Weak and strong moments of ℓ_r-norms of log-concave vectors *

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revised version

Abstract

We show that for $p \geq 1$ and $r \geq 1$ the p-th moment of the ℓ_r-norm of a log-concave random vector is comparable to the sum of the first moment and the weak p-th moment up to a constant proportional to r. This extends the previous result of Paouris concerning Euclidean norms.

1 Introduction and Main Results

A measure μ on a locally convex linear space F is called logarithmically concave (log-concave in short) if for any compact nonempty sets $K, L \subset F$ and $\lambda \in [0,1]$, $\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda}\mu(L)^{1-\lambda}$. A random vector with values in F is called log-concave if its distribution is logarithmically concave. The class of log-concave measures is closed under linear transformations, convolutions and weak limits. By the result of Borell [3] a d-dimensional vector with a full dimensional support is log-concave iff it has a log-concave density, i.e. a density of the form e^{-h}, where h is a convex function with values in $(-\infty, \infty]$. A typical example of a log-concave vector is a vector uniformly distributed over a convex body. Various results and conjectures about log-concave measures are discussed in the recently published monograph [4].

One of the fundamental properties of log-concave vectors is the Paouris inequality [9] (see also [1] for a shorter proof). It states that for a log-concave vector X in $\mathbb{R}^n$,

$$(\mathbb{E}\|X\|_2^p)^{1/p} \leq C_1 \left((\mathbb{E}\|X\|_2^2)^{1/2} + \sigma_X(p) \right) \qquad \text{for } p \geq 1, \tag{1}$$

where

$$\sigma_X(p) := \sup_{\|t\|_2 \leq 1} \left(\mathbb{E} \left| \sum_{i=1}^n t_i X_i \right|^p \right)^{1/p}.$$

Here and in the sequel by $C_1, C_2, \ldots$ we denote absolute constants.

It is natural to ask whether inequality (1) may be generalized to non-Euclidean norms. In [6] the following conjecture was formulated and discussed.

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Conjecture 1. *There exists a universal constant C such that for any log-concave vector X with values in a finite dimensional normed space $(F, \|\ \|)$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*, \|\varphi\|_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \qquad \text{for } p \ge 1.$$

Our main result states that the conjecture holds for spaces that may be embedded in ℓ_r for some $r \ge 1$.

Theorem 2. *Let X be a log-concave vector with values in a normed space $(F, \|\ \|)$ which may be isometrically embedded in ℓ_r for some $r \in [1, \infty)$. Then for $p \ge 1$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C_2 r\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*, \|\varphi\|_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\right).$$

Remark 3. Let X and F be as above. Then by Chebyshev's inequality we obtain large deviation estimate for $\|X\|$:

$$\mathbb{P}(\|X\| \ge 2eC_2 rt\mathbb{E}\|X\|) \le \exp\left(-\sigma_{X,F}^{-1}(t\mathbb{E}\|X\|)\right) \qquad \text{for } t \ge 1,$$

where

$$\sigma_{X,F}(p) := \sup_{\varphi\in F^*, \|\varphi\|_*\le 1} (\mathbb{E}\varphi(X)^p)^{1/p} \qquad \text{for } p \ge 1$$

denotes the weak p-th moment of $\|X\|$.

Remark 4. If $i\colon F \to \ell_r$ is a nonisometric embedding and $\lambda = \|i\|_{F\to\ell_r}\|i^{-1}\|_{i(F)\to F}$, then we may define another norm on F by $\|x\|' := \|i(x)\|/\|i\|_{F\to\ell_r}$. Obviously $(F, \|\ \|')$ isometrically embeds in ℓ_r, moreover $\|x\|' \le \|x\| \le \lambda\|x\|'$ for $x \in F$. Hence Theorem 2 gives

$$\begin{aligned}(\mathbb{E}\|X\|^p)^{1/p} &\le \lambda(\mathbb{E}(\|X\|')^p)^{1/p} \le C_2 r\lambda\left(\mathbb{E}\|X\|' + \sup_{\varphi\in F^*, \|\varphi\|'_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\right) \\ &\le C_2 r\lambda\left(\mathbb{E}\|X\| + \sup_{\varphi\in F^*, \|\varphi\|_*\le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\right).\end{aligned}$$

Since log-concavity is preserved under linear transformations and, by the Hahn-Banach theorem, any linear functional on a subspace of ℓ_r is a restriction of a functional on the whole ℓ_r with the same norm, it is enough to prove Theorem 2 for $F = \ell_r$. An easy approximation argument shows that we may consider finite dimensional spaces ℓ_r^n. To simplify the notation for an n-dimensional vector X and $p \ge 1$ we write

$$\sigma_{r,X}(p) := \sup_{\|t\|_{r'}\le 1}\left(\mathbb{E}\left|\sum_{i=1}^n t_i X_i\right|^p\right)^{1/p},$$

where r' denotes the Hölder's dual of r, i.e. $r' = \frac{r}{r-1}$ for $r > 1$ and $r' = \infty$ for $r = 1$.

Theorem 5. *Let X be a finite dimensional log-concave vector and $r \in [1, \infty)$. Then*

$$(\mathbb{E}\|X\|_r^p)^{1/p} \leq C_2 r\,(\mathbb{E}\|X\|_r + \sigma_{r,X}(p)) \qquad \text{for } p \geq 1.$$

To show the above theorem we follow the approach from [7] and establish the following result.

