Separability of Reachability Sets of Vector Addition Systems*

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Abstract

Given two families of sets $\mathcal{F}$ and $\mathcal{G}$, the $\mathcal{F}$-separability problem for $\mathcal{G}$ asks whether for two given sets $U, V \in \mathcal{G}$ there exists a set $S \in \mathcal{F}$, such that U is included in S and V is disjoint with S. We consider two families of sets $\mathcal{F}$: modular sets $S \subseteq \mathbb{N}^d$, defined as unions of equivalence classes modulo some natural number $n \in \mathbb{N}$, and unary sets, which extend modular sets by requiring equality below a threshold n, and equivalence modulo n above n. Our main result is decidability of modular and unary separability for the class $\mathcal{G}$ of reachability sets of Vector Addition Systems, Petri Nets, Vector Addition Systems with States, and for sections thereof.

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1 Introduction

We investigate separability problems for sets of vectors from $\mathbb{N}^d$. We say that a set U is *separated from* a set V by a set S if $U \subseteq S$ and $V \cap S = \emptyset$. For two families of sets $\mathcal{F}$ and $\mathcal{G}$, the *$\mathcal{F}$-separability problem for $\mathcal{G}$* asks for two given sets $U, V \in \mathcal{G}$ whether U is separated from V by some set from $\mathcal{F}$. Concretely, we consider $\mathcal{F}$ to be modular sets or unary sets[1], and $\mathcal{G}$ to be reachability sets of Vector Addition Systems or generalizations thereof.

Motivation. The separability problem is a classical problem in theoretical computer science. It was investigated most extensively in the area of formal languages, for $\mathcal{G}$ being the family of all regular word languages. Since regular languages are effectively closed under complement, the $\mathcal{F}$-separability problem is a generalization of the $\mathcal{F}$-characterization problem, which asks whether a given language belongs to $\mathcal{F}$. Indeed, $L \in \mathcal{F}$ if and only if L is separated from its complement by some language from $\mathcal{F}$. Separability problems for regular languages attracted recently a lot of attention, which resulted in establishing the decidability of $\mathcal{F}$-separability for the family $\mathcal{F}$ of separators being the piecewise testable languages [2, 22] (recently generalized to finite ranked trees [5]), the locally and locally threshold testable languages [21], the

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[1] Since S separates U from V iff its complement separates V from U, and since $\mathcal{F}$ is closed under complement, we could equally well have defined a symmetric version of the separability problem by saying that S *separates* $\{U, V\}$ iff $U \subseteq S$ and $V \cap S = \emptyset$.

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languages definable in first order logic [24], and the languages of certain higher levels of the first order hierarchy [23], among others.

Separability of nonregular languages attracted little attention till now. The reasons for this are twofold. First, for regular languages one can use standard algebraic tools, like syntactic monoids, and indeed most of the results have been obtained with the help of such techniques. Second, some strong intractability results have been known already since the 70's, when Szymanski and Williams proved that regular separability of context-free languages is undecidable [25]. Later Hunt [10] generalized this result: he showed that $\mathcal{F}$-separability of context-free languages is undecidable for every class $\mathcal{F}$ which is closed under finite boolean combinations and contains all languages of the form $w\Sigma^*$ for $w \in \Sigma^*$. This is a very weak condition, so it seemed that nothing nontrivial can be done outside regular languages with respect to separability problems. Furthermore, Kopczyński has recently shown that regular separability is undecidable even for languages of visibly pushdown automata [12], thus strengthening the result by Szymanski and Williams. On the positive side, piecewise testable separability has been shown decidable for context-free languages, languages of Vector Addition Systems (VAS languages), and some other classes of languages [3]. This inspired us to start a quest for decidable cases beyond regular languages.

To the best of our knowledge, beside [3], separability problems for VAS languages have not been investigated before.

Our contribution. In this paper, we make a substantial step towards solving regular separability of VAS languages. Instead of VAS languages themselves (i.e., subsets of Σ^*), in this paper we investigate their commutative closures, or, alternatively, subsets of $\mathbb{N}^d$ represented as reachability sets of VASes, VASes with states, or Petri nets. A VAS reachability set is just the set of configurations of a VAS which can be reached from a specified initial configuration. Towards a unified treatment, instead of considering separately VASes, VASes with states, and Petri nets, we consider *sections* of VAS reachability sets (abbreviated as VAS sections below), which turn out to be expressive enough to represent sections of VASes with states and Petri nets, and thus being a convenient subsuming formalism. A *section* of a set of vectors $X \subseteq \mathbb{N}^d$ is the set obtained by first fixing a value for certain coordinates, and then projecting the result to the remaining coordinates. For example, if X is the set of pairs $\{(x, y) \in \mathbb{N}^2 \mid x \text{ divides } y\}$, then the section of X obtained by fixing the first coordinate to 3 is the set $\{0, 3, 6, \dots\}$. It can be easily shown that VAS sections are strictly more general than VAS reachability sets themselves, and they are equiexpressive with sections of VASes with states and Petri nets.

We study the separability problem of VAS sections by simpler classes, namely, modular and unary sets. A set $X \subseteq \mathbb{N}^d$ is *modular* if there exists a modulus $n \in \mathbb{N}$ s.t. X is closed under the congruence modulo n on every coordinate, and it is *unary* if there exists a threshold $n \in \mathbb{N}$ s.t. it is closed under the congruence modulo n above the threshold n on every coordinate. Clearly, VAS sections are more general than both unary and modular sets, and unary sets are more general than modular sets. Moreover, unary sets are tightly connected with commutative regular languages, in the sense that the Parikh image[2] of a commutative regular language is a unary set, and vice versa, the inverse Parikh image of a unary set is a commutative regular language. As our main result, we show that the modular and unary separability problems are decidable for VAS sections (and thus for sections of

[2] The Parikh image of a language of words $L \subseteq \{a_1, \dots, a_k\}$ is the subset of $\mathbb{N}^k$ obtained by counting occurrences of letters in L.

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VASes with states and Petri nets). Both proofs use similar techniques, and invoke two semi-decision procedures: the first one (positive) enumerates witnesses of separability, and the second one (negative) enumerates witnesses of nonseparability. A separability witness is just a modular (or unary) set, and verifying that it is indeed a separator easily reduces to the VAS reachability problem. Thus, the hard part of the proof is to invent a finite and decidable witness of nonseparability, i.e., a finite object whose existence proves that none of infinitely many modular (resp. unary) sets is a separator. Our main technical observation is that two nonseparable VAS reachability sets always admit two *linear* subsets thereof that are already nonseparable.

From our result, thanks to the tight connection between unary sets and commutative regular languages mentioned above, we can immediately deduce decidability of regular separability for *commutative closures of VAS languages*, and *commutative regular* separability for VAS languages. This constitutes a first step towards determining the status of regular separability for languages of VASes.

Related research. Choffrut and Grigorieff have shown decidability of separability of rational relations by recognizable relations in $\Sigma^* \times \mathbb{N}^d$ [1]. Rational subsets of $\mathbb{N}^d$ are precisely the semilinear sets, and recognizable (by morphism into a monoid) subsets of $\mathbb{N}^d$ are precisely the unary sets. Thus, by ignoring the Σ^* component, one obtains a very special case of our result, namely decidability of the unary separability problem for semilinear sets. Moreover, since modular sets are subsets of $\mathbb{N}^d$ which are recognizable by a morphism into a monoid which happens to be a group, we also obtain a new result, namely, decidability of separability of rational subsets of $\mathbb{N}^d$ by subsets of $\mathbb{N}^d$ recognized by a group.

From a quite different angle, our research seems to be closely related to the VAS reachability problem. Leroux [15] has shown a highly nontrivial result: the reachability sets of two VASes are disjoint if, and only if, they can be separated by a semilinear set. In other words, semilinear separability for VAS reachability sets is equivalent to the VAS (non-)reachability problem. This connection suggests that modular and unary separability are interesting problems in themselves, enriching our understanding of VASes. Finally, we show that VAS reachability reduces to unary separability, thus the problem does not become easier by considering the simpler class of unary sets as opposed to semilinear sets. For modular separability we have a weaker complexity lower bound, i.e. ExpSpace-hardness, by a reduction from control state reachability for VASSes.

