The reachability problem for Petri nets is not elementary

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RP'19, Brussels, 2019.09.11

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but the proof is so:)

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The reachability ... blem
f ...
is ... tary

crash course in counter programming (without zero tests)

... the proof is so:)

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Many faces of Petri nets

- Petri nets **[Petri 1962]**
- vector addition systems **VAS** **[Karp, Miller 1969]**
- vector addition systems with states **VASS** **[Hopcroft, Pansiot 1979]**
- automata with counters without zero tests
- **counter programs without zero tests**
- multiset rewriting
- …

Counter programs

a sequence of commands of the form:

$\mathsf{x} \mathrel{+}= 1$	(increment counter x)
$\mathsf{x} \mathrel{-}= 1$	(decrement counter x)
goto L **or** L'	(jump to either line L or line L')
zero? x	(continue if counter x equals 0)

counters are nonnegative

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except for the very last command which is of the form:

halt if $\mathsf{x}_1, \ldots, \mathsf{x}_l = 0$	(terminate provided all the listed counters are zero)

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Example:

1: $\mathsf{x}' \mathrel{+}= 100$
2: **goto** $\boxed{5}$ **or** $\boxed{3}$
3: $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{y} \mathrel{+}= 2$
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initially all counters 0:
x' = x = y = 0

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finally:
x' = 0 x = 100 y = 200

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no zero tests

initially all counters 0:
x' = x = y = 0

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Minsky machines

the conditional jump of Minsky machines

$$\textbf{if } \mathsf{x} = 0 \textbf{ then goto } L \textbf{ else } \mathsf{x} \mathrel{-}= 1$$

is simulated by counter program **with** zero tests:

$$\begin{array}{l}
1\text{:}\ \ \textbf{goto } \boxed{2} \textbf{ or } \boxed{4} \\
2\text{:}\ \ \textbf{zero? } \mathsf{x} \\
3\text{:}\ \ \textbf{goto } L \\
4\text{:}\ \ \mathsf{x} \mathrel{-}= 1
\end{array}$$

Many faces of Petri nets

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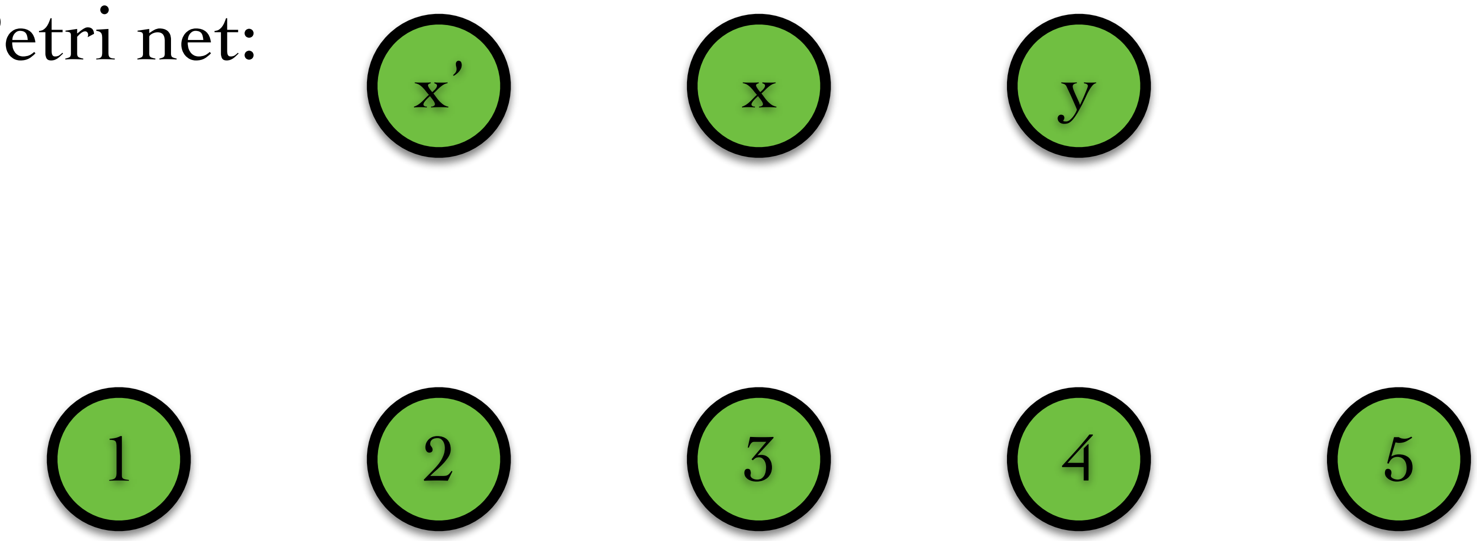
Petri net:



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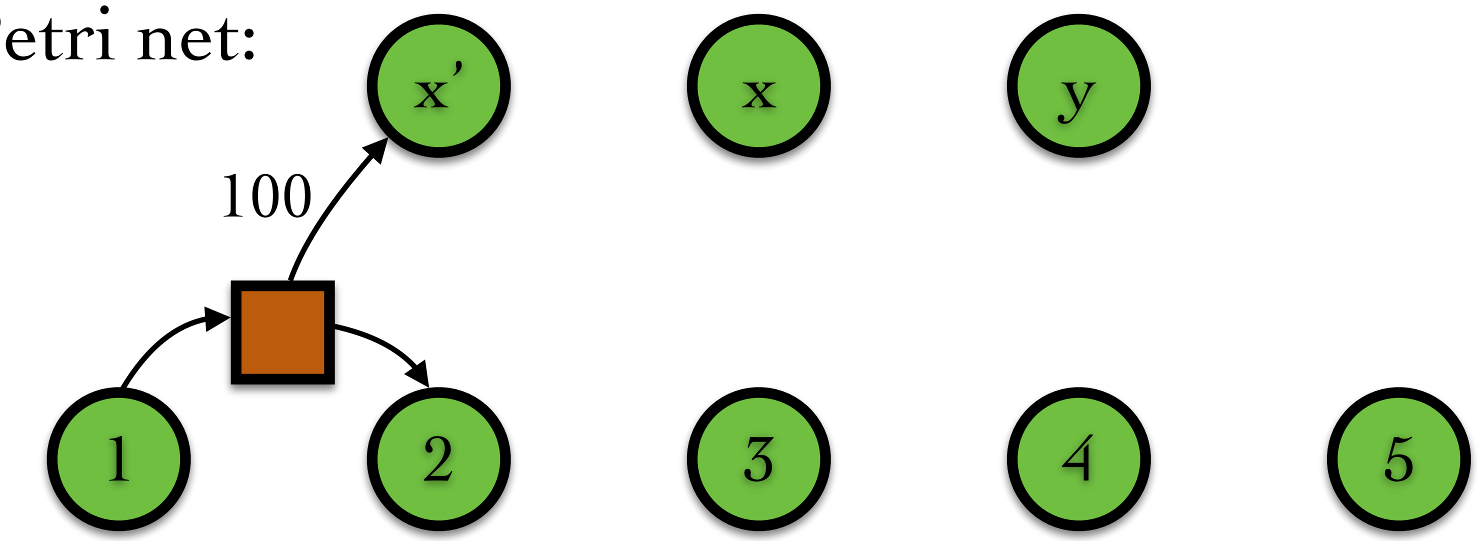