Theorem 6. *Suppose that $r \in [1, \infty)$ and X is a log-concave n-dimensional random vector. Let*

$$d_i := (\mathbb{E}X_i^2)^{1/2}, \quad d := \left(\sum_{i=1}^n d_i^r\right)^{1/r}. \tag{2}$$

Then for $p \geq r$,

$$\mathbb{E}\left(\sum_{i=1}^n |X_i|^r \mathbf{1}_{\{|X_i| \geq t d_i\}}\right)^{p/r} \leq (C_3 r \sigma_{r,X}(p))^p \qquad \text{for } t \geq C_4 r \log\left(\frac{d}{\sigma_{r,X}(p)}\right). \tag{3}$$

Remark 7. Any finite dimensional space embeds isometrically in ℓ_∞, so to show Conjecture 1 it is enough to establish Theorem 2 (with a universal constant in place of $C_2 r$) for $r = \infty$. Such a result was shown for isotropic log-concave vectors (i.e. log-concave vectors with mean zero and identity covariance matrix), cf. [8, Corollary 3.8]. However a linear image of an isotropic vector does not have to be isotropic, so to establish the conjecture we need to consider either isotropic vectors and an arbitrary norm or vectors with a general covariance structure and the standard ℓ_∞-norm.

2 Proofs

Let us first discuss the notation. By C we denote universal constants, the value of C may differ at each occurrence. Whenever we want to fix the value of an absolute constant we use letters $C_1, C_2, \ldots$. We may always assume that $C_i \geq 1$. By $|I|$ we denote the cardinality of a set I. For an n-dimensional random vector Z and $a \in \mathbb{R}^n$ we write aZ for the vector $(a_i Z_i)_i$. Observe that $\mathbb{E}\|aZ\|_2^2 = \sum_i a_i^2 \mathbb{E}Z_i^2$.

Let us recall some useful facts about log-concave vectors (for details see [7]). If Z is log-concave real random variable then

$$\mathbb{P}(|Z| \geq t) \leq \exp\left(2 - \frac{t}{2e(\mathbb{E}Z^2)^{1/2}}\right) \quad \text{for } t \geq 0.$$

Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ is a seminorm, $(\mathbb{E}f(Z)^p)^{1/p} \leq C_5 \frac{p}{q} (\mathbb{E}f(Z)^q)^{1/q}$ for $p \geq q \geq 1$ (see [4, Theorem 2.4.6]). Therefore for any log-concave vector X and any r,

$$\sigma_{r,X}(\lambda p) \leq C_5 \lambda \sigma_{r,X}(p) \quad \text{for } \lambda \geq 1,\ p \geq 2.$$

The Paouris inequality (1) together with Chebyshev's inequality imply

$$\mathbb{P}\left(\|X\|_2 \geq eC_1\left((\mathbb{E}\|X\|_2^2)^{1/2}+\sigma_X(p)\right)\right) \leq e^{-p} \quad \text{for } p \geq 1. \tag{4}$$

The next proposition generalizes Proposition 4 from [7].

Proposition 8. *Let X, r, d_i, and d be as in Theorem 6 and $A := \{X \in K\}$, where K is a convex set in $\mathbb{R}^n$ satisfying $0 < \mathbb{P}(A) \leq 1/e$. Then*
(i) for every $t \geq r$,

$$\sum_{i=1}^n \mathbb{E}|X_i|^r \mathbf{1}_{A\cap\{X_i\geq td_i\}} \leq C_6^r \mathbb{P}(A)\left(r^r\sigma_{r,X}^r(-\log(\mathbb{P}(A))) + (dt)^r e^{-t/C_7}\right). \tag{5}$$

(ii) for every $t > 0$, $u \geq 1$,

$$\begin{aligned}\sum_{k=0}^\infty 2^{kr}\sum_{i=1}^n d_i^r \mathbf{1}_{\{\mathbb{P}(A\cap\{X_i\geq 2^k td_i\})\geq e^{-u}\mathbb{P}(A)\}} \\ \leq \frac{(C_8 u)^r}{t^r}\left(\sigma_{r,X}^r(-\log(\mathbb{P}(A))) + d^r\mathbf{1}_{\{t\leq uC_9\}}\right).\end{aligned} \tag{6}$$

Proof. Let Y be a random vector defined by

$$\mathbb{P}(Y\in B) = \frac{\mathbb{P}(A\cap\{X\in B\})}{\mathbb{P}(A)} = \frac{\mathbb{P}(X\in B\cap K)}{\mathbb{P}(X\in K)},$$

i.e. Y is distributed as X conditioned on A. Clearly, for every measurable set B one has $\mathbb{P}(X\in B)\geq \mathbb{P}(A)\mathbb{P}(Y\in B)$. It is easy to see that Y is log-concave.

To simplify the notation set

$$p_A := -\log\mathbb{P}(A) \quad \text{and} \quad c_i := (\mathbb{E}Y_i^2)^{1/2},\ i=1,\ldots,n.$$

Let

$$I = I(v) := \{i\leq n\colon\ \mathbb{E}Y_i^2 \geq v^2 d_i^2\},$$

where v is an absolute constant to be chosen later. Let us also fix a sequence $(a_i)_{i\leq n}$.