2 Preliminaries

Vectors. By $\mathbb{N}$ and $\mathbb{Z}$ we denote the set of natural and integer numbers, respectively. For a vector $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$ and for a coordinate $i \in \{1, \ldots, d\}$, we denote by $u[i]$ its i-th component u_i. The zero vector is denoted by 0. The order $\leq$ and the sum operation $+$ naturally extend to vectors pointwise. Moreover, if $n \in \mathbb{Z}$, then nu is the vector $(nu_1, \ldots, nu_d)$. These operations extend to sets element-wise in the natural way: For two sets of vectors $U, V \subseteq \mathbb{Z}^d$ we denote by $U + V$ its Minkowski sum $\{u + v \mid u \in U, v \in V\}$. For a (possibly infinite) set of vectors $S \subseteq \mathbb{Z}^d$, let $\textsc{Lin}(S)$ and $\textsc{Lin}^{\geq 0}(S)$ be the set of *linear combinations* and *non-negative linear combinations* of vectors from S, respectively, i.e.,

$$\textsc{Lin}(S) = \{a_1 v_1 + \ldots + a_k v_k \mid v_1, \ldots, v_k \in S, a_1, \ldots, a_k \in \mathbb{Z}\}, \text{ and}$$
$$\textsc{Lin}^{\geq 0}(S) = \{a_1 v_1 + \ldots + a_k v_k \mid v_1, \ldots, v_k \in S, a_1, \ldots, a_k \in \mathbb{N}\}.$$

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When the set $S = \{v_1, \ldots, v_k\}$ is finite, we alternatively write $\textsc{Lin}(v_1, \ldots, v_k)$ instead of $\textsc{Lin}(\{v_1, \ldots, v_k\})$, and similarly for $\textsc{Lin}^{\geq 0}(v_1, \ldots, v_k)$.

Modular, unary, linear, and semilinear sets. Two vectors $x, y \in \mathbb{Z}^d$ are *n-modular equivalent*, written $x \equiv_n y$, if, for all $i \in \{1, \ldots, d\}$, we have $x[i] \equiv y[i] \mod n$. Moreover, two non-negative vectors $x, y \in \mathbb{N}^d$ are *n-unary equivalent*, written $x \cong_n y$, if $x \equiv_n y$ and $x[i] \geq n \iff y[i] \geq n$ for all $i \in \{1, \ldots, d\}$. A d-dimensional set $S \subseteq \mathbb{N}^d$ is *modular* if there exists a number $n \in \mathbb{N}$, s.t. S is a union of n-modular equivalence classes. *Unary* sets $S \subseteq \mathbb{N}^d$ are defined similarly w.r.t. n-unary equivalence classes.

A set $S \subseteq \mathbb{N}^d$ is *linear* if it is of the form $S = \{b\} + \textsc{Lin}^{\geq 0}(p_1, \ldots, p_k)$ for some *base* $b \in \mathbb{N}^d$ and some *periods* $p_1, \ldots, p_k \in \mathbb{N}^d$. A set is *semilinear* if it is a finite union of linear sets. Note that a modular set is also unary (since $\cong_n$ is finer than $\equiv_n$), and that unary set is in turn a semilinear set, which can be presented as a finite union of linear sets in which all the periods are parallel to the coordinate axes, i.e., they have exactly one non-zero entry.

Separability. For $S, U, V \subseteq \mathbb{N}^d$, we say that S *separates* U from V if $U \subseteq S$ and $V \cap S = \emptyset$. The set S is also called a *separator* of U, V. For a family $\mathcal{F}$ of sets, we say that U is $\mathcal{F}$ separable from V if U is separated from V by a set $S \in \mathcal{F}$. In this paper, the set of separators $\mathcal{F}$ will be the modular sets and the unary ones. Since both classes are closed under complement, the notion of $\mathcal{F}$ separability is symmetric: U is $\mathcal{F}$ separable from V iff V is $\mathcal{F}$ separable from U. Thus we use also a symmetric notation, in particular we say that U and V are $\mathcal{F}$ separable instead of saying that U is $\mathcal{F}$ separable from V. For two families of sets $\mathcal{F}$ and $\mathcal{G}$, the $\mathcal{F}$ *separability problem for* $\mathcal{G}$ asks whether two given sets $U, V \in \mathcal{G}$ are $\mathcal{F}$ separable. In this paper we mainly consider two instances of $\mathcal{F}$, namely modular sets and unary sets, and thus we speak of *modular* and *unary separability* problems, respectively.

Vector Addition Systems. A d-dimensional *Vector Addition System* (VAS) is a pair $V = (s, T)$, where $s \in \mathbb{N}^d$ is the *source* configuration and $T \subseteq_{\text{FIN}} \mathbb{Z}^d$ is the set of finitely many *transitions*. A *partial run* ρ of a VAS $V = (s, T)$ is a sequence $(v_0, t_0, v_1), \ldots, (v_{n-1}, t_{n-1}, v_n) \in \mathbb{N}^d \times T \times \mathbb{N}^d$ such that for all $i \in \{0, \ldots, n-1\}$ we have $v_i + t_i = v_{i+1}$. The *source* of this partial run is the configuration v_0 and the *target* of this partial run is the configuration v_n, we write $\textsc{source}(\rho) = v_0$, $\textsc{target}(\rho) = v_n$. The *labeling* of ρ is the sequence $t_0 \ldots t_{n-1} \in T^*$, we write $\textsc{label}(\rho) = t_0 \ldots t_{n-1}$. For a sequence $\alpha \in T^*$ and a partial run ρ such that $\textsc{label}(\rho) = \alpha$, $\textsc{source}(\rho) = u$ and $\textsc{target}(\rho) = v$ we write $u \xrightarrow{\alpha} v$ to denote this unique partial run. A partial run ρ of (s, T) with $\textsc{source}(\rho) = s$ is called a *run*. The set of all runs of a VAS V is denoted as $\textsc{Runs}(V)$. The *reachability set* $\textsc{Reach}(V)$ of a VAS V is the set of targets of all its runs; the sets $\textsc{Reach}(V)$ we call *VAS reachability sets* in the sequel. The family of all VAS reachability sets we denote as $\textsc{Reach}(\text{VAS})$.

▶ **Example 1.** Consider a VAS $V = (s, T)$, for a source configuration $s = (1, 0, 0)$ and a set of transitions $T = \{(-1, 2, 1), (2, -1, 1)\}$. One easily proves that

$$\textsc{Reach}(V) \;=\; \{(a, b, c) \in \mathbb{N}^3 \mid a + b = c + 1 \;\wedge\; a - b \equiv 1 \mod 3\}.$$

Vector Addition Systems with states. A d-dimensional *VAS with states* (VASS) is a triple $V = (s, T, Q)$, where Q is a finite set of *states*, $s \in Q \times \mathbb{N}^d$ is the *source* configuration and $T \subseteq_{\text{FIN}} Q \times \mathbb{Z}^d \times Q$ is a finite set of *transitions*. Similarly as in case of VASes, a *run* ρ of a VASS $V = (s, T, Q)$ is a sequence $(q_0, v_0, s_0, q_1, v_1), \ldots, (q_{n-1}, v_{n-1}, s_{n-1}, q_n, v_n) \in Q \times \mathbb{N}^d \times \mathbb{Z}^d \times Q \times \mathbb{N}^d$ such that $(q_0, v_0) = s$ and for all $i \in \{0, \ldots, n-1\}$ we have

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$(q_i, s_i, q_{i+1}) \in T$ and $v_i + s_i = v_{i+1}$. We write $\textsc{target}(\rho) = (q_n, v_n)$. The *reachability set* of a VASS V in state q is $\textsc{Reach}_q(V) = \{v \in \mathbb{N}^d \mid (q, v) = \textsc{target}(\rho) \text{ for some run } \rho\}$. The family of all such reachability sets of all VASSes we denote as $\textsc{Reach}(\text{VASS})$.

▶ **Example 2** (cf. [8]). Let V be a 3-dimensional VASS with two states, p and p', the source configuration $(p, (1, 0, 0))$, and four transitions:

$$(p, (-1, 1, 0), p), \quad (p, (0, 0, 0), p'), \quad (p', (2, -1, 0), p'), \quad (p', (0, 0, 1), p).$$

Then $\textsc{Reach}_p(V) = \{(a, b, c) \in \mathbb{N}^3 \mid 1 \leq a + b \leq 2^c\}$.

3 Sections

VAS reachability sets are central for this paper. However, in order to make this family of sets more robust, we prefer to consider the slightly larger family of *sections* of VAS reachability sets. The intuition about a section is that we fix values on a subset of coordinates in vectors, and collect all the values that can occur on the other coordinates. For a vector $u \in \mathbb{N}^d$ and a subset $I \subseteq \{1, \ldots, d\}$ of coordinates, by $\pi_I(u) \in \mathbb{N}^{|I|}$ we denote the *I-projection* of u, i.e., the vector obtained from u by removing coordinates not belonging to I. The projection extends element-wise to sets of vectors $S \subseteq \mathbb{N}^d$, denoted $\pi_I(S)$. For a set of vectors $S \subseteq \mathbb{N}^d$, a subset $I \subseteq \{1, \ldots, d\}$, and a vector $u \in \mathbb{N}^{d-|I|}$, the *section* of S w.r.t. I and u is the set

$$\textsc{sec}_{I,u}(S) := \pi_I(\{v \in S \mid \pi_{\{1,\ldots,d\} \setminus I}(v) = u\}) \subseteq \mathbb{N}^{|I|}.$$

We denote by $\textsc{SecReach}(\text{VAS})$ the family of all sections of VAS reachability sets, which we abbreviate as *VAS sections below.* Similarly, the family of all sections of VASS-reachability sets we denote by $\textsc{SecReach}(\text{VASS})$.