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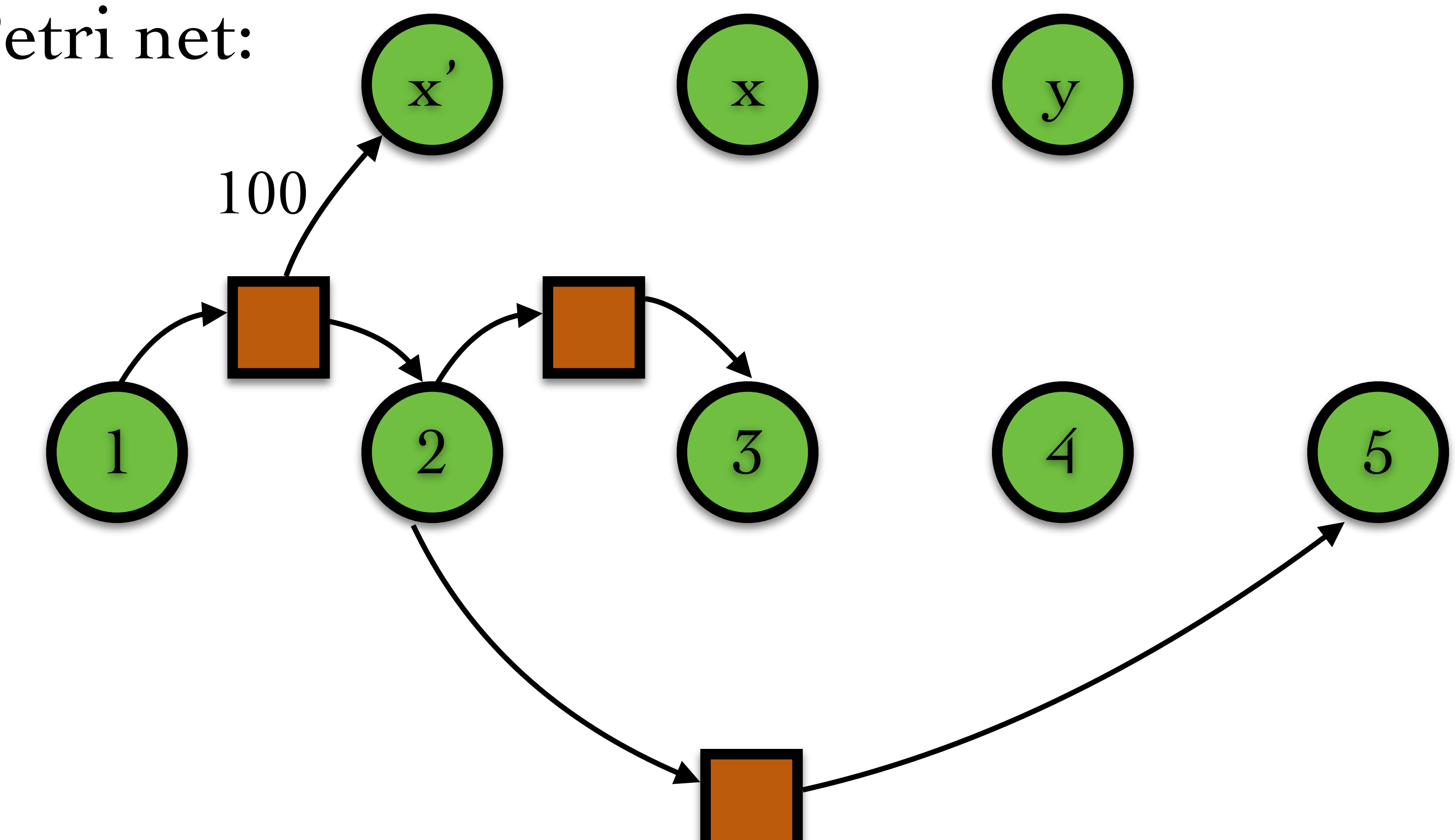

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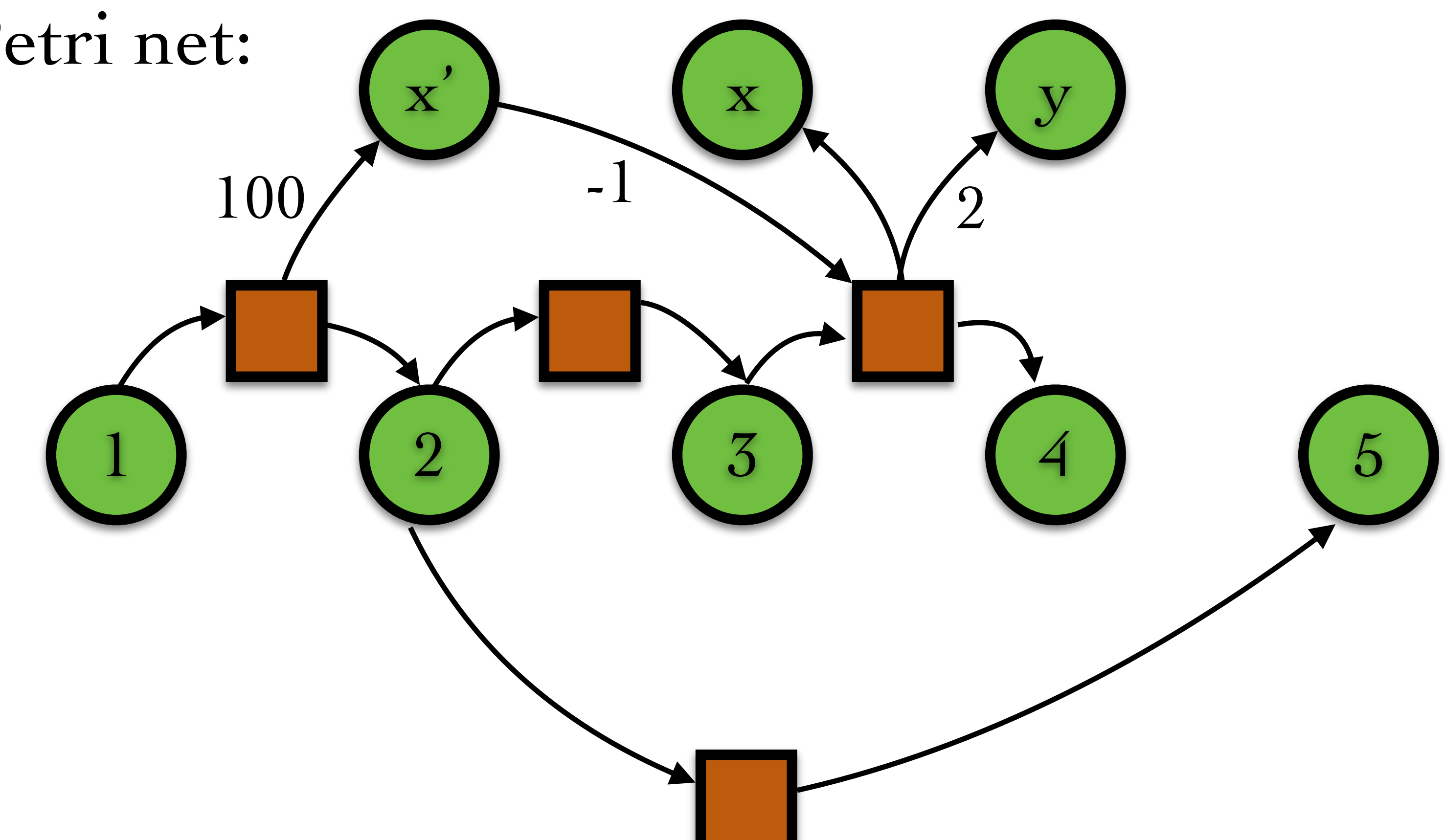

