Sudakov minoration principle and supremum of some processes

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Abstract

Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We find a new version of Sudakov minoration principle for estimating from below $E\sup_{t\in T}\sum t_i X_i$. If we moreover assume that tails of X_i are of moderate growth this enables us to give geometric conditions on set T equivalent to boundedness of the process $(\sum t_i X_i)_{t\in T}$. This generalize previous results of Talagrand [12].

Introduction. Let X_i be a sequence of symmetric, independent random variables with $\sup EX_i^2 < \infty$. To each $t \in l^2$ we may then associate a random variable $X_t = \sum t_i X_i$. It is quite important to understand for which subsets $T \subset l^2$ the process X_t is bounded a.e.. In many important cases (such as Gaussian and Bernoulli r.v. and in the case of all processes considered in this paper) the boundedness of this process is equivalent to $E\sup_{t\in T} X_t < \infty$. Since T may be uncountable, to avoid measurability problems we define

$$E\sup_{t\in T} X_t = \sup_{F\subset T, \#F<\infty} E\sup_{t\in F} X_t.$$

In general, the question of boundedness seems to be very hard, however it was answered in many special cases. The particular important case of X_i Gaussian random variables was for a long time a subject of research. A major break through was done by Michel Talagrand in [9] (with

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much simpler proof in [11]). In [12] Talagrand discovered some new methods, which enabled him to treat the case of identically distributed X_i with $\ln P(|X_i| \geq t) = -|t|^p, 1 \leq p < \infty$. At the moment, the most important unsolved case seems to be Bernoulli process i.e. when $X_i = \varepsilon_i$ is the Bernoulli sequence (for some partial results and conjectures see [13]).

In this paper we generalize some previous results of Talagrand. Theorem 3 formulates geometric conditions on the set T, which are equivalent to the boundedness in the case when the functions $N_i(t) = -\ln P(|X_i| \geq t)$ are convex and of moderate growth (precise definitions are given below). We do not know if the assumptions about the growth of N_i are necessary. If one can get rid of them it would give a solution for Bernoulli processes.

In the paper we strictly follow the ideas and methods from [12]. The main new ingredient needed to prove Theorem 3 is a generalization of Sudakov minoration principle, given in Theorem 1, which is of its own interest. Despite that some parts of its proof are similar to those of Talagrand, we think that our approach to the minoration problem (especially Lemma 1) is new.

Definitions and notation 1. Let X_i be a sequence of independent symmetric real random variables such that the functions $N_i : [0, \infty) \to [0, \infty]$, given by

$$N_i(t) = -\ln P(|X_i| \geq t), \quad t \geq 0$$

are convex. Since it is only matter of normalization we may and will assume that

$$\inf\{t > 0 : N_i(t) \geq 1\} = 1. \tag{1}$$

Let us define the functions $\hat{N}_i(t)$ by the formula

$$\hat{N}_i(t) = \begin{cases} t^2 & \text{for} \quad |t| \leq 1 \\ 2N_i(|t|) - 1 & \text{for} \quad |t| > 1. \end{cases}$$

For sequences (a_i) of real numbers we define

$$\|(a_i)\|_{\mathcal{N},u} = \sup\{\sum a_i b_i : \sum \hat{N}_i(b_i) \leq u\}$$

and we put

$$B_{\mathcal{N}}(u) = \{(a_i) : \|(a_i)\|_{\mathcal{N},u} \leq u\}.$$

We also define for $u > 0$

$$B_\varepsilon(u) = \{(a_i) : \sum \min(|a_i|, a_i^2) \leq u\}$$

and for $p > 0$

$$B_p = \{(a_i) : \sum |a_i|^p \leq 1\}.$$

For $t = (t_i) \in l^2$ we denote by X_t the random variable $\sum t_i X_i$.