Put $S = \sum_{i\in I}|a_i|c_i^{-1}Y_i^2$. Observe that $S = \|((|a_i|/c_i)^{1/2}Y_i)_{i\in I}\|_2^2$, hence by the log-concavity of Y, $\mathbb{E}S^2 \leq (2C_5)^4(\mathbb{E}S)^2$, and the Paley-Zygmund inequality yields

$$\mathbb{P}\left(\sum_{i\in I}|a_i|c_i^{-1}Y_i^2 \geq \frac{1}{2}\sum_{i\in I}|a_i|c_i\right) = \mathbb{P}\left(S\geq\frac{1}{2}\mathbb{E}S\right) \geq \frac{1}{4}\frac{(\mathbb{E}S)^2}{\mathbb{E}S^2} \geq \frac{1}{(2\sqrt{2}C_5)^4}. \tag{7}$$

We have $\mathbb{E}Y_i^4 \leq (2C_5c_i)^4$, so by Chebyshev's inequality we get

$$\mathbb{P}\left(\sum_{i\in I}|a_i|c_i^{-3}Y_i^4 \geq (2C_5)^4 s\sum_{i\in I}|a_i|c_i\right) \leq \frac{1}{s} \quad \text{for } s>0. \tag{8}$$

Combining (7) and (8) we conclude that there exist constants C_{10}, C_{11} such that

$$\mathbb{P}\left(\sum_{i\in I}|a_i|c_i^{-1}Y_i^2 \ge \frac{1}{2}\sum_{i\in I}|a_i|c_i,\ \sum_{i\in I}|a_i|c_i^{-3}Y_i^4 \le C_{10}\sum_{i\in I}|a_i|c_i\right) \ge \frac{1}{C_{11}}$$

and therefore

$$\mathbb{P}\left(\sum_{i\in I}|a_i|c_i^{-1}X_i^2 \ge \frac{1}{2}\sum_{i\in I}|a_i|c_i,\ \sum_{i\in I}|a_i|c_i^{-3}X_i^4 \le C_{10}\sum_{i\in I}|a_i|c_i\right) \ge \frac{1}{C_{11}}\mathbb{P}(A) \ge e^{-C_{12}p_A}.$$

Let $\tilde{X}$ be the vector $(|a_i|^{1/2}c_i^{-1/2}X_i)_{i\in I}$ conditioned on the set

$$B := \left\{\sum_{i\in I}|a_i|c_i^{-3}X_i^4 \le C_{10}\sum_{i\in I}|a_i|c_i\right\}.$$

Then

$$\mathbb{P}\left(\|\tilde{X}\|_2^2 \ge \frac{1}{2}\sum_{i\in I}|a_i|c_i\right) \ge \frac{1}{\mathbb{P}(B)}e^{-C_{12}p_A} \ge e^{-C_{12}p_A}. \tag{9}$$

The random vector $\tilde{X}$ is log-concave and $\mathbb{P}(B) \ge 1/2$ if v is sufficiently large (since $\mathbb{E}X_i^4 \le Cd_i^4 \le Cv^{-4}c_i^4$ for $i \in I$). Thus

$$\mathbb{E}\|\tilde{X}\|_2^2 = \frac{1}{\mathbb{P}(B)}\mathbb{E}\left(\sum_{i\in I}|a_i|c_i^{-1}X_i^2\mathbf{1}_B\right) \le 2\sum_{i\in I}\mathbb{E}|a_i|c_i^{-1}d_i^2 \le 2v^{-2}\sum_{i\in I}|a_i|c_i. \tag{10}$$

Now we will estimate $\sigma_{\tilde{X}}(p)$. To this end fix $t \in \mathbb{R}^I$ with $\|t\|_2 \le 1$. Let $\alpha, s > 0$ be numbers to be chosen later and

$$J_\alpha := \{i \in I\colon\ |t_i|(|a_i|c_i)^{-1/2} \le \alpha\}.$$

We have

$$\left\|\sum_{i\in J_\alpha}t_i\tilde{X}_i\right\|_p \le \mathbb{P}(B)^{-1/p}\left\|\sum_{i\in J_\alpha}t_i(|a_i|c_i)^{-1/2}|a_i|X_i\right\|_p \le 2\alpha\sigma_{1,aX}(p).$$

Moreover

$$\left\|\sum_{i\notin J_\alpha}t_i\tilde{X}_i\mathbf{1}_{\{|\tilde{X}_i|\le s(|a_i|c_i)^{1/2}\}}\right\|_p \le \sum_{i\notin J_\alpha}s|t_i|(|a_i|c_i)^{1/2} \le s\sum_{i\notin J_\alpha}\frac{|t_i|^2}{|t_i|(|a_i|c_i)^{-1/2}} \le \frac{s}{\alpha}\sum_{i\in I}t_i^2 \le \frac{s}{\alpha}.$$

