

In consequence, there is essentially one algorithm for both BPA and BPP

Number of iterations

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$$B = (P, E) \qquad \mathrm{ref}(B) = (P', E')$$

$$\equiv_{\mathrm{ref}(B)} \subseteq \equiv_B \implies P \subseteq P'$$

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correctness: $\sim \;\subseteq\; \equiv_B$

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Refinement

Outline of the algorithm:

1. Compute the initial base B
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unique decomposition of $\equiv_B$

Outline

- Background
  - Context-free graphs and commutative context-free graphs
  - Bisimulation equivalence problem
  - Norm
  - History of the problem
  - Unique decomposition
  - Naive algorithm

- Efficient algorithm for BPA and BPP
  - Outline of the algorithm
  - **Refinement**
  - Efficient computation of refinement for BPA
  - Time-cost analysis
  - Partially-commutative context-free graphs

What refinement?

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$\equiv$ is a congruence with unique decomposition property

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$$\equiv \; \xcancel{\mapsto \; \equiv} \; \cap \exp(\equiv)$$

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Example:

$$
\begin{array}{cccccccccccc}
X_1 & \xrightarrow{a} & \varepsilon & X_2 & \xrightarrow{a} & X_1 & X_2 & \xrightarrow{b} & X_1X_1 & X_3 & \xrightarrow{a} & X_2 \\
X_4 & \xrightarrow{a} & X_3X_2 & X_4 & \xrightarrow{b} & X_4 & X_5 & \xrightarrow{a} & X_4 & X_5 & \xrightarrow{a} & X_3X_3
\end{array}
$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

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$$X_1 \text{ and } X_2 \text{ are primes wrt. } \;\equiv \cap \exp(\equiv)$$

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let $\alpha \equiv \beta \iff |\alpha| = |\beta|$

X_1 and X_2 are primes wrt. $\equiv \cap \exp(\equiv)$ $\qquad (X_2,\, X_1 X_1) \notin \exp(\equiv)$

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X_1 and X_2 are primes wrt. $\equiv \cap \exp(\equiv)$

$$(X_1\, X_1\, X_1,\ X_1\, X_2) \in \equiv \cap \exp(\equiv)$$

Right refinement

$$\boxed{\equiv \;\mapsto\; \mathbf{gnrb}(\equiv \cap \exp(\equiv))}$$

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greatest
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Norm-reducing bisimulation

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norm-reducing transitions:

$$\alpha \xrightarrow{a} \beta \qquad |\beta| < |\alpha| \qquad\qquad |\beta| = |\alpha| - 1$$

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norm-reducing bisimulation expansion n-r-exp(_)

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R is a n-r-bisimulation if $R \subseteq$ n-r-exp(R)

Norm-reducing bisimulation

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norm-reducing bisimulation expansion n-r-exp(_)

R is a n-r-bisimulation if $R \subseteq$ n-r-exp(R)

Greatest norm-reducing bisimulation contained in $\equiv$:

$$\mathbf{gnrb}(\equiv) \stackrel{\text{def}}{=} \bigcup \{R : R \text{ is a norm-reducing bisimulation, } R \subseteq \equiv\}$$

Right refinement

$$\mathbf{gnrb}(\equiv \ \cap \exp(\equiv))$$

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Adaptation of Bisimulation Game:

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- positions always belong to $\equiv$

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Adaptation of Bisimulation Game:

- positions always belong to $\equiv$ (otherwise Spoiler wins)

$$\alpha \equiv \beta$$

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Adaptation of Bisimulation Game:

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Adaptation of Bisimulation Game:

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  (otherwise Spoiler wins)
- Spoiler may freely use norm-reducing transitions
- Spoiler may use an arbitrary transition once,

  thus declaring the move to be the last one

Correctness of refinement

$$\boxed{\equiv \;\mapsto\; \mathbf{gnrb}(\equiv \cap \exp(\equiv))}$$

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Theorem:

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Proof (both for BPA and BPP)

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a crucial technical observation

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Proof for BPA:

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Proof for BPA:

Existence of prime decomposition by induction on norm

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$\mathbf{gnrb}(\equiv\ \cap \exp(\equiv))$ has unique decomposition property

Proof for BPA:

Existence of prime decomposition by induction on norm

Cancellation: $\alpha\gamma \sim \beta\gamma \implies \alpha \sim \beta$

Lemma:

$\mathbf{gnrb}(\equiv \ \cap \exp(\equiv))$ has unique decomposition property

Proof for BPA:

Existence of prime decomposition by induction on norm

Cancellation: $\alpha\gamma \ \sim \ \beta\gamma \quad \Longrightarrow \quad \alpha \ \sim \ \beta$

$\{(\alpha, \beta) \ : \ (\alpha\gamma, \beta\gamma) \ \in \ \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))\}$

is a norm-reducing bisimulation contained in $\equiv \ \cap \exp(\equiv)$

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Proof for BPA:

Existence of prime decomposition by induction on norm

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Uniqueness of prime decomposition by induction on norm

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Uniqueness of prime decomposition by induction on norm

$(P_1 \ldots P_n, Q_1 \ldots Q_m) \in \mathrm{gnrb}(\equiv\ \cap \exp(\equiv))$

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Uniqueness of prime decomposition by induction on norm

$(P_1 \ldots P_n, Q_1 \ldots Q_m) \;\in\; \mathrm{gnrb}(\equiv \; \cap \exp(\equiv))$

$n, m \neq 1$

Lemma:

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Proof for BPA:

Existence of prime decomposition by induction on norm

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$n, m \neq 1$

$P_n = Q_m$ by induction assumption

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Proof for BPA:

Existence of prime decomposition by induction on norm

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Uniqueness of prime decomposition by induction on norm

$(P_1 \ \ldots \ P_n, Q_1 \ \ldots \ Q_m) \ \in \ \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))$

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the lemma follows by cancellation and induction assumption

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$\mathbf{gnrb}(\equiv\ \cap \exp(\equiv))$ has unique decomposition property

Proof for BPP:

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Existence of prime decomposition by induction on norm

Uniqueness of prime decomposition by induction on norm

$$(X_1^{a_1} \ldots X_n^{a_n}, X_1^{b_1} \ldots X_n^{b_n}) \ \in \ \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))$$

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Existence of prime decomposition by induction on norm

Uniqueness of prime decomposition by induction on norm

$(X_1^{a_1} \ldots X_n^{a_n}, X_1^{b_1} \ldots X_n^{b_n}) \in \text{gnrb}(\equiv\ \cap \exp(\equiv))$ $\quad |X_1| \leq |X_2| \leq \ldots \leq |X_n|$

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Uniqueness of prime decomposition by induction on norm

$(X_1^{a_1} \dots X_n^{a_n}, X_1^{b_1} \dots X_n^{b_n}) \ \in \ \mathrm{gnrb}(\equiv\ \cap \exp(\equiv))$ $\qquad |X_1| \ \leq \ |X_2| \ \leq \ \dots \ \leq \ |X_n|$

let $m = \max\{i \ : \ a_i \neq b_i\}$

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Proof for BPP:

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$(X_1^{a_1} \ \ldots \ X_n^{a_n}, X_1^{b_1} \ \ldots \ X_n^{b_n}) \ \in \ \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))$ $\quad |X_1| \ \leq \ |X_2| \ \leq \ \ldots \ \leq \ |X_n|$

let $m = \max\{i \ : \ a_i \neq b_i\}$ $\quad a_m > b_m$

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$(X_1^{a_1}\ \ldots X_n^{a_n}, X_1^{b_1}\ \ldots X_n^{b_n}) \in \text{gnrb}(\equiv\ \cap \exp(\equiv))$ $\quad |X_1| \leq |X_2| \leq \ldots \leq |X_n|$

let $m = \max\{i\ :\ a_i \neq b_i\}$ $\quad a_m > b_m$

Case I: $\quad a_i > 0$ for some $i < m$

Lemma:

$\mathbf{gnrb}(\equiv\ \cap \exp(\equiv))$ has unique decomposition property

Proof for BPP:

Existence of prime decomposition by induction on norm

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let $m = \max\{i : a_i \neq b_i\}$ $\quad a_m > b_m$

Case I: $\quad a_i > 0$ for some $i < m$ $\quad$ Spoiler moves on the left

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$(X_1^{a_1} \ \ldots X_n^{a_n}, X_1^{b_1} \ \ldots X_n^{b_n}) \ \in \ \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))$ $\quad |X_1| \ \leq \ |X_2| \ \leq \ \ldots \ \leq \ |X_n|$

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Case I: $\quad a_i > 0$ for some $i < m$ $\quad$ Spoiler moves on the left

Case II: $\quad a_i > 0$ for some $i > m$

Lemma:

$\mathbf{gnrb}(\equiv \ \cap \exp(\equiv))$ has unique decomposition property

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Existence of prime decomposition by induction on norm

Uniqueness of prime decomposition by induction on norm

$(X_1^{a_1} \ldots X_n^{a_n}, X_1^{b_1} \ldots X_n^{b_n}) \in \text{gnrb}(\equiv \ \cap \exp(\equiv))$ $\quad |X_1| \leq |X_2| \leq \ldots \leq |X_n|$

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Existence of prime decomposition by induction on norm

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$(X_1^{a_1} \ldots X_n^{a_n}, X_1^{b_1} \ldots X_n^{b_n}) \ \in \ \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))$ $\quad |X_1| \ \leq \ |X_2| \ \leq \ \ldots \ \leq \ |X_n|$

let $m = \max\{i \ : \ a_i \neq b_i\}$ $\quad a_m > b_m$

Case I: $a_i > 0$ for some $i < m$ Spoiler moves on the left

Case II: $a_i > 0$ for some $i > m$ Spoiler moves on the left

Case III: $a_m \geq 2$

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Existence of prime decomposition by induction on norm

Uniqueness of prime decomposition by induction on norm

$(X_1^{a_1} \ldots X_n^{a_n}, X_1^{b_1} \ldots X_n^{b_n}) \in \text{gnrb}(\equiv\ \cap \exp(\equiv))$ $\quad |X_1| \leq |X_2| \leq \ldots \leq |X_n|$

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Case II: $a_i > 0$ for some $i > m$ Spoiler moves on the left

Case III: $a_m \geq 2$

$b_m = 0$

Lemma:

$\mathbf{gnrb}(\equiv \ \cap \exp(\equiv))$ has unique decomposition property

Proof for BPP:

Existence of prime decomposition by induction on norm

Uniqueness of prime decomposition by induction on norm

$(X_1^{a_1} \ldots X_n^{a_n}, X_1^{b_1} \ldots X_n^{b_n}) \in \mathrm{gnrb}(\equiv \ \cap \exp(\equiv))$ $\quad |X_1| \leq |X_2| \leq \ldots \leq |X_n|$

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One common algorithm.

What about one common proof?

$$\begin{tikzcd} \text{unique decomposition} \arrow[d, Rightarrow, shift left=3, "\text{BPP}"] \\ \text{cancellation} \arrow[u, Rightarrow, shift left=3, "\text{BPA}"] \end{tikzcd}$$

One common algorithm.

What about one common proof?

possible in the setting of
partially-commmutative
context-free graphs

unique decomposition

BPA $\Uparrow$ $\Downarrow$ BPP

cancellation

Outline

- Background
    - Context-free graphs and commutative context-free graphs
    - Bisimulation equivalence problem
    - Norm
    - History of the problem
    - Unique decomposition
    - Naive algorithm
- Efficient algorithm for BPA and BPP
    - Outline of the algorithm
    - Refinement
    - **Efficient computation of refinement for BPA**
    - Time-cost analysis
    - Partially-commutative context-free graphs

Computation of refinement

$$\boxed{\equiv \;\mapsto\; \mathbf{gnrb}(\equiv \,\cap\, \exp(\equiv))}$$

Fact: $(\alpha, \beta) \in \mathbf{gnrb}(\equiv) \iff \alpha \equiv \beta$ and $(\alpha, \beta) \in \text{n-r-exp}(\mathbf{gnrb}(\equiv))$

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Corollary:

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$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

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initialise $B = (P, E)$ as the base of $\equiv_0$;
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Outline of the algorithm:

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$X_i \equiv_{B'} \alpha$

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Until now we dealt with both BPA and BPP

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From now on we restrict ourselves to BPA

Left-most prime factor (lpf)

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Def: $X_j \in P$ is left-most prime factor of X_i wrt. B

iff

$$X_i \;\equiv_B\; X_j\,\alpha, \quad \text{for some } \alpha \in (P \;\cap\; \{X_1 \ldots X_{i-1}\})^*$$

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Proof: Bisimulation Game for $X_i \equiv_B X_j \, \alpha$

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Spoiler's move: $X_i \xrightarrow{a} \alpha_i$

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$$\mathrm{dec}_B(\alpha_i) = \mathrm{dec}_B(\gamma)\,\mathrm{dec}_B(\alpha)$$

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$\texttt{lpftest}_{B'}(X_i, X_j)$ subroutine on the next slide

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$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$\texttt{lpftest}_{B'}(X_i, X_j)$ subroutine on the next slide

Fact: $\mathrm{lpf}_B(X_i) \leq \mathrm{lpf}_{B'}(X_i) < X_i$

Small number of candidates

When computing *B'*, restrict to candidate prime decompositions of the following form:

$$X_i = X_j\,\alpha \qquad \text{where} \quad \alpha = \text{suffix}_{|X_i|-|X_j|}(\text{dec}_{B'}(\alpha_i))$$
$$\text{dec}_{B'}(X_i) = X_j\,\alpha$$

$\alpha_i \in \{X_1 \ldots X_{i-1}\}^*$
$\text{dec}_{B'}(\alpha_i)$ defined

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Proof:

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$$\begin{aligned} X_i &= X_j\,\alpha \qquad \text{where} \quad \alpha = \text{suffix}_{|X_i|-|X_j|}(\text{dec}_{B'}(\alpha_i)) \\ \text{dec}_{B'}(X_i) &= X_j\,\alpha \end{aligned}$$

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$\text{lpf}_B(\text{lpf}_{B'}(X_i)) = \text{lpf}_B(X_i)$

Algorithm 2 Efficient algorithm for CFG

initialise $B = (P, E)$ as the base of $\equiv_0$;
$P' := P$;
repeat
  $E' := \emptyset$;
  for all $X_i \in (\{X_2, \ldots, X_n\} \setminus P)$ **do**
    if $\neg \texttt{lpftest}_{(P',E')}(X_i, \texttt{lpf}_{(P,E)}(X_i))$ **then**
      for all $X_j \in \{X_{\texttt{lpfindex}_{(P,E)}+1}, \ldots, X_{i-1}\} \cap (P' \setminus P)$ **do**
        if $\texttt{lpftest}_{(P',E')}(X_i, X_j)$ **then**
          $s := \texttt{dec}_{(P',E')}(\alpha_i)$;
          $E' := E' \cup \{(X_i, X_j\ \texttt{suffix}_{|X_i|-|X_j|}(s))\}$;
          break the inner for loop;
        end if
      end for
      if the inner for loop not broken **then**
        $P' := P' \cup \{X_i\}$;
      end if
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      $X := \texttt{lpf}_{(P,E)}(X_i)$;
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    end if
  end for
  $P := P'$; $E := E'$;
until P does not change