▶ **Example 3.** Consider the VAS V from Example 1. For $I = \{1, 2\}$ and $u = 7 \in \mathbb{N}^1$ we have $\textsc{sec}_{I,u}(\textsc{Reach}(V)) = \{(0, 8), (3, 5), (6, 2)\}$.

Note that in a special case of $I = \{1, \ldots, d\}$, when u is necessarily the empty vector, $\textsc{sec}_{I,u}(S) = S$. Thus $\textsc{Reach}(\text{VAS})$ is a subfamily of $\textsc{SecReach}(\text{VAS})$, and likewise for VASSes. We argue that VAS sections are a more robust class than VAS reachability sets. Indeed, as shown below VAS sections are closed under positive boolean combinations, which is not the case for VAS reachability sets.

Reachability sets of VASes are a strict subfamily of reachability sets of VASes with states, which in turn are a strict subfamily of sections of reachability sets of VASes. However, when sections of reachability set are compared, there is no difference between VASes and VASes with states, which motivates considering sections in this paper. These observations are summarized in the following two propositions:

▶ Proposition 4. $\textsc{Reach}(\text{VAS}) \subsetneq \textsc{Reach}(\text{VASS}) \subsetneq \textsc{SecReach}(\text{VAS})$.

▶ Proposition 5. $\textsc{SecReach}(\text{VAS}) = \textsc{SecReach}(\text{VASS})$.

▶ Remark. In a similar vein one shows that reachability sets of Petri nets include $\textsc{Reach}(\text{VAS})$ and are included in $\textsc{Reach}(\text{VASS})$. Therefore, as long as sections are considered, there is no difference between VASes, Petri nets, and VASSes. In consequence, our results apply not only to VASes, but to all the three models.

We conclude this section by stating closure property of VAS sections. By *positive boolean combination* we mean sets obtained by taking only intersections and unions, but not complements.

▶ Proposition 6. The family of VAS sections is closed under positive boolean combinations.

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4 Results

As our main technical contribution, we prove decidability of the modular and unary separability problems for the class of sections of VAS reachability sets.

▶ **Theorem 7.** *The modular separability problem for VAS sections is decidable.*

▶ **Theorem 8.** *The unary separability problem for VAS sections is decidable.*

The proofs are postponed to Sections 5–7. Furthermore, as a corollary of Theorem 8 we derive decidability of two commutative variants of the regular separability of VAS languages (formulated in Theorems 9 and 10 below).

To consider languages instead of reachability sets, we need to assume that transitions of a VAS are *labeled* by elements of an alphabet Σ, and thus every run is labeled by a word over Σ obtained by concatenating labels of consecutive transitions of a run. We allow for silent transitions labeled by ε, i.e., transitions that do not contribute to the labeling of a run. The language $L(V)$ of a VAS V contains labels of those runs of V that end in an *accepting* configuration. Our results work for several variants of acceptance; for instance, for a given fixed configuration v_0,

- we may consider a configuration v accepting if $v \geq v_0$ (this choice yields so called *coverability languages*); or
- we may consider a configuration v accepting if $v = v_0$ (this choice yields *reachability languages*).

The Parikh image of a word $w \in \Sigma^*$, for a fixed total ordering $a_1 < \ldots < a_d$ of Σ, is a vector in $\mathbb{N}^d$ whose ith coordinate stores the number of occurrences of a_i in w. We lift the operation element-wise to languages, thus the Parikh image of a language L, denoted $\Pi(L)$, is a subset of $\mathbb{N}^d$. Two words w, v over Σ are *commutatively equivalent* if their Parikh images are equal. The *commutative closure* of a language $L \subseteq \Sigma^*$, denoted $\textsc{cc}(L)$, is the language containing all words $w \in \Sigma^*$ commutatively equivalent to some word $v \in L$. A language L is *commutative* if it is invariant under commutatively equivalence, i.e., $L = \textsc{cc}(L)$. Note that a commutative language is uniquely determined by its Parikh image. The Parikh image of any commutative regular language is unary: A finite automaton recognizing a commutative language can only count modulo n above threshold n for each letter in the alphabet independently. Moreover, all unary set can be obtained as the Parikh image of a commutative regular language. Similarly, the inverse Parikh image of a unary set is a commutative regular language, and all commutative regular languages can be obtained in this way. In this sense, commutative regular languages and unary sets are in correspondence with each other.

As a corollary of Theorem 8 we deduce decidability of the following two commutative variants of the regular separability of VAS languages:

- *commutative regular separability of VAS languages*: given two VASes V, V', decide whether there is a commutative regular language R that includes $L(V)$ and is disjoint from $L(V')$;
- *regular separability for commutative closures of VAS languages*: given two VASes V, V', decide whether there is a regular language R that includes $\textsc{cc}(L(V))$ and is disjoint from $\textsc{cc}(L(V'))$.

▶ **Theorem 9.** *Commutative regular separability is decidable for VAS languages.*

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Indeed, given two VASes V, W one easy constructs another two VASes V', W' s.t. $\Pi(L(V))$ is a section of $\textsc{Reach}(V')$, and similarly for W'. By the tight correspondence between commutative regular languages and unary sets, we observe that $L(V)$ and $L(W)$ are separated by a commutative regular language if, and only if, their Parikh images $\Pi(L(V))$ and $\Pi(L(W))$ are separated by a unary set, which is is decidable by Theorem 8.

▶ **Theorem 10.** *Regular separability is decidable for commutative closures of VAS languages.*

Similarly as above, we reduce to unary separability of VAS reachability sets (which is decidable once again by Theorem 8), which is immediate once one proves the following crucial observation.

▶ **Lemma 11.** *Two commutative languages $L, K \subseteq \Sigma^*$ are regular separable if, and only if, their Parikh images are unary separable.*

Proof. For the "if" direction, let $\Pi(K)$ and $\Pi(L)$ be separable by some unary set $U \subseteq \mathbb{N}^d$. Let $S = \{w \in \Sigma^* \mid \Pi(w) \in U\}$. It is easy to see that S is (commutative) regular since U is unary, and that S separates K and L. For the "only if" direction, let K and L be separable by a regular language S, say $K \subseteq S$ and $S \cap L = \emptyset$. Let M be the syntactic monoid of S and ω be its idempotent power, i.e., a number such that $m^\omega = m^{2\omega}$ for every $m \in M$. In particular, for every word $u \in \Sigma^*$ we have (P) $uv^\omega w \in L \iff uv^{2\omega}w \in S$; in other words, one can substitute v^ω by $v^{2\omega}$ and vice versa in every context. Let $\Sigma = \{a_1, \ldots, a_d\}$. For $u = (u_1, \ldots, u_d) \in \mathbb{N}^d$ define the word $w_u = a_1^{u_1} \cdots a_d^{u_d}$. For every $u, v \in \mathbb{N}^d$ such that $u \cong_\omega v$, by repetitive application of (P) we get $w_u \in S$ iff $w_v \in S$. As K is commutative and $K \subseteq S$, we have $w_u \in S$ for all $u \in \Pi(K)$; similarly, we have $w_v \notin S$ for all $v \in \Pi(L)$. Thus, for all $u \in \Pi(K)$, $v \in \Pi(L)$ we have $u \not\cong_\omega v$. Let $U = \{x \in \mathbb{N}^d \mid \exists_{y \in \Pi(K)}\, x \cong_\omega y\}$. The set U separates $\Pi(K)$ and $\Pi(L)$ and, being a union of $\cong_\omega$ equivalence classes, it is unary. ◀

5 Modular separability of linear sets

The rest of the paper is devoted to the proofs of Theorems 7 and 8. In this section we prove two preliminary results that will be later used in Section 6, where the proof of Theorems 7 is completed. First, we prove a combinatorial result on linear combinations (cf. Lemma 12 below). Second, we prove that modular separability of linear sets is decidable (cf. Corollary 15). While this second result follows from [1] and is thus not a new result, we provide here another simple proof to make the paper self-contained.