Observe that by the definition of the set B and the vector $\tilde{X}$ we have

$$\sum_{i\in I}(|a_i|c_i)^{-1}\tilde{X}_i^4 \le C_{10}\sum_{i\in I}|a_i|c_i.$$

Thus

$$\left\|\sum_{i\notin J_\alpha} t_i\tilde{X}_i\mathbf{1}_{\{|\tilde{X}_i|>s(|a_i|c_i)^{1/2}\}}\right\|_p \le \left\|\left(\sum_{i\notin J_\alpha}\tilde{X}_i^2\mathbf{1}_{\{|\tilde{X}_i|>s(|a_i|c_i)^{1/2}\}}\right)^{1/2}\right\|_p$$
$$\le \left\|\frac{1}{s}\left(\sum_{i\in I}(|a_i|c_i)^{-1}\tilde{X}_i^4\right)^{1/2}\right\|_p \le \frac{1}{s}\left(C_{10}\sum_{i\in I}|a_i|c_i\right)^{1/2}.$$

Combining the above estimates we obtain

$$\left\|\sum_{i\in I} t_i\tilde{X}_i\right\|_p \le 2\alpha\sigma_{1,aX}(p) + \frac{s}{\alpha} + \frac{1}{s}\left(C_{10}\sum_{i\in I}|a_i|c_i\right)^{1/2}.$$

Taking the supremum over t and optimizing over $\alpha > 0$ we get

$$\sigma_{\tilde{X}}(p) \le 4(s\sigma_{1,aX}(p))^{1/2} + \frac{1}{s}\left(C_{10}\sum_{i\in I}|a_i|c_i\right)^{1/2} \qquad \text{for } s > 0. \tag{11}$$

Paouris' inequality (4) (applied to $\tilde{X}$ instead of X) together with (10) and (11) implies that

$$\mathbb{P}\left(\|\tilde{X}\|_2 \ge eC_1\left(\left(\frac{2}{v^2}\sum_{i\in I}|a_i|c_i\right)^{1/2} + 4(s\sigma_{1,aX}(C_{12}p_A))^{1/2} + \frac{1}{s}\left(C_{10}\sum_{i\in I}|a_i|c_i\right)^{1/2}\right)\right) < e^{-C_{12}p_A}$$

Comparing the above with (9) we get

$$eC_1\left(\left(\frac{2}{v^2}\sum_{i\in I}|a_i|c_i\right)^{1/2} + 4(s\sigma_{1,aX}(C_{12}p_A))^{1/2} + \frac{1}{s}\left(C_{10}\sum_{i\in I}|a_i|c_i\right)^{1/2}\right) \ge \left(\frac{1}{2}\sum_{i\in I}|a_i|c_i\right)^{1/2}.$$

If we choose s and v to be sufficiently large absolute constants we will get

$$\sum_{i\in I}|a_i|(\mathbb{E}Y_i^2)^{1/2} = \sum_{i\in I}|a_i|c_i \le C\sigma_{1,aX}(C_{12}p_A) \le C\sigma_{1,aX}(p_A).$$

Put $a_i := (\mathbb{E}|Y_i|^2)^{(r-1)/2}\mathbf{1}_{i\in I}$. If $\|t\|_\infty \leq 1$, then $(\sum |t_i a_i|^{r'})^{1/r'} \leq \|a\|_{r'}$. Thus the previous inequality implies

$$\sum_{i\in I} \left(\mathbb{E}|Y_i|^2\right)^{r/2} \leq C\sigma_{1,aX}(p_A) \leq C\|a\|_{r'}\sigma_{r,X}(p_A) = C\left(\sum_{i\in I}\left(\mathbb{E}|Y_i|^2\right)^{r/2}\right)^{1/r'}\sigma_{r,X}(p_A).$$

This gives

$$\sum_{i\in I}(\mathbb{E}|Y_i|^2)^{r/2} \leq C^r\sigma_{r,X}^r(p_A).$$

Since $\|Y_i\|_r \leq \max\{1, C_5 r/2\}\|Y_i\|_2$ we also get

$$\sum_{i\in I}\mathbb{E}|Y_i|^r \leq (Cr)^r\sigma_{r,X}^r(p_A).$$

To prove (5) note that if $i \notin I$, then $\mathbb{P}(|Y_i| \geq sd_i) \leq 2e^{-s/C}$ for $s \geq 0$, hence for $t \geq r$, $\mathbb{E}|Y_i|^r\mathbf{1}_{\{Y_i\geq td_i\}} \leq (Ctd_i)^r e^{-t/C}$ and

$$\sum_{i\notin I}\mathbb{E}|Y_i|^r\mathbf{1}_{\{Y_i\geq t\}} \leq (Ctd)^r e^{-t/C}.$$

Hence

$$\begin{aligned}\frac{1}{\mathbb{P}(A)}\sum_{i=1}^n \mathbb{E}|X_i|^r\mathbf{1}_{A\cap\{X_i\geq td_i\}} &= \sum_{i=1}^n\mathbb{E}|Y_i|^r\mathbf{1}_{\{Y_i\geq td_i\}}\\ &\leq C^r\left(r^r\sigma_{r,X}^r(-\log(\mathbb{P}(A))) + (dt)^r e^{-t/C}\right).\end{aligned}$$

To show (6) note first that for every i the random variable Y_i is log-concave, hence for $s \geq 0$,

$$\frac{\mathbb{P}(A\cap\{X_i\geq s\})}{\mathbb{P}(A)} = \mathbb{P}(Y_i \geq s) \leq \exp\left(2 - \frac{s}{2e\|Y_i\|_2}\right).$$