$\mathrm{lpf}_B(X_i) \leq \mathrm{lpf}_{B'}(X_i) < X_i$

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    end for
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?

before explaining lpftest,
let's look at an example

Example:

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

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$$X_i \xrightarrow{a_i} \alpha_i$$

Example:

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

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$X_i \xrightarrow{a_i} \alpha_i$

Initial base:

Example:

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

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Fix $\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$$X_i \xrightarrow{a_i} \alpha_i$$

Initial base:

$$P = \{ X_1 \}$$

$$\begin{aligned} E = \{ X_2 &= X_1 X_1 \\ X_3 &= X_1 X_1 X_1 \\ X_4 &= X_1 X_1 X_1 X_1 X_1 X_1 \\ X_5 &= X_1 X_1 X_1 X_1 X_1 X_1 X_1 \} \end{aligned}$$

Example:

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

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$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

first iteration

Example: first iteration

$X_1 \xrightarrow{a} \varepsilon \quad X_2 \xrightarrow{a} X_1 \quad X_2 \xrightarrow{b} X_1X_1 \quad X_3 \xrightarrow{a} X_2$

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$\alpha_1 = \varepsilon \quad \alpha_2 = X_1 \quad \alpha_3 = X_2 \quad \alpha_4 = X_3X_2 \quad \alpha_5 = X_3X_3$

$$P = \{\, X_1 \,\}$$

$$\begin{aligned} E = \{\, X_2 &= X_1\, X_1 \\ X_3 &= X_1\, X_1\, X_1 \\ X_4 &= X_1\, X_1\, X_1\, X_1\, X_1\, X_1 \\ X_5 &= X_1\, X_1\, X_1\, X_1\, X_1\, X_1\, X_1 \} \end{aligned}$$

Example:

first iteration

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$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$$\begin{aligned} P &= \{\, X_1 \,\} \\ E &= \{\, X_2 = X_1\,X_1 \\ &\quad\; X_3 = X_1\,X_1\,X_1 \\ &\quad\; X_4 = X_1\,X_1\,X_1\,X_1\,X_1\,X_1 \\ &\quad\; X_5 = X_1\,X_1\,X_1\,X_1\,X_1\,X_1\,X_1 \} \end{aligned}$$

$$\begin{aligned} P' &= \{\, X_1, X_2, X_5 \,\} \\ E' &= \{\, X_3 = X_1\,X_2 \\ &\quad\; X_4 = X_2\,X_2\,X_2 \,\} \end{aligned}$$

Example:

first iteration

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$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$$P = \{ X_1 \}$$

$$\begin{aligned} E = \{ X_2 &= X_1\,X_1 \\ X_3 &= X_1\,X_1\,X_1 \\ X_4 &= X_1\,X_1\,X_1\,X_1\,X_1\,X_1 \\ X_5 &= X_1\,X_1\,X_1\,X_1\,X_1\,X_1\,X_1 \} \end{aligned}$$

$$X_2 \equiv_{B'} X_1\,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1\,X_1 \;?$$

$$P' = \{ X_1, X_2, X_5 \}$$

$$\begin{aligned} E' = \{ X_3 &= X_1\,X_2 \\ X_4 &= X_2\,X_2\,X_2 \} \end{aligned}$$

Example:

first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

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$$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1 \; ?$$

$X_i \equiv_{B'} \alpha$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2, \; X_4 = X_2 X_2 X_2 \}$$

Example:

first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

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$$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1 \ ?$$ **no!** (b)

$X_i \equiv_{B'} \alpha$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

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$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2, \; X_4 = X_2 X_2 X_2 \}$$

Example:

first iteration

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

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$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$P = \{ X_1 \}$

$E = \{ X_2 = X_1\,X_1$
$X_3 = X_1\,X_1\,X_1$
$X_4 = X_1\,X_1\,X_1\,X_1\,X_1\,X_1$
$X_5 = X_1\,X_1\,X_1\,X_1\,X_1\,X_1\,X_1 \}$

$X_2 \equiv_{B'} X_1\,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1\,X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1\,\mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1\,X_2$?

$X_i \equiv_{B'} \alpha$
$(a) \quad (X_i, \alpha) \in \equiv_B$
$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$
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$P' = \{ X_1, X_2, X_5 \}$

$E' = \{ X_3 = X_1\,X_2$
$X_4 = X_2\,X_2\,X_2 \}$

Example: first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$
$$E = \{ X_2 = X_1 X_1, \; X_3 = X_1 X_1 X_1, \; X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \; X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 \, X_1 \; ?$$ no! (b)

$$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 \, X_2 \; ?$$ yes!

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr\!-\!exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$
$$E' = \{ X_3 = X_1 \, X_2, \; X_4 = X_2 \, X_2 \, X_2 \}$$

Example: first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$

$$E = \{ X_2 = X_1 X_1, \quad X_3 = X_1 X_1 X_1, \quad X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \quad X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$X_2 \equiv_{B'} X_1 \,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? no! (b)

$X_3 \equiv_{B'} X_1 \,\mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? yes!

$X_3 \equiv_{B'} X_2 \,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$?

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr\!-\!exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2, \quad X_4 = X_2 X_2 X_2 \}$$

Example:

first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$

$$E = \{ X_2 = X_1 X_1, \quad X_3 = X_1 X_1 X_1, \quad X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \quad X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 \, X_1 \; ?$$ no! (b)

$$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 \, X_2 \; ?$$ yes!

$$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3)) \; ?$$ undefined!

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2, \quad X_4 = X_2 X_2 X_2 \}$$

Example: first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$
$$E = \{ X_2 = X_1 X_1, \; X_3 = X_1 X_1 X_1, \; X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \; X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 \, X_1$? no! (b)

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 \, X_2$? yes!

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_4 \equiv_{B'} X_1 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 \, X_1 \, X_2 \, X_2$?

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$
$$E' = \{ X_3 = X_1 \, X_2, \; X_4 = X_2 \, X_2 \, X_2 \}$$

Example: **first iteration**

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$P = \{ X_1 \}$

$E = \{ X_2 = X_1 X_1$
$X_3 = X_1 X_1 X_1$
$X_4 = X_1 X_1 X_1 X_1 X_1 X_1$
$X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$

$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? **yes!**

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? **undefined!**

$X_4 \equiv_{B'} X_1 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 X_1 X_2 X_2$? **no!** (b)

$X_i \equiv_{B'} \alpha$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$P' = \{ X_1, X_2, X_5 \}$

$E' = \{ X_3 = X_1 X_2$
$X_4 = X_2 X_2 X_2 \}$

Example:

first iteration

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$P = \{ X_1 \}$

$E = \{ X_2 = X_1 X_1$
$X_3 = X_1 X_1 X_1$
$X_4 = X_1 X_1 X_1 X_1 X_1 X_1$
$X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$

$X_2 \equiv_{B'} X_1 \,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1 \,\mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? **yes!**

$X_3 \equiv_{B'} X_2 \,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? **undefined!**

$X_4 \equiv_{B'} X_1 \,\mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 X_1 X_2 X_2$? **no!** (b)

$X_4 \equiv_{B'} X_2 \,\mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$?

