Computation theory with atoms

I. Sets with atoms
II. Computation models with atoms

Sławomir Lasota
University of Warsaw

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I. Sets with atoms

- sets with atoms
- orbit-finite sets
- definable sets

} recap

- representation theorem
- homogeneous atoms

Atoms

Atoms are a fixed logical structure

atoms	atom automorphisms
equality atoms $(N, =)$	all bijections
integer atoms $(Z, <)$	translations
integers with successor $(Z, +1)$	translations
total order atoms $(Q, <)$	monotonic bijections
timed atoms $(Q, <, +1)$	monotonic bijections preserving integer differences
vector space $(Q^n, +, q\cdot\_)$	linear bijections
...	...

Atoms are a parameter in the following

look here

Sets with atoms

Classical sets are built using $\varnothing$ and { }

e.g. $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing,\{\varnothing\}\}\}$

an atom contains
no elements

Sets with atoms are built using $\varnothing$ and { } and atoms

Examples:

- $\varnothing$
- three atoms {a, b, c},
- a pair (a, b) of atoms, encoded eg. as {a, {a,b}}
- atoms ∖ {a, b, c}
- ordered pairs of atoms
- finite words over atoms
- finite subsets of atoms
- ~~all subsets of atoms~~
- ....

illegal for (N, =)

legality depends on
atom automorphisms

Support

Extend atom automorphisms π to all sets element-wise, e.g.

$\pi(\{a, b, c\}) = \{\pi(a), \pi(b), \pi(c)\}$

$\pi(\text{atoms} \setminus \{a, b, c\}) = \text{atoms} \setminus \{\pi(a), \pi(b), \pi(c)\}$

$\pi(\{\{a\}, \{a,b\}\}) = \{\{\pi(a)\}, \{\pi(a), \pi(b)\}\}$

A set X is supported by a finite set S of atoms, if every atom S-automorphism (= identity on S) preserves X:

$$(\pi(a) = a \text{ for all } a \in S) \implies \pi(X) = X$$

A set X is **legal** if it is hereditarily finitely supported:

- X is finitely supported,
- its elements are finitely supported,
- and so on...

Sets supported by $\emptyset$ are called equivariant

Most often we work with equivariant sets

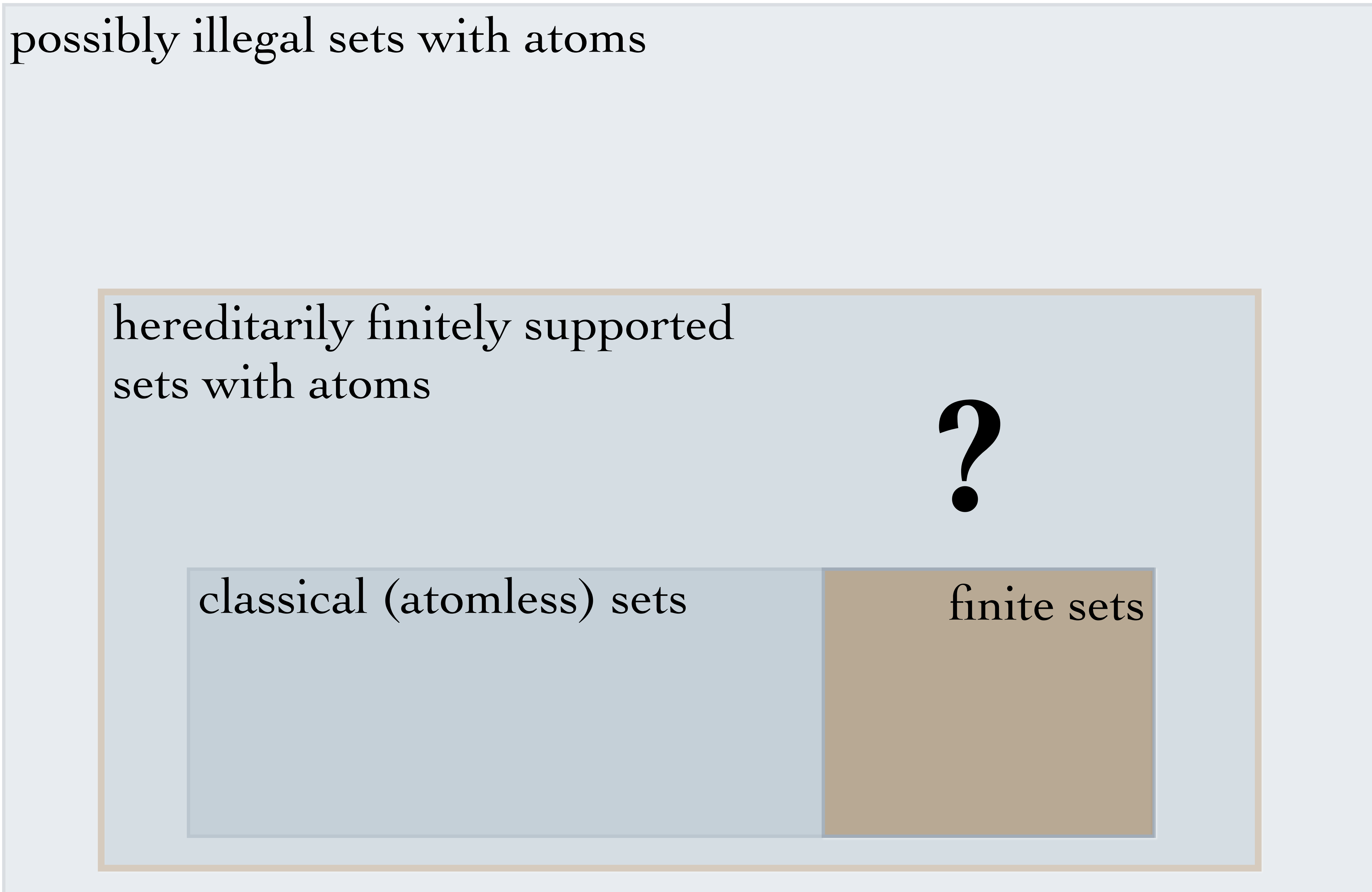
Support

Examples:

• $\emptyset$	$\emptyset$
• three atoms $\{1, 3, 6\}$	$\{1, 3, 6\}$,
• a pair $(3, 7)$ of atoms	$\{3, 7\}$,
• atoms $\setminus \{2, 5, 1\}$	$\{2, 5, 1\}$
• ordered pairs of atoms	$\emptyset$
• finite words over atoms	$\emptyset$
• finite subsets of atoms	$\emptyset$
• all subsets of atoms	$\emptyset$

support = atoms that you use in order to "define" a set

Legal sets with atoms

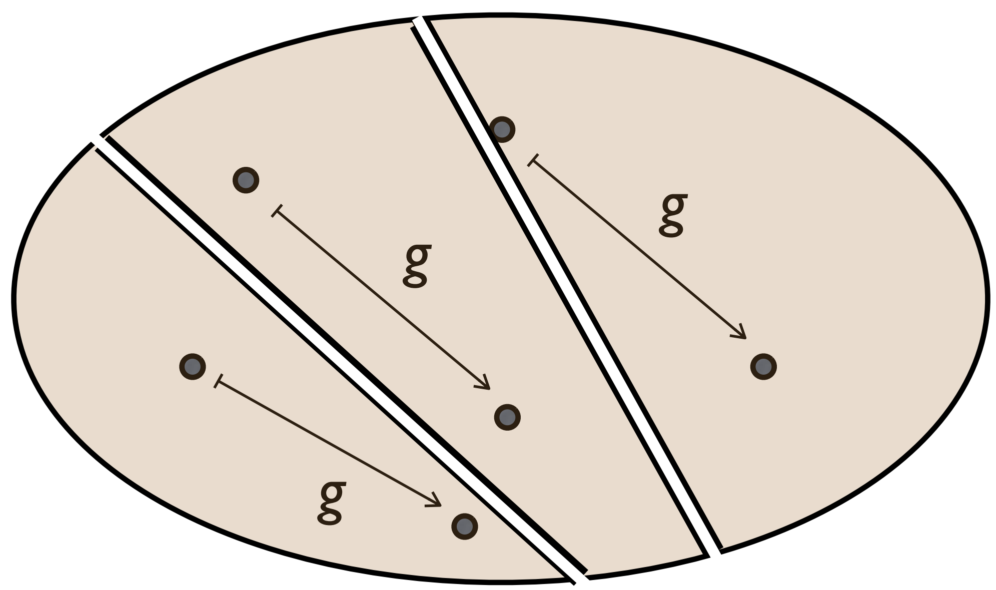


Orbits

x, y are in the same **S-orbit**

if

π(x) = y for an S-automorphism π



∅**-orbits** we call orbits

Oligomorphic atoms

a structure $\mathcal{A}$ is **oligomorphic**

if

$\mathcal{A}^{(n)}$ split into finitely many $\emptyset$-orbits for every n.