Thus, if $\mathbb{P}(A\cap\{X_i \geq 2^k td_i\}) \geq e^{-u}\mathbb{P}(A)$ and $u \geq 1$, then $\|Y_i\|_2 \geq 2^k td_i/(2e(u+2)) \geq 2^k td_i/(6eu)$. In particular this cannot happen if $i \notin I$, $k \geq 0$ and $u \leq t/C_9$ with C_9 large enough. Therefore

$$
\begin{aligned}
&\sum_{k=0}^{\infty} 2^{kr} \sum_{i=1}^{n} d_i^r \mathbf{1}_{\{\mathbb{P}(A\cap\{X_i\geq 2^k t d_i\})\geq e^{-u}\mathbb{P}(A)\}} \\
&\qquad\qquad \leq \left(\sum_{i\in I} + \mathbf{1}_{\{t\leq uC_9\}} \sum_{i\notin I}\right) d_i^r \sum_{k=0}^{\infty} 2^{kr} \mathbf{1}_{\{(\mathbb{E}Y_i^2)^{1/2}\geq 2^k t d_i/(6eu)\}} \\
&\qquad\qquad \leq \left(\sum_{i\in I} + \mathbf{1}_{\{t\leq uC_9\}} \sum_{i\notin I}\right) d_i^r \frac{(Cu)^r}{(td_i)^r} (\mathbb{E}Y_i^2)^{r/2} \\
&\qquad\qquad \leq \frac{(Cu)^r}{t^r} \left(\sum_{i\in I} (\mathbb{E}Y_i^2)^{r/2} + \mathbf{1}_{\{t\leq uC_9\}} \sum_{i\notin I} d_i^r\right) \\
&\qquad\qquad \leq \frac{(Cu)^r}{t^r} \left(\sigma_{r,X}^r(-\log(\mathbb{P}(A))) + d^r \mathbf{1}_{\{t\leq uC_9\}}\right). \qquad \square
\end{aligned}
$$

We will also use the following simple combinatorial lemma (Lemma 11 in [5]).

Lemma 9. *Let $l_0 \geq l_1 \geq \ldots \geq l_s$ be a fixed sequence of positive integers and*

$$
\mathcal{F} := \{f \colon \{1,2,\ldots,l_0\} \to \{0,1,2,\ldots,s\} \colon \ \forall_{1\leq i\leq s} \ |\{r \colon f(r) \geq i\}| \leq l_i\}.
$$

Then

$$
|\mathcal{F}| \leq \prod_{i=1}^{s} \left(\frac{e l_{i-1}}{l_i}\right)^{l_i}.
$$

Proof of Theorem 6. Observe that we may assume that $t \geq C_4 r$. Indeed, if $e\sigma_{r,X}(p) \leq d$ then by our assumption $t \geq C_4 r$. If $e\sigma_{r,X}(p) > d$ then

$$
\begin{aligned}
&\left(\mathbb{E}\left(\sum_{i=1}^{n} |X_i|^r \mathbf{1}_{\{|X_i|\geq t d_i\}}\right)^{p/r}\right)^{1/p} \\
&\qquad\qquad \leq C_4 r \left(\sum_{i=1}^{n} d_i^r\right)^{1/r} + \left(\mathbb{E}\left(\sum_{i=1}^{n} |X_i|^r \mathbf{1}_{\{|X_i|\geq \max\{t, C_4 r\} d_i\}}\right)^{p/r}\right)^{1/p} \\
&\qquad\qquad \leq e C_4 r \sigma_{r,X}(p) + \left(\mathbb{E}\left(\sum_{i=1}^{n} |X_i|^r \mathbf{1}_{\{|X_i|\geq \max\{t, C_4 r\} d_i\}}\right)^{p/r}\right)^{1/p}.
\end{aligned}
$$

Moreover, the vector $-X$ is also log-concave, has the same values of d_i and $\sigma_{r,-X} = \sigma_{r,X}$. Hence it is enough to show that

$$
\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbf{1}_{\{X_i\geq t d_i\}}\right)^{p/r} \leq (C r \sigma_{r,X}(p))^p \quad \text{for } t \geq C_4 r \max\left\{1, \log\left(\frac{d}{\sigma_{r,X}(p)}\right)\right\}.
$$

Observe that for $l = 1, 2, \ldots,$

$$\mathbb{E}\left(\sum_{i=1}^{n} X_i^r \mathbf{1}_{\{X_i \geq td_i\}}\right)^l \leq \mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=0}^{\infty} 2^{(k+1)r}(td_i)^r \mathbf{1}_{\{X_i \geq 2^k td_i\}}\right)^l$$
$$= (2t)^{rl} \sum_{i_1,\ldots,i_l=1}^{n} \sum_{k_1,\ldots,k_l=0}^{\infty} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}),$$

where

$$B_{i_1,k_1\ldots,i_l,k_l} := \{X_{i_1} \geq 2^{k_1} td_{i_1}, \ldots, X_{i_l} \geq 2^{k_l} td_{i_l}\}.$$

Define a positive integer l by

$$\frac{p}{r} < l \leq 2\frac{p}{r} \quad \text{and} \quad l = 2^M \text{ for some positive integer } M.$$

