Orbit-finite linear programming

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I. Overview

II. The problem, formally

III. An algorithm, sketched ideas

IV. Open questions

$$\begin{array}{cc} & \overbrace{\hphantom{0 \quad 1 \quad 1 \quad \cdots}}^{\mathbb{A}} \\ \mathbb{A}\left\{\vphantom{\begin{matrix}0\\1\\1\\\vdots\end{matrix}}\right. & \begin{bmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} \end{array} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

$$\sum_{a \in \mathbb{A} \setminus b} a \geq 1 \qquad (b \in \mathbb{A})$$

Invariant under all permutations $\pi : \mathbb{A} \to \mathbb{A}$

$$\sum_{a \in \mathbb{A} \setminus b} a \geq 1 \longmapsto \sum_{a \in \mathbb{A} \setminus \pi(b)} a \geq 1$$

- Is it solvable?

$$\begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix} \leftarrow \text{minimize}$$

$$\begin{bmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

$S = \{a_1 \dots a_n\} \subseteq \mathbb{A}$

$$x_n(a) = \begin{cases} \frac{1}{n-1} & a \in S \\ 0 & a \notin S \end{cases}$$

- What is the minimal value of objective function?
- Is the minimal value achieved?

Results :

Orbit-finite linear programming is computable.

Orbit-finite <u>integer</u> linear programming is not.

$$\begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix} \leftarrow \text{minimize}$$

$$\begin{bmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \geqslant \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

$$\longmapsto$$

$$\begin{bmatrix} n \end{bmatrix} \leftarrow \text{minimize}$$

$$\begin{bmatrix} n-1 \\ n \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} \geqslant \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

An <u>invariant</u> linear program always has an <u>invariant</u> solution/optimum!

Motivation

- Data-enriched models:
  - Petri nets with data
  - weighted register automata

    [Bojańczyk, Klin, Moerman 2021]

- Solvability of equations

  [Ghosh, Hofman, L. 2022]

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

- Linear algebra in orbit-finite dimension

II. The problem, formally

Fix <u>atoms</u> $\mathbb{A} = (\{1,2,\ldots\}, =)$

orbit-finite sets $\stackrel{\text{def.}}{=}$ **sets finite up to automorphisms of $\mathbb{A}$ ***

$\quad\quad\quad\quad\quad\quad\quad\quad ||$

(disjoint) **finite unions of orbits**

<u>Examples:</u>

- $\mathbb{A}^2$ is the union of 2 orbits:

  $\{aa : a \in \mathbb{A}\}$

  $\mathbb{A}^{(2)} = \{ab : a, b \in \mathbb{A},\ a \neq b\}$

- $\mathbb{A}^3$ is the union of 5 orbits
- 2-sets $\binom{\mathbb{A}}{2}$ is 1 orbit
- $\{\emptyset\}$

support

*up to automorphisms of $\mathbb{A}$ that fix some finite subset $S \subseteq \mathbb{A}$

Orbit-finite sets

$\mathbb{A} = (\{1,2,\dots\}, =)$

$S \subseteq_{fin} \mathbb{A}$ support ($S = \emptyset$ – equivariant sets)

(disjoint) finite unions of S-orbits

Examples:

- $\{\underline{1}\, b : a, b \in \mathbb{A}, \underline{1} \neq b \neq \underline{2}\}$ $S = \{\underline{1}, \underline{2}\} \subseteq \mathbb{A}$
- $\{\underline{1}\,\underline{2}, \underline{2}\,\underline{3}, \underline{3}\,\underline{1}\}$

sets with atoms [Bojańczyk, Klin, L. 2011]

nominal sets [Gabbay, Pitts 1999] [Pistore 1999]

Fraenkel-Mostowski sets [Fraenkel 1922]

Orbit-finite vectors

$\mathbb{A} = (\{1, 2, \dots\}, =)$ fixed

$v : B \to \mathbb{R}$

orbit-finite index set

orbit-finite function

$v : B \to \mathbb{Z}$

integer vectors

Examples:

$B = \mathbb{A}^{(2)}$

orbit-finite $v : \begin{array}{ll} (1, b) \mapsto 0.5 & \\ (a, b) \mapsto 0 & a \neq 1 \end{array}$ $\quad S = \{1\} \subseteq \mathbb{A}$

finite $v : \begin{array}{ll} (1, 2) \mapsto 0.5 & \\ (a, b) \mapsto 0 & (a, b) \neq (1, 2) \end{array}$ $\quad S = \{1, 2\} \subseteq \mathbb{A}$