Linear combinations modulo n. We start with some preliminary results from linear algebra. For $n \in \mathbb{N}$, let $\textsc{Lin}_n^{\geq 0}(v_1, \ldots, v_k)$ be the closure of $\textsc{Lin}^{\geq 0}(v_1, \ldots, v_k)$ modulo n, i.e.,

$$\textsc{Lin}_n^{\geq 0}(v_1, \ldots, v_k) = \{v \in \mathbb{N}^d \mid \exists_{u \in \textsc{Lin}^{\geq 0}(v_1, \ldots, v_k)}\, v \equiv_n u\}.$$

Similarly one defines $\textsc{Lin}_n(v_1, \ldots, v_k)$ be the closure of $\textsc{Lin}(v_1, \ldots, v_k)$ modulo n. Observe however that $\textsc{Lin}_n(v_1, \ldots, v_k) = \textsc{Lin}_n^{\geq 0}(v_1, \ldots, v_k)$. Indeed, if $v \equiv_n l_1 v_1 + \ldots + l_k v_k$ for $l_1, \ldots, l_k \in \mathbb{Z}$ then $v \equiv_n (l_1 + Nn)v_1 + \ldots + (l_k + Nn)v_k$ for any $N \in \mathbb{N}$. The following observation connects linear combinations to the modular closure of non-negative linear combinations.

▶ **Lemma 12.** $\textsc{Lin}(v_1, \ldots, v_k) = \bigcap_{n>0} \textsc{Lin}_n^{\geq 0}(v_1, \ldots, v_k)$.

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Modular separability. In the rest of the paper, we heavily rely on the following straightforward characterization of modular separability.

▶ Proposition 13. Two sets $U, V \subseteq \mathbb{N}^d$ are modular separable if, and only if, there exists $n \in \mathbb{N}$ such that for all $u \in U$, $v \in V$ we have $u \not\equiv_n v$.

In the special case of linear sets, the characterization above boils down to the following property:

▶ **Lemma 14.** *Two linear sets $\{b\} + \textsc{Lin}^{\geq 0}(P)$ and $\{c\} + \textsc{Lin}^{\geq 0}(Q)$ are not modular separable if, and only if, $b - c \in \textsc{Lin}(P \cup Q)$.*

Since the condition in the lemma above is effectively testable being an instance of solvability of systems of linear Diophantine equations, we get the following corollary:

▶ **Corollary 15.** *Modular separability of linear sets is decidable.*

▶ Remark. Since linear Diophantine equations are solvable in polynomial time, we obtain the same complexity for modular separability of linear sets. This observation however will not be useful in the sequel.

▶ Remark. The unary separability problem is decidable for linear sets, as shown in [1], but we will not need this fact in the sequel. Moreover, it follows from our stronger decidability result stated in Theorem 8, since linear sets are included in VAS sections.

6 Modular separability of VAS sections

In this section we prove Theorem 7, and thus provide an algorithm to decide modular separability for VAS reachability sets. Given two VAS sections U and V, the algorithm runs in parallel two semi-decision procedures: one (positive) which looks for a witness of separability, and another one (negative) which looks for a witness of nonseparability. Directly from the characterization of Proposition 13, the positive semi-decision procedure simply enumerates all moduli $n \in \mathbb{N}$ and checks whether $u \not\equiv_n v$ for all $u \in U$ and $v \in V$. The latter condition can be decided by reduction to the VAS (non)reachability problem [20, 17].

▶ **Lemma 16.** *For two VAS sections U and V and a modulus $n \in \mathbb{N}$, it is decidable whether there exist $u \in U$ and $v \in V$ s.t. $u \equiv_n v$.*

It remains to design the negative semi-decision procedure, which is the nontrivial part. In Lemma 22, we show that if two VAS reachability sets are not modular separable, then in fact they already contain two *linear* subsets which are not modular separable. In order to construct such linear witnesses of nonseparablity, we use the theory of well quasi orders and some elementary results in algebra, which we present next.

The order on runs. In this section, we define a certain well quasi order on runs $\trianglelefteq$ which will prove useful in the following; a weaker version of this order was defined in [11].

A quasi order $(X, \preccurlyeq)$ is a *well quasi order* (wqo) if for every infinite sequence $x_0, x_1, \ldots \in X$ there exist indices $i, j \in \mathbb{N}, i < j$, such that $x_i \preccurlyeq x_j$. It is folklore that if $(X, \preccurlyeq)$ is a wqo, then in every infinite sequence $x_0, x_1, \ldots \in X$ there even exists an infinite monotonically non-decreasing subsequence $x_{i_1} \preccurlyeq x_{i_2} \preccurlyeq \ldots$. We will use Dickson's [4] and Higman's [7] Lemmas to define new wqo's on pairs and sequences. For two qos $(X, \leq_X)$ and $(Y, \leq_Y)$, let the product $(X \times Y, \leq_{X \times Y})$ be ordered componentwise by $(x, y) \leq_{X \times Y} (x', y')$ if $x \leq_X x'$ and $y \leq_Y y'$.

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▶ **Lemma 17** (Dickson [4]). *If $(X, \leq_X)$ and $(Y, \leq_Y)$ are wqos, then $(X \times Y, \leq_{X\times Y})$ is a wqo.*

As a corollary, if two qos $(X, \leq_1)$ and $(X, \leq_2)$ on the same domain are wqos, then also their intersection is a wqo. For a qo $(X, \leq)$, let $(X^*, \leq_*)$ be quasi ordered by the subsequence order $\leq_*$, defined as $x_1x_2\cdots x_k \leq_* y_1y_2\ldots y_m$ if there exist $1 \leq i_1 < \ldots < i_k \leq m$ such that $x_j \leq y_{i_j}$ for all $j \in \{1, \ldots, k\}$.

▶ **Lemma 18** (Higman [7]). *If $(X, \leq)$ is a wqo then $(X^*, \leq_*)$ is a wqo.*

By considering the finite set of transitions T well quasi ordered by equality, we define the order $\leq^1$ on triples $\mathbb{N}^d \times T \times \mathbb{N}^d$ componentwise as $(u, s, u') \leq^1 (v, t, v')$ if $u \leq v$, $s = t$, and $u' \leq v'$, which is a wqo by Dickson's Lemma. We further extend $\leq^1$ to an order $\trianglelefteq$ on runs by defining, for two runs ρ and σ in $(\mathbb{N}^d \times T \times \mathbb{N}^d)^*$, $\rho \trianglelefteq \sigma$ if $\rho \leq^1_* \sigma$ and $\textsc{target}(\rho) \leq \textsc{target}(\sigma)$; Here, $\leq^1_*$ is the extension of $\leq^1$ to sequences, and thus a wqo by Higman's Lemma, which implies that $\trianglelefteq$ is itself a wqo. Our order $\trianglelefteq$ is very similar to the weaker order defined in [11], which is the same as $\trianglelefteq$, except that it does not include target configurations.

▶ Proposition 19. $\trianglelefteq$ is a well quasi order.

The following lemma is a quantitative version of monotonicity of VASes, and it says not only that larger runs can do more than smaller runs, but also that all nonnegative linear combinations of increments of larger runs can in fact be realized.

▶ **Lemma 20.** *Let $\rho_0, \rho_1, \ldots, \rho_k$ be runs of a VAS s.t., for all $i \in \{1, \ldots, k\}$, $\rho_0 \trianglelefteq \rho_i$, and let $\delta_i := \textsc{target}(\rho_i) - \textsc{target}(\rho_0) \geq 0$. For any $\delta \in \textsc{Lin}^{\geq 0}(\delta_1, \ldots, \delta_k)$, there exists a run ρ s.t. $\rho_0 \trianglelefteq \rho$ and $\delta = \textsc{target}(\rho) - \textsc{target}(\rho_0)$.*

We conclude this part by showing that any (possibly infinite) subset of $\mathbb{Z}^d$ can be overapproximated by taking linear combinations of a *finite* subset thereof. This will be important below in order to construct linear sets as witnesses of nonseparability.

▶ **Lemma 21.** *For every (possibly infinite) set of vectors $S \subseteq \mathbb{Z}^d$, there exist finitely many vectors $v_1, \ldots, v_k \in S$ s.t. $S \subseteq \textsc{Lin}(v_1, \ldots, v_k)$.*

Proof. Treat $\mathbb{Z}^d$ as a freely finitely generated abelian group, and consider the subgroup $\textsc{Lin}(S)$ of $\mathbb{Z}^d$ generated by S, i.e., the subgroup containing all linear combinations of finitely many elements of S. We use the following result in algebra: every subgroup of a finitely generated abelian group is finitely generated (see for instance Corollary 1.7, p. 74, in [9]). By this result applied to $\textsc{Lin}(S)$ we get a finite set of generators $F \subseteq \textsc{Lin}(S)$ s.t. $\textsc{Lin}(F) = \textsc{Lin}(S)$. Every element of F is a linear combination of finitely many elements of S. Thus let $v_1, \ldots, v_k$ be all the elements of S appearing as a linear combination of some element from F. Then clearly $F \subseteq \textsc{Lin}(v_1, \ldots, v_k)$, and thus $S \subseteq \textsc{Lin}(S) = \textsc{Lin}(F) \subseteq \textsc{Lin}(\textsc{Lin}(v_1, \ldots, v_k)) = \textsc{Lin}(v_1, \ldots, v_k)$, as required. ◀

▶ Remark. In fact one can show that the generating set F has at most d elements. However, no upper bound on k follows, and even for $d = 1$ the number of vectors k can be arbitrarily large. Indeed, let $p_1, \ldots, p_k$ be different prime numbers, let $u_i = (p_1 \cdot \ldots \cdot p_k)/p_i$ and $S = \{u_1, \ldots, u_k\}$. Then for every $i \in \{1, \ldots, k\}$, the number u_i is not a linear combination of numbers u_j, $j \neq i$, as u_i is not divisible by p_i, while all the others are. Therefore we need all the elements of S in the set $\{v_1, \ldots, v_k\}$.