Then $\sigma_{r,X}(p) \leq \sigma_{r,X}(rl) \leq \sigma_{r,X}(2p) \leq 2C_5\sigma_{r,X}(p)$. Since for any nonnegative r.v. Z we have $(\mathbb{E}Z^{p/r})^{r/p} \leq (\mathbb{E}Z^l)^{1/l}$, it is enough to show that

$$m(l) \leq \left(\frac{Cr\sigma_{r,X}(rl)}{t}\right)^{rl} \qquad \text{for } t \geq C_4 r \max\left\{1, \log\left(\frac{d}{\sigma_{r,X}(rl)}\right)\right\}, \tag{12}$$

where

$$m(l) := \sum_{k_1,\ldots,k_l=0}^{\infty} \sum_{i_1,\ldots,i_l=1}^{n} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}).$$

We divide the sum in $m(l)$ into several parts. Define sets

$$I_0 := \left\{(i_1, k_1, \ldots, i_l, k_l) \colon \ \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) > e^{-rl}\right\},$$

and for $j = 1, 2, \ldots,$

$$I_j := \left\{(i_1, k_1, \ldots, i_l, k_l) \colon \ \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) \in (e^{-rl2^j}, e^{-rl2^{j-1}}]\right\}.$$

Then $m(l) = \sum_{j\geq 0} m_j(l)$, where

$$m_j(l) := \sum_{(i_1,k_1,\ldots,i_l,k_l)\in I_j} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1\ldots,i_l,k_l}).$$

To estimate $m_0(l)$ define for $1 \leq s \leq l$,

$$P_s I_0 := \{(i_1, k_1, \ldots, i_s, k_s) \colon (i_1, k_1, \ldots, i_l, k_l) \in I_0 \text{ for some } i_{s+1}, \ldots, k_l\}.$$

We have (since t is assumed to be large)

$$\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s}) \leq \mathbb{P}(B_{i_1,k_1}) \leq \exp(2 - 2^{k_1-1}t/e) \leq e^{-1}.$$

Thus for $s = 1, \dots, l-1$,

$$\sum_{(i_1,k_1,\dots,i_{s+1},k_{s+1})\in P_{s+1}I_0} 2^{(k_1+\dots+k_{s+1})r} d_{i_1}^r \dots d_{i_{s+1}}^r \mathbb{P}(B_{i_1,k_1,\dots,i_{s+1},k_{s+1}}) \leq \sum_{(i_1,k_1,\dots,i_s,k_s)\in P_sI_0} 2^{(k_1+\dots+k_s)r} d_{i_1}^r \cdots d_{i_s}^r F(i_1,k_1,\dots,i_s,k_s),$$

where

$$\begin{aligned}
F(i_1,k_1,\dots,i_s,k_s) &:= \sum_{i=1}^{n}\sum_{k=0}^{\infty} 2^{kr} d_i^r \mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s} \cap \{X_i \geq 2^k t d_i\}) \\
&\leq \sum_{i=1}^{n} \mathbb{E} 2t^{-r}|X_i|^r \mathbf{1}_{B_{i_1,k_1,\dots,i_s,k_s}\cap\{X_i\geq td_i\}} \\
&\leq 2t^{-r} C_6^r \mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s}) \left(r^r \sigma_{r,X}^r(-\log \mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s})) + (dt)^r e^{-t/C_7} \right),
\end{aligned}$$

where the last inequality follows by (5). Note that for $(i_1,k_1,\dots,i_s,k_s) \in P_sI_0$ we have $\mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s}) > e^{-rl}$. Moreover, by our assumptions on t (if C_4 is sufficiently large with respect to C_7),

$$(dt)^r e^{-t/C_7} \leq t^r e^{-t/(2C_7)} d^r e^{-t/(2C_7)} \leq r^r \sigma_{r,X}^r(rl).$$

Therefore

$$\sum_{(i_1,k_1,\dots,i_{s+1},k_{s+1})\in P_{s+1}I_0} 2^{(k_1+\dots+k_{s+1})r} d_{i_1}^r \dots d_{i_{s+1}}^r \mathbb{P}(B_{i_1,k_1,\dots,i_{s+1},k_{s+1}}) \leq 4t^{-r}(C_6 r \sigma_{r,X}(rl))^r \sum_{(i_1,k_1,\dots,i_s,k_s)\in P_sI_0} 2^{(k_1+\dots+k_s)r} d_{i_1}^r \dots d_{i_s}^r \mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s}).$$

By induction we get

$$\begin{aligned}
m_0(l) &= \sum_{(i_1,k_1,\dots,i_l,k_l)\in I_0} 2^{(k_1+\dots+k_l)r} d_{i_1}^r \cdots d_{i_l}^r \mathbb{P}(B_{i_1,k_1,\dots,i_l,k_l}) \\
&\leq \left(\frac{4C_6 r\sigma_{r,X}(rl)}{t}\right)^{r(l-1)} \sum_{(i_1,k_1)\in P_1I_0} 2^{k_1 r} d_{i_1}^r \mathbb{P}(B_{i_1,k_1}).
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{(i_1,k_1)\in P_1I_0} 2^{k_1 r} d_{i_1}^r \mathbb{P}(B_{i_1,k_1}) &\leq \sum_{i_1=1}^{n} d_{i_1}^r \sum_{k_1=0}^{\infty} 2^{k_1 r} e^{2-2^{k_1-1}t/e} \\
&\leq \sum_{i_1=1}^{n} d_{i_1}^r 2e^{2-t/(2e)} \leq \left(\frac{Cr\sigma_{r,X}(rl)}{t}\right)^r,
\end{aligned}$$