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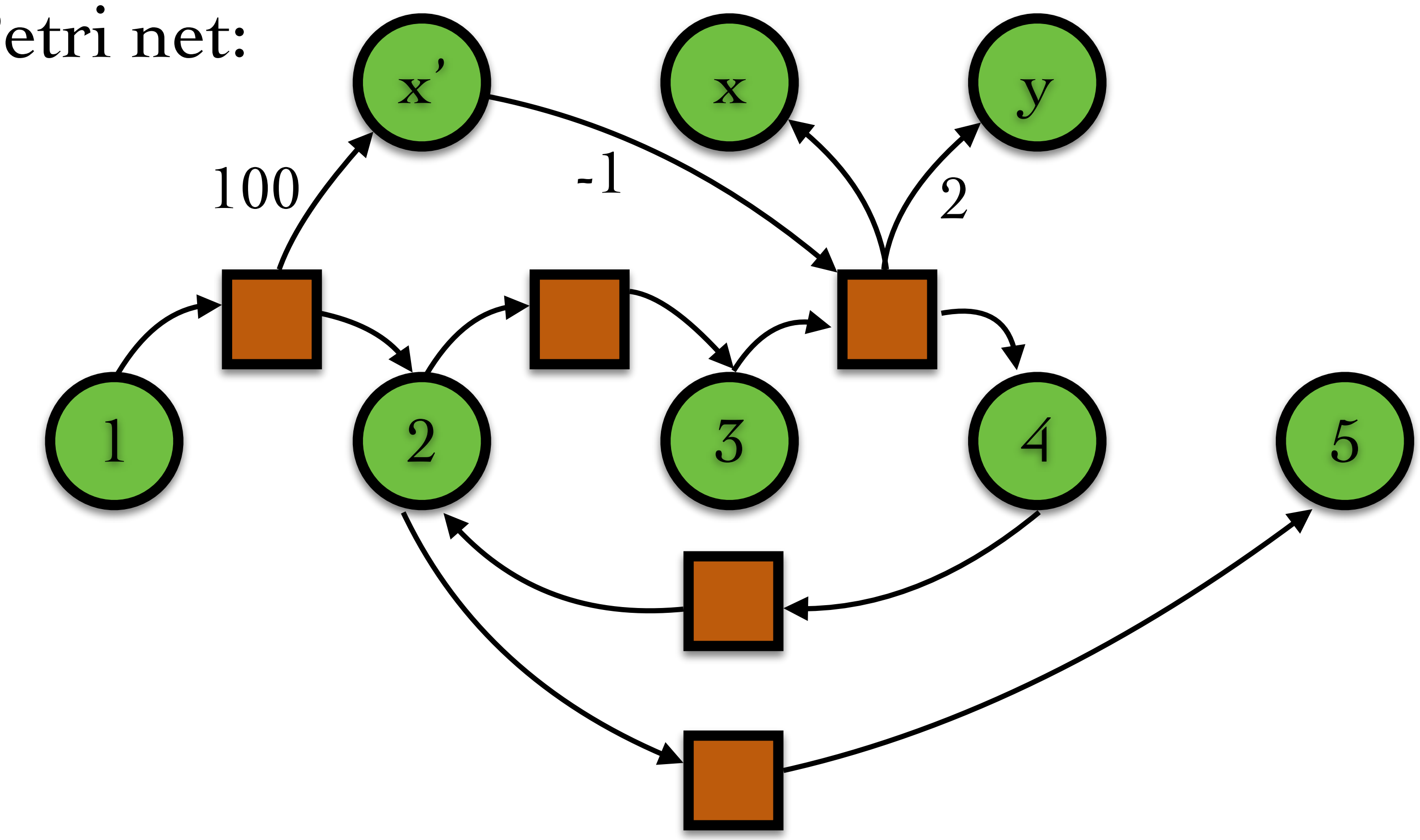

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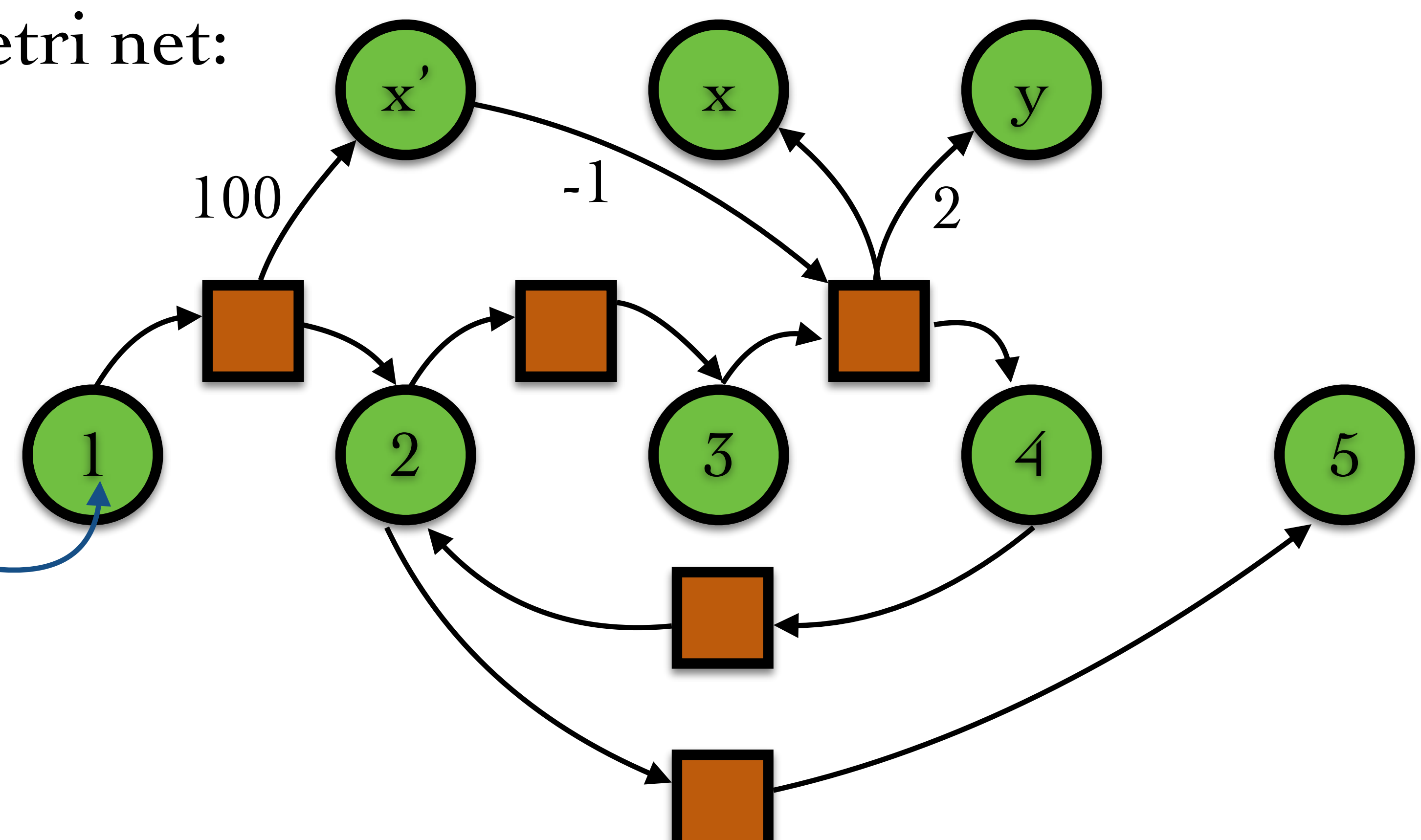