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Modular nonseparability witness. We now concentrate on the negative semi-decision procedure. Let $U, V \subseteq \mathbb{N}^d$ be two VAS sections:

$$U = \text{SEC}_{I,\bar{u}}(R_U) \subseteq \mathbb{N}^d \quad \text{and} \quad V = \text{SEC}_{J,\bar{v}}(R_V) \subseteq \mathbb{N}^d,$$

where $R_U \subseteq \mathbb{N}^{d_U}$ and $R_V \subseteq \mathbb{N}^{d_V}$ are the reachability sets of the two VASes W_U and W_V, and $I \subseteq \{1, \ldots, d_U\}$ and $J \subseteq \{1, \ldots, d_V\}$ with $|I| = |J| = d$ are projecting coordinates, and $\bar{u} \in \mathbb{N}^{d_U - d}, \bar{v} \in \mathbb{N}^{d_V - d}$ are two sectioning vectors.

Observe that by padding coordinates we can assume w.l.o.g. that the two input VASes have the same dimension $d' = d_U = d_V$. Furthermore, we can also assume w.l.o.g. that $\bar{u} = \bar{v} = 0$. Indeed, one can add an additional coordinate, such that for performing any transition it is necessary that this coordinate is nonzero and a special, final transition, which causes the additional coordinate to be equal zero and subtracts $\bar{u}$ (or $\bar{v}$) from the other coordinates. The result of adding this gadget is that now we can assume $\bar{u} = \bar{v} = 0$, but the section itself does not change.

Finally, by reordering coordinates we can guarantee that the coordinates that are projected away appear on the same positions in both VASes, i.e., $I = J$. With these assumptions, we observe that modular separability of sets $U, V \subseteq \mathbb{N}^d$ is equivalent to modular separability of sets $U', V' \subseteq \mathbb{N}^{d'}$, defined as U, V but *without* projecting onto the subset I of coordinates:

$$U' = \{v \in R_U \mid \pi_{\{1,\ldots,d'\}\setminus I}(v) = 0\} \qquad V' = \{v \in R_V \mid \pi_{\{1,\ldots,d'\}\setminus I}(v) = 0\}.$$

We call the set U' (resp. V') the *expansion* of U (resp. V).

We say that a linear set $L = \{b\} + \text{LIN}^{\geq 0}(p_1, \ldots, p_k) \subseteq \mathbb{N}^{d'}$ is a U*-witness* if W_U admits runs $\rho, \rho_1, \ldots, \rho_k$ s.t. $\rho \trianglelefteq \rho_1, \ldots, \rho \trianglelefteq \rho_k$, and

$$\begin{aligned} b &= \text{TARGET}(\rho) \in U' \\ b + p_i &= \text{TARGET}(\rho_i) \in U' \quad \text{for } i \in \{1, \ldots, k\} \end{aligned} \tag{1}$$

Analogously one defines V-witnesses, but with respect to W_V.

▶ **Lemma 22.** *For two VAS sections $U, V \subseteq \mathbb{N}^d$, the following conditions are equivalent:*

1. *U, V are not modular separable;*
2. *the expansions U', V' of U, V are not modular separable;*
3. *there exist linear subsets $L \subseteq U'$, $M \subseteq V'$ that are not modular separable;*
4. *there exist a U-witness L and a V-witness M that are not modular separable.*

Proof. Equivalence of points 1 and 2 follows by the definition of expansion. Point 4 implies 3, as a U-witness is necessarily a subset of the expansion U' by Lemma 20. Point 3 implies 2, since if two sets are separable, also subsets thereof are separable (moreover, the separator remains the same). It remains to show that 2 implies 4.

Let $U', V' \subseteq \mathbb{N}^{d'}$ be the expansions of two VAS sections $U, V \subseteq \mathbb{N}^d$, as above, and assume that they are not modular separable. We construct two linear sets $L, M \subseteq \mathbb{N}^{d'}$ constituting a U-witness and a V-witness, respectively. By Proposition 13, there exists an infinite sequence of pairs of reachable configurations $(u_0, v_0), (u_1, v_1), \ldots \in U' \times V'$ s.t. $u_n \equiv_n v_n$ for all $n \in \mathbb{N}$. By taking an appropriate infinite subsequence we can ensure that even $u_n \equiv_{n!} v_n$ for all $n \in \mathbb{N}$. Let us fix for every $n \in \mathbb{N}$ runs ρ_n and σ_n such that $u_n = \text{TARGET}(\rho_n)$ and $v_n = \text{TARGET}(\sigma_n)$. Since $\trianglelefteq$ is a wqo by Proposition 19, we can extract a monotone non-decreasing subsequence, and thus we can ensure that even $\rho_0 \trianglelefteq \rho_1 \trianglelefteq \cdots$ and $\sigma_0 \trianglelefteq \sigma_1 \trianglelefteq \cdots$. Here we use the fact that $u_n \equiv_{n!} v_n$ in the original sequence, and thus $u_n \equiv_i v_n$ for every

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$i \in \{1, \ldots, n\}$, consequently the new subsequence still has $u_n \equiv_n v_n$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let $\delta_n := u_n - u_0$ and $\gamma_n := v_n - v_0$, and consider the set of corresponding differences $S_{\inf} := \{\delta_n - \gamma_n \mid n \in \mathbb{N}\}$. By Lemma 21, there exists a finite subset thereof $S := \{\delta_{i_1} - \gamma_{i_1}, \ldots, \delta_{i_k} - \gamma_{i_k}\}$ such that $S_{\inf} \subseteq \textsc{Lin}(S)$, and thus there exist two finite subsets $P := \{\delta_{i_1}, \ldots, \delta_{i_k}\}$ and $Q := \{\gamma_{i_1}, \ldots, \gamma_{i_k}\}$ such that

$$S_{\inf} \subseteq \textsc{Lin}(P - Q) \subseteq \textsc{Lin}(P) - \textsc{Lin}(Q) \subseteq \textsc{Lin}_n^{\geq 0}(P) - \textsc{Lin}_n^{\geq 0}(Q), \tag{2}$$

where the last inclusion follows from Lemma 12. Let the two linear sets L and M be defined as

$$L := \{u_0\} + \textsc{Lin}^{\geq 0}(P) \quad \text{and} \quad M := \{v_0\} + \textsc{Lin}^{\geq 0}(Q).$$

By construction, L is a U-witness and M a V-witness. It thus only remains to show that L and M are not modular separable. For any n, by Eq. 2 we have $\delta_n - \gamma_n \equiv_n \delta'_n - \gamma'_n$ for some $\delta'_n \in \textsc{Lin}^{\geq 0}(P)$ and $\gamma'_n \in \textsc{Lin}^{\geq 0}(Q)$. Consider now the two new infinite sequences $u'_1, u'_2, \cdots \in L$ and $v'_1, v'_2, \cdots \in M$ defined, for every $n \geq 1$, as $u'_n := u_0 + \delta'_n$ and $v'_n := v_0 + \gamma'_n$. Then,

$$\begin{aligned}
u'_n - v'_n &= (u_0 + \delta'_n) - (v_0 + \gamma'_n) \\
&= (u_0 - v_0) + (\delta'_n - \gamma'_n) && \text{(by def. of } \delta'_n, \gamma'_n) \\
&\equiv_n (u_0 - v_0) + (\delta_n - \gamma_n) \\
&= (u_0 + \delta_n) - (v_0 + \gamma_n) \\
&= u_n - v_n \equiv_n 0 && \text{(by def. of } u_n, v_n) ,
\end{aligned}$$

and thus $u'_n \equiv_n v'_n$. This, thanks to the characterization of Proposition 13, implies that L and M are not modular separable. ◀

▶ Remark. Note that a modular nonseparability witness exists even in the case when the two reachability sets U, V have nonempty intersection. In this case, it is enough to consider two runs ρ_0 and σ_0 ending up in the same configuration $\textsc{target}(\rho_0) = \textsc{target}(\sigma_0)$, and considering the linear sets $L := M := \{\textsc{target}(\rho_0)\}$.