where the last two inequalities follow from the assumptions on t. Thus

$$m_0(l) \le \left(\frac{C r \sigma_{r,X}(rl)}{t} \right)^{rl}.$$

Now we estimate $m_j(l)$ for $j > 0$. Fix $j > 0$ and define a positive integer ρ_1 by

$$r 2^{\rho_1 - 1} < \frac{t}{C_9} \le r 2^{\rho_1}.$$

For all $(i_1, k_1, \dots, i_l, k_l) \in I_j$ define a function $f_{i_1,k_1,\dots,i_l,k_l} \colon \{1, \dots, \ell\} \to \{0, 1, \dots\}$ by

$$f_{i_1,k_1,\dots,i_l,k_l}(s) := \begin{cases} 0 & \text{if } \frac{\mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\dots,i_{s-1},k_{s-1}})} > e^{-r}, \\ \rho & \text{if } e^{-r2^{\rho}} < \frac{\mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\dots,i_{s-1},k_{s-1}})} \le e^{-r2^{\rho-1}}, \ \rho \ge 1. \end{cases}$$

Note that for every $(i_1, k_1, \dots, i_l, k_l) \in I_j$ one has

$$1 = \mathbb{P}(B_\emptyset) \ge \mathbb{P}(B_{i_1,k_1}) \ge \mathbb{P}(B_{i_1,k_1,i_2,k_2}) \ge \ldots \ge \mathbb{P}(B_{i_1,k_1,\dots,i_l,k_l}) > \exp(-rl2^j).$$

Denote

$$\mathcal{F}_j := \{ f_{i_1,k_1,\dots,i_l,k_l} \colon \ (i_1, k_1, \dots, i_l, k_l) \in I_j \}.$$

Then for $f = f_{i_1,k_1,\dots,i_l,k_l} \in \mathcal{F}_j$ and $\rho \ge 1$ one has

$$\begin{aligned} \exp(-r2^j l) < \mathbb{P}(B_{i_1,k_1,\dots,i_l,k_l}) &= \prod_{s=1}^{\ell} \frac{\mathbb{P}(B_{i_1,k_1,\dots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\dots,i_{s-1},k_{s-1}})} \\ &\le \exp(-r2^{\rho-1} |\{s \colon \ f(s) \ge \rho\}|). \end{aligned}$$

Hence for every $\rho \ge 1$ one has

$$|\{s \colon \ f(s) \ge \rho\}| \le \min\{2^{j+1-\rho} l, l\} =: l_\rho. \tag{13}$$

In particular f takes values in $\{0, 1, \dots, j + 1 + \log_2 l\}$. Clearly, $\sum_{\rho \ge 1} l_\rho = (j+2)l$ and $l_{\rho-1}/l_\rho \le 2$, so by Lemma 9

$$|\mathcal{F}_j| \le \prod_{\rho=1}^{j+1+\log_2 l} \left(\frac{e l_{\rho-1}}{l_\rho} \right)^{l_\rho} \le e^{2(j+2)l}.$$

Now fix $f \in \mathcal{F}_j$ and define

$$I_j(f) := \{ (i_1, k_1, \dots, i_l, k_l) \colon \ f_{i_1,k_1,\dots,i_l,k_l} = f \}$$

and for $s \le l$,

$$I_{j,s}(f) := \{ (i_1, k_1, \dots, i_s, k_s) \colon \ f_{i_1,k_1,\dots,i_l,k_l} = f \text{ for some } i_{s+1}, k_{s+1} \dots, i_l, k_l \}.$$

Recall that for $s \geq 1$, $\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s}) \leq e^{-1}$. Moreover for $s \leq l$,

$$\begin{aligned}\sigma_X(-\log \mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s})) \leq \sigma_X(-\log \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l})) &\leq \sigma_X(rl2^j)\\ &\leq C_5 2^j \sigma_X(rl).\end{aligned}$$

Hence estimate (6) applied with $u = r2^{f(s+1)}$ implies for $1 \leq s \leq l-1$,

$$\begin{aligned}&\sum_{(i_1,k_1,\ldots,i_{s+1},k_{s+1}) \in I_{j,s+1}(f)} 2^{(k_1+\ldots+k_{s+1})r} d_{i_1}^r \ldots d_{i_{s+1}}^r \mathbb{P}(B_{i_1,k_1,\ldots,i_{s+1},k_{s+1}})\\ &\qquad \leq g(f(s+1)) \sum_{(i_1,k_1,\ldots,i_s,k_s) \in I_{j,s}(f)} 2^{(k_1+\ldots+k_s)r} d_{i_1}^r \ldots d_{i_s}^r \mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s}),\end{aligned}$$

where

$$g(\rho) := \begin{cases} (C_8C_5r)^r t^{-r} 2^{jr} \sigma_{r,X}(rl)^r & \text{for } \rho = 0,\\ (C_8C_5r)^r t^{-r} 2^{r(\rho+j)} \sigma_{r,X}(rl)^r \exp(-r2^{\rho-1}) & \text{for } 1 \leq \rho < \rho_1,\\ (C_8C_5r)^r t^{-r} 2^{r\rho} (2^{rj} \sigma_{r,X}(rl)^r + d^r) \exp(-r2^{\rho-1}) & \text{for } \rho \geq \rho_1.\end{cases}$$