Example: for atoms $(\mathbb{Q}, \leq)$, atoms$^{(n)}$ has n! orbits

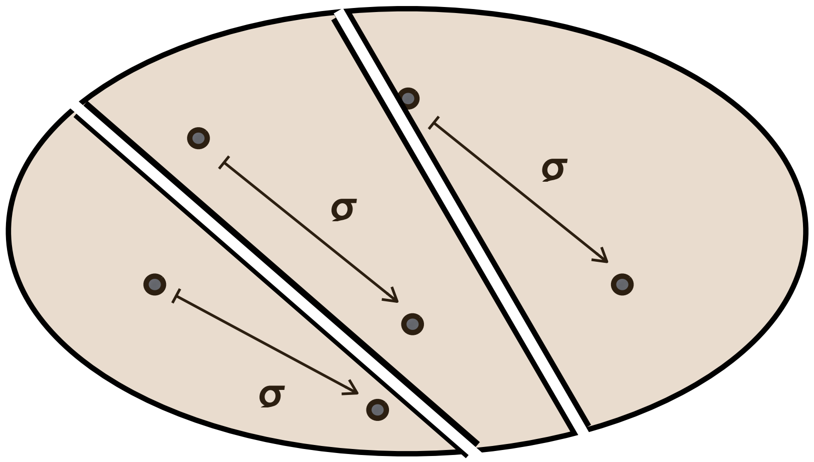
Example: for atoms $(\mathbb{Q}, \leq, +1)$, atoms$^{(2)}$ has infinitely many orbits

$(7,6⅓)$ $(7,7⅓)$ $(7,8)$ $(7,8⅓)$ ...

Orbit-finite sets

x, y are in the same S-orbit if $\pi(x) = y$ for an S-automorphism π

A set is orbit-finite if its partition into orbits is finite



If atoms are oligomorphic,
orbit-finiteness does not depend on S

Examples:

- $\emptyset$
- three atoms {1, 3, 6}
- a pair (3, 7) of atoms

} finite

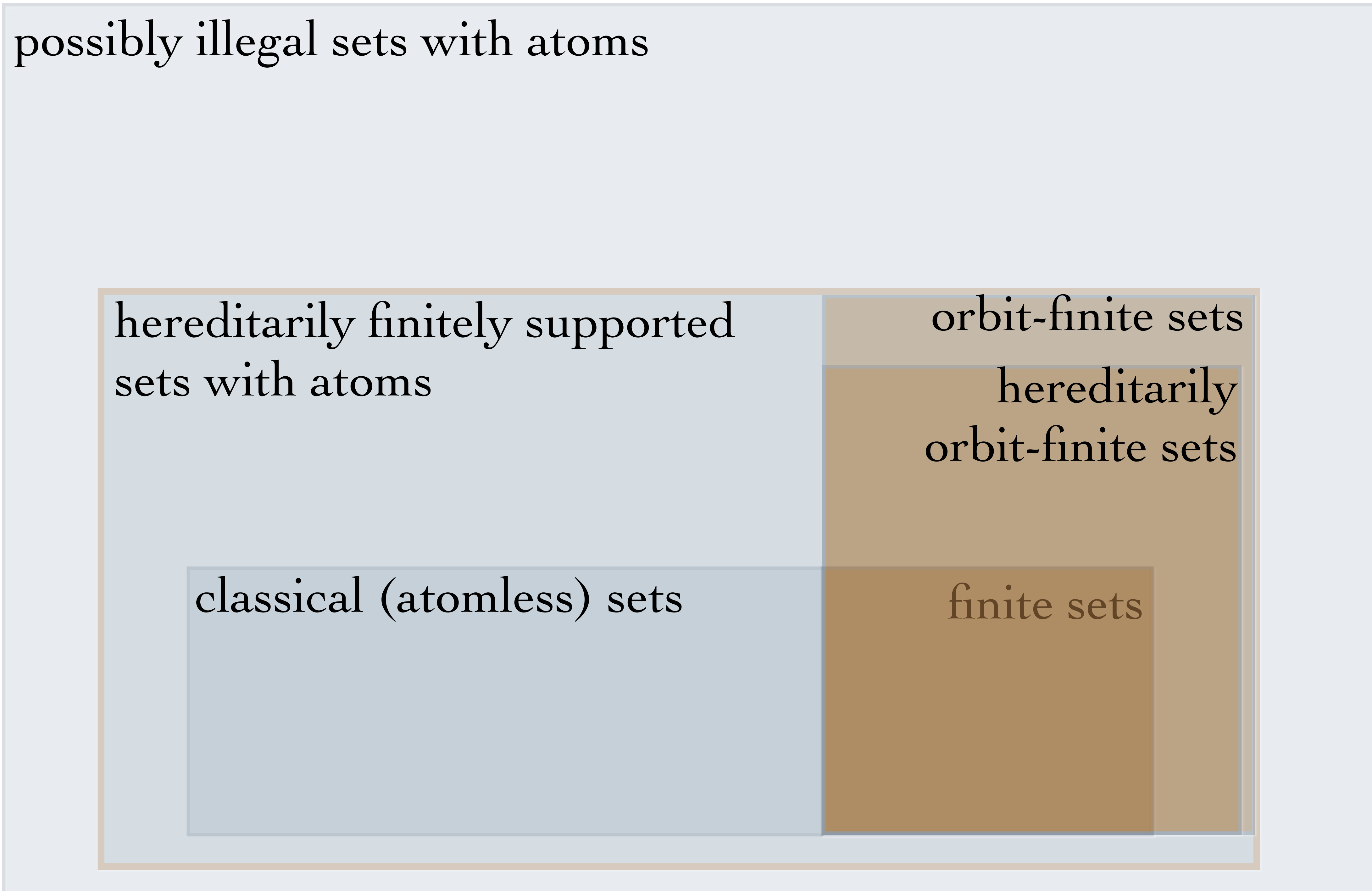
- atoms ∖ {2, 5, 1}
- ordered pairs of atoms

} orbit-finite for oligomorphic atoms

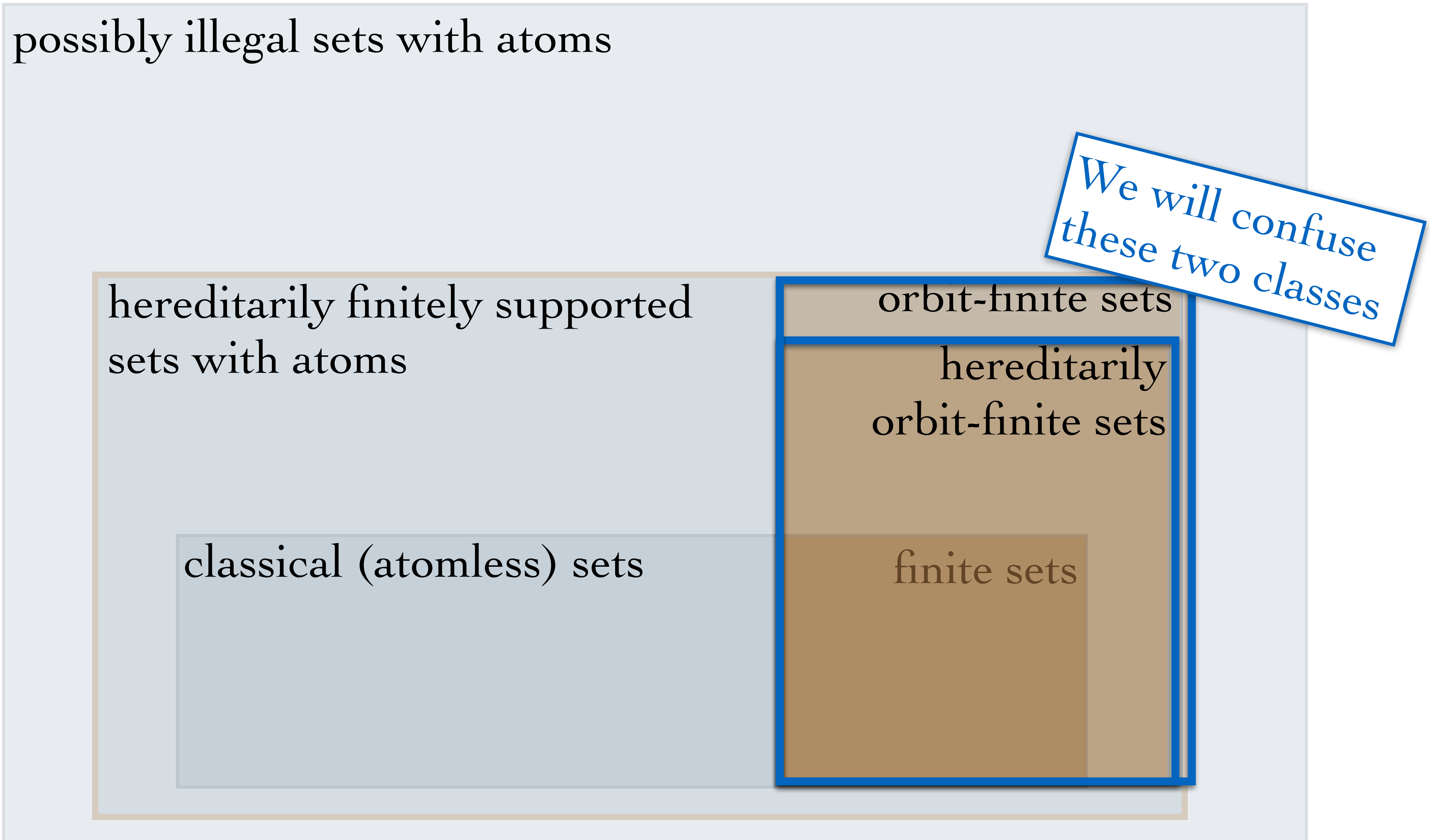
- finite words over atoms
- finite subsets of atoms
- all subsets of atoms