$X_i \equiv_{B'} \alpha$
$(a) \quad (X_i, \alpha) \in \equiv_B$
$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$
$(c) \quad (X_i, \alpha) \in \mathtt{nr\!-\!exp}(\equiv_{B'})$

$P' = \{ X_1, X_2, X_5 \}$

$E' = \{ X_3 = X_1 X_2$
$X_4 = X_2 X_2 X_2 \}$

Example: first iteration

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$P = \{ X_1 \}$

$E = \{ X_2 = X_1 X_1$
$X_3 = X_1 X_1 X_1$
$X_4 = X_1 X_1 X_1 X_1 X_1 X_1$
$X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$

$X_2 \equiv_{B'} X_1\, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? no! (b)

$X_3 \equiv_{B'} X_1\, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? yes!

$X_3 \equiv_{B'} X_2\, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_4 \equiv_{B'} X_1\, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 X_1 X_2 X_2$? no! (b)

$X_4 \equiv_{B'} X_2\, \mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$? yes!

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$P' = \{ X_1, X_2, X_5 \}$

$E' = \{ X_3 = X_1 X_2$
$X_4 = X_2 X_2 X_2 \}$

Example:

first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$P = \{ X_1 \}$

$E = \{ X_2 = X_1 X_1$
$X_3 = X_1 X_1 X_1$
$X_4 = X_1 X_1 X_1 X_1 X_1 X_1$
$X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$

$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 \, X_1$? no! (b)

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 \, X_2$? yes!

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_4 \equiv_{B'} X_1 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 \, X_1 \, X_2 \, X_2$? no! (b)

$X_4 \equiv_{B'} X_2 \, \mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 \, X_2 \, X_2$? yes!

$X_5 \equiv_{B'} X_1 \, \mathrm{suffix}_6(\mathrm{dec}_{B'}(\alpha_5)) = X_1 \, X_1 \, X_2 \, X_1 \, X_2$?

$X_i \equiv_{B'} \alpha$
$(a) \quad (X_i, \alpha) \in \equiv_B$
$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$
$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$P' = \{ X_1, X_2, X_5 \}$

$E' = \{ X_3 = X_1 \, X_2$
$X_4 = X_2 \, X_2 \, X_2 \}$

Example: first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$P = \{ X_1 \}$
$E = \{ X_2 = X_1 X_1$
$X_3 = X_1 X_1 X_1$
$X_4 = X_1 X_1 X_1 X_1 X_1 X_1$
$X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$

$X_2 \equiv_{B'} X_1 \,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? no! (b)

$X_3 \equiv_{B'} X_1 \,\mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? yes!
$X_3 \equiv_{B'} X_2 \,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_4 \equiv_{B'} X_1 \,\mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 X_1 X_2 X_2$? no! (b)
$X_4 \equiv_{B'} X_2 \,\mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$? yes!

$X_5 \equiv_{B'} X_1 \,\mathrm{suffix}_6(\mathrm{dec}_{B'}(\alpha_5)) = X_1 X_1 X_2 X_1 X_2$? no! (c)

$X_i \equiv_{B'} \alpha$
(a) $(X_i, \alpha) \in \equiv_B$
(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$
(c) $(X_i, \alpha) \in \mathtt{nr}-\mathtt{exp}(\equiv_{B'})$

$P' = \{ X_1, X_2, X_5 \}$
$E' = \{ X_3 = X_1 X_2$
$X_4 = X_2 X_2 X_2 \}$

Example: first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$

$$E = \{ X_2 = X_1 X_1, \; X_3 = X_1 X_1 X_1, \; X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \; X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$X_2 \equiv_{B'} X_1\,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1\,X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1\,\mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1\,X_2$? **yes!**

$X_3 \equiv_{B'} X_2\,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? **undefined!**

$X_4 \equiv_{B'} X_1\,\mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1\,X_1\,X_2\,X_2$? **no!** (b)

$X_4 \equiv_{B'} X_2\,\mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2\,X_2\,X_2$? **yes!**

$X_5 \xrightarrow{a} X_4$

$X_5 \equiv_{B'} X_1\,\mathrm{suffix}_6(\mathrm{dec}_{B'}(\alpha_5)) = X_1\,X_1\,X_2\,X_1\,X_2$? **no!** (c)

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1\,X_2, \; X_4 = X_2\,X_2\,X_2 \}$$

Example: **first iteration**

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$

$$E = \{ X_2 = X_1 X_1,\ X_3 = X_1 X_1 X_1,\ X_4 = X_1 X_1 X_1 X_1 X_1 X_1,\ X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$X_2 \equiv_{B'} X_1\,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1\,X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1\,\mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1\,X_2$? **yes!**

$X_3 \equiv_{B'} X_2\,\mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? **undefined!**

$X_4 \equiv_{B'} X_1\,\mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1\,X_1\,X_2\,X_2$? **no!** (b)

$X_4 \equiv_{B'} X_2\,\mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2\,X_2\,X_2$? **yes!**

$$X_5 \xrightarrow{a} X_4$$

$$X_4 \not\equiv_{B'} X_1\,X_2\,X_1\,X_2$$

$X_5 \equiv_{B'} X_1\,\mathrm{suffix}_6(\mathrm{dec}_{B'}(\alpha_5)) = X_1\,X_1\,X_2\,X_1\,X_2$? **no!** (c)

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1\,X_2,\ X_4 = X_2\,X_2\,X_2 \}$$

Example: **first iteration**

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$
$$E = \{ X_2 = X_1 X_1, \; X_3 = X_1 X_1 X_1, \; X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \; X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? **yes!**

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? **undefined!**

$X_4 \equiv_{B'} X_1 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 X_1 X_2 X_2$? **no!** (b)

$X_4 \equiv_{B'} X_2 \, \mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$? **yes!**

$$X_5 \xrightarrow{a} X_4$$
$$X_4 \not\equiv_{B'} X_1 X_2 X_1 X_2$$

$X_5 \equiv_{B'} X_1 \, \mathrm{suffix}_6(\mathrm{dec}_{B'}(\alpha_5)) = X_1 X_1 X_2 X_1 X_2$? **no!** (c)

$X_5 \equiv_{B'} X_2 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_5)) = X_2 X_2 X_1 X_2$?