Remark 1. In some situations it is more convienient to use a different expresion then $\|(a_i)\|_{\mathcal{N},u}$. Namely let

$$M_i(t) = (\hat{N}_i)^*(t) = \begin{cases} \frac{t^2}{4} & \text{for} \quad |t| \leq 2 \\ |t| - 1 & \text{for} \quad 2 < |t| < t_0 \\ 2N_i^*(\frac{|t|}{2}) + 1 & \text{for} \quad |t| \geq t_0, \end{cases}$$

where $F^*(t) = \sup\{ts - F(s) : s \geq 0\}$ are Lagrange transforms of a function F and $t_0 = 2(N_i)'_+(1) \geq 2$. We define the following Orlicz norm

$$\|(a_i)\|'_{\mathcal{N},u} = \inf\{t > 0 : \sum M_i(ua_i/t) \leq u\}.$$

Next proposition shows that both definitions gives equivalent norms.

Proposition 1 *The following three statements hold*

$$\lambda B_{\mathcal{N}}(u) \subset B_{\mathcal{N}}(\lambda u) \subset B_{\mathcal{N}}(u) \text{ for any } u > 0 \text{ and } \lambda \in (0,1), \tag{2}$$

$$B_{\mathcal{N}}(u) = B_{\varepsilon}(u) = \sqrt{u} B_2(u) \text{ for any } u \in (0,1), \tag{3}$$

and

$$\|(a_i)\|'_{\mathcal{N},u} \leq \|(a_i)\|_{\mathcal{N},u} \leq 2\|(a_i)\|'_{\mathcal{N},u} \text{ for any sequence } (a_i). \tag{4}$$

Proof. Obviously $\|\lambda(a_i)\|_{\mathcal{N},\lambda u} = \lambda\|(a_i)\|_{\mathcal{N},\lambda u} \leq \lambda\|(a_i)\|_{\mathcal{N},u}$, so $\lambda B_{\mathcal{N}}(u) \subset B_{\mathcal{N}}(\lambda u)$. On the other hand, since the functions $\hat{N}_i$ are convex and $\hat{N}_i(0) = 0$ we have $\hat{N}_i(\lambda t) \leq \lambda \hat{N}_i(t)$ and therefore $B_{\mathcal{N}}(\lambda u) \subset B_{\mathcal{N}}(u)$. The equalities in (3) are obvious. Finally, since $(\hat{N}_i)^* = M_i$, (4) follows by simple properties of Orlicz norms (see [8] ,Theorem 2.5).

Proposition 2 *There exists a universal constant K_1 such that*

$$P(|X_t| \geq K_1\|t\|_{\mathcal{N},u}) \leq e^{-u} \text{ for } u \geq 1.$$

In the case, when X_i are identically distributed this was proved in [1]. The general case follows by Corollary 1 of [4] or Example 2 in [5] and Chebyshev's inequality.

Corollary 1 *If $T \subset B_{\mathcal{N}}(u)$ for some $u \geq 1$ then*

$$E \sup_{t \in T} X_t \leq 2K_1 \max(u, \ln \#T).$$

Proof. Let $r = K_1 \max(u, \ln \#T)$. Let us notice that if $t \in B_{\mathcal{N}}(u)$ then for $s \geq u$, $\|(t_i)\|_{\mathcal{N},s} \leq s$ and therefore by Proposition 2, $P(|X_t| \geq s) \leq e^{-s/K_1}$ for $s \geq r$. Hence integrating by parts

$$E \sup_{t \in T} X_t \leq r + \sum_{t \in T} \int_r^\infty P(|X_t| \geq s) ds \leq r + \#T \int_r^\infty e^{-s/K_1} ds$$

$$\leq r + K_1 \#T e^{-r/K_1} \leq r + K_1 \leq 2K_1 \max(u, \ln \#T).$$

The next Lemma is a crusial new ingredient in the proof of Theorem 1. The function w in it is added for strictly technical reasons, it makes the induction possible, but in the sequel we will use Lemma 1 with $w \equiv 0$.