initially one token here

Many faces of Petri nets

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halt requires no token here

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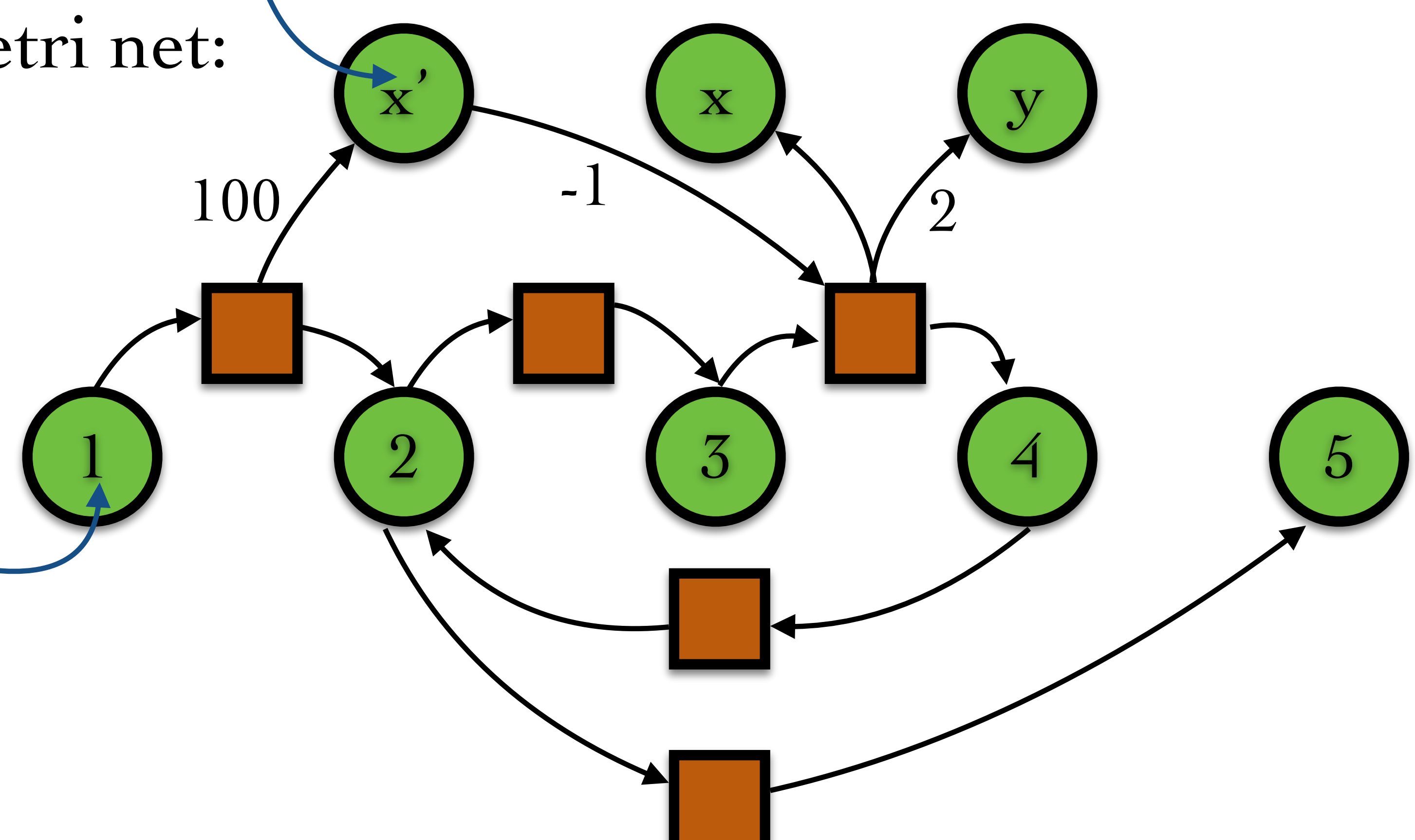

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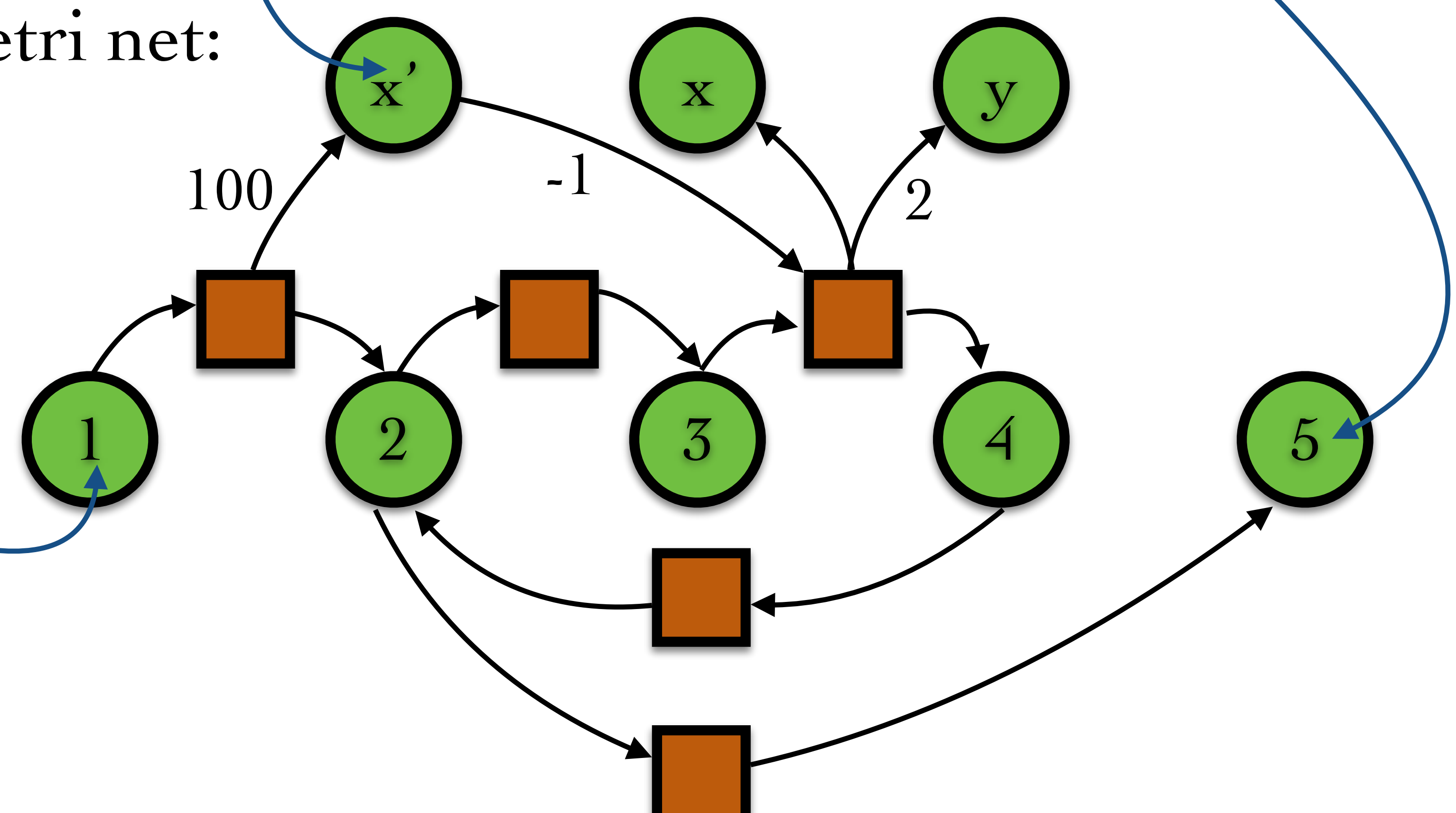
halt requires no token here
and one token there

Petri net:



initially one token here

Reachability and coverability

Reachability problem: given a counter program **without zero tests**

1: $x' \mathrel{+}= 100$
2: **goto** 5 **or** 3
3: $x \mathrel{+}= 1 \quad x' \mathrel{-}= 1 \quad y \mathrel{+}= 2$
4: **goto** 2
5: **halt if** $x' = 0$.

can it terminate (execute its halt command)?