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$
$$E' = \{ X_3 = X_1 X_2, \; X_4 = X_2 X_2 X_2 \}$$

Example: first iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1 \}$$

$$E = \{ X_2 = X_1 X_1, \; X_3 = X_1 X_1 X_1, \; X_4 = X_1 X_1 X_1 X_1 X_1 X_1, \; X_5 = X_1 X_1 X_1 X_1 X_1 X_1 X_1 \}$$

$X_2 \equiv_{B'} X_1 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_2)) = X_1 X_1$? **no!** (b)

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? **yes!**

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? **undefined!**

$X_4 \equiv_{B'} X_1 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_4)) = X_1 X_1 X_2 X_2$? **no!** (b)

$X_4 \equiv_{B'} X_2 \, \mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$? **yes!**

$$X_5 \xrightarrow{a} X_4 \qquad X_4 \not\equiv_{B'} X_1 X_2 X_1 X_2$$

$X_5 \equiv_{B'} X_1 \, \mathrm{suffix}_6(\mathrm{dec}_{B'}(\alpha_5)) = X_1 X_1 X_2 X_1 X_2$? **no!** (c)

$X_5 \equiv_{B'} X_2 \, \mathrm{suffix}_5(\mathrm{dec}_{B'}(\alpha_5)) = X_2 X_2 X_1 X_2$? **no!** (b)

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2, \; X_4 = X_2 X_2 X_2 \}$$

Example:

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

Example:

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

second iteration

Example: second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1, X_2, X_5 \}$$

$$E = \{ X_3 = X_1 X_2$$

$$X_4 = X_2 X_2 X_2 \}$$

Example:

second iteration

$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$

$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$

$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$

$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$

$$P = \{ X_1, X_2, X_5 \}$$

$$E = \{ X_3 = X_1 X_2, \quad X_4 = X_2 X_2 X_2 \}$$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2 \}$$

Example: second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1, X_2, X_5 \}$$
$$E = \{ X_3 = X_1 X_2 \quad X_4 = X_2 X_2 X_2 \}$$

$$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2 \ ?$$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$
$$E' = \{ X_3 = X_1 X_2 \}$$

Example:

second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{\, X_1, X_2, X_5 \,\}$$

$$E = \{\, X_3 = X_1\, X_2, \quad X_4 = X_2\, X_2\, X_2 \,\}$$

$$X_3 \equiv_{B'} X_1\, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1\, X_2 \ ?$$

$X_i \equiv_{B'} \alpha$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{\, X_1, X_2, X_4, X_5 \,\}$$

$$E' = \{\, X_3 = X_1\, X_2 \,\}$$

Example: second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1, X_2, X_5 \}$$

$$E = \{ X_3 = X_1 X_2, \quad X_4 = X_2 X_2 X_2 \}$$

$$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2 \ ?$$ yes!

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2 \}$$

Example: **second iteration**

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1, X_2, X_5 \}$$
$$E = \{ X_3 = X_1 X_2, \quad X_4 = X_2 X_2 X_2 \}$$

$$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2 \ ?$$ yes!

$$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3)) \ ?$$

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$
$$E' = \{ X_3 = X_1 X_2 \}$$

Example: second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1 X_1 \qquad X_3 \xrightarrow{a} X_2$$
$$X_4 \xrightarrow{a} X_3 X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3 X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3 X_2 \qquad \alpha_5 = X_3 X_3$$

$$P = \{ X_1, X_2, X_5 \}$$
$$E = \{ X_3 = X_1 X_2, \quad X_4 = X_2 X_2 X_2 \}$$

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? yes!

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$
$$E' = \{ X_3 = X_1 X_2 \}$$

Example:

second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1, X_2, X_5 \}$$
$$E = \{ X_3 = X_1 X_2 \quad X_4 = X_2 X_2 X_2 \}$$

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? yes!

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_4 \equiv_{B'} X_2 \, \mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$?

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$
$$E' = \{ X_3 = X_1 X_2 \}$$

Example:

second iteration

$$X_1 \xrightarrow{a} \varepsilon \qquad X_2 \xrightarrow{a} X_1 \qquad X_2 \xrightarrow{b} X_1X_1 \qquad X_3 \xrightarrow{a} X_2$$

$$X_4 \xrightarrow{a} X_3X_2 \qquad X_4 \xrightarrow{b} X_4 \qquad X_5 \xrightarrow{a} X_4 \qquad X_5 \xrightarrow{a} X_3X_3$$

$$|X_1| = 1 \quad |X_2| = 2 \quad |X_3| = 3 \quad |X_4| = 6 \quad |X_5| = 7$$

$$\alpha_1 = \varepsilon \qquad \alpha_2 = X_1 \qquad \alpha_3 = X_2 \qquad \alpha_4 = X_3X_2 \qquad \alpha_5 = X_3X_3$$

$$P = \{ X_1, X_2, X_5 \}$$

$$E = \{ X_3 = X_1 X_2 \quad X_4 = X_2 X_2 X_2 \}$$

$X_3 \equiv_{B'} X_1 \, \mathrm{suffix}_2(\mathrm{dec}_{B'}(\alpha_3)) = X_1 X_2$? yes!

$X_3 \equiv_{B'} X_2 \, \mathrm{suffix}_1(\mathrm{dec}_{B'}(\alpha_3))$? undefined!

$X_4 \equiv_{B'} X_2 \, \mathrm{suffix}_4(\mathrm{dec}_{B'}(\alpha_4)) = X_2 X_2 X_2$? no! (b) and (c)

$$X_4 \xrightarrow{b} X_4 \equiv_B X_2 X_2 X_2$$

$$X_2 X_2 X_2 \xrightarrow{b} X_1 X_1 X_2 X_2$$

$X_i \equiv_{B'} \alpha$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$P' = \{ X_1, X_2, X_4, X_5 \}$$

$$E' = \{ X_3 = X_1 X_2 \}$$

Outline

- Background
  - Context-free graphs and commutative context-free graphs
  - Bisimulation equivalence problem
  - Norm
  - History of the problem
  - Unique decomposition
  - Naive algorithm
- Efficient algorithm for BPA and BPP
  - Outline of the algorithm
  - Refinement
  - Efficient computation of refinement for BPA
  - **Time-cost analysis**
  - Partially-commutative context-free graphs

Algorithm 2 Efficient algorithm for CFG

initialise $B = (P, E)$ as the base of $\equiv_0$;
$P' := P$;
repeat
$E' := \emptyset$;
for all $X_i \in (\{X_2, \ldots, X_n\} \setminus P)$ **do**
if $\neg \texttt{lpftest}_{(P',E')}(X_i, \texttt{lpf}_{(P,E)}(X_i))$ **then**
for all $X_j \in \{X_{\texttt{lpfindex}_{(P,E)}+1}, \ldots, X_{i-1}\} \cap (P' \setminus P)$ **do**
if $\texttt{lpftest}_{(P',E')}(X_i, X_j)$ **then**
$s := \texttt{dec}_{(P',E')}(\alpha_i)$;
$E' := E' \cup \{(X_i, X_j\ \texttt{suffix}_{|X_i|-|X_j|}(s))\}$;
break the inner for loop;
end if
end for
if the inner for loop not broken **then**
$P' := P' \cup \{X_i\}$;
end if
else
$X := \texttt{lpf}_{(P,E)}(X_i)$;
$s := \texttt{dec}_{(P',E')}(\alpha_i)$;
$E' := E' \cup \{(X_i, X\ \texttt{suffix}_{|X_i|-|X|}(s))\}$;
end if
end for
$P := P'$; $E := E'$;
until P does not change

$\mathcal{O}(n^2)$ invocations

Algorithm 2 Efficient algorithm for CFG

initialise $B = (P, E)$ as the base of $\equiv_0$;
$P' := P$;
repeat
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  for all $X_i \in (\{X_2, \dots, X_n\} \setminus P)$ **do**
    if $\neg \texttt{lpftest}_{(P',E')}(X_i, \texttt{lpf}_{(P,E)}(X_i))$ **then**
      for all $X_j \in \{X_{\texttt{lpfindex}_{(P,E)}+1}, \dots, X_{i-1}\} \cap (P' \setminus P)$ **do**
        if $\texttt{lpftest}_{(P',E')}(X_i, X_j)$ **then**
          $s := \texttt{dec}_{(P',E')}(\alpha_i)$;
          $E' := E' \cup \{(X_i, X_j\ \texttt{suffix}_{|X_i|-|X_j|}(s))\}$;
          break the inner for loop;
        end if
      end for
      if the inner for loop not broken **then**
        $P' := P' \cup \{X_i\}$;
      end if
    else
      $X := \texttt{lpf}_{(P,E)}(X_i)$;
      $s := \texttt{dec}_{(P',E')}(\alpha_i)$;
      $E' := E' \cup \{(X_i, X\ \texttt{suffix}_{|X_i|-|X|}(s))\}$;
    end if
  end for
  $P := P'$; $E := E'$;
until P does not change