Orbit-finite linear programming

column index is orbit-finite = set of variables

orbit-finite

$$\begin{bmatrix} \quad c \quad \end{bmatrix} \leftarrow \text{maximize}$$

$$\text{row index is orbit-finite}\left\{\overbrace{\begin{bmatrix} \quad A \quad \end{bmatrix}}^{\text{column index is orbit-finite}}\right. \cdot \left.\begin{bmatrix} x \end{bmatrix}\right\}\text{set of variables} \quad \le \quad \left.\begin{bmatrix} b \end{bmatrix}\right\}$$

matrix and rhs vector are orbit-finite

Example:

$$
\begin{array}{ccc}
 & \mathbb{A}^{(2)} & \mathbb{A} \\
 & \left[\begin{array}{cccc|ccc} 0 & 0 & 0 \cdots & 0\,0 & 1 & 1\cdots & 1 \end{array}\right] & \leftarrow \text{maximize}
\end{array}
$$

$$
\begin{array}{c}
\mathbb{A}
\end{array}
\left[\begin{array}{cccc|ccc}
1\,1\,1 & -1-1-1 & & 0 & 1 & & 0 \\
 & 1\,1\,1 & -1-1-1 & & & 1 & \\
 & & \ddots & & & & \ddots \\
0 & & & & 0 & & 1 \\
\hline
1\,1\,1 & \cdots & \cdots & 1\,1 & & & \\
\hline
-1-1-1 & \cdots & \cdots & -1-1 & -1-1 & \cdots & -1
\end{array}\right]
\cdot
\left[\begin{array}{c} \\ x \\ \hline \\ \end{array}\right]
\leq
\left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 1 \\ \hline -2 \end{array}\right]
$$

$$\sum_{b \neq a} ba + a \leq \sum_{b \neq a} ab \quad (a \in \mathbb{A})$$

$$\sum_{b \neq a} ab \leq 1$$

$$\sum_{b \neq a} ab + \sum_{a} a \geq 2$$

$$
\begin{tikzcd}
b \arrow[dr] & & b \\
c \arrow[r] & a \arrow[ur] \arrow[r] \arrow[dr] & c \\
d \arrow[ur] & & d \\
\vdots & & \vdots
\end{tikzcd}
$$

Orbit-finite linear programming

Input: integer vectors b, c and matrix A • how represented?

$$\begin{bmatrix} \quad c \quad \end{bmatrix} \leftarrow \text{maximize}$$

$$\begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} \leq \begin{bmatrix} b \end{bmatrix}$$

Output: the supremum

$\mathbb{Q}$ is sufficient

$$\sup \left\{ c \cdot x \; : \; \begin{array}{l} A \cdot x \leq b \\ x \text{ orbit-finite} \end{array} \right\} \in \mathbb{Q} \cup \{-\infty, +\infty\}$$

in this case, whether the supremum is achieved

optimisation / solvability

Representation of input

Support

orbit-finite set $=$ (disjoint) finite union of S-orbits

atom dimension

$$S\text{-orbit} \approx (\mathbb{A} \setminus S)^{(d)} / G \qquad G \leq \mathrm{Sym}_d$$

$$\boxed{\text{think of: } \mathbb{A}^{(d)} \quad \text{(non-repeating } d\text{-tuples)}}$$

definable sets [Bojańczyk, Toruńczyk 2018]

Variants:

$$\begin{bmatrix} \quad c \quad \end{bmatrix} \leftarrow \text{maximize}$$

$$\begin{bmatrix} \\ \quad A \quad \\ \\ \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

$\mathbb{R}$ / $\mathbb{Z}$

orbit-finite / **finite**

$\leq$ / $=$ and $x \geq 0$

Orbit-finite (integer) linear programming

Results :

Orbit-finite linear programming is computable.

Orbit-finite <u>integer</u> linear programming is not.