Lemma 1 *Let $\infty > n \geq 0$, $u \geq 1$ and $r_i, p_i \geq 1, q_i \in \mathbb{R}$, $i = 1, \ldots, n$ be given. Let $Y_1, \ldots, Y_n$ be a sequence of independent symmetric real random variables such that*

$$P(|Y_i| \geq x) = \begin{cases} e^{-x} & \text{for} \quad 0 \leq x \leq p_i \\ 0 & \text{for} \quad x > p_i. \end{cases}$$

Assume that a set $S \subset l^2$ with $\ln \#S \leq u$ and a function $w : S \to [0, \infty)$ satisfy the following two conditions

$$\forall_{t \in S} \forall_{i \leq n} \quad t_i \in \{q_i - r_i, q_i + r_i\}$$

and

$$\forall_{t,s \in S, t \neq s} \quad \sum_{i=1}^n r_i p_i \delta_{\{t_i \neq s_i\}} + w(t) + w(s) \geq u, \tag{5}$$

then

$$E \sup_{t \in S} (\sum_{i=1}^n t_i Y_i + w(t)) \geq \frac{1}{4} \ln(\sum_{t \in S} e^{2w(t)}).$$

Proof. We will prove Lemma 1 by the induction over n. To simplify the notation let

$$b(S) = E \sup_{t \in S} (\sum_{i=1}^n t_i Y_i + w(t)).$$

Obviously we may assume that $\#S \geq 2$. It is also enough to consider the case of $q_i = 0$ for all i.

If $n = 0$ then by (5), $b(S) = \max_{t \in S} w(t) \geq u/2$, so

$$\frac{1}{4}\ln(\sum_{t\in S} e^{2w(t)}) \leq \frac{1}{4}\ln \#S + \frac{1}{2}\max_{t\in S} w(t) \leq \frac{u}{4} + \frac{b(S)}{2} \leq b(S).$$

Let us assume that Lemma holds for $n-1$, we will prove it for n. Let

$$A = \{t \in S : t_n = r_n\} \text{ and } B = \{t \in S : t_n = -r_n\}.$$

If $A = \emptyset$ or $B = \emptyset$ the induction step follows trivially, because $Er_nY_n = 0$. Otherwise we define

$$a = \sum_{t\in A} e^{2w(t)}, \quad b = \sum_{t\in B} e^{2w(t)},$$

$$Y_A = \sup_{t\in A}(\sum_{i=1}^{n-1} t_iY_i + w(t)) \quad \text{and} \quad Y_B = \sup_{t\in B}(\sum_{i=1}^{n-1} t_iY_i + w(t)).$$

We have to show that $b(S) \geq 4^{-1}\ln(a+b)$. By the symmetry of Y_i we may assume that $a \geq b$. We will consider few cases.

Case I: $a/b < e^{4r_np_n}$. We have for any number x

$$b(S) \geq E(r_nY_n + Y_A)I_{\{Y_n \geq -x\}} + E(-r_nY_n + Y_B)I_{\{Y_n < -x\}}$$

$$= r_nEY_nI_{\{Y_n\geq -x\}} - r_nEY_nI_{\{Y_n<-x\}} + EY_AP(Y_n \geq -x) + EY_BP(Y_n < -x).$$

So by the induction assumptions, for $0 \leq x < p_n$, we obtain

$$b(S) \geq 2r_nEY_nI_{\{Y_n>x\}} + \frac{1}{4}\ln a - \frac{1}{8}e^{-x}\ln\frac{a}{b}$$

$$= r_n(xe^{-x} + e^{-x} - e^{-p_n}) + \frac{1}{4}\ln a - \frac{1}{8}e^{-x}\ln\frac{a}{b}.$$

Choosing $x = (4r_n)^{-1}\ln(a/b)$ we get

$$b(S) \geq \frac{1}{2}r_nxe^{-x} + r_n(e^{-x} - e^{-p_n}) + \frac{1}{4}\ln a.$$

Since $r_n, p_n \geq 1$ and $x \in [0, p_n]$, we have

$$\frac{1}{2} r_n x e^{-x} + r_n(e^{-x} - e^{-p_n}) \geq \frac{1}{2} r_n x e^{-x} + e^{-r_n x} - e^{-r_n p_n} \geq \frac{1}{2} e^{-r_n x}.$$

Therefore

$$b(S) \geq \frac{1}{2}(\frac{b}{a})^{1/4} + \frac{1}{4}\ln a \geq \frac{1}{4}(\frac{b}{a} + \ln a) \geq \frac{1}{4}\ln(a+b).$$