$\mathcal{O}(n^2)$ invocations

lpftest

$$X_j \; = \; \mathrm{lpf}_{B'}(X_i)$$

$$X_i \;\; \equiv_{B'} \;\; \alpha$$

$$\alpha \;\; = \;\; X_j \, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$(a) \quad (X_i, \alpha) \in \, \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string "a"
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
- $\textsc{Equal}(s_1, s_2)$: equality test

$$X_i \equiv_{B'} \alpha$$
$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string "a"
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$$X_i \equiv_{B'} \alpha$$
$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

Example:

$$\begin{aligned}
s_0 &:= \textsc{String}(a)\\
tmp &:= \textsc{String}(b)\\
s_1 &:= \textsc{Concatenate}(s_0, tmp)\\
s_2 &:= \textsc{Concatenate}(s_1, s_0)\\
s_3 &:= \textsc{Concatenate}(s_2, s_1)\\
&\ldots
\end{aligned}$$

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string "a"
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
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$$X_i \equiv_{B'} \alpha$$
$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

Example:

$$\begin{aligned} s_0 &:= \textsc{String}(a) \\ tmp &:= \textsc{String}(b) \\ s_1 &:= \textsc{Concatenate}(s_0, tmp) \\ s_2 &:= \textsc{Concatenate}(s_1, s_0) \\ s_3 &:= \textsc{Concatenate}(s_2, s_1) \\ &\dots \end{aligned}$$

ab

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$
$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string "a"
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
- $\textsc{Equal}(s_1, s_2)$: equality test

Example:

$$\begin{aligned}
s_0 &:= \textsc{String}(a)\\
tmp &:= \textsc{String}(b)\\
s_1 &:= \textsc{Concatenate}(s_0, tmp) && \text{ab}\\
s_2 &:= \textsc{Concatenate}(s_1, s_0) && \text{aba}\\
s_3 &:= \textsc{Concatenate}(s_2, s_1)\\
&\dots
\end{aligned}$$

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$
$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string "a"
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
- $\textsc{Equal}(s_1, s_2)$: equality test

Example:

$$\begin{aligned}
s_0 &:= \textsc{String}(a) \\
tmp &:= \textsc{String}(b) \\
s_1 &:= \textsc{Concatenate}(s_0, tmp) && \text{ab} \\
s_2 &:= \textsc{Concatenate}(s_1, s_0) && \text{aba} \\
s_3 &:= \textsc{Concatenate}(s_2, s_1) && \text{abaab} \\
&\dots
\end{aligned}$$

lpftest

$$X_j \;=\; \mathrm{lpf}_{B'}(X_i)$$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string $"a"$
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
- $\textsc{Equal}(s_1, s_2)$: equality test

$$X_i \;\equiv_{B'}\; \alpha$$
$$\alpha \;=\; X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $\quad (X_i, \alpha) \in \equiv_B$

(b) $\quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $\quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

Example:

$$\begin{array}{rcll} s_0 & := & \textsc{String}(a) & \\ tmp & := & \textsc{String}(b) & \\ s_1 & := & \textsc{Concatenate}(s_0, tmp) & \text{ab} \\ s_2 & := & \textsc{Concatenate}(s_1, s_0) & \text{aba} \\ s_3 & := & \textsc{Concatenate}(s_2, s_1) & \text{abaab} \\ \ldots & & & \text{abaababa} \end{array}$$

lpftest

$$X_j \;=\; \mathrm{lpf}_{B'}(X_i)$$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string $''a''$
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
- $\textsc{Equal}(s_1, s_2)$: equality test

$$X_i \;\equiv_{B'}\; \alpha$$
$$\alpha \;=\; X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$(a) \quad (X_i, \alpha) \in\; \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

Example:

$$\begin{array}{rcl} s_0 & := & \textsc{String}(a) \\ tmp & := & \textsc{String}(b) \\ s_1 & := & \textsc{Concatenate}(s_0, tmp) \\ s_2 & := & \textsc{Concatenate}(s_1, s_0) \\ s_3 & := & \textsc{Concatenate}(s_2, s_1) \\ \ldots & & \end{array}$$

ab
aba
abaab
abaababa
abaababaabaab

lpftest

$$X_j \;=\; \mathrm{lpf}_{B'}(X_i)$$

$X_i \;\equiv_{B'}\; \alpha$

$\alpha \;=\; X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$

$(a) \quad (X_i, \alpha) \in\, \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

- $\textsc{String}(a)$ for $a \in \Sigma$: creates a string "a"
- $\textsc{Concatenate}(s_1, s_2)$:
- $\textsc{Split}(s, i)$: splits into a prefix and suffix
- $\textsc{Equal}(s_1, s_2)$: equality test

Example:

$$\begin{array}{rcll}
s_0 & := & \textsc{String}(a) & \\
tmp & := & \textsc{String}(b) & \\
s_1 & := & \textsc{Concatenate}(s_0, tmp) & \text{ab} \\
s_2 & := & \textsc{Concatenate}(s_1, s_0) & \text{aba} \\
s_3 & := & \textsc{Concatenate}(s_2, s_1) & \text{abaab} \\
\ldots & & & \text{abaababa} \\
 & & & \text{abaababaabaab} \\
 & & & \ldots
\end{array}$$

lpftest

$$X_j \; = \; \mathrm{lpf}_{B'}(X_i)$$

$$X_i \;\; \equiv_{B'} \;\; \alpha$$

$$\alpha \;\; = \;\; X_j\, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr}{-}\mathtt{exp}(\equiv_{B'})$

lpftest

$$X_j \;=\; \mathrm{lpf}_{B'}(X_i)$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N \text{ polylog}(N))$;
- SPLIT: $\mathcal{O}(N \text{ polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$

[Alstrup, Brodal, Rauhe 2000]

$$X_i \;\equiv_{B'}\; \alpha$$
$$\alpha \;=\; X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N \text{ polylog}(N))$;
- SPLIT: $\mathcal{O}(N \text{ polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$ [Alstrup, Brodal, Rauhe 2000]

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- exponential compression

lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N \text{ polylog}(N))$;
- SPLIT: $\mathcal{O}(N \text{ polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$

$X_i \equiv_{B'} \alpha$

$\alpha = X_j \, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

[Alstrup, Brodal, Rauhe 2000]

- exponential compression
- efficient manipulation without decompression

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$(a) \quad (X_i, \alpha) \in \equiv_B$$

$$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$$

$$(c) \quad (X_i, \alpha) \in \mathtt{nr}-\mathtt{exp}(\equiv_{B'})$$

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N \text{ polylog}(N))$;
- Split: $\mathcal{O}(N \text{ polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N \text{ polylog}(N))$;
- SPLIT: $\mathcal{O}(N \text{ polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N \text{ polylog}(N))$;
- Split: $\mathcal{O}(N \text{ polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

EQUAL

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N \text{ polylog}(N))$;
- SPLIT: $\mathcal{O}(N \text{ polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

EQUAL $\mathcal{O}(\log N)$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N \text{ polylog}(N))$;
- SPLIT: $\mathcal{O}(N \text{ polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$X_i \equiv_{B'} \alpha$

$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$

$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$

Split, Concatenate

Equal $\mathcal{O}(\log N)$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N \text{ polylog}(N))$;
- Split: $\mathcal{O}(N \text{ polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

SPLIT, CONCATENATE	$\mathcal{O}(N\,\mathrm{polylog}(N))$
EQUAL	$\mathcal{O}(\log N)$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

SPLIT, CONCATENATE $\mathcal{O}(N\,\mathrm{polylog}(N))$

EQUAL $\mathcal{O}(\log N)$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

SPLIT, CONCATENATE $\mathcal{O}(N\,\mathrm{polylog}(N))$

EQUAL $\mathcal{O}(\log N)$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

SPLIT, CONCATENATE $\mathcal{O}(N\,\mathrm{polylog}(N))$

EQUAL $\mathcal{O}(\log N)$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\, \mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

Split, Concatenate $\mathcal{O}(N\,\mathrm{polylog}(N))$

Equal $\mathcal{O}(\log N)$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- Split: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$.