} orbit-infinite

Hereditarily orbit-finite sets



Hereditarily orbit-finite = definable



Definable sets

equality atoms (N, =)
total order atoms (Q, <)

Examples:

- ∅ — ∅
- three atoms {1, 3, 6} — {1, 3, 6},
- a pair (3, 7) of atoms — {{3}, {3,7}} = 3 7
- atoms ∖ {2, 5, 1} — { d : d atom, d ≠ 2, d ≠ 5, d ≠ 1 }
- ordered pairs of atoms — { ab : a,b atoms, a ≠ b }
- finite words over atoms
- finite subsets of atoms
- all subsets of atoms

} orbit-infinite

atoms$^{(3)}$ modulo cyclic shift

{ {abc, bca, cab} : a,b,c atoms, a ≠ b, b ≠ c, c ≠ a }

{ {ab, cd} : a,b,c,d atoms, pairwise different }

{ a 1 : a in atoms, a ≠ 2 }

{ a : a in atoms, 4.5 < a < 6.1 }

∅-definable ?

equality atoms (N, =)
total order atoms (Q, <)

∅-definable	$\emptyset$
{1,3,6}-definable	$\{1, 3, 6\}$,
{3,7}-definable	$\{\{3\}, \{3,7\}\} = 3\ 7$
{2,5,1}-definable	$\{\, d : d \text{ atom}, d \neq 2, d \neq 5, d \neq 1 \,\}$
∅-definable	$\{\, ab : a,b \text{ atoms}, a \neq b \,\}$

∅-definable	$\{\, \{abc, bca, cab\} : a,b,c \text{ atoms}, a \neq b, b \neq c, c \neq a \,\}$
	$\{\, \{ab, cd\} : a,b,c,d \text{ atoms, pairwise different} \,\}$
{1,2}-definable	$\{\, a\ 1 : a \text{ in atoms}, a \neq 2 \,\}$
{4.5, 6.1}-definable	$\{\, a : a \text{ in atoms}, 4.5 < a < 6.1 \,\}$

- sets with atoms
- orbit-finite sets
- definable sets

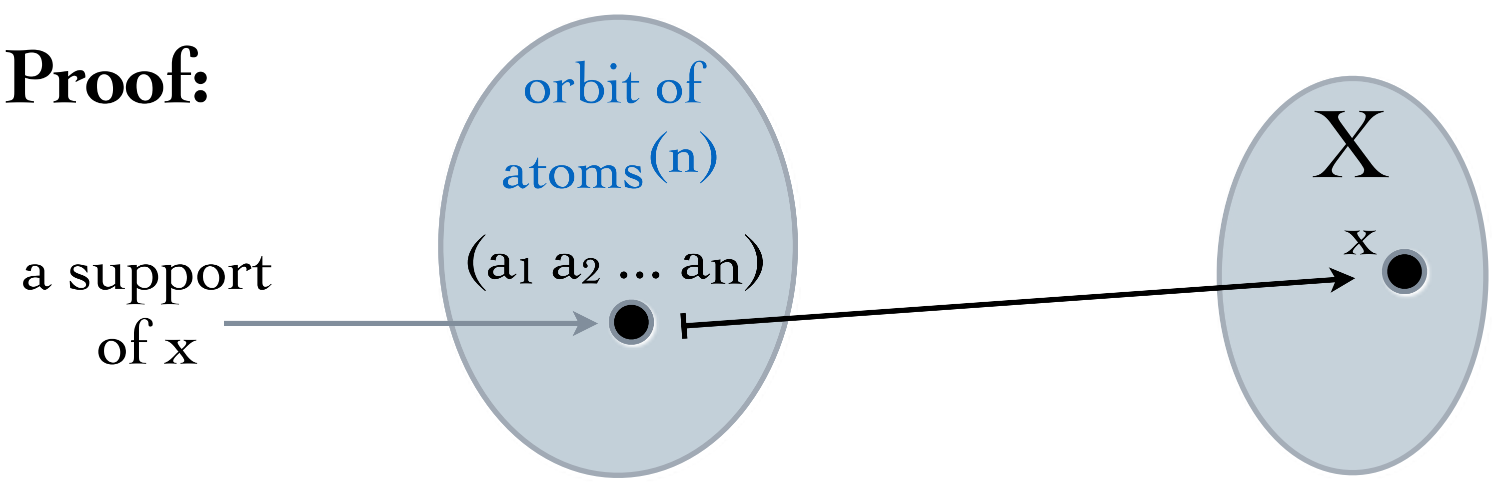
} recap

- **representation theorem**
- homogeneous atoms

Representation theorem

Theorem: Every equivariant orbit admits a surjective equivariant function from an orbit of atoms$^{(n)}$, for some n.

Proof:

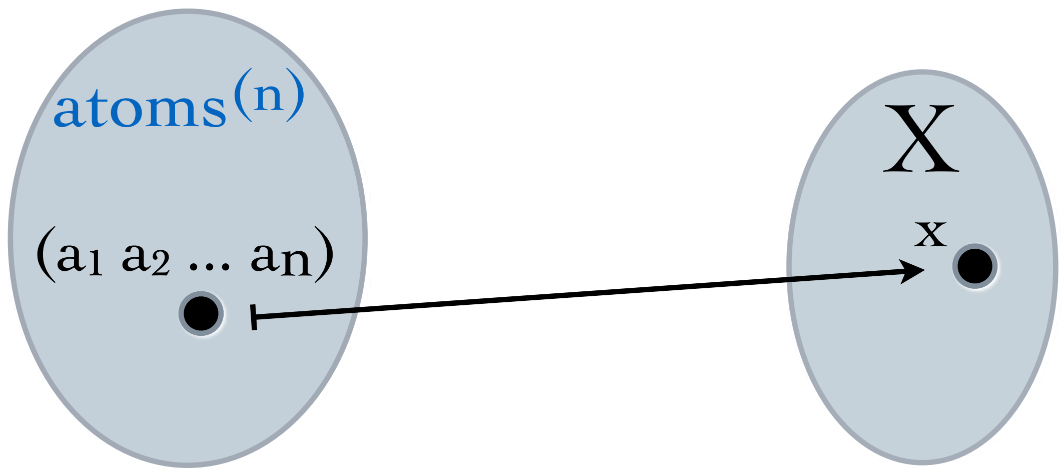


Surjective function = the orbit of the pair $((a_1\ a_2\ \ldots\ a_n), x)$

Representation theorem

Theorem: Every equivariant orbit admits a surjective equivariant function from $atoms^{(n)}$, for some n, s.t.

$$f(a_1\ a_2 \ldots a_n) = f(b_1\ b_2 \ldots b_n) \implies \{a_1\ a_2 \ldots a_n\} = \{b_1\ b_2 \ldots b_n\}.$$



n defines **dimension** of an orbit

Theorem: Every equivariant orbit is isomorphic to $atoms^{(n)}$ modulo G, for some n and G a group of permutations of {1…n}.

equivariant bijection

Examples: $atoms^{(2)} /_{(1\ 2)} = P_2(\text{atoms})$

$atoms^{(3)} /_{(1\ 2\ 3)} = atoms^{(3)}$ modulo cyclic shift

$atoms^{(5)} /_{(1\ 2\ 3)(4\ 5)}$

Straight sets: every orbit isomorphic to $atoms^{(n)}$ for some n

Least support

Theorem: Every equivariant orbit is isomorphic to $\text{atoms}^{(n)}$ modulo G, for some n and G a group of permutations of {1…n}.

Examples: $\text{atoms}^{(2)} /_{(1\ 2)} = P_2(\text{atoms})$
$\text{atoms}^{(3)} /_{(1\ 2\ 3)} = \text{atoms}^{(3)}$ modulo cyclic shift
$\text{atoms}^{(5)} /_{(1\ 2\ 3)(4\ 5)}$

Straight sets: every orbit isomorphic to $\text{atoms}^{(n)}$ for some n

Corollary: Every set (element) x has the least support **supp**(x), i.e., support included in every support of x.

$\textbf{supp}(\ (3, 6, 7, 2) /_{(1\ 2\ 3)}\ ) = \{3, 6, 7, 2\}$

- sets with atoms
- orbit-finite sets
- definable sets

} recap

- representation theorem
- **homogeneous atoms**

Oligomorphic atoms

a structure $\mathcal{A}$ is oligomorphic

if

$\mathcal{A}^{(n)}$ is orbit-finite for every n.