Case II: $r_n(e^{-p_n/2} - e^{-p_n})P(Y_A \leq Y_B + r_n p_n) \geq (4a)^{-1}b$**.** We get

$$b(S) = E\max(r_n Y_n + Y_A, -r_n Y_n + Y_B) = E(Y_A + r_n Y_n) + E(-2r_n Y_n + Y_B - Y_A)^+$$

$$\geq EY_A + E(-2r_n Y_n - r_n p_n)^+ P(Y_A \leq Y_B + r_n p_n).$$

Since $p_n \geq 1$ we obtain

$$E(-2r_n Y_n - r_n p_n)^+ = 2r_n E Y_n I_{\{2Y_n \geq p_n\}} - \frac{r_n p_n}{2} e^{-p_n/2} = r_n(e^{-p_n/2} - e^{-p_n}).$$

Hence by the induction assumptions

$$b(S) \geq \frac{1}{4}\ln a + r_n(e^{-p_n/2} - e^{-p_n})P(Y_A \leq Y_B + r_n p_n) \geq \frac{1}{4}(\ln a + \frac{b}{a}) \geq \frac{1}{4}\ln(a+b).$$

Case III: $b/a \leq e^{-4r_n p_n}$ **and** $r_n(e^{-p_n/2} - e^{-p_n})P(Y_A \leq Y_B + r_n p_n) \leq (4a)^{-1}b$**.** Let us define in this case the function $\tilde{w}$ on S by the formula

$$\tilde{w}(t) = \begin{cases} w(t) & \text{for} \quad t \in A \\ w(t) + r_n p_n & \text{for} \quad t \in B. \end{cases}$$

Then clearly $\tilde{w}$ satisfy (5) for $n-1$ and by the induction assumption we have

$$E\sup_{t \in S}(\sum_{i=1}^{n-1} t_i X_i + \tilde{w}(t)) \geq \frac{1}{4}\ln(a + e^{2r_n p_n} b).$$

We have

$$E\sup_{t \in S}(\sum_{i=1}^{n-1} t_i X_i + \tilde{w}(t)) = E\max(Y_A, Y_B + r_n p_n)$$

$$\leq E\max(Y_A, Y_B) + r_n p_n P(Y_A \leq Y_B + r_n p_n).$$

Hence

$$b(S) = E\max(Y_A, Y_B) \geq \frac{1}{4}\ln(a + e^{2r_n p_n} b) - \frac{p_n b}{4a(e^{-p_n/2} - e^{-p_n})}.$$

Since $\ln(1+x) \geq x - x^2/2$ for $x \geq 0$, we have

$$b(S) \geq \frac{1}{4}\ln a + \frac{b}{4a}(e^{2r_n p_n}(1 - \frac{e^{-2r_n p_n}}{2}) - p_n(e^{-p_n/2} - e^{-p_n})^{-1})$$

$$\geq \frac{1}{4}\ln a + \frac{b}{4a}e^{p_n/2}(e^{3p_n/2}(1 - \frac{e^{-2}}{2}) - p_n(1 - e^{-1/2})^{-1})$$

$$\geq \frac{1}{4}\ln a + \frac{b}{4a} \geq \frac{1}{4}\ln(a+b).$$

This completes the proof of the last case and of the whole Lemma.

Corollary 2 *Let numbers* $l_i \in I\!R$ *and a set* $T \subset B_\varepsilon(u/20)$ *satisfy conditions*

$$\forall_{t\in T}\forall_i \quad t_i \in \{0, l_i\} \tag{6}$$

and

$$\forall_{t,s\in T, t\neq s} \quad \|t - s\|_{\mathcal{N},u} \geq u. \tag{7}$$

Then the following inequality holds

$$E\sup_{t\in T} X_t \geq \frac{1}{32}\min(u, \ln\#T).$$

Proof. By the symmetry of X_i we may assume that $l_i \geq 0$. For $\#T = 1$, $\ln\#T = 0$ and there is nothing to prove. Otherwise by (3) we have that $u > 1$ and we may choose $T_1 \subset T$ such that

$$\min(u, \ln\#T) \geq \ln\#T_1 \geq \frac{\ln 2}{\ln 3}\min(u, \ln\#T).$$

From (7) and (4) we get that for any $s, t \in T, s \neq t$

$$\sum M_i(2(t_i - s_i)) \geq u.$$

Let

$$J=\{i: l_i\geq 1, \quad M_i(2l_i)>2l_i-1\}. \tag{8}$$

If $i\notin J$ then

$$\begin{aligned} M_i(2(t_i-s_i)) = M_i(2l_i)\delta_{\{t_i\neq s_i\}} &\leq \min(l_i^2, 2l_i-1)\delta_{\{t_i\neq s_i\}} \\ &\leq 2\min(|t_i|,t_i^2)+2\min(|s_i|,s_i^2). \end{aligned}$$