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

SPLIT, CONCATENATE $\mathcal{O}(N\,\mathrm{polylog}(N))$

EQUAL $\mathcal{O}(\log N)$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$.

$\mathcal{O}(N^2)$ invocations

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

Split, Concatenate $\mathcal{O}(N\,\mathrm{polylog}(N))$

Equal $\mathcal{O}(\log N)$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N\ \mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- Split: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

SPLIT, CONCATENATE $\mathcal{O}(N\,\mathrm{polylog}(N))$ — (a) $(X_i, \alpha) \in \equiv_B$

EQUAL $\mathcal{O}(\log N)$

$\mathcal{O}(N^2\,\mathrm{polylog}(N))$ — (b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

SPLIT, CONCATENATE	$\mathcal{O}(N\,\mathrm{polylog}(N))$		
EQUAL	$\mathcal{O}(\log N)$	(a)	$(X_i, \alpha) \in \equiv_B$
	$\mathcal{O}(N^2\,\mathrm{polylog}(N))$	(b)	$(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$
		(c)	$(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

SPLIT, CONCATENATE $\mathcal{O}(N\,\mathrm{polylog}(N))$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

EQUAL $\mathcal{O}(\log N)$ (a) $(X_i, \alpha) \in \equiv_B$

$\mathcal{O}(N^2\,\mathrm{polylog}(N))$ (b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

similarly (c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Time cost of lpftest

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(X_i) = \mathrm{dec}_B(\alpha)$$

SPLIT, CONCATENATE	$\mathcal{O}(N\,\mathrm{polylog}(N))$	(a) $(X_i, \alpha) \in \equiv_B$
EQUAL	$\mathcal{O}(\log N)$	(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$
dominating	$\mathcal{O}(N^2\,\mathrm{polylog}(N))$	
	similarly	(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- STRING: $\mathcal{O}(N \log N)$;
- CONCATENATE: $\mathcal{O}(N\ \mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- SPLIT: $\mathcal{O}(N\ \mathrm{polylog}(N))$;
- EQUAL: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Total time cost

Total time cost

BPA: $\mathcal{O}(N^4 \, \mathrm{polylog}(N))$

Total time cost

BPA: $\mathcal{O}(N^4 \operatorname{polylog}(N))$

Simple grammars: $\mathcal{O}(N^3 \operatorname{polylog}(N))$

Simple grammars

$$X_j = \text{lpf}_{B'}(X_i)$$

$\mathcal{O}(N \,\text{polylog}(N))$ $\quad X_i \equiv_{B'} \alpha$

$\alpha = X_j \,\text{suffix}_{|X_i|-|X_j|}(\text{dec}_{B'}(\alpha_i))$

$\mathcal{O}(\log N)$ $\quad (a) \quad (X_i, \alpha) \in \equiv_B$

$\mathcal{O}(N^2 \,\text{polylog}(N))$ $\quad (b) \quad (X_i, \alpha) \in \texttt{exp}(\equiv_B)$

similarly $\quad (c) \quad (X_i, \alpha) \in \texttt{nr-exp}(\equiv_{B'})$

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta \,\text{suffix}_{|X_i|-|X_j|}(\text{dec}_{B'}(\alpha_i))$$

$$\text{dec}_B(\gamma) = \text{dec}_B(\delta)\,\text{suffix}_{|X_i|-|X_j|}(\text{dec}_{B'}(\alpha_i))$$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N \,\text{polylog}(N))$; $\quad \mathcal{O}(N)$ invocations
- Split: $\mathcal{O}(N \,\text{polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$. $\quad \mathcal{O}(N^2)$ invocations

Simple grammars

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

(a) $(X_i, \alpha) \in \equiv_B$

(b) $(X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

(c) $(X_i, \alpha) \in \mathtt{nr{-}exp}(\equiv_{B'})$

$\mathcal{O}(N^2 \,\mathrm{polylog}(N))$

similarly

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta)\,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N \,\mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- Split: $\mathcal{O}(N \,\mathrm{polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Simple grammars

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

assume $\{a : X_i \xrightarrow{a} \gamma\} = \{a : X_j \xrightarrow{a} \delta\}$

$$X_i \equiv_{B'} \alpha$$

$$\alpha = X_j \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$(a) \quad (X_i, \alpha) \in \equiv_B$

$(b) \quad (X_i, \alpha) \in \mathtt{exp}(\equiv_B)$

$(c) \quad (X_i, \alpha) \in \mathtt{nr-exp}(\equiv_{B'})$

$\mathcal{O}(N^2 \,\mathrm{polylog}(N))$

similarly

$$X_i \xrightarrow{a} \gamma \qquad X_j \xrightarrow{a} \delta$$

$$\gamma \equiv_B \delta \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

$$\mathrm{dec}_B(\gamma) = \mathrm{dec}_B(\delta) \,\mathrm{suffix}_{|X_i|-|X_j|}(\mathrm{dec}_{B'}(\alpha_i))$$

- String: $\mathcal{O}(N \log N)$;
- Concatenate: $\mathcal{O}(N \,\mathrm{polylog}(N))$; $\mathcal{O}(N)$ invocations
- Split: $\mathcal{O}(N \,\mathrm{polylog}(N))$;
- Equal: $\mathcal{O}(\log N)$. $\mathcal{O}(N^2)$ invocations

Simple grammars

$$X_j = \mathrm{lpf}_{B'}(X_i)$$

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$\mathcal{O}(N^2)$ invocations in all lpftest's

Total time cost

BPA: $\mathcal{O}(N^4 \,\mathrm{polylog}(N))$

Simple grammars: $\mathcal{O}(N^3 \,\mathrm{polylog}(N))$

BPP: $\mathcal{O}(N^3)$ [Jancar 2003]

Outline

- Background
  - Context-free graphs and commutative context-free graphs
  - Bisimulation equivalence problem
  - Norm
  - History of the problem
  - Unique decomposition
  - Naive algorithm
- Efficient algorithm for BPA and BPP
  - Outline of the algorithm
  - Refinement
  - Efficient computation of refinement for BPA
  - Time-cost analysis
  - **Partially-commutative context-free graphs**

One algorithm for both BPA and BPP?

One algorithm for both BPA and BPP?