Reachability and coverability

Reachability problem: given a counter program **without zero tests**

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can it terminate (execute its halt command)?

Coverability problem: given a counter program **without zero tests** with trivial halt command

1: $\mathsf{x}' \mathrel{+}= 100$
2: **goto** $\boxed{5}$ **or** $\boxed{3}$
3: $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{y} \mathrel{+}= 2$
4: **goto** $\boxed{2}$
5: **halt**.

can it terminate (reach its halt command)?

Reachability and coverability

Reachability problem: given a counter program **without zero tests**

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configuration reachability

can it terminate (execute its halt command)?

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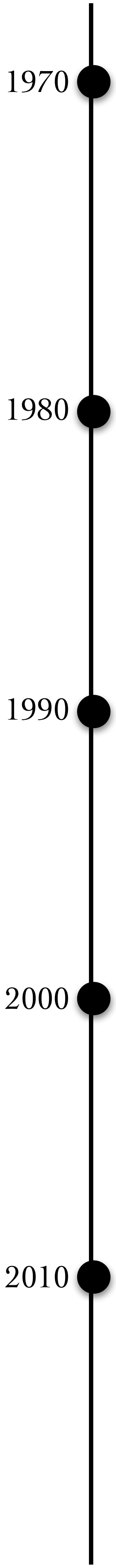
control-state reachability

can it terminate (reach its halt command)?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1990
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1969 decidability of coverability **[Karp, Miller]**

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$\left.\right\}$ KLMST decomposition (1977–1992)

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2019 — TOWER lower bound F_3 **[Czerwiński, L., Lazic, Leroux, Mazowiecki]**

} $F_3 \ldots F_\omega$ gap

TOWER lower bound

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Theorem: The reachability problem for Petri nets is TOWER-hard

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- Petri nets with h+16 places

Computing large numbers

[Mayr, Meyer 1981]: Petri net of size $O(n)$ can **weakly** compute

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Computing large numbers ... or long shortest runs

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has the **shortest** run of length 2^{2^n}

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[Lipton 1976]: Petri net of size $O(\mathrm{n}^2)$ can **exactly** compute 2^{2^n}

has the **shortest** run of length 2^{2^n}

We prove that Petri net of size $O(\mathrm{n})$ can **exactly** compute $\mathrm{TOWER}(n)$

has the **shortest** run of length $\mathrm{TOWER}(n)$

Computing large numbers **... or long shortest runs**

[Mayr, Meyer 1981]: Petri net of size O(n) can **weakly** compute

$$\mathit{Ackermann}(\mathrm{n}) = F_\omega(\mathrm{n}) = \mathrm{F_n}(\mathrm{n})$$

has the **longest** run of length $\mathit{Ackermann}$(n)

[Lipton 1976]: Petri net of size $O(\mathrm{n}^2)$ can **exactly** compute 2^{2^n}

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We prove that Petri net of size O(n) can **exactly** compute $\mathrm{TOWER}(n)$

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    - logics over data words
    - fragments of linear logic
    - process calculi
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  - logics over data words
  - fragments of linear logic
  - process calculi
  - solvability of linear equations with ordered data
- makes obsolete previously known TOWER lower bounds for:
  - branching VASS
  - pushdown VASS

let's embark on the proof…

Loop programs

1: $x' \mathrel{+}= 100$

2: **goto** $\boxed{5}$ **or** $\boxed{3}$

3: $x \mathrel{+}= 1 \quad x' \mathrel{-}= 1 \quad y \mathrel{+}= 2$

4: **goto** $\boxed{2}$

5: **halt if** $x' = 0$.

Loop programs

```
1: x′ += 100
2: goto 5 or 3
3: x += 1    x′ −= 1    y += 2
4: goto 2
5: halt if x′ = 0.
```

```
1: x′ += 100
2: loop
3:     x += 1    x′ −= 1    y += 2
4: halt if x′ = 0.
```

EXPSPACE lower bound for coverability

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- simulation of 2^{2^n}-bounded counter machine **with zero tests**

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- how to implement $\mathbf{Dec}_n$?

Implementation of $\mathbf{Dec}_n$:

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- iterated squaring

$$\overbrace{((2^2)^2\ldots)^2}^{n\text{ times}} \quad = \quad 2^{\overbrace{2\cdot 2\cdot \ldots \cdot 2}^{n\text{ times}}} \quad = \quad 2^{2^n}$$

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- subroutine $\mathbf{Dec}_i\ \mathsf{x}_i$ that decrements x_i exactly 2^{2^i} times, $i = 1 \dots n$

or aborts

- the code of $\mathbf{Dec}_{i+1}\ \hat{\mathsf{x}}_{i+1}$:

```
loop
    x_i += 1    x̂_i −= 1
    loop
        y_i += 1    ŷ_i −= 1
        x̂_{i+1} −= 1    x_{i+1} += 1
    Dec_i y_i
Dec_i x_i.
```

iterated exactly $2^{2^{i+1}}$ times

EXPSPACE lower bound for coverability

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- $\boxed{\text{key idea: compute exactly } 2^{2^n} \text{ due to } \textbf{iterated squaring}\text{:}}$

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TOWER lower bound for reachability

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- simulation of 2^{2^n}-bounded counter program **with zero tests**

TOWER lower bound for reachability

- $\boxed{\text{key idea: compute a pair of numbers with ratio } 3!^n \text{ due to } \textbf{iterated factorial}\text{:}}$

$$3!^n \quad = \quad \overbrace{((3!)!\ldots)!}^{n\text{ times}}$$

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Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

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Let R - fixed positive integer.

Suppose some 3 counters b, c, d are initially set **nondeterministically** to:

$$\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R$$

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How to simulate R-bounded counter program **with zero tests**?

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How to simulate R-bounded counter program **with zero tests**?

The idea:

loop

$\quad \mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$

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at most R iterations

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$$\left.\begin{array}{l}
\textbf{loop} \\
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\quad \mathsf{d} \mathrel{-}= 1 \\
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\end{array}\right\} \text{at most } R \text{ iterations}$$

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The idea:

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$ forward invariant

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at most R iterations

$\mathsf{d} = \mathsf{c} \cdot R$

implied by **halt if** $\ldots, \mathsf{d} = 0.$

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loop
$\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
$\mathsf{d} \mathrel{-}= 1$
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exactly ~~at most~~ R iterations

$\mathsf{d} = \mathsf{c} \cdot R$

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Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

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How to simulate R-bounded counter program **with zero tests**?

The idea:

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$ forward invariant

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1 \\ \quad \boxed{\begin{array}{l} \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1. \end{array}} \end{array}\right\} \text{exactly } \cancel{\text{at most}}\ R \text{ iterations}$$

$\mathsf{d} = \mathsf{c} \cdot R$ backward invariant

implied by **halt if** $\ldots, \mathsf{d} = 0.$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$
with zero tests

Let R - fixed positive integer.

Suppose some 3 counters b, c, d are initially set **nondeterministically** to:

$$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R} \quad \text{ratio } R$$

How to simulate R-bounded counter program **with zero tests**?