Since $T\subset B_\varepsilon(u/20)$, for any $t,s\in T, t\neq s$ we have

$$\frac{4u}{5}\leq \sum_{i\in J} M_i(2(t_i-s_i)) \leq \sum_{i\in J}(2N_i^*(l_i)+1)\delta_{\{t_i\neq s_i\}}. \tag{9}$$

Without loss of generality we may assume that J is finite, say $J=\{1,\ldots,n\}$. For each $i\leq n$, by Remark 1 $l_i\geq (N_i)'_+(1)$, so we may then find $s_i\geq 1$ such that

$$N_i^*(l_i)=l_is_i-N_i(s_i). \tag{10}$$

Let $p_i=N_i(s_i), r_i=5l_is_i/2N_i(s_i)$ and Y_i be as in Lemma 1. By (8) and (10) we obtain that $p_i,r_i\geq 1$, and by (9) and (10)

$$\sum_{i=1}^n r_ip_i\delta_{\{t_i\neq s_i\}}\geq u \text{ for } s,t\in T, t\neq s. \tag{11}$$

Since the functions N_i are convex and $N_i=0$ we have $P(|X_i|\geq ts_i)\geq e^{-tN_i(s_i)}$ for $t\in[0,1]$. Hence

$$P(|5l_iX_i|\geq x)\geq P(|2r_iY_i|\geq x) \text{ for } x\geq 0.$$

So by the contraction principle (cf. [3], Corollary 1.2.1)

$$5E\sup_{t\in T}X_t\geq E\sup_{t\in T}\sum_{i=1}^n \delta_{\{t_i\neq 0\}}2r_iY_i.$$

By Lemma 1 (with $w\equiv 0$) and (11) follows that

$$E\sup_{t\in T}\sum_{i=1}^n \delta_{\{t_i\neq 0\}}2r_iY_i\geq \frac{1}{4}\ln\#T_1,$$

therefore

$$E\sup_{t\in T}X_t\geq \frac{\ln 2}{20\ln 3}\min(u,\ln\#T)\geq\frac{1}{32}\min(u,\ln\#T).$$

Proposition 3 *For any $T \subset l^2$ and $u > 0$ the inequality*

$$E \sup_{t\in T} \sum t_i \varepsilon_i \geq \frac{1}{K_2} \min(u, \ln N(T, B_\varepsilon(u)))$$

holds with universal constant K_2.

Proposition 3 follows from Theorem 4.15 of [6] and an easy observation that

$$uB_1 + \sqrt{u}B_2 \subset 2B_\varepsilon(u),$$

where B_1 and B_2 are unit balls in l^1 and l^2 respectively.

Theorem 1 *There exists a universal constant K_3 such that for each $u > 0$ and each set $T \subset l^2$, which satisfy the condition*

$$\forall_{t,s\in T, t\neq s} \quad \|t - s\|_{\mathcal{N},u} \geq u \tag{12}$$

the following inequality holds

$$E \sup_{t\in T} X_t \geq \frac{1}{K_3} \min(u, \ln \#T).$$

Proof. We will follow the methods of [12], reducing the proof to the case of special sets T, such as in Corollary 2.

By approximation argument we may assume that $T \subset \mathbb{R}^m$ with $m < \infty$ and T is finite. Let

$$\delta = (\max(512K_1, 8))^{-1},$$

where K_1 is the same as in Corollary 1. We will divide the proof into several simpler steps.