Using the characterization of Lemma 22, the negative semi-decision procedure enumerates all pairs L, M, where L is a U-witness and M is a V-witness and checks whether L and M are modular separable, which is decidable due to Corollary 15. Enumerating U-witnesses (and V-witnesses) amounts of enumerating finite sets of runs $\{\rho, \rho_1, \ldots, \rho_k\}$ satisfying (1).

▶ Remark. It is also possible to design another negative semi-decision procedure using Lemma 22. This one enumerates all linear sets L and M (not necessarily only those in the special form of U- or V- witnesses) and checks whether they are modular separable *and* included in U and V, respectively. While this procedure is conceptually simpler than the one we presented, we now need the two extra inclusion checks $L \subseteq U$ and $M \subseteq V$. Indeed, U- and V-witnesses were designed in such a way that the two inclusions above hold by construction and do not have to be checked. The problem whether a given linear set is included in a given VAS reachability set is decidable [14], however we chose to present the previous semi-decision procedure in order to be self contained.

7 Unary separability of VAS sections

The proof of Theorem 8 goes along the same lines as the proof of Theorem 7. It uses an immediate characterization of unary separability, which is the same as Proposition 13, with unary equivalence $\cong_n$ in place of modular equivalence $\equiv_n$.

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▶ Proposition 23. Two sets $U, V \subseteq \mathbb{N}^d$ are unary separable if, and only if, there exists $n \in \mathbb{N}$ such that, for all $u \in U$ and $v \in V$, we have $u \ncong_n v$.

As before, basing on the characterization of Proposition 23, the positive semi-decision procedure enumerates all $n \in \mathbb{N}$ and checks whether the $\cong_n$-closures of the two reachability sets are disjoint, which is effective thanks to the following fact:

▶ **Lemma 24.** *For two VAS sections U and V and $n \in \mathbb{N}$, it is decidable whether there exist $u \in U$ and $v \in V$ such that $u \cong_n v$.*

This can be proved in a way similar to Lemma 16, with the adjustment that we allow on every coordinate a decrement by n only if the value is above $2n$. The negative semi-decision procedure enumerates nonseparability witnesses, and bases on the exact copy of Lemma 22, except that "modular" is replaced by "unary":

▶ **Lemma 25.** *For two VAS sections $U, V \subseteq \mathbb{N}^d$, the following conditions are equivalent:*

1. *U, V are not unary separable;*
2. *the expansions U', V' of U, V are not unary separable;*
3. *there exist linear subsets $L \subseteq U'$, $M \subseteq V'$ that are not unary separable;*
4. *there exist a U-witness L and a V-witness M that are not unary separable.*

8 Final remarks

We have shown decidability of modular and unary separability for sections of VAS reachability sets, which include (sections of) reachability sets of VASes with states and Petri nets. As a corollary, we have derived decidability of regular separability of commutative closures of VAS languages, and of commutative regular separability of VAS languages. The decidability status of regular separability for VAS languages remains an intriguing open problem.

Complexity. Most of the problems shown decidable in this paper are easily shown to be at least as hard as the VAS reachability problem. In particular, this applies to unary separability of VAS reachability sets, and to regular separability of commutative closures of VAS languages. Indeed, for unary separability, it suffices to notice that a configuration u cannot reach a configuration v if, and only if, the set reachable from u can be unary separated from the singleton set $\{v\}$, also a VAS reachability set. When the separator exists, it can be taken to be the complement of $\{v\}$ itself, which is unary.

While the problem of modular separability is ExpSpace-hard, we do not know whether it is as hard as the VAS reachability problem. The hardness can be shown by reduction from the control state reachability problem in VASSes, which is ExpSpace-hard [19]. For a VASS V and a target control state q thereof, we construct two new VASSes V_0 and V_1, which are copies of V with one additional coordinate, which at the beginning is zero for V_0 and one for V_1. We also add one new transition from control state q, which allows V_1 to decrease the additional coordinate by one. One can easily verify that the two VASS reachability sets definable by V_0 and V_1 are modular separable if, and only if, the control state q is not reachable in V, which finishes the proof of ExpSpace-hardness.

The unarity and modularity characterization problems. Closely related problems to separability are the modularity and unarity characterization problems: is a given section of a VAS reachability set modular, resp., unary? We focus here on the unarity problem, but the other one can be dealt in the same way. Decidability of the unarity problem would

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follow immediately from Theorem 8, if sections of VAS reachability sets were (effectively) closed under complement. This is however not the case. Indeed, if the complement of a VAS reachability set is a section of another VAS reachability set, then both sets are necessarily a section of a Presburger invariant [15], hence semilinear. But we know that VAS reachability sets can be non-semilinear, and thus they are not closed under complement. However, the unarity problem can be shown to be decidable directly, at least for VAS reachability sets, by using the following two facts: first, it is decidable if a given VAS reachability set U is semilinear (see the unpublished works [6, 13]); second, when a VAS reachability set is semilinar, a concrete representation thereof as a semilinear set is effectively computable [16]. Indeed, if a given U is not semilinear, it is not unary either; otherwise, compute a semilinear representation, and check if it is unary. The latter can be checked directly, or can be reduced to unary separability of semilinear sets.

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A Proofs from Sec. 3

▶ **Proposition 4.** $\textsc{Reach}(\text{VAS}) \subsetneq \textsc{Reach}(\text{VASS}) \subsetneq \textsc{SecReach}(\text{VAS})$.

Proof. In order to prove strictness of the first inclusion, consider the VASS V from Example 2. The reachability set $\textsc{Reach}_p(V)$ is not semilinear; on the other hand the reachability sets of of 3-dimensional VASes are always semilinear [8].

Now we turn to the second inclusion. It is folklore that for a d-dimensional VASS V with n states and m transitions one can construct a $(d+n+m)$-dimensional VAS V' simulating V. Among the new coordinates, n correspond to states and m to transitions. For a transition $t = (q, v, q')$ of V there are two transitions in V': the first one subtracts 1 on the coordinate corresponding to state q and adds 1 on the coordinate corresponding to t; the second one subtracts 1 on the coordinate corresponding to t, adds 1 on the coordinate corresponding to q', and adds v on the original d coordinates. Finally, if (q_0, v_0) is the initial configuration of V, then the initial configuration of V' is a copy of v_0 on the original d dimensions, equals 1 on the coordinate corresponding to q_0, and equals 0 on the rest of the new coordinates. Then the reachability set $\textsc{Reach}_q(V)$ equals the section of $\textsc{Reach}(V')$ obtained by fixing the coordinate corresponding to q to 1 and all other new coordinates to 0.

For strictness of the second inclusion, apply the above-mentioned transformation to the VASS V from Example 2, in order to obtain a 9-dimensional VAS V'. The section of $\textsc{Reach}(V')$ that fixes the second original coordinate to 0, the coordinate corresponding to state p to 1, and all the other new coordinates to 0 is $S := \{(a, b) \in \mathbb{N}^2 \mid 0 \leq a \leq 2^b\}$. This 2-dimensional set is not semilinear, while reachability sets of 2-dimensional VASSes are always semilinear [8]. Thus S is not a 2-dimensional VAS reachability set. ◀

▶ **Proposition 5.** $\textsc{SecReach}(\text{VAS}) = \textsc{SecReach}(\text{VASS})$.

Proof. One inclusion is obvious, since VASSes are more general than VASes, and the same holds when taking sections. For the other directions, consider a VASS V and a section thereof $S := \textsc{sec}_{I,v}(\textsc{Reach}_q(V))$. Reconsider the folklore construction of a VAS V' that simulates V (cf. the proof of the previous Proposition 5). The section of the reachability set of $\textsc{Reach}(V')$ that fixes the coordinate corresponding to q to 1, all the other new coordinates to 0, and all the original coordinates not belonging to the set I as in vector v, equals S. ◀

▶ **Proposition 6.** The family of VAS sections is closed under positive boolean combinations.

Proof. We only sketch the proof. For closure under union, we just use nondeterminism to guess which VAS to run. Dealing with sections is straightforward since 1) we can assume w.l.o.g. that sections are done w.r.t. the 0 vector, 2) by padding coordinates we can assume that the two input VASes have the same dimension, and 3) by reordering coordinates we can guarantee that the coordinates that are projected away appear all together on the right (the same simplifying assumptions will be made in Sections 6 and 7; cf. the details just before Lemma 22). For closure under intersection, we proceed under similar assumptions, and the intuition is to run the first VAS forward in two identical copies, and then to run backward the second VAS only in the second copy, using a section to make sure that the second VAS is accepting, and then project away the second copy. ◀

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B Proofs from Sec. 5

▶ **Lemma 12.** $\textsc{Lin}(v_1, \ldots, v_k) \;=\; \bigcap_{n>0} \textsc{Lin}_n^{\geq 0}(v_1, \ldots, v_k)$.