The idea:

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} > \mathsf{c} \cdot R$ forward invariant

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$ forward invariant

loop

$\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$

$\mathsf{d} \mathrel{-}= 1$

$\mathsf{c} \mathrel{-}= 1.$

exactly ~~at most~~ R iterations

$\mathsf{d} = \mathsf{c} \cdot R$ backward invariant

implied by **halt if** $\ldots, \mathsf{d} = 0.$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them <u>to at most R</u>:

loop

$\quad \hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \ldots$

$\quad \mathsf{d} \mathrel{-}= 1 \quad \boxed{\mathsf{b} \mathrel{-}= 1}$

$\mathsf{c} \mathrel{-}= 1$

(only for zero-tested counters)

Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

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loop
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$\mathsf{c} \mathrel{-}= 1$

(only for zero-tested counters)

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

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(only for zero-tested counters)

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
- $\mathsf{x} \mathrel{-}= 1$ replace by $\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

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- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
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- **zero?** x replace by

$$\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1 \\ \quad \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \\ \textbf{loop} \\ \quad \mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1 \\ \quad \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}$$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

```
loop
    x̂ += 1    ŷ += 1    . . .
    d −= 1    b −= 1
c −= 1
```

(only for zero-tested counters)

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
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```
loop
    x += 1    x̂ −= 1
    d −= 1
c −= 1
loop
    x −= 1    x̂ += 1
    d −= 1
c −= 1
```

Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

```
loop
    x̂ += 1    ŷ += 1    ...
    d −= 1    b −= 1
c −= 1
```

(only for zero-tested counters)

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
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```
loop
    x += 1    x̂ −= 1
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loop
    x −= 1    x̂ += 1
    d −= 1
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```

forward invariant

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

$$\begin{array}{l} \textbf{loop} \\ \quad \hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \ldots \\ \quad \mathsf{d} \mathrel{-}= 1 \quad \mathsf{b} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}$$

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forward invariant

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$

- extend halt: **halt if** $\ldots, \mathsf{d} = 0.$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$ with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

$$\begin{array}{l} \textbf{loop} \\ \quad \hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \ldots \\ \quad \mathsf{d} \mathrel{-}= 1 \quad \mathsf{b} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}$$

(only for zero-tested counters)

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
- $\mathsf{x} \mathrel{-}= 1$ replace by $\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$
- **zero?** x replace by

$$\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1 \\ \quad \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \\ \textbf{loop} \\ \quad \mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1 \\ \quad \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}$$

forward invariant

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$

- extend halt: **halt if** $\ldots, \mathsf{d} = 0.$

backward invariant

$\mathsf{d} = \mathsf{c} \cdot R$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \cdots \\ \quad \boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1} \quad \boxed{\mathsf{b} \mathrel{-}= 1} \end{array}\right\} \text{exactly } R \text{ iterations}$$

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
- $\mathsf{x} \mathrel{-}= 1$ replace by $\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$
- **zero?** x replace by

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1 \\ \quad \boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1} \end{array}\right\} \text{exactly } R \text{ iterations}$$

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1 \\ \quad \boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1} \end{array}\right\} \text{exactly } R \text{ iterations}$$

- extend halt: **halt if** $\ldots, \mathsf{d} = 0.$

forward invariant

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$

backward invariant

$\mathsf{d} = \mathsf{c} \cdot R$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

loop
$\hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \cdots$
$\boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1} \quad \boxed{\mathsf{b} \mathrel{-}= 1}$

exactly R iterations

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
- $\mathsf{x} \mathrel{-}= 1$ replace by $\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$
- **zero?** x replace by

loop
$\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
$\boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1}$

exactly R iterations

loop
$\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$
$\boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1}$

exactly R iterations

forward invariant
$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$

violation punished at the end

backward invariant
$\mathsf{d} = \mathsf{c} \cdot R$

- extend halt: **halt if** $\ldots, \mathsf{d} = 0.$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

  loop
  $\hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \cdots$
  $\boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1} \quad \boxed{\mathsf{b} \mathrel{-}= 1}$

  exactly R iterations

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
- $\mathsf{x} \mathrel{-}= 1$ replace by $\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$
- **zero?** x replace by

  loop
  $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
  $\boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1}$

  exactly R iterations

  loop
  $\mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1$
  $\boxed{\mathsf{d} \mathrel{-}= 1 \quad \mathsf{c} \mathrel{-}= 1}$

  exactly R iterations

- extend halt: **halt if** $\ldots, \mathsf{d} = 0.$

forward invariant
$\mathsf{x} + \hat{\mathsf{x}} \le R$ and $\mathsf{d} \ge \mathsf{c} \cdot R$

violation punished at the end

backward invariant
$\mathsf{d} = \mathsf{c} \cdot R$

Using ratio R to simulate R-bounded counter program $\mathcal{P}$

with zero tests

$\boxed{\mathsf{b} = R \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R}$

- introduce shadow counters and initiate them to at most R:

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \hat{\mathsf{x}} \mathrel{+}= 1 \quad \hat{\mathsf{y}} \mathrel{+}= 1 \quad \cdots \\ \quad \mathsf{d} \mathrel{-}= 1 \quad \mathsf{b} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}\right\} \text{exactly } R \text{ iterations}$$

- $\mathsf{x} \mathrel{+}= 1$ replace by $\mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1$
- $\mathsf{x} \mathrel{-}= 1$ replace by $\mathsf{x} \mathrel{-}= 1$
- **zero?** x replace by

the construction doesn't depend on R

forward invariant

$\mathsf{x} + \hat{\mathsf{x}} \leq R$ and $\mathsf{d} \geq \mathsf{c} \cdot R$

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{+}= 1 \quad \hat{\mathsf{x}} \mathrel{-}= 1 \\ \quad \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}\right\} \text{exactly } R \text{ iterations}$$

$$\left.\begin{array}{l} \textbf{loop} \\ \quad \mathsf{x} \mathrel{-}= 1 \quad \hat{\mathsf{x}} \mathrel{+}= 1 \\ \quad \mathsf{d} \mathrel{-}= 1 \\ \mathsf{c} \mathrel{-}= 1 \end{array}\right\} \text{exactly } R \text{ iterations}$$

violation punished at the end

backward invariant

$\mathsf{d} = \mathsf{c} \cdot R$

- extend halt: **halt if** $\ldots, \mathsf{d} = 0$.

Computing and using ratio

$$\mathcal{A} \triangleright \mathcal{P}$$

Computing and using ratio

$$\mathcal{A} \triangleright \mathcal{P}$$

R-bounded counter program **with zero tests**
which is simulated using ratio R

Computing and using ratio

$$\mathcal{A} \triangleright \mathcal{P}$$

counter program **without zero tests** that computes ratio R

R-bounded counter program **with zero tests** which is simulated using ratio R

Computing and using ratio

$$\overbrace{\mathcal{A} \triangleright \mathcal{P}}^{\text{counter program } \textbf{without zero tests}}$$

counter program **without zero tests** that computes ratio R

R-bounded counter program **with zero tests** which is simulated using ratio R

Computing and using ratio

$$\overbrace{\mathcal{A} \triangleright \mathcal{P}}^{\substack{\text{counter program}\\ \textbf{without zero tests}}}$$

counter program **without zero tests**
that computes ratio R

R-bounded counter program **with zero tests**
which is simulated using ratio R

merged halts of $\mathcal{A}$ and $\mathcal{P}$

- extend halt: **halt if** $\ldots, \mathsf{d} = 0.$

How to compute ratio?

$$3!^n = \overbrace{((3!)!\ldots)!}^{n\ \text{times}}$$

How to compute ratio?

$$3!^n = \overbrace{((3!)!\ldots)!}^{n\text{ times}}$$

- ratio 3:

```
b += 3
c += 1    d += 3
loop
      c += 1    d += 3
halt.
```

How to compute ratio?

$$3!^n \quad = \quad \overbrace{((3!)!\ldots)!}^{n\text{ times}}$$

- ratio 3:

```
b += 3
c += 1    d += 3
loop
    c += 1    d += 3
halt.
```

- we define a counter program that, using ratio R, computes ratio $R!$

factorial amplifier

How to compute ratio?

$$3!^n = \overbrace{((3!)!\ldots)!}^{n\text{ times}}$$

- ratio 3:

```
b += 3
c += 1    d += 3
loop
     c += 1    d += 3
halt.
```

hece the amplifier can use *R*-bounded zero-tested counters

- we define a counter program that, using ratio R, computes ratio $R!$

factorial amplifier

How to compute ratio?

$$3!^n = \overbrace{((3!)!\ldots)!}^{n \text{ times}}$$

- ratio 3:

```
1: b += 3
2: c += 1    d += 3
3: loop
4:     c += 1    d += 3
5: halt.
```

hece the amplifier can use
R-bounded zero-tested
counters

- we define a counter program that, using ratio R, computes ratio $R!$

factorial amplifier

- and self-compose it sufficiently many times: $\overbrace{((\mathcal{A}_3 \triangleright \mathcal{F}) \triangleright \mathcal{F}) \triangleright \cdots \mathcal{F}}^{n \text{ compositions}}$

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio R!

$$\frac{2}{1} \quad \cdot \quad \frac{3}{2} \quad \cdot \quad \ldots \quad \cdot \quad \frac{R}{R-1} \quad = \quad R$$

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \quad \cdot \quad \frac{3}{2} \quad \cdot \quad \ldots \quad \cdot \quad \frac{R}{R-1} \quad = \quad R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x −= i    x′ += i + 1
    loop
        x′ −= 1    x += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

a zero test

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x -= i    x' += i + 1
    loop
        x' -= 1    x += 1
    i += 1
zero? î
loop
    x -= i    y -= 1
halt if y = 0
```

initially equal R

a zero test

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x -= i    x' += i + 1
    loop
        x' -= 1    x += 1
    i += 1
zero? i
loop
    x -= i    y -= 1
halt if y = 0
```

initially equal R

a zero test

```
loop
    i -= 1    i' += 1    x -= 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

further zero tests

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x −= i    x′ += i + 1
    loop
        x′ −= 1    x += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

initially equal R

a zero test

```
loop
    i −= 1    i′ += 1    x −= 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

further zero tests

```
x′ += 1
loop
    i −= 1    i′ += 1    x′ += 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop                    nondeterministic init
    x += 1    y += 1
loop
    loop
        x −= i    x′ += i + 1
    loop
        x′ −= 1    x += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

initially equal R

a zero test

```
loop
    i −= 1    i′ += 1    x −= 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

further zero tests

```
x′ += 1
loop
    i −= 1    i′ += 1    x′ += 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop            nondeterministic init
    x += 1    y += 1
loop
    loop
        x -= i    x' += i + 1     weak multiplication by (i+1)/i
    loop
        x' -= 1    x += 1
    i += 1
zero? i
loop
    x -= i    y -= 1
halt if y = 0
```

initially equal R

a zero test

```
loop
    i -= 1    i' += 1    x -= 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

further zero tests

```
x' += 1
loop
    i -= 1    i' += 1    x' += 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x -= i    x' += i + 1
    loop
        x' -= 1    x += 1
    i += 1
zero? i
loop
    x -= i    y -= 1
halt if y = 0
```

nondeterministic init

weak multiplication by $\frac{i+1}{i}$

initially equal R

a zero test

tests if $x \geq y \cdot R$

```
loop
    i -= 1    i' += 1    x -= 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

further zero tests

```
x' += 1
loop
    i -= 1    i' += 1    x' += 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

Factorial amplifier - the idea

counter program that, using ratio R, computes ratio $R!$

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x -= i    x' += i + 1
    loop
        x' -= 1    x += 1
    i += 1
zero? î
loop
    x -= i    y -= 1
halt if y = 0
```

nondeterministic init

exact ~~weak~~ multiplication by $\frac{i+1}{i}$

initially equal R

a zero test

tests if $x \geq y \cdot R$

```
loop
    i -= 1    i' += 1    x -= 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

further zero tests

```
x' += 1
loop
    i -= 1    i' += 1    x' += 1
zero? i
loop
    i' -= 1    i += 1
zero? i'
```

Factorial amplifier - the idea

counter program th
ratio R

fine, but where is the factorial?

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1} = R$$

```
i += 1    x += 1    y += 1
loop
    x += 1    y += 1
loop
    loop
        x −= i    x′ += i + 1
    loop
        x′ −= 1    x += 1
    i += 1
zero? i
loop
    x −= i    y −= 1
halt if y = 0
```

nondeterministic init

initially equal R

a zero test

exact ~~weak~~ multiplication by $\frac{i+1}{i}$

tests if $x \geq y \cdot R$

```
loop
    i −= 1    i′ += 1    x −= 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

further zero tests

```
x′ += 1
loop
    i −= 1    i′ += 1    x′ += 1
zero? i
loop
    i′ −= 1    i += 1
zero? i′
```

Factorial amplifier - the idea

counter program th
ratio R

fine, but where is the factorial?

$$\boxed{\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{R}{R-1}} = R$$

```
i += 1   x += 1   y += 1
loop
    x += 1   y += 1
loop
    loop
        x −= i   x′ += i + 1
    loop
        x′ −= 1   x += 1
    i += 1
zero? î
loop
    x −= i   y −= 1
halt if y = 0
```

nondeterministic init

exact ~~weak~~ multiplication by $\frac{i+1}{i}$

initially equal R

a zero test

tests if $x \geq y \cdot R$

```
loop
    i −= 1   i′ += 1   x −= 1
zero? i
loop
    i′ −= 1   i += 1
zero? i′
```

further zero tests

```
x′ += 1
loop
    i −= 1   i′ += 1   x′ += 1
zero? i
loop
    i′ −= 1   i += 1
zero? i′
```

Factorial amplifier $\mathsf{b} = R! \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

Factorial amplifier $b = R! \quad c > 0 \quad d = c \cdot R!$

counter program that, using ratio R, computes ratio $R!$

```
i += 1   x += 1   y += 1   b += 1   c += 1   d += 1
loop
    x += 1   y += 1   c += 1   d += 1
loop
    loop
        c -= i   c' += 1
        loop at most b times
            x -= i   d -= i   x' += i + 1
    loop
        b -= 1   b' += i + 1
    loop
        b' -= 1   b += 1
    loop
        c' -= 1   c += 1
        loop at most b times
            x' -= 1   x += 1   d += 1
    i += 1
zero? î
loop
    x -= i   y -= 1
halt if y = 0
```

```
loop
    b -= 1   b' += 1
loop
    b' -= 1   b += 1
    <body>
```

Factorial amplifier $\mathsf{b} = R!$ $\quad \mathsf{c} > 0$ $\quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

invariant
$\boxed{\mathsf{d} = \mathsf{c} \cdot \mathsf{b}}$

```
i += 1   x += 1   y += 1   b += 1   c += 1   d += 1
loop
    x += 1   y += 1   c += 1   d += 1
loop
    loop
        c −= i   c′ += 1
        loop at most b times
            x −= i   d −= i   x′ += i + 1
    loop
        b −= 1   b′ += i + 1
    loop
        b′ −= 1   b += 1
    loop
        c′ −= 1   c += 1
        loop at most b times
            x′ −= 1   x += 1   d += 1
    i += 1
zero? î
loop
    x −= i   y −= 1
halt if y = 0
```