Step 1. By the Jensen inequality and the contraction principle we have

$$E \sup_{t\in T} X_t \geq E \sup_{t\in T} \sum t_i E|X_i|\varepsilon_i \geq e^{-1} E \sup_{t\in T} \sum t_i \varepsilon_i, \tag{13}$$

because by (1) we have $E|X_i| \geq e^{-1}$ for each i. Hence Proposition 3 yields

$$E \sup_{t\in T} X_t \geq e^{-1} E \sup_{t\in T} \sum t_i \varepsilon_i \geq (eK_2)^{-1} \min(\delta^2 u, \ln N(T, B_\varepsilon(\delta^2 u))).$$

Since

$$\#T \leq N(T, B_\varepsilon(\delta^2 u)) \sup_{x\in \mathbb{R}^m} \#(T \cap (x + B_\varepsilon(\delta^2 u)))$$

it is enough to consider the case when $T \subset x + B_\varepsilon(\delta^2 u)$ for some $x \in I\!R^m$. Moreover, since $E\sup_{t\in T} X_t = E\sup_{t\in T-x} X_t$ we may assume that

$$T \subset B_\varepsilon(\delta^2 u). \tag{14}$$

Step 2. It suffices to prove that under (12) and (14)

$$E\sup_{t\in T} X_t \geq \frac{u}{512} \quad \text{if } \ u \geq 1 \text{ and } \ln \#T \geq (\ln\delta^{-1} + 2\delta + 1)u.$$

Indeed, suppose that we have proved this. Then for arbitrary u let $C = \ln\delta^{-1} + 2\delta + 1$ and

$$\tilde{u} = \min(u, C^{-1}\ln \#T).$$

If $\tilde{u} < 1$ then $B_{\mathcal{N}}(\tilde{u}) = B_\varepsilon(\tilde{u})$, so by (13) and Proposition 3

$$E\sup_{t\in T} X_t \geq (eK_2)^{-1}\tilde{u} \geq (eK_2C)^{-1}\min(u, \ln \#T).$$

If $\tilde{u} \geq 1$ then

$$E\sup_{t\in T} X_t \geq \tilde{u}/512 \geq (512C)^{-1}\min(u, \ln \#T).$$

Step 3. Let λ be the probability measure on $I\!R$ given by the density $e^{-|x|}/2$ and let $\mu = \otimes_{i=1}^m \lambda$ be the product measure on $I\!R^m$. For any $x \in I\!R^m$ and $t \in T$ we define

$$T_x = \{t \in T : t_i \in (x_i - \delta, x_i + \delta) \cup (-\delta, \delta), i = 1, \ldots, m\}$$

and

$$A_t = \{x \in I\!R^m : t_i \in (x_i - \delta, x_i + \delta) \cup (-\delta, \delta), i = 1, \ldots, m\}.$$

Let us notice that

$$\lambda(\{x_i : t_i \in (x_i - \delta, x_i + \delta) \cup (-\delta, \delta)\}) \geq \delta \min_{x \in (t_i - \delta, t_i + \delta)} e^{-|x|} \geq \delta e^{-|t_i| - \delta} \tag{15}$$

and

$$\lambda(\{x_i : t_i \in (x_i - \delta, x_i + \delta) \cup (-\delta, \delta)\}) = 1 \text{ if } \ |t_i| < \delta. \tag{16}$$

We have

$$\sum_{|t_i| \geq \delta} |t_i| \leq \delta^{-1} \sum \min(t_i^2, |t_i|) \leq \delta u$$

and

$$\#\{i : |t_i| \geq \delta\} \leq \delta^{-2} \sum \min(t_i^2, |t_i|) \leq u.$$

Therefore by (15) and (16)

$$\mu(A_t) \geq \delta^u e^{-2\delta u}.$$

Hence

$$\int \sum_{t \in T} I_{A_t}(x) d\mu(x) \geq \#T \delta^u e^{-2\delta u} \geq e^u.$$

So there exists $x \in I\!R^m$ such that

$$\ln \#T_x \geq u.$$

Step 4. Let us choose $S \subset T_x$ with $\ln \#S \in [u/2, u]$. For any $s \in S$ we define $\tilde{s} \in I\!R^m$ by the formula

$$\tilde{s}_i = \begin{cases} 0 & \text{if} \quad |s_i| < \delta \\ x_i & \text{if} \quad |s_i| \geq \delta. \end{cases}$$

Let $s, t \in S$, we have then by (4), (12) and (14)

$$u \leq \sum M_i(2(s_i - t_i)) \leq \sum_{2|s_i - t_i| \geq 1} M_i(2(s_i - t_i)) + 16\delta^2 u$$