Proof. The left-to-right inclusion is immediate: for any $n \in \mathbb{N}$ we have

$$\textsc{Lin}(v_1, \ldots, v_k) \;\subseteq\; \textsc{Lin}_n(v_1, \ldots, v_k) \;=\; \textsc{Lin}_n^{\geq 0}(v_1, \ldots, v_k).$$

For the right-to-left inclusion we take an algebraic perspective, and treat $S := \textsc{Lin}(v_1, \ldots, v_k)$ as a subgroup of $\mathbb{Z}^d$ generated by $F = \{v_1, \ldots, v_k\}$. Let I be the set of all d unit vectors in $\mathbb{Z}^d$. For every $n \in \mathbb{N}_{\geq 0}$, let $n\mathbb{Z}^d$ denote the subgroup of $\mathbb{Z}^d$ generated by nI, and let S_n be the subgroup of $\mathbb{Z}^d$ generated by $F \cup (nI)$. In algebraic terms, our obligation is to show that

$$\bigcap_{n \in \mathbb{N}_{\geq 0}} S_n \;\subseteq\; S. \tag{3}$$

Let $G := \mathbb{Z}^d/S$ be the quotient group and consider the quotient group homomorphism $h : \mathbb{Z}^d \to G$. It is legal, as every subgroup of an abelian group is normal, thus we can consider a quotient with respect to it. We have thus $\ker(h) = \{x \in \mathbb{Z}^d \mid h(x) = 0_G\} = S$, where 0_G is the zero element of G. Now (3) is equivalent to

$$h\Big(\bigcap_{n \in \mathbb{N}_{\geq 0}} S_n\Big) \;=\; \{0_G\},$$

which will immediately follow, once we manage to show

$$\bigcap_{n \in \mathbb{N}_{\geq 0}} h(S_n) \;=\; \{0_G\}.$$

Observe that $h(S_n) = h(n\mathbb{Z}^d)$, for every $n \in \mathbb{N}_{\geq 0}$, and hence we may equally well demonstrate:

$$\bigcap_{n \in \mathbb{N}_{\geq 0}} h(n\mathbb{Z}^d) \;=\; \{0_G\}. \tag{4}$$

The group G, being a finitely generated abelian group, is isomorphic to the direct product of a finite group G_1 (let l be its order, i.e., the number of its elements) and $G_2 = \mathbb{Z}^k$, for some $k \in \mathbb{N}$ (see for instance Theorem 2.2, p. 76, in [9]). For showing (4), consider an element $g \in G$ which belongs to $h(n\mathbb{Z}^d)$ for all $n \in \mathbb{N}_{\geq 0}$, and its two projections g_1 and g_2 in G_1 and G_2, respectively. As $g \in h(l\mathbb{Z}^d)$, then necessarily $g_1 = l \cdot g'$ for some $g' \in G_1$, and since the order of every element divides the order of the group l, we have $g_1 = 0_{G_1}$. Similarly, we deduce that $g_2 = 0_{G_2}$; indeed, this is implied by the fact that for every $n \in \mathbb{N}_{\geq 0}$, $g_2 = ng'$ for some $g' \in G_2$. Thus $g = 0_G$ as required. ◀

▶ **Lemma 14.** *Two linear sets $\{b\} + \textsc{Lin}^{\geq 0}(P)$ and $\{c\} + \textsc{Lin}^{\geq 0}(Q)$ are not modular separable if, and only if, $b - c \in \textsc{Lin}(P \cup Q)$.*

Proof. Let $L = \{b\} + \textsc{Lin}^{\geq 0}(P)$ and $M = \{c\} + \textsc{Lin}^{\geq 0}(Q)$, with $P = \{p_1, \ldots, p_m\}$ and $Q = \{q_1, \ldots, q_n\}$. First we show the "if" direction. By Proposition 13, it is enough to show that, for every $n \in \mathbb{N}$, there exist two vectors $u \in L$ and $v \in M$ s.t. $u \equiv_n v$. Fix an $n \in \mathbb{N}$. By assumption, we have $b - c \in \textsc{Lin}(P \cup Q)$, and thus $c - b \in \textsc{Lin}(P \cup Q) = \textsc{Lin}(P \cup -Q)$. By Lemma 12, $c - b \in \textsc{Lin}_n^{\geq 0}(P \cup -Q)$, i.e., there exist $\delta \in \textsc{Lin}^{\geq 0}(P)$ and $\gamma \in \textsc{Lin}^{\geq 0}(Q)$ such that $c - b \equiv_n \delta - \gamma$. Thus, if we take $u = b + \delta$ and $v = c + \gamma$ we clearly have $u - v = (b - c) + (\delta - \gamma) \equiv_n (b - c) + (c - b) = 0$, and thus $u \equiv_n v$.

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For the "only if" direction, assume that L and M as above are not modular separable. By Proposition 13, for every $n \geq 0$ there exist vectors $u_n \in L$ and $v_n \in M$ s.t. $u_n \equiv_n v_n$. By definition, $u_n = b + \delta_n$ and $v_n = c + \gamma_n$, for some $\delta_n \in \textsc{Lin}^{\geq 0}(P)$ and $\gamma_n \in \textsc{Lin}^{\geq 0}(Q)$. Since $u_n \equiv_n v_n$, we have $b - c \equiv_n \gamma_n - \delta_n \in \textsc{Lin}(P \cup Q)$, and thus $b - c \in \textsc{Lin}_n^{\geq 0}(P \cup Q)$. Since n was arbitrary, by Lemma 12 we have $b - c \in \textsc{Lin}(P \cup Q)$, as required. ◀

We say that a set of vectors $U \subseteq \mathbb{N}^d$ is *diagonal* if, for every threshold $x \in \mathbb{N}$, there exists a vector $u \in U$ which is strictly larger than x in every component. Let $I \subseteq \{1, \ldots, d\}$ be a set of coordinates. Two set of vectors $U, V \subseteq \mathbb{N}^d$ are *I-linked* if there exists a sectioning vector $u \in \mathbb{N}^{d-|I|}$ s.t. $\pi_{\{1,\ldots,d\}\setminus I}(U) = \pi_{\{1,\ldots,d\}\setminus I}(V) = \{u\}$ and $\pi_I(U)$, $\pi_I(V)$ are diagonal. The sets U, V are *linked* if they are I-linked for some $I \subseteq \{1, \ldots, d\}$.

C Proofs for Sec. 6

▶ **Lemma 16.** *For two VAS sections U and V and a modulus $n \in \mathbb{N}$, it is decidable whether there exist $u \in U$ and $v \in V$ s.t. $u \equiv_n v$.*

Proof. Recall that U is obtained from the reachability set of a VAS by fixing values $\bar{u}$ on some coordinates, and projecting to the remaining coordinates; and likewise V is obtained, by fixing values $\bar{v}$ on some coordinates. We modify the two VASes by allowing each non-fixed coordinate to be decremented by n, and we check whether the two thus modified VASes admit a pair of reachable vectors u, v that agree on fixed coordinates with $\bar{u}$ and $\bar{v}$, respectively, and on all the non-fixed coordinates are equal and smaller than n. ◀

▶ **Lemma 20.** *Let $\rho_0, \rho_1, \ldots, \rho_k$ be runs of a VAS s.t., for all $i \in \{1, \ldots, k\}$, $\rho_0 \trianglelefteq \rho_i$, and let $\delta_i := \textsc{target}(\rho_i) - \textsc{target}(\rho_0) \geq 0$. For any $\delta \in \textsc{Lin}^{\geq 0}(\delta_1, \ldots, \delta_k)$, there exists a run ρ s.t. $\rho_0 \trianglelefteq \rho$ and $\delta = \textsc{target}(\rho) - \textsc{target}(\rho_0)$.*

Lemma 20 follows as an immediate corollary of the following simpler property.