Yes, in the framework of partially-commutative context-free graphs

Partial commutation

partially-commutative CFGs :

instead of words or multisets, consider partial orders

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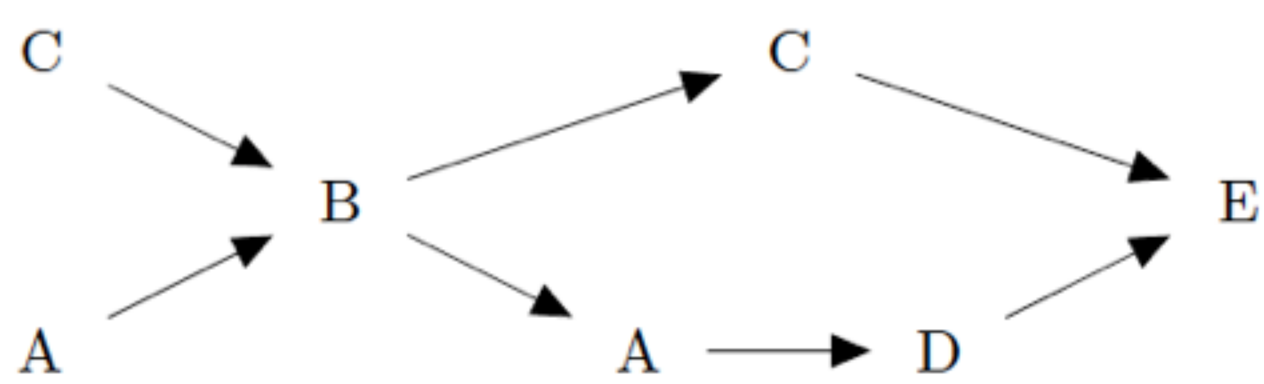
configuration:



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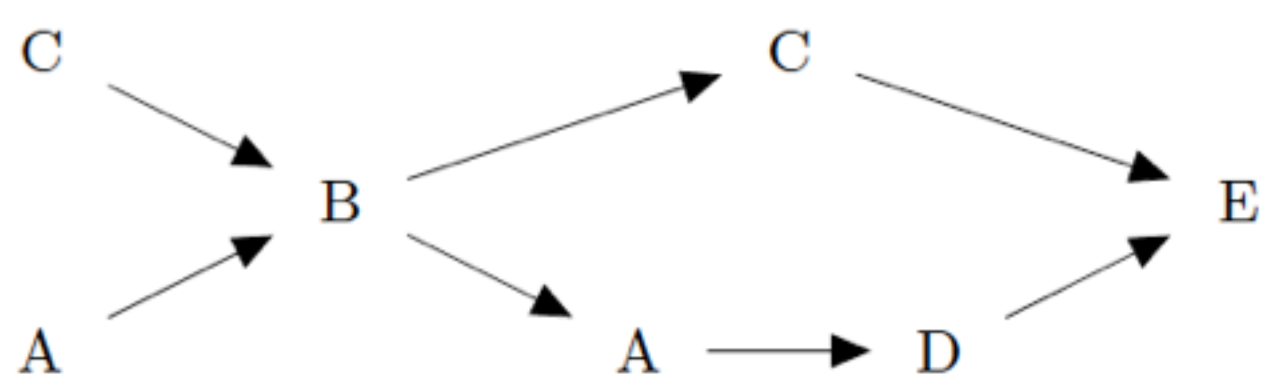

BPA and BPP are special cases

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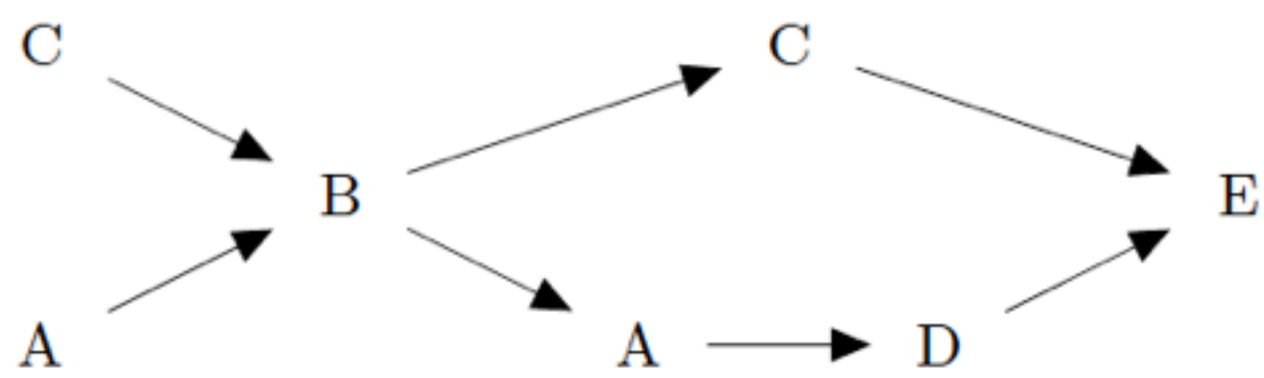

no independence: $C\ A\ B\ A\ C\ D\ E$

BPA and BPP are special cases

Partial commutation

partially-commutative CFGs :
instead of words or multisets, consider partial orders

Example:

variables: $\{\ A, B, C, D, E\ \}$

dependence: (A, B), (B, C), (C, E), (D, E)

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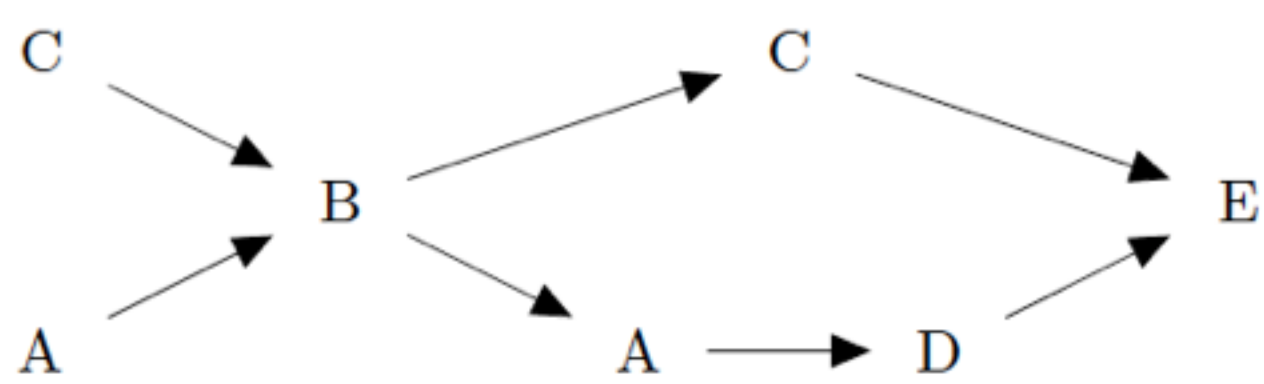

BPA and BPP are special cases

no independence: $C\,A\,B\,A\,C\,D\,E$

no dependence: $A^2\,B\,C^2\,DE$

Open question

	normed	unnormed
BPA	P-complete	EXPTIME-hard in 2-EXPTIME
BPP	P-complete	PSPACE-complete

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- the Prover-Refuter Game may be restricted to configurations of exponential length

Open question

	normed	unnormed
BPA	P-complete	EXPTIME-hard in 2-EXPTIME
BPP	P-complete	PSPACE-complete

- the Prover-Refuter Game may be restricted to configurations of exponential length
- can these configurations be compressed exponentially?

attractive PhD positions

at the University of Warsaw

thank you!