By an easy observation

$$|\tilde{s}_i - \tilde{t}_i| \geq |s_i - t_i| - 2\delta \geq |s_i - t_i|/2 \quad \text{if} \quad 2|s_i - t_i| \geq 1,$$

so

$$\sum M_i(4(\tilde{s}_i - \tilde{t}_i)) \geq \sum_{2|s_i - t_i| \geq 1} M_i(2(s_i - t_i)) \geq u(1 - 16\delta^2) \geq u/2.$$

Therefore by (4) and Corollary 2 we have

$$E \sup_{s \in S} X_{\tilde{s}} \geq \frac{1}{128} \min(\frac{u}{2}, \ln \#S) \geq \frac{u}{256}.$$

Step 5. Let us notice that for any $s \in S$ we have by (14)

$$\sum M_i(\delta^{-1}(s_i - \tilde{s}_i)) = \delta^{-2} \sum (s_i - \tilde{s}_i)^2 \leq$$

$$\delta^{-2}\sum \min(\delta^2, s_i^2) \le \delta^{-2}\sum \min(|s_i|, s_i^2) \le u.$$

Therefore by (4) and Corollary 1

$$E\sup_{s\in S}(X_{\tilde{s}} - X_s) = E\sup_{s\in S} X_{\tilde{s}-s} \le \delta K_1 \max(u, \ln \# S) = \delta K_1 u \le u/512.$$

Since

$$E\sup_{s\in S} X_s \ge E\sup_{s\in S} X_{\tilde{s}} - E\sup_{s\in S}(X_{\tilde{s}} - X_s)$$

we finally get

$$E\sup_{s\in S} X_s \ge \frac{u}{256} - \frac{u}{512} = \frac{u}{512}.$$

and this completes the proof.

Theorem 2 *If $T \subset B_{\mathcal{N}}(u)$ for some $u \ge 1$ and the additional condition*

$$\forall_i \; N_i(t) = t \quad \text{for} \quad 0 \le t \le 1 \tag{17}$$

is satisfied, then

$$\forall_{s > K_4 u} \quad P(|\sup_{t\in T} X_t - E\sup_{t\in T} X_t| \ge s) \le e^{-s/K_4}, \tag{18}$$

where K_4 is an universal constant.

Proof. Let λ, μ be as in Step 3 of the proof of Theorem 1 and let $F_i : \mathbb{R} \to \mathbb{R}$ be an odd function, such that its restriction to $\mathbb{R}^+$ is the inverse of N_i. Then X_i has the distribution equal to $F_i(\lambda)$. In particular this implies

$$P(\sup_{t\in T} X_t \le a) = \mu(x : \sup_{t\in T} \sum t_i F_i(x_i) \le a),$$

By the result of Talagrand [10] (see [7] for a simpler proof) we have for each mesurable set $A \subset \mathbb{R}^m$ with $\mu(A) > 0$ and $b > 0$

$$\mu(A + B_\varepsilon(36b)) \ge 1 - (\mu(A))^{-1} e^{-b}.$$

Let

$$A = \{x \in \mathbb{R}^m : \sup_{t\in T} \sum t_i F_i(x_i) \le a\}.$$

For each $y \in A$, $z \in B_\varepsilon(36b)$ and $x = y + z$, by the convexity of N_i we have $|F_i(x_i) - F_i(y_i)| \le 2F_i(|x_i - y_i|)$, so that

$$\sup_{t \in T} \sum t_i F_i(x_i) \le a + \sup_{t \in T} \sum t_i (F_i(x_i) - F_i(y_i))$$

$$\le a + 2 \sup_{t \in T} \sum |t_i| F_i(|z_i|) \le a + 2 \sup_{t \in T} \sup\{\sum |t_i| c_i : \sum \hat{N}_i(c_i) \le 72b\}$$

$$\le a + 2 \sup_{t \in T} \|t\|_{\mathcal{N},72b}.$$

Since for $r \ge u$, by (2) $T \subset B_{\mathcal{N}}(u) \subset B_{\mathcal{N}}(r)$, we have $\|t\|_{\mathcal{N},