▶ **Lemma 26.** *Let ρ, ρ_1, and ρ_2 be runs of a VAS s.t. $\rho \trianglelefteq \rho_1, \rho_2$. There exists a run ρ' s.t. $\rho \trianglelefteq \rho'$ and $\textsc{target}(\rho') - \textsc{target}(\rho) = (\textsc{target}(\rho_1) - \textsc{target}(\rho)) + (\textsc{target}(\rho_2) - \textsc{target}(\rho))$.*

Proof. The proof is almost identical to the proof of Proposition 5.1. in [18]. Let the VAS be (s, T), and let $\rho = v_0 \xrightarrow{t_0} v_1 \xrightarrow{t_1} \cdots \xrightarrow{t_{n-1}} v_n$, where $v_0 = s$. Then ρ_i, for $i \in \{1, 2\}$ is of the form

$$\begin{aligned}\rho_i = \quad & v_0 \xrightarrow{\rho_0^i} v_0 + \delta_0^i \xrightarrow{t_0} v_1 + \delta_0^i \xrightarrow{\rho_1^i} v_1 + \delta_1^i \xrightarrow{t_1} v_1 + \delta_2^i \xrightarrow{\rho_2^i} \cdots \\ & \xrightarrow{\rho_{n-1}^i} v_{n-1} + \delta_{n-1}^i \xrightarrow{t_{n-1}} v_n + \delta_{n-1}^i \xrightarrow{\rho_n^i} v_n + \delta_n^i,\end{aligned}$$

where for all $i \in \{1, 2\}$ and $j \in \{0, \ldots, n\}$ we have $\delta_j^i \geq 0$. Thus by letting $\rho' := \rho_0^1 \rho_0^2 t_0 \rho_1^1 \rho_1^2 t_1 \rho_2^1 \rho_2^2 \cdots \rho_{n-1}^1 \rho_{n-1}^2 t_{n-1} \rho_n^1 \rho_n^2$ we clearly have a run $v_0 \xrightarrow{\rho'} v_n + \delta_n^1 + \delta_n^2$ which indeed looks like

$$\begin{aligned} v_0 & \xrightarrow{\rho_0^1} v_0 + \delta_0^1 \xrightarrow{\rho_0^2} v_0 + \delta_0^1 + \delta_0^2 \xrightarrow{t_0} v_1 + \delta_0^1 + \delta_0^2 \\ & \xrightarrow{\rho_1^1} v_1 + \delta_1^1 + \delta_0^2 \xrightarrow{\rho_1^2} v_1 + \delta_1^1 + \delta_1^2 \xrightarrow{t_1} v_2 + \delta_1^1 + \delta_1^2 \\ & \xrightarrow{\rho_2^1} \cdots \xrightarrow{t_{n-1}} v_n + \delta_{n-1}^1 + \delta_{n-1}^2 \\ & \xrightarrow{\rho_n^1} v_n + \delta_n^1 + \delta_{n-1}^2 \xrightarrow{\rho_n^2} v_n + \delta_n^1 + \delta_n^2.\end{aligned}$$

This finishes the proof of Lemma 26. ◀

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▶ **Lemma 21.** *For every (possibly infinite) set of vectors $S \subseteq \mathbb{Z}^d$, there exist finitely many vectors $v_1, \ldots, v_k \in S$ s.t. $S \subseteq \textsc{Lin}(v_1, \ldots, v_k)$.*

D Proofs for Sec. 7

▶ **Lemma 27.** *Let $U, V \subseteq \mathbb{N}^d$ be two linked linear sets. Then, U and V are unary separable if, and only if, they are modular separable.*

Proof. Let U and V be two linked linear sets. One direction is obvious since modular separability implies unary separability. For the other direction, let U and V be modular nonseparable, and we show that they are unary nonseparable either. By Lemma 13, there exists a sequence of vectors $u_n \in U$ and $v_n \in V$ s.t. $u_n \equiv_n v_n$. We construct a new sequence $u'_n \in U$ and $v'_n \in V$ s.t. $u'_n \cong_n v'_n$, which will then show that U and V are not unary separable by Lemma 23. Since U and V are linked, there exist a set of coordinates $I \subseteq \{1, \ldots, d\}$ and a sectioning vector for the remaining coordinates $u \in \mathbb{N}^{d-|I|}$ s.t. 1) $\pi_{\{1,\ldots,d\}\setminus I}(U) = \pi_{\{1,\ldots,d\}\setminus I}(V) = \{u\}$ and 2) $\pi_I(U), \pi_I(V)$ are diagonal. In particular, by 1) the two sequences u_n and v_n project to u on the complement of I, i.e., $\pi_{\{1,\ldots,d\}\setminus I}(u_n) = \pi_{\{1,\ldots,d\}\setminus I}(v_n) = \{u\}$. Moreover, for any $n \in \mathbb{N}$, since $\pi_I(u_n) \in \pi_I(U)$, and the latter set is diagonal by 2), there exists an increment $\delta_n \in \mathbb{N}^{|I|}$ s.t. $\pi_I(u_n) \leq \pi_I(u_n) + \delta_n \in \pi_I(U)$. Moreover, since U is a linear set, δ_n can be chosen to have its components multiple of n. Let u'_n be $\pi_I(u_n) + \delta_n$ on coordinates I, and u on the remaining ones. By the choice of δ_n, $u'_n \equiv_n u_n$, and, moreover, u'_n is larger than n on coordinates I. The vector v'_n can be constructed similarly from v_n. We thus have $u'_n \cong_n v'_n$, since on coordinates I both u'_n and v'_n are above n, and on the remaining coordinates they are equal to u. ◀

▶ **Lemma 25.** *For two VAS sections $U, V \subseteq \mathbb{N}^d$, the following conditions are equivalent:*

1. *U, V are not unary separable;*
2. *the expansions U', V' of U, V are not unary separable;*
3. *there exist linear subsets $L \subseteq U'$, $M \subseteq V'$ that are not unary separable;*
4. *there exist a U-witness L and a V-witness M that are not unary separable.*

Proof. We only concentrate on showing that 2 implies 4. Assume that the expansions U' and V' are not unary separable, for two sections U and V represented as (recall the simplifying assumptions about VAS sections from Section 6)

$$U = \textsc{sec}_{I,0}(R_U) \subseteq \mathbb{N}^d \quad \text{and} \quad V = \textsc{sec}_{I,0}(R_V) \subseteq \mathbb{N}^d,$$

where $R_U, R_V \subseteq \mathbb{N}^{d'}$ are the reachability sets of two VASes and $I \subseteq \{1, \ldots, d'\}$ with $|I| = d$ are projecting coordinates. Since U' and V' are not unary separable, by Proposition 23, there exists an infinite sequence of pairs of reachable configurations $(u_0, v_0), (u_1, v_1), \ldots \in U' \times V'$ s.t. $u_n \cong_n v_n$ for all $n \in \mathbb{N}$. It means that for every $n \in \mathbb{N}$ there exist runs ρ_n and σ_n in the two VASes ending up in reachable configurations $u_n := \textsc{target}(\rho_n) \in R_U$ and $v_n := \textsc{target}(\sigma_n) \in R_V$. Define $\delta_n := u_n - u_0$ and $\gamma_n := v_n - v_0$ for all $n \in \mathbb{N}$. Since $\trianglelefteq$ is a wqo, by reasoning as in the proof of Lemma 22, we can assume w.l.o.g. that $\rho_0 \trianglelefteq \rho_1 \trianglelefteq \cdots$, and similarly for the σ_i's.

Since $u_n \cong_n v_n$, the two sequences $u_0 \leq u_1 \leq \cdots$ and $v_0 \leq v_1 \leq \cdots$ are unbounded on the same set of coordinates. Let $F \subseteq \{1, \ldots, d'\}$ be this set; note that $F \subseteq I$. By eliminating a sufficiently long prefix of these two sequences, we can further assume that

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bounded coordinates are in fact constant, and again from $u_n \cong_n v_n$ it follows that this constant is the same vector for both sequences. Consequently,

$$\pi_{\{1,\ldots,d'\}\setminus F}(u_0) = \pi_{\{1,\ldots,d'\}\setminus F}(v_0), \text{ and} \tag{5}$$

$$\forall_{n\in\mathbb{N}}\, \pi_{\{1,\ldots,d'\}\setminus F}(\delta_n) = \pi_{\{1,\ldots,d'\}\setminus F}(\gamma_n) = 0. \tag{6}$$

By proceeding as in the proof of Lemma 22, there exist two finite sets $P := \{\delta_{i_1}, \ldots, \delta_{i_k}\}$ and $Q := \{\gamma_{i_1}, \ldots, \delta_{i_k}\}$ such that the linear sets $L := \{u_0\} + \textsc{Lin}^{\geq 0}(P) \subseteq U$ is a U-witness, the linear set $M := \{v_0\} + \textsc{Lin}^{\geq 0}(Q) \subseteq V$ is a V-witness, and L, M are not modular separable. It remains to show that L and M are not *unary* separable either. While unary nonseparability is a stronger property than modular nonseparability in general, by Lemma 27 the two conditions are in fact equivalent when the two sets are *linked*. We make use of the set F as chosen before, and we show that L and M are F-linked. Indeed, if $j \in F$ then w.l.o.g. we may assume that the two sequences $\pi_j(u_0) < \pi_j(u_1) < \ldots$ and $\pi_j(v_0) < \pi_j(v_1) < \ldots$ are strictly increasing. Thus, $\pi_j(\delta_n), \pi_j(\gamma_n) > n$ for every $n \in \mathbb{N}$, which implies that $\pi_F(L)$ and $\pi_F(M)$ are diagonal. On the other hand, if $j \in \{1, \ldots, d'\} \setminus F$, from properties (5) and (6) above, we have $\pi_{\{1,\ldots,d\}\setminus F}(L) = \pi_{\{1,\ldots,d\}\setminus F}(M) = \{\pi_{\{1,\ldots,d\}\setminus F}(u_0)\}$. Thus L and M are indeed F-linked. ◀

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