Theorem: orbit-finite sets are stable under Caertesian products and subsets

Homogeneous atoms (relational case)

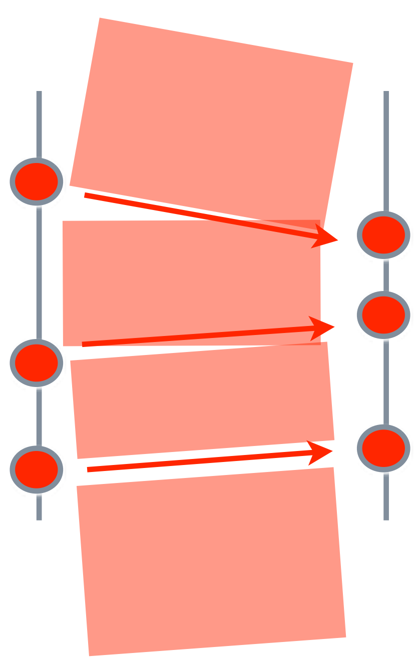
a relational structure A is homogeneous

if

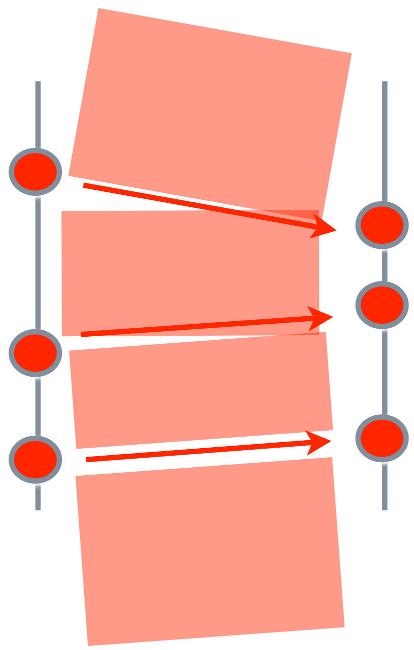
every isomorphism of **finite substructures** of A extends to an automorphism of the whole structure

induced substructures

Example: (Q, ≤)



Homogeneous atoms (relational case)

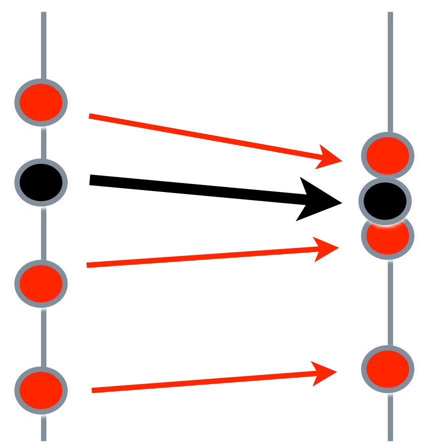


Examples:

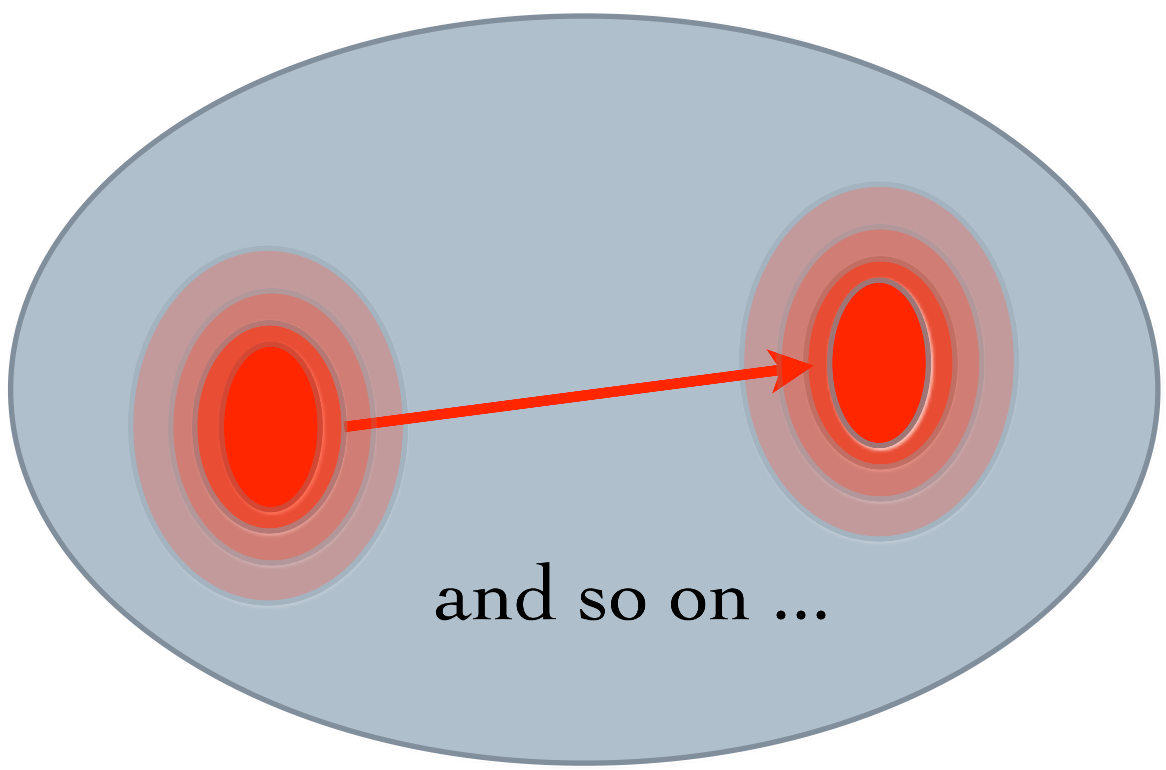
total order atoms $(Q, <)$
~~integer atoms $(Z, <)$~~
~~$(Q, <, +1)$~~
equality atoms $(N, =)$
random graph
…

random graph = countable infinite graph yielded almost surely if every pair of nodes is connected by an edge with independent probability ⅓

Homogeneous atoms (relational case)



extension property



Homogeneous atoms (general case)

a structure $\mathcal{A}$ is homogeneous

if

every isomorphism of **finitely generated substructures** of $\mathcal{A}$ extends to an automorphism of the whole structure

Example: bit vectors (V, +)

V = infinite-dimensional linear space over Z_2 =
infinite sequences over {0,1} with finitely many 1's

0101001101000011101110000000 …

substructure generated by {01010…, 01100…} =
{01010…, 01100…, 00110…, 00000…}

substructure generated by X = ? **subspace spanned by X**

Example: (Z, +1)

substructure generated by {7} = {7, 8, …}

Least support?

bit-vector atoms (V, +)

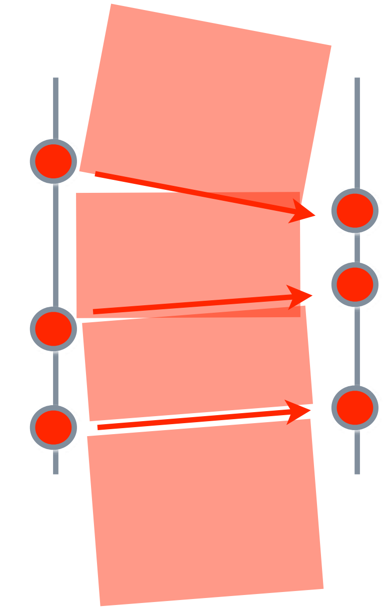
0101001101000011101111000 …

supp((01010…, 01100…)) = **?**

Theorem: Every set x has the least **closed** support **supp**(x), i.e., closed support included in every closed support of x.

supp((01010…, 01100…)) = {01010…, 01100…, 00110…, 00000…}

Quantifier elimination



Observation: When atoms are homogeneous

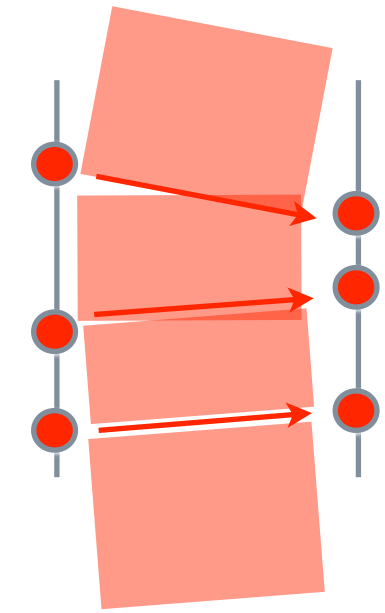
two tuples in $\text{atoms}^{(n)}$ are in the same orbit **iff** the tuples generate isomorphic substructures

there is a function **b** such that substructures generated by n atoms have size bounded by **b**(n)

Corollary: When atoms are homogeneous, have finite vocabulary and bounded substructures,

- atoms are oligomorphic
- legal subsets of atoms^n = **quantifier-free** definable subsets of atoms^n

Example: For bit-vector atoms (V, =), what is **b**(n)? **b**(n) = 2^n
Integer atoms (Z, +1) ?