Suppose that $(i_1, k_1) \in I_1(f)$ and $f(1) = \rho$. Then

$$\exp(-r2^\rho) \leq \mathbb{P}(X_{i_1} \geq 2^{k_1} t d_{i_1}) \leq \exp(2 - 2^{k_1-1} t/e),$$

hence $2^{k_1} t \leq er2^{\rho+2}$. W.l.o.g. $C_9 > 4e$, therefore $\rho \geq \rho_1$. Moreover, $2^{rk_1} \leq (4er)^r 2^{r\rho} t^{-r}$, hence

$$\sum_{(i_1,k_1) \in I_{j,1}(f)} 2^{rk_1} d_{i_1}^r \mathbb{P}(B_{i_1,k_1}) \leq d^r (8er)^r t^{-r} 2^{r\rho} \exp(-r2^{\rho-1}) \leq g(\rho) = g(f(1)),$$

since w.l.o.g. $C_8C_5 \geq 8e$. Thus an easy induction shows that

$$\begin{aligned}m_j(f) &:= \sum_{(i_1,\ldots,k_l) \in I_j(f)} 2^{(k_1+\ldots+k_l)r} d_{i_1}^r \ldots d_{i_l}^r \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l})\\ &\leq \prod_{s=1}^{l} g(f(s)) = \prod_{\rho=0}^{\infty} g(\rho)^{n_\rho},\end{aligned}$$

where $n_\rho := |f^{-1}(\rho)|$.

Observe that

$$e^{-r2^{j-1}l} \geq \mathbb{P}(B_{i_1,k_1,\ldots,i_l,k_l}) = \prod_{s=1}^{l} \frac{\mathbb{P}(B_{i_1,k_1,\ldots,i_s,k_s})}{\mathbb{P}(B_{i_1,k_1,\ldots,i_{s-1},k_{s-1}})} \geq e^{-lr} \prod_{s\colon f(s)\geq 1} e^{-r2^{f(s)}}.$$

Therefore

$$r\sum_{\rho=1}^{\infty} n_\rho 2^{\rho-1} = \frac{r}{2} \sum_{s\colon f(s)\geq 1} 2^{f(s)} \geq \frac{r}{2} l(2^{j-1} - 1).$$

Moreover
$$\sum_{\rho\geq 1}\rho n_\rho \leq (j+1)l+\sum_{\rho\geq j+2}\rho l_\rho=(2j+4)l.$$
Thus
$$\prod_{\rho=0}^{\infty} g(\rho)^{n_\rho}\leq \left(\frac{C_8C_5r2^j\sigma_{r,X}(rl)}{t}\right)^{rl}2^{rl(2j+4)}\left(1+\frac{d^r}{\sigma_{r,X}(rl)^r}\right)^m\exp\left(-\frac{rl}{2}(2^{j-1}-1)\right),$$
where $m=\sum_{\rho\geq\rho_1}n_\rho\leq l_{\rho_1}\leq 2^{j+1-\rho_1}l$. By the assumption on t we have $1+d^r/\sigma_{r,X}(rl)^r\leq 2\exp(t/C_4)\leq\exp(r2^{\rho_1-4})$ if C_4 is large enough (with respect to C_9). Hence
$$m_j(l)\leq|\mathcal{F}_j|\left(\frac{\sqrt{e}C_8C_52^{(3j+4)}r\sigma_{r,X}(rl)}{t}\right)^{rl}\exp(-rl2^{j-3}).$$

We get
$$m(l)=\sum_{j=0}^{\infty}m_j(l)\leq\left(\frac{Cr\sigma_{r,X}(rl)}{t}\right)^{rl}+\sum_{j=1}^{\infty}\left(\frac{C2^{5j}r\sigma_{r,X}(rl)}{t}\right)^{rl}\exp(-rl2^{j-3}).$$
To finish the proof of (12), note that
$$\sum_{j=1}^{\infty}\left(2^{5j}\right)^{rl}\exp(-rl2^{j-3})\leq C^{rl}\sum_{j=1}^{\infty}\exp(-rl2^{j-4})\leq C^{rl}. \qquad \square$$

Proof of Theorem 5. Since $(\mathbb{E}\|X\|_r^p)^{1/p}\leq C_5p\mathbb{E}\|X\|_r$, we may assume that $p\geq r$. Let d_i and d be as in Theorem 6. Then
$$d=\|(\mathbb{E}X_i^2)^{1/2}\|_r\leq 2C_5\|(\mathbb{E}|X_i|)\|_r\leq 2C_5\mathbb{E}\|X\|_r.$$
Set
$$\tilde{p}:=\inf\{q\geq p\colon\ \sigma_{r,X}(q)\geq d\}.$$
Theorem 6 applied with $\tilde{p}$ instead of p and $t=0$ yields
$$\begin{aligned}(\mathbb{E}\|X\|_r^p)^{1/p}&\leq(\mathbb{E}\|X\|_r^{\tilde{p}})^{1/\tilde{p}}\leq C_3r\sigma_{r,X}(\tilde{p})=C_3r\max\{d,\sigma_{r,X}(p)\}\\&\leq Cr(\mathbb{E}\|X\|_r+\sigma_{r,X}(p)).\end{aligned} \qquad \square$$

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