```
loop
    b −= 1   b′ += 1
loop
    b′ −= 1   b += 1
    <body>
```

Factorial amplifier $\mathsf{b} = R! \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

invariant
$\boxed{\mathsf{d} = \mathsf{c} \cdot \mathsf{b}}$

```
i += 1   x += 1   y += 1   b += 1   c += 1   d += 1
loop                                  nondeterministic init
    x += 1   y += 1   c += 1   d += 1
loop
    loop
        c −= i   c′ += 1
        loop at most b times
            x −= i   d −= i   x′ += i + 1
    loop
        b −= 1   b′ += i + 1
    loop
        b′ −= 1   b += 1
    loop
        c′ −= 1   c += 1
        loop at most b times
            x′ −= 1   x += 1   d += 1
    i += 1
zero? î
loop
    x −= i   y −= 1
halt if y = 0
```

```
loop
    b −= 1   b′ += 1
loop
    b′ −= 1   b += 1
    <body>
```

Factorial amplifier $\mathsf{b} = R!$ $\quad \mathsf{c} > 0$ $\quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

invariant
$\boxed{\mathsf{d} = \mathsf{c} \cdot \mathsf{b}}$

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop                                        nondeterministic init
    x += 1    y += 1    c += 1    d += 1
loop
    loop
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1
    loop
        b −= 1    b′ += i + 1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

weak multiplication by $\frac{\mathsf{i}+1}{\mathsf{i}}$

Factorial amplifier $\mathsf{b} = R!$ $\quad \mathsf{c} > 0$ $\quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

invariant
$\mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop                                        nondeterministic init
    x += 1    y += 1    c += 1    d += 1
loop
    loop
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1      weak multiplication by (i+1)/i
    loop
        b −= 1    b′ += i + 1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

tests if $\mathsf{x} \geq \mathsf{y} \cdot R$

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

Factorial amplifier $\mathsf{b} = R!$ $\quad \mathsf{c} > 0$ $\quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

$$\boxed{\text{invariant} \quad \mathsf{d} = \mathsf{c} \cdot \mathsf{b}}$$

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop                                        nondeterministic init
    x += 1    y += 1    c += 1    d += 1
loop
    loop
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1
    loop
        b −= 1    b′ += i + 1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

exact ~~weak~~ multiplication by $\frac{\mathsf{i}+1}{\mathsf{i}}$

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

tests if $\mathsf{x} \geq \mathsf{y} \cdot R$

Factorial amplifier $\mathsf{b} = R!$ $\quad \mathsf{c} > 0$ $\quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

$\boxed{\text{invariant } \mathsf{d} = \mathsf{c} \cdot \mathsf{b}}$

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop                                          nondeterministic init
    x += 1    y += 1    c += 1    d += 1
loop
    loop      weak division by i
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1      exact ~~weak~~ multiplication by (i+1)/i
    loop
        b −= 1    b′ += i + 1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
zero? î
loop
    x −= i    y −= 1
halt if y = 0
```

tests if $\mathsf{x} \geq \mathsf{y} \cdot R$

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

Factorial amplifier $b = R! \quad c > 0 \quad d = c \cdot R!$

counter program that, using ratio R, computes ratio $R!$

```
i += 1   x += 1   y += 1   b += 1   c += 1   d += 1
loop                                   nondeterministic init
    x += 1   y += 1   c += 1   d += 1
loop
    loop
        c −= i   c′ += 1
        loop at most b times
            x −= i   d −= i   x′ += i + 1
    loop
        b −= 1   b′ += i + 1
    loop
        b′ −= 1   b += 1
    loop
        c′ −= 1   c += 1
        loop at most b times
            x′ −= 1   x += 1   d += 1
    i += 1
zero? î
loop
    x −= i   y −= 1
halt if y = 0
```

exact ~~weak~~ division by i

exact ~~weak~~ multiplication by $\frac{i+1}{i}$

tests if $x \geq y \cdot R$

invariant
$d = c \cdot b$

```
loop
    b −= 1   b′ += 1
loop
    b′ −= 1   b += 1
    <body>
```

Factorial amplifier $\mathsf{b} = R!\quad \mathsf{c} > 0\quad \mathsf{d} = \mathsf{c}\cdot R!$

counter program that, using ratio R, computes ratio $R!$

invariant
$\mathsf{d} = \mathsf{c}\cdot\mathsf{b}$

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop                                        nondeterministic init
    x += 1    y += 1    c += 1    d += 1
loop
    loop                 exact ~~weak~~ division by i
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1      exact ~~weak~~ multiplication by (i+1)/i
    loop
        b −= 1    b′ += i + 1                    weak multiplication by i+1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
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loop
    x −= i    y −= 1
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tests if $\mathsf{x} \geq \mathsf{y}\cdot R$

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

Factorial amplifier $\mathsf{b} = R! \quad \mathsf{c} > 0 \quad \mathsf{d} = \mathsf{c} \cdot R!$

counter program that, using ratio R, computes ratio $R!$

invariant $\boxed{\mathsf{d} = \mathsf{c} \cdot \mathsf{b}}$

```
i += 1    x += 1    y += 1    b += 1    c += 1    d += 1
loop                                        nondeterministic init
    x += 1    y += 1    c += 1    d += 1
loop
    loop                  exact ~~weak~~ division by i
        c −= i    c′ += 1
        loop at most b times
            x −= i    d −= i    x′ += i + 1      exact ~~weak~~ multiplication by (i+1)/i
    loop                                          exact ~~weak~~ multiplication by i+1
        b −= 1    b′ += i + 1
    loop
        b′ −= 1    b += 1
    loop
        c′ −= 1    c += 1
        loop at most b times
            x′ −= 1    x += 1    d += 1
    i += 1
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loop
    x −= i    y −= 1
halt if y = 0
```

tests if $\mathsf{x} \geq \mathsf{y} \cdot R$

```
loop
    b −= 1    b′ += 1
loop
    b′ −= 1    b += 1
    <body>
```

Factorial amplifier $b = R!$ $\quad c > 0 \quad d = c \cdot R!$

counter program that, using ratio R, computes ratio $R!$

```
i += 1   x += 1   y += 1   b += 1   c += 1   d += 1
loop                                   nondeterministic init
    x += 1   y += 1   c += 1   d += 1
loop
    loop            exact ~~weak~~ division by i
        c -= i   c' += 1
        loop at most b times
            x -= i   d -= i   x' += i + 1      exact ~~weak~~ multiplication by (i+1)/i
    loop                                       exact ~~weak~~ multiplication by i+1
        b -= 1   b' += i + 1
    loop
        b' -= 1   b += 1
    loop
        c' -= 1   c += 1
        loop at most b times
            x' -= 1   x += 1   d += 1
    i += 1
zero? î
loop
    x -= i   y -= 1
halt if y = 0
```

tests if $x \geq y \cdot R$

```
loop
    b -= 1   b' += 1
loop
    b' -= 1   b += 1
    <body>
```

invariant
$\boxed{d = c \cdot b}$

$\boxed{b = R!}$

Future work

Future work

- TOWER…ACKERMANN gap $\qquad$ F3…Fω gap

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- TOWER…ACKERMANN gap $F_3 \ldots F_\omega$ gap
- improving the lower bound? TOWER amplifier?

Future work

- TOWER…ACKERMANN gap

F3…Fω gap

- improving the lower bound? TOWER amplifier?
- better lower bounds for
  - branching VASS
  - pushdown VASS
  - VASS with 1 zero test
  - VASS with hierarchical zero tests

Future work

- TOWER…ACKERMANN gap

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- refined analysis for fixed dimension

Future work

- TOWER…ACKERMANN gap $\qquad$ $F_3 \ldots F_\omega$ gap
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Future work

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thank you!