In the sequel, atoms are **well-behaved**:

- have finite vocabulary
- are homogeneous
- have bounded substructures
- are effective

hence oligomorphic and FO = quantifier free logic

orbits of atoms$^{(n)}$ = substructures generated by n atoms

there is a function **b** such that substructures generated by n atoms have size bounded by **b**(n)

hence quantifier-free logic decidable

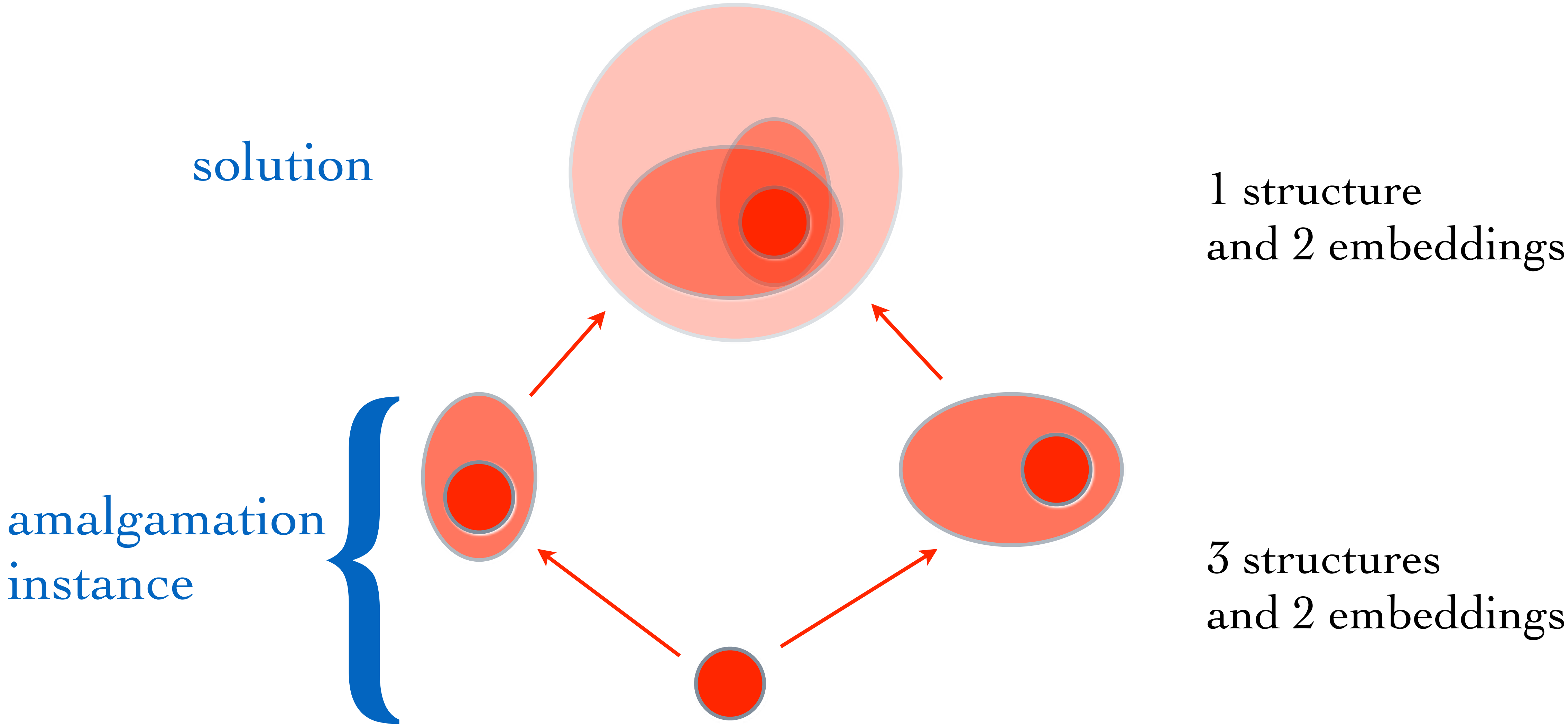
finitely generated substructures of atoms are computable

Age = finitely generated substructures

atoms	finitely generated substructures
equality atoms (N, =)	finite pure sets
integer atoms (Z, <)	finite total orders
total order atoms (Q, <)	finite total orders
vector space (Q^n, +, q·_)	vector spaces over Q of dim ≤ n
bit vectors (V, +)	finite vector spaces over Z_2
?	**finite graphs**
?	finite trees
?	finite partial orders

amalgamation classes

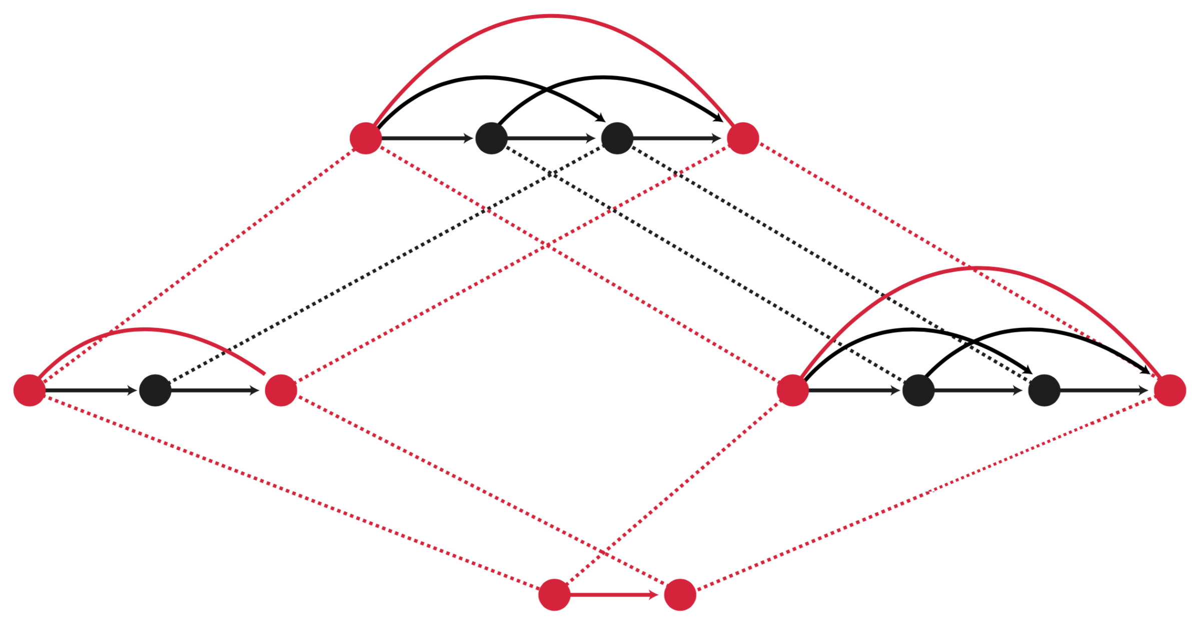
Amalgamation class



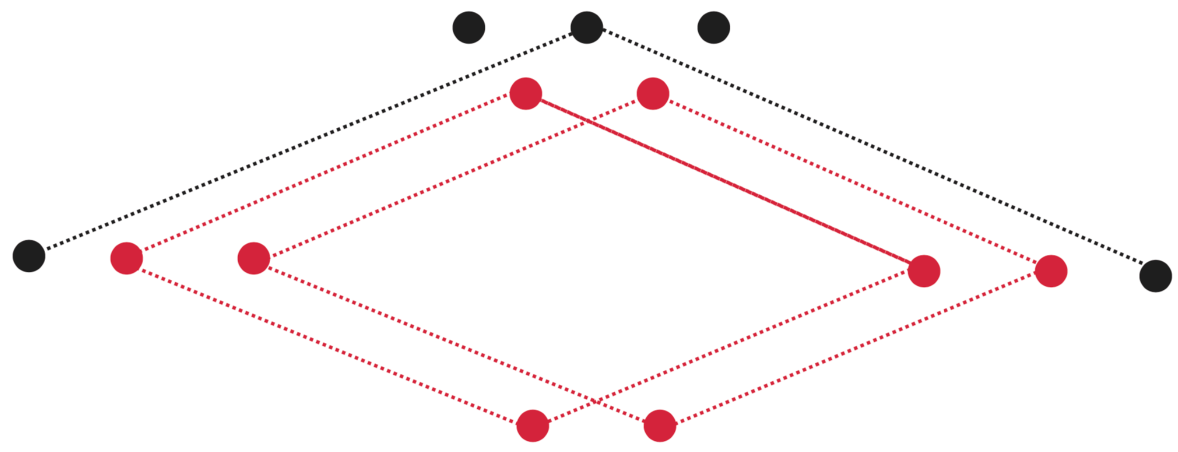
- class is closed under amalgamation if every instance has a solution
- amalgamation class = class of finitely generated structures closed under iso, substructures and amalgamation