Previous work

[Klin, Kopczyński, Ochremiak, Toruńczyk 2015]

restrictions on row index

[Hofman, Leroux, Totzke 2015]

[Hofman, Różycki 2021]

$$\begin{bmatrix} \; A \; \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

rows finite

columns finite

finite solvability over $\mathbb{Q}$, $\mathbb{Z}$

unrestricted solvability over finite fields

[Ghosh, Hofman, L. 2022]

orbit-finite solvability over any commutative ring

III. An algorithm, sketched ideas

Give me a support (size) of solution,
and I will solve an orbit-finite system
of linear inequalities

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

An algorithm:

orbit-finite linear programming

finite polynomially-parametrized linear programming

blow up exponential in atom dimension

by support size $n = |S|$ of solution

first-order real arithmetic

EXPTIME

Poly-parametrized linear programming

$n \in \mathbb{N}$

$$\begin{bmatrix} n^2-1 & 2n^3 & \cdots \end{bmatrix} \leftarrow \text{maximize}$$

$$\begin{bmatrix} n+2 & n^3 & \cdots \\ n^2 & 2n-3 & \\ \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} -n \\ 3n^3 \\ \vdots \end{bmatrix}$$

- n fixed $\longmapsto$ finite linear program
- monotonicity
- solvability : <u>for some</u> $n \in \mathbb{N}$ ...
- optimisation : compute $\sup\{\dots\}$ <u>over all</u> $n \in \mathbb{N}$

Encode into real arithmetic

$n \in \mathbb{N}$

$\begin{bmatrix} n^2-1 & 2n^3 & \cdots \end{bmatrix}$ ← maximize

$$\begin{bmatrix} n+2 & n^3 & \cdots \\ n^2 & 2n-3 & \\ \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} -n \\ 3n^3 \\ \vdots \end{bmatrix}$$

- solvability : for some $n \in \mathbb{N}$ ...
- optimisation : compute $\sup\{\ldots\}$ over all $n \in \mathbb{N}$

for all sufficiently large $n \in \mathbb{N}$

$$\exists r \in \mathbb{R}.\ \forall n \in \mathbb{N}.\ n > r \Rightarrow \ldots$$
$$\Vert\!\!\!\wr$$
$$\exists r \in \mathbb{R}.\ \forall n \in \mathbb{R}.\ n > r \Rightarrow \ldots$$

$\boxed{\text{KEY OBSERVATION}}$

An algorithm:

orbit-finite linear programming

⇝

finite polynomially-parametrized linear programming

by support size $n = |S|$ of solution

blow up exponential in atom dimension

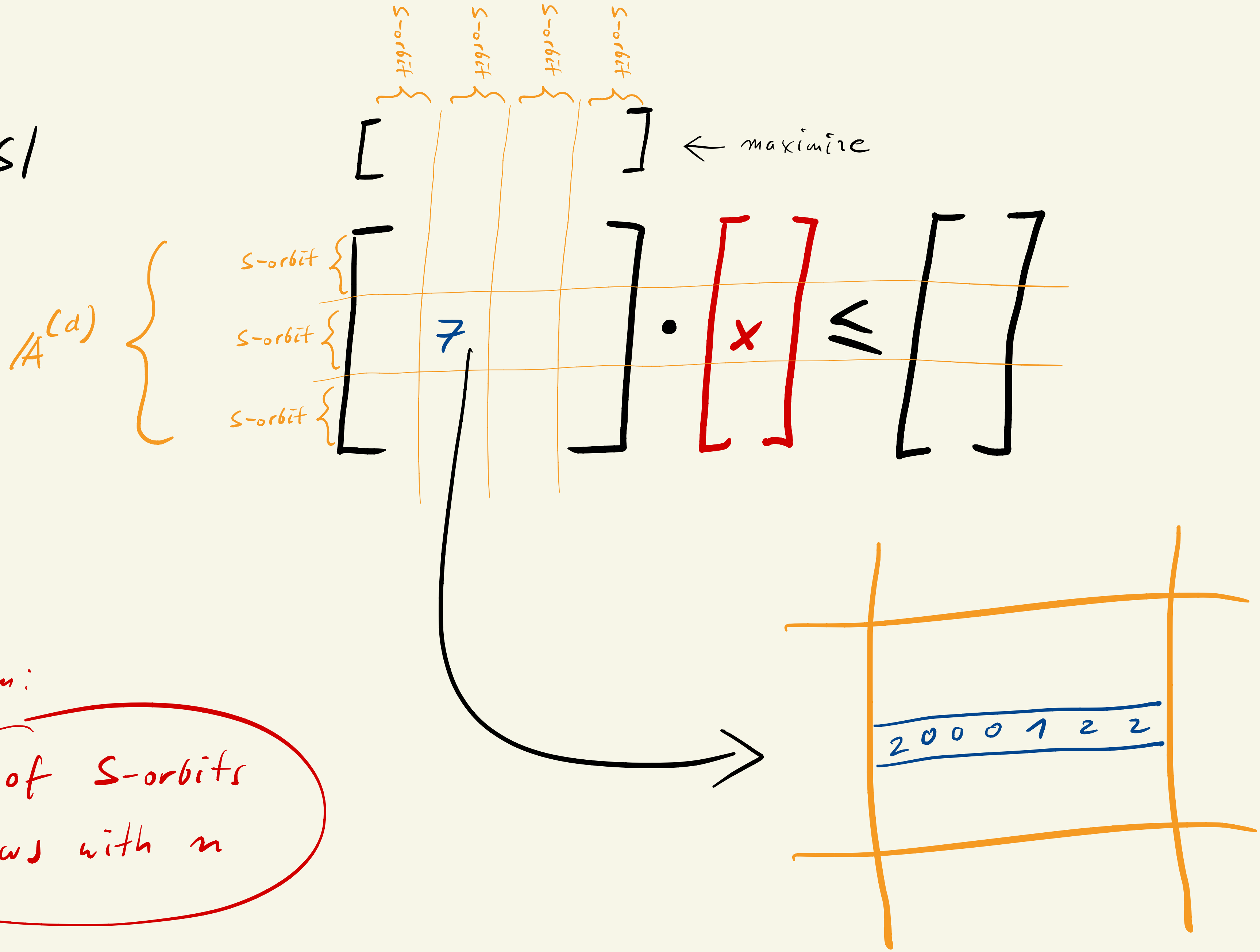
⇝

first-order real arithmetic

EXPTIME

Aggregate ?

$n = |S|$



Problem:

nr of S-orbits grows with n

Solution: Modify the concept of S-orbit

$S = \{ \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9} \}$ $\quad n = 9$

S-orbits in $\mathbb{A}^{(5)}$ = invariant under S-permutations

fixes every element of S

$\rightsquigarrow$

fixes S as a set

$(\_, \underline{1}, \underline{2}, \_, \underline{3})$

$(\_, \underline{4}, \underline{5}, \_, \underline{6})$

$(\underline{1}, \_, \underline{2}, \_, \_)$

$(\_, \underline{4}, \underline{8}, \_, \_)$

...

$\rightsquigarrow$

$(\_, \in S, \in S, \_, \in S)$

$(\in S, \_, \in S, \_, \_)$

$(\_, \in S, \in S, \_, \_)$

...

