Topological Spaces Related to Linear Algebra

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Multiplications on the cartesian spaces

A structure of a normed associative and commutative $\mathbb{R}$-algebra (i.e. a sensible multiplication) can be defined only on the caretesian spaces $\mathbb{R}^n$ for $n = 1, 2$ where $\mathbb{R}^1 = \mathbb{R}$ - real numbers ,$\mathbb{R}^2 = \mathbb{C}$ - complex numbers (pairs of real numbers). A structure of an associative but not commutative $\mathbb{R}$-algebra can be defined for $n = 4$ on pairs of complex numbers $\mathbb{R}^4 = \mathbb{H}$ - quaternions. Further, if we omit the associativity assumption, then a $\mathbb{R}$-bilinear multiplication can be defined for $n = 8$ on pairs of quaternions $\mathbb{R}^8 = \mathbb{O}$-octonions. Moreover, we have inclusions of $\mathbb{R}$-algebras $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$ and a conjugation on $\mathbb{O}$ restricts to the usual conjugation of quaternions and complex numbers. A norm on $\mathbb{O}$ is defined as usual $\|o\|^2 = oo^*$ thus $\|oo'\| = \|o\|\|o'\|$. If $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ then any cartesian space $\mathbb{F}^n$ carries a structure of an $\mathbb{F}$-module (vector space) via multiplications of coordinates and an $\mathbb{F}$-valued (hermitian) scalar product is defined in the usual way: $(v, w) := \Sigma\ v_i w_i^*$, where $*$ denotes conjugation. Vectors of lenght 1 constitute a group for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and an H-space for $\mathbb{F} = \mathbb{O}$. If not otherwise stated, in problems below $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (not $\mathbb{O}$).

The Stiefel manifolds

Noncompact Stiefel manifold For $k \leqslant n$ denote by $\bar{V}_k(\mathbb{F}^n)$ the set of k-tuples of $\mathbb{F}$-lineary independent vectors in $\mathbb{F}^n$. Clearly $\bar{V}_k(\mathbb{F}^n)$ is an open subspace of $(\mathbb{F}^n)^k = \mathbb{F}^{nk}$ thus it is a topological space, even a noncompact nk-dimensional manifold.
Compact Stiefel manifold For $k \leqslant n$ denote by $V_k(\mathbb{F}^n)$ the set of k-orthonormal tuples vectors in $\mathbb{F}^n$. Clearly $V_k(\mathbb{F}^n) \subset \bar{V}_k(\mathbb{F}^n)$. Compact Stiefel manifolds are called Stiefel manifolds, for short.

Zad. 1. Note that $\bar{V}_1(\mathbb{F}^n)$ consists of all nonzero vectors, $V_1(\mathbb{F}^n)$ is a sphere, $\bar{V}_n(\mathbb{F}^n)$ is homeomorphic to the full linear group denoted by $GL(n, \mathbb{F})$, and $V_n(\mathbb{F}^n)$ is homeomorphic to the group of the norm preserving automorphisms of $\mathbb{F}^n$, denoted by $O(n, \mathbb{F})$.

Zad. 2. Prove that the compact Stiefiel manifold $V_k(\mathbb{F}^n) \subset \bar{V}_k(\mathbb{F}^n)$ is indeed compact and it is a strong deformation retract of the noncompact Stiefiel manifold.

Zad. 3. How many connected components does $V_k(\mathbb{F}^n)$ have?

Zad. 4. Prove that the compact Stiefiel manifold $V_k(\mathbb{F}^n)$ is indeed a smooth manifold and its dimension is $2nk - 1/2(d_{\mathbb{F}}k^2 - d_F k + 2k)$ where $d = \dim_{\mathbb{R}} \mathbb{F}$

Zad. 5. The natural action of $GL(n, \mathbb{F})$ on $\mathbb{F}^n$ induces an action on $\bar{V}_k(\mathbb{F}^n)$. Show that the action is transitive and identify the isotropy group (in particular the stabilizer of the tuple $(e_1, .., e_k)$). Similarly for the compact Stiefel manifold and the group of isometries $O(n, \mathbb{F})$.

Zad. 6. Let $r \leqslant s \leqslant n$ then there exist a projection $V_s(\mathbb{F}^n) \to V_r(\mathbb{F}^n)$. Show that it is locally trivial and identify its fiber (also a Stiefel manifold.)

The Grassmann manifolds

Grassmann manifold (or Grassmannian.) Let $k \leqslant n$ the Grassmann manifold $G_k(\mathbb{F}^n)$ is a set of k-dimensional subspaces in $\mathbb{F}^n$. There is an obvious projection $p : \bar{V}_k(\mathbb{F}^n) \to G_k(\mathbb{F}^n)$ which assigns to each k-tuple a linear subspace generated by it. The map p defines a quotient topology on $G_k(\mathbb{F}^n)$.

Zad. 7. Prove that $G_k(\mathbb{F}^n)$ is Hausdorff, thus a compact space.

Zad. 8. Prove that there exist a homeomorphism $G_k(\mathbb{F}^n) \simeq G_{n-k}(\mathbb{F}^n)$

Zad. 9. Prove that $G_k(\mathbb{F}^n)$ is a connected, compact manifold of dimension $dk(n-k)$.

Zad. 10. There is an embedding of the Grassmannian $G_k(\mathbb{F}^n)$ in the cartesian space $\mathbb{F}^{n^2} = \operatorname{Hom}(F^n, F^n)$ which assigns to every subsapce the orthogonal projection on it. The embedding defines a natural (operator) metric on $G_k(\mathbb{F}^n)$.

Zad. 11. The Grassmannians $G_1(\mathbb{F}^n)$ are the well-known projective spaces, denoted $\mathbb{F}P(n)$. Note that $G_1(\mathbb{F}^2) = S^d$ and if we identify S^d with the one-point compactification of $\mathbb{F}$ the projection p corresponds to the map $p_d : S^{2d-1} \to S^d$ given by $p_d(z_0, z_1) = z_0/z_1$ where $z_i \in \mathbb{F}$. Note, that the same formula works for $\mathbb{F} = \mathbb{O}$, however the higher dimensional projective spaces over octonions do not exist. The maps $p_d : S^{2d-1} \to S^d$ for $d = 1, 2, 4, 8$ are called the Hopf maps and they play a very important role in homotopy theory; a fiber of p_d is a sphere S^{d-1}. Check directly that the Hopf maps are locally trivial, thus fibrations.

Zad. 12. The natural action of $GL(n, \mathbb{F})$ (resp. $O(n, \mathbb{F})$) on $\mathbb{F}^n$ induces an action on $G_k(\mathbb{F}^n)$. Show that the actions are transitive and describe the isotropy groups (in particular of the canonical subspace $F^k \subset F^n$)

Zad. 13. Prove that there is a free action of the group $O(k, \mathbb{F})$ on $V_k(\mathbb{F}^n)$ such that the orbit space is homeomorphic to $G_k(\mathbb{F}^n)$. Similarly for the noncompact Stiefel manifold.

Zad. 14. Prove that the map $p : V_k(\mathbb{F}^n) \to G_k(\mathbb{F}^n)$ is locally trivial (even a principal $O(k, \mathbb{F})$-bundle), thus a fibration.