Amalgamation classes

- finite total orders

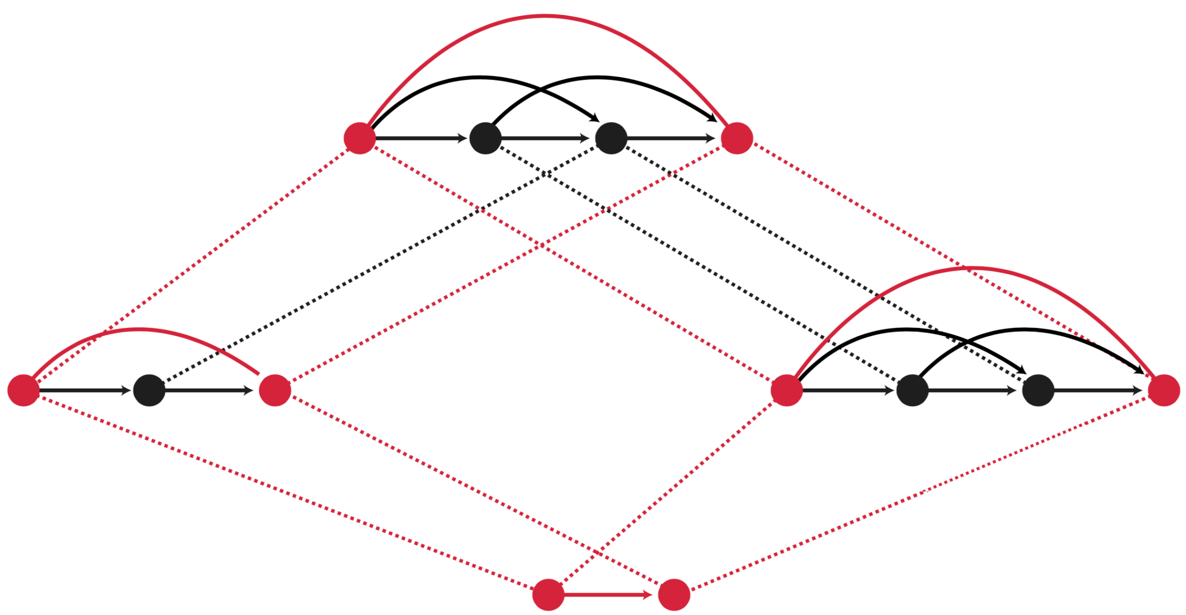


- finite pure sets

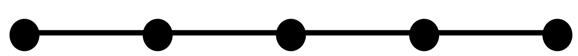


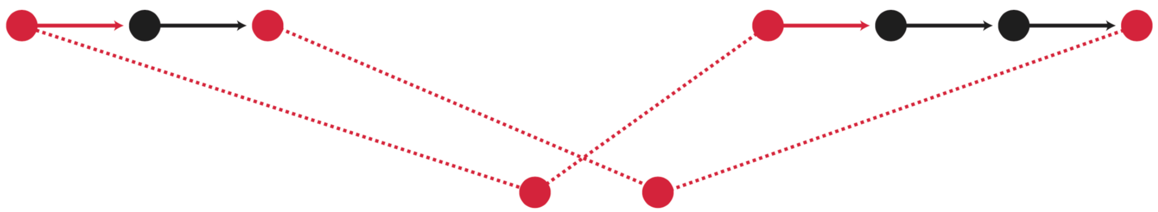
Amalgamation classes

- finite total orders



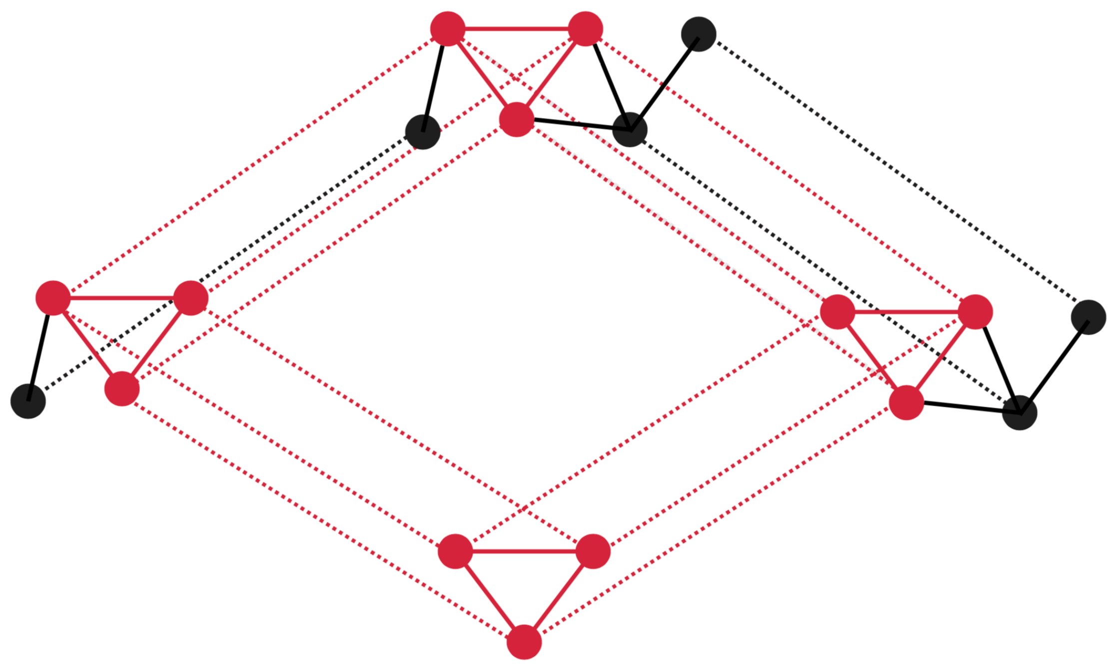
- paths?



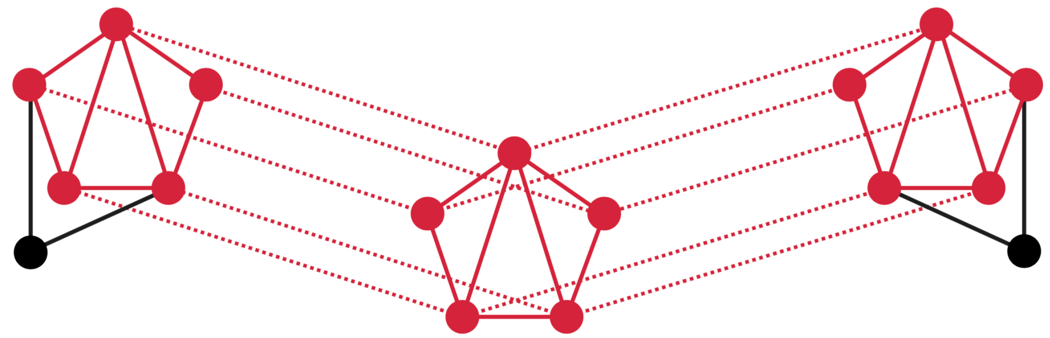


Amalgamation classes

- finite graphs

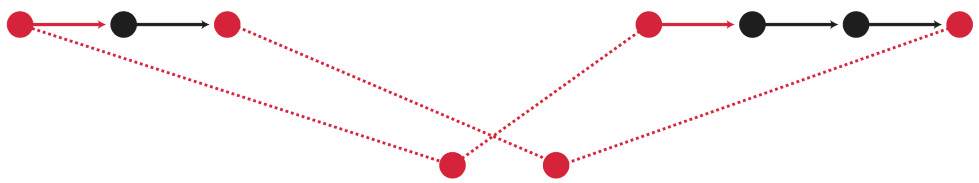


- finite planar graphs?

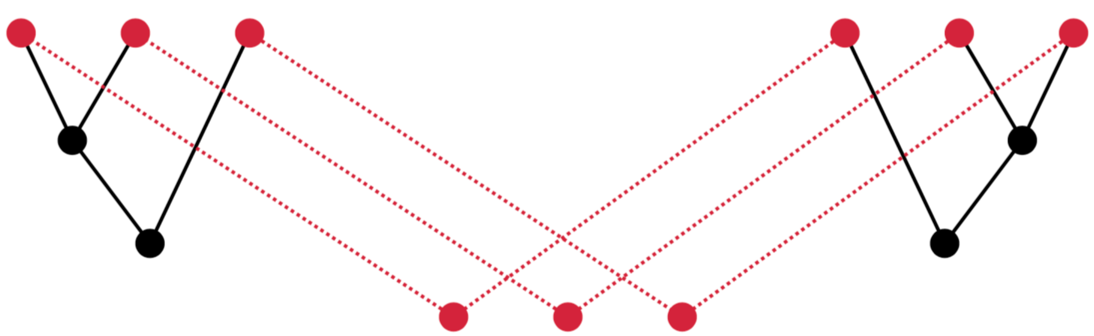


Amalgamation classes

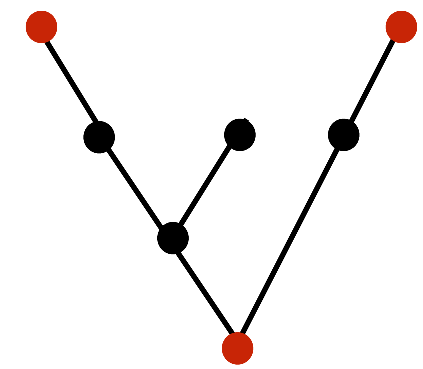
- finite trees (child relation)?

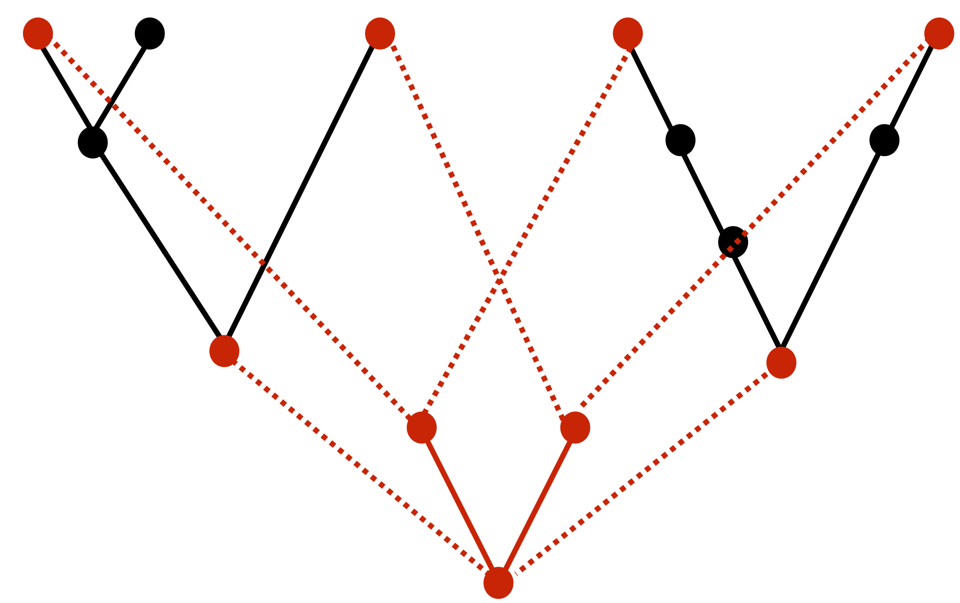


- finite trees (descendant relation)?



- finite trees (lca function)



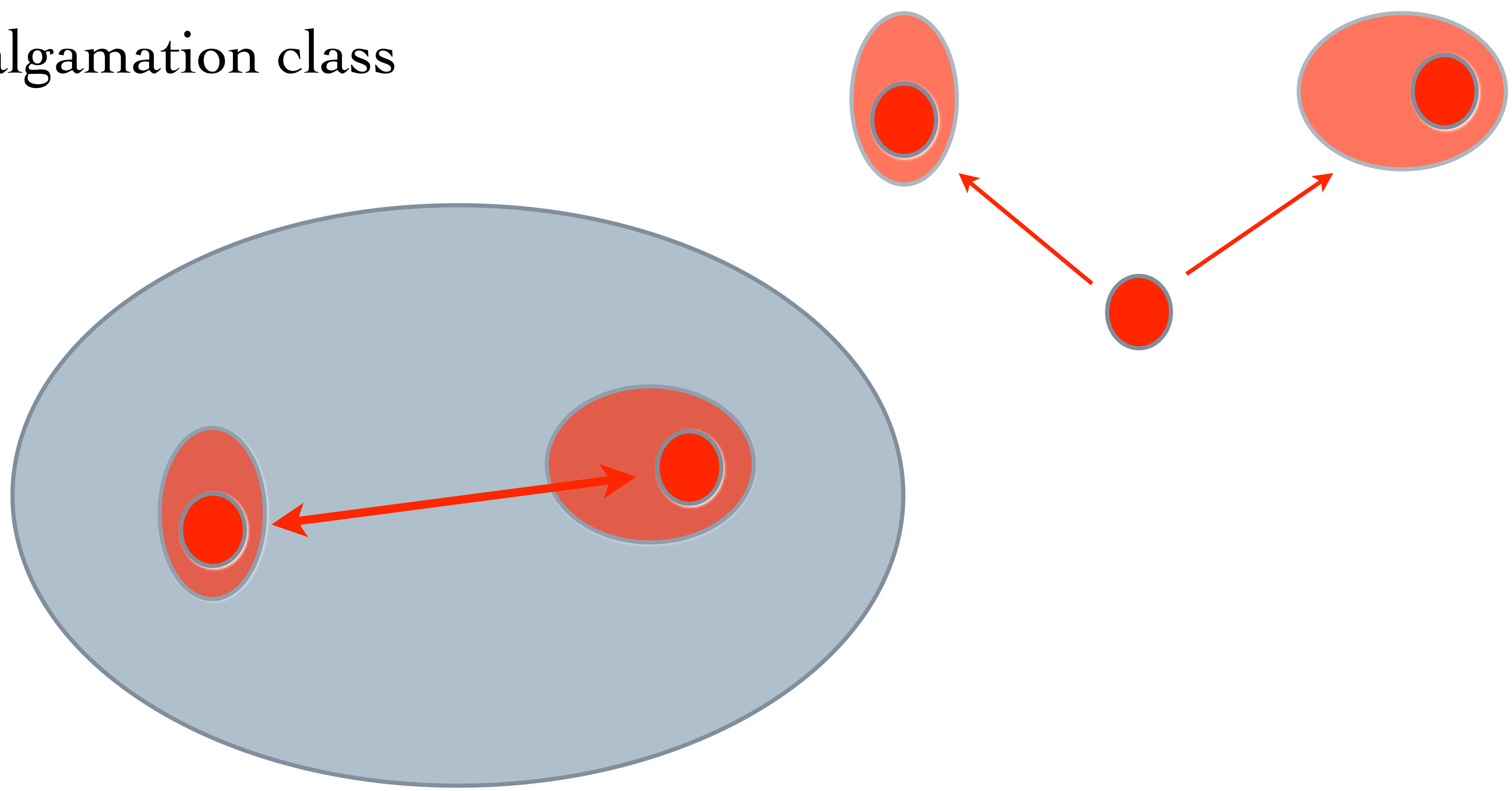


Homogeneity vs amalgamation

Theorem (Freïssé):

homogeneous structures → (*Age* = finitely generated substructures) → amalgamation classes of finitely generated structures

- *Age* yields an amalgamation class



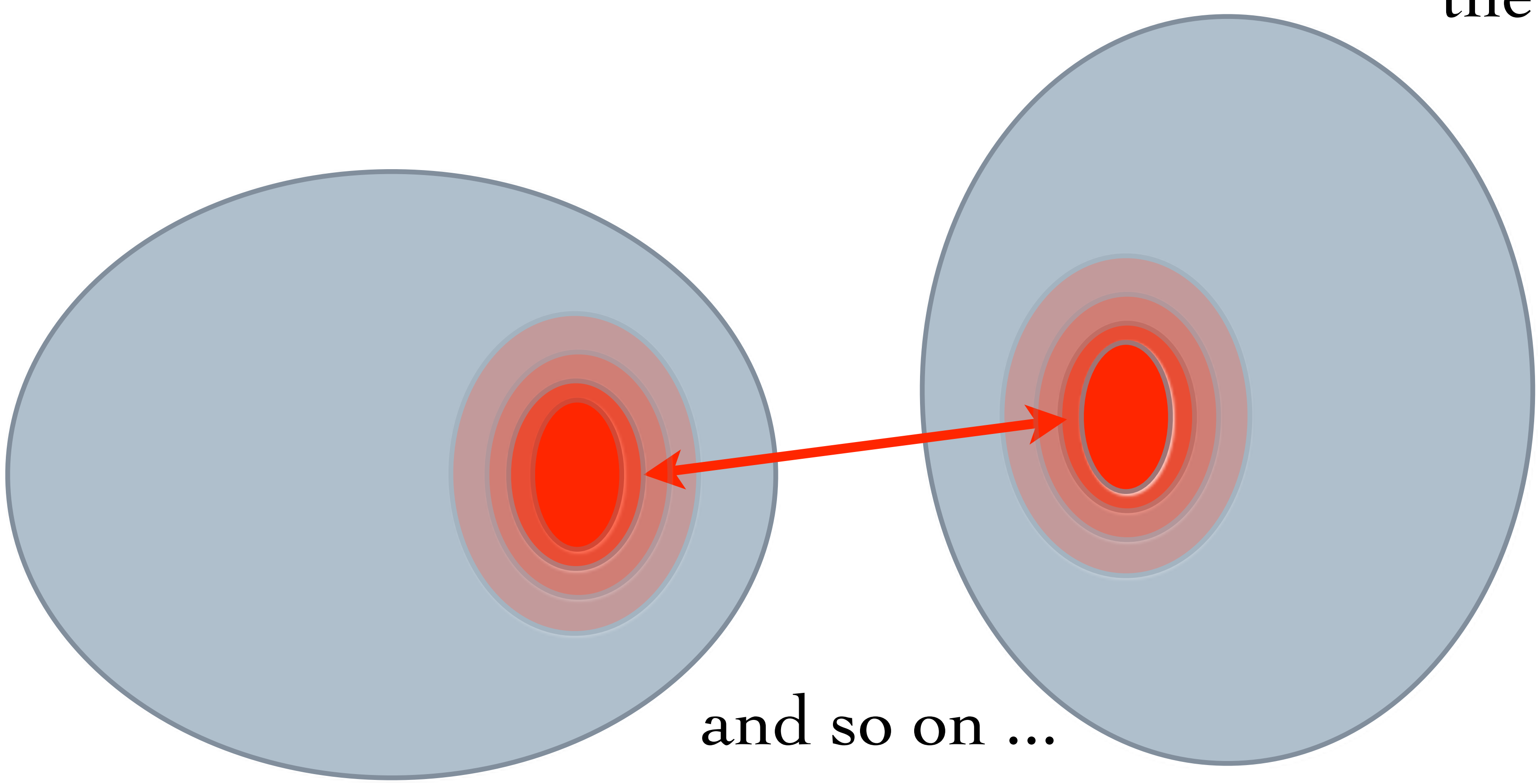
Homogeneity vs amalgamation

Theorem (Freïssé):

homogeneous structures —— *Age* = finitely generated substructures ——→ amalgamation classes of finitely generated structures

a homogeneous structure is uniquely determined by its finitely generated substructures

- *Age* is injective (up to iso): consider 2 homogeneous structures with the same age



and so on ...

Homogeneity vs amalgamation

Theorem (Freïssé):

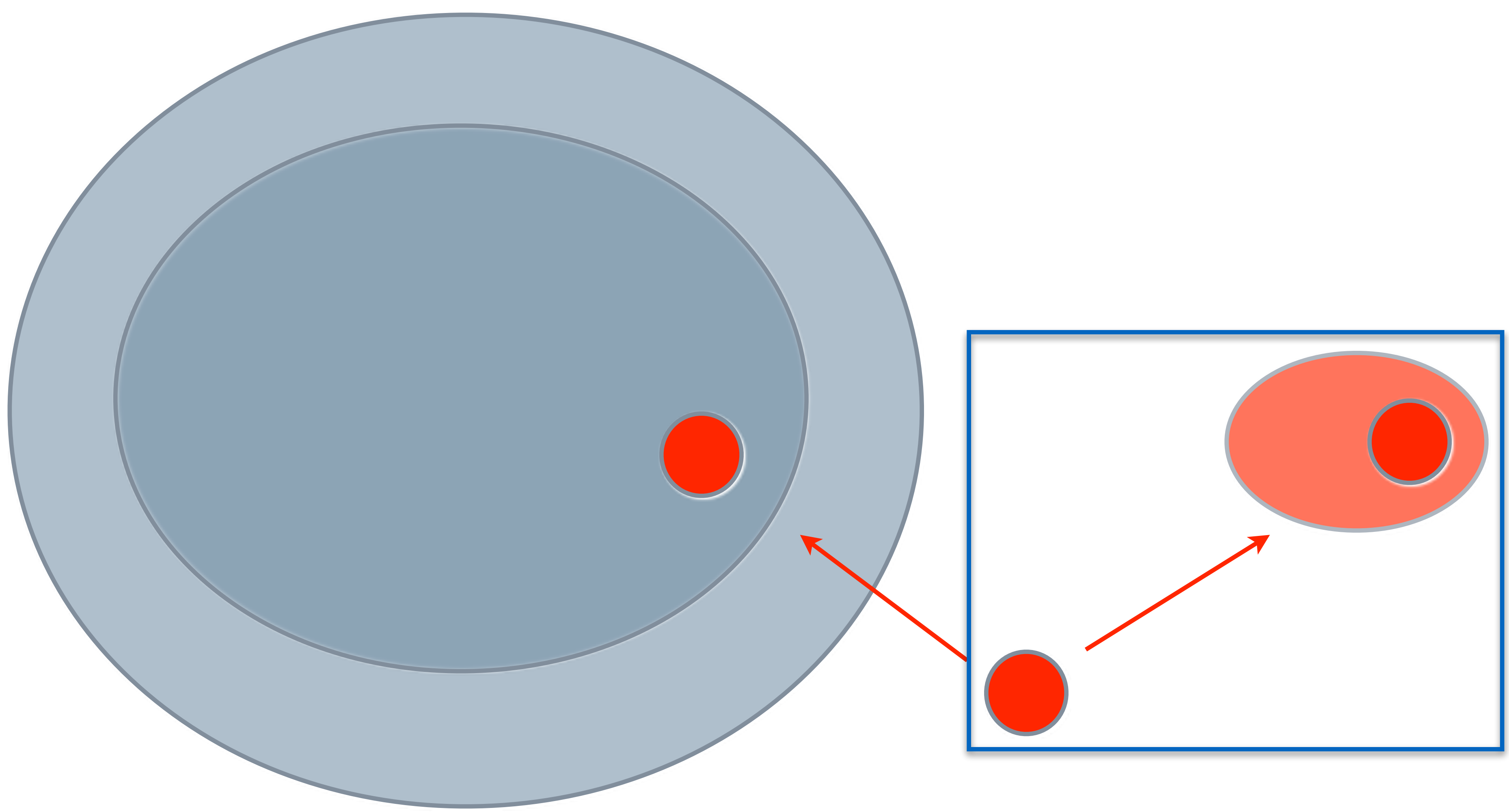
homogeneous structures

Age = finitely generated substructures →

amalgamation classes of finitely generated structures

← Freïssé limit

- *Age* is surjective:



enumerate all such triples

Homogeneity vs amalgamation

Theorem (Freïssé):

homogeneous structures → (*Age* = finitely generated substructures) → amalgamation classes of finitely generated structures

amalgamation classes of finitely generated structures → (Freïssé limit) → homogeneous structures

atoms	amalgamation class
equality atoms (N, =)	finite pure sets
~~integer atoms (Z, <)~~	finite total orders
total order atoms (Q, <)	finite total orders
bit vectors (V, +)	finite vector spaces over Z_2
random (universal) **graph**	**finite graphs**
universal tree	finite trees
universal partial order	finite partial orders
…	…

classification challenge

Theorem: [Lachlan, Woodrow'80] Let $\mathbb{A}$ be an infinite countable homogeneous graph. Then either $\mathbb{A}$ or its complement is isomorphic to one of:

- universal (random) graph
- universal graph excluding n-clique, for some n
- disjoint union of cliques of the same (finite or infinite) size

An analogous (but more complex) classification exists for directed graphs [Cherlin'98].

A classification of all homogeneous structures remains a great challenge.

WQO solvable problems:

- emptiness of 1-dim alternating automata
- coverability of Petri nets

WQO Dichotomy Conjecture:

For a homogeneous structure *A*, exactly one of the following conditions holds:

- *Age(*A), ordered by embeddings, is a WQO
- WQO solvable problems are undecidable.

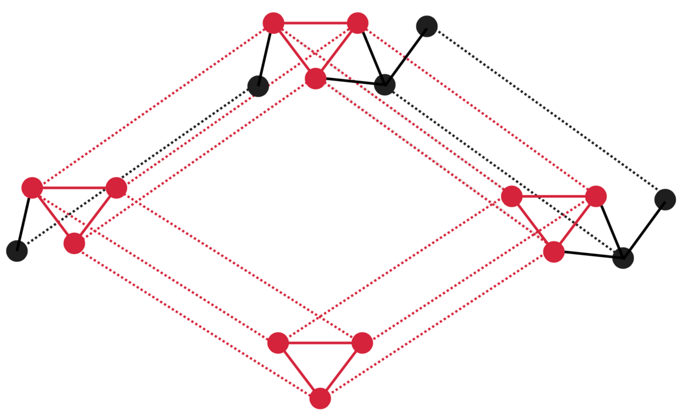
The conjecture confirmed for:

- graphs
- directed graphs
- 2-colored graphs
- 5-colored finitely bounded graphs
- when all relations are equivalences
- ...

?

Random graph = universal graph?

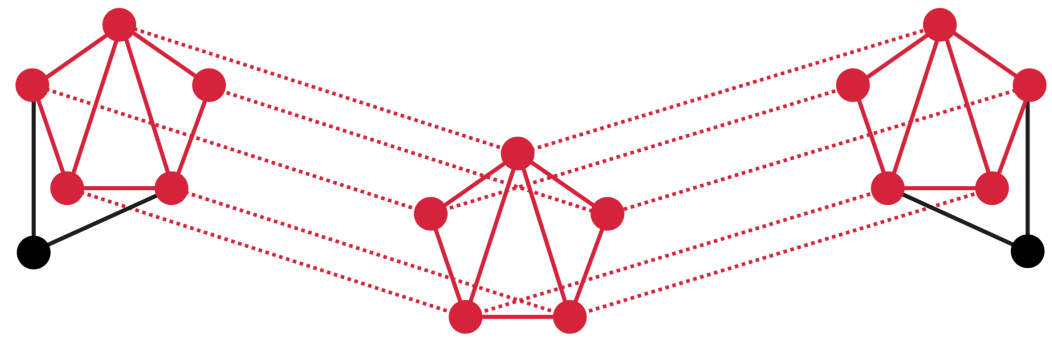
Question: Why *Age*(the random graph) = all finite graphs?
Why is the random graph homogeneous?



?

Assume atoms to be a relational structure.

Question: Closure under singleton amalgamation implies closure under (arbitrary) amalgamation?



?

Consider *Age*(atoms) ordered by embeddings.

Question: In which case below *Age*(atoms) is a WQO?
What about **colored** *Age*(atoms)?

- equality atoms (N, =)
- total order atoms (Q, <)
- universal graph atoms
- universal partial order atoms