$n(n-1)(n-2)$ many S-orbits aggregated together

KEY OBSERVATION:
the modification preserves solvability/optimum

S-permutations

fixes every element of S

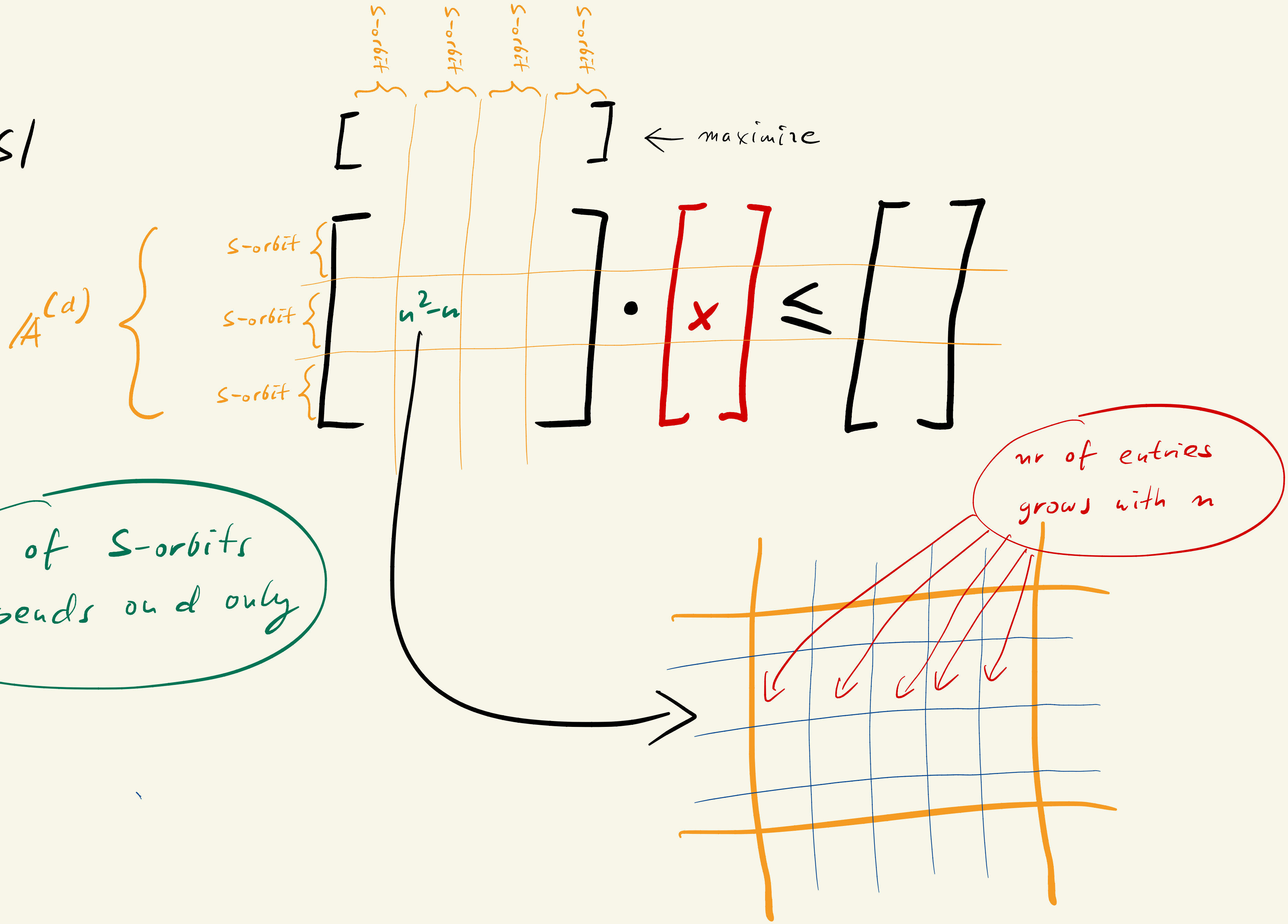
fixes S as a set

KEY OBSERVATION:
the modification preserves solvability/optimum

An invariant linear program always has an invariant solution/optimum!

Aggregate well

$n = |S|$



nr of S-orbits depends on d only

Example:

An invariant linear program always has an invariant solution/optimum!

$$\begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix} \leftarrow \text{minimize}$$

$$\begin{bmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \geqslant \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

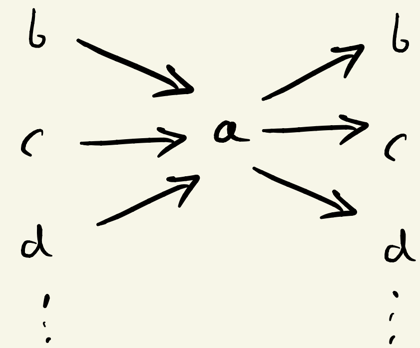
$$\begin{bmatrix} n \end{bmatrix} \leftarrow \text{minimize}$$

$$\begin{bmatrix} n-1 \\ n \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} \geqslant \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example:

An invariant linear program always has an invariant solution/optimum!

$$
\begin{array}{c}
\begin{array}{cc} \mathbb{A}^{(2)} & \mathbb{A} \end{array} \\
\mathbb{A}\left[\begin{array}{c|c}
1\,1\,1 - 1 - 1 - 1 & 1 \\
\quad 1\,1\,1 - 1 - 1 - 1 & \quad 1 \\
\qquad \ddots & \qquad \ddots \\
 & \qquad\qquad 1 \\
\hline
1\,1\,1 \cdots\cdots 1\,1 & \\
\hline
1\,1\,1 \cdots\cdots 1\,1 & 1\,1 \cdots 1
\end{array}\right]
\end{array}
\cdot \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 2 \end{bmatrix}
$$



$$\sum_{b \neq a} ba + a = \sum_{b \neq a} ab \quad (a \in \mathbb{A})$$

$$\sum_{b \neq a} ab = 1$$

$$\sum_{b \neq a} ab + \sum_{a} a = 2$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ n(n-1) & n \\ n(n-1) & 0 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

IV. Open questions

- richer atoms, e.g. $\mathbb{A} = (\mathbb{Q}, \leq)$
- representation of the whole solution set
- duality
- non-orbit-finite solutions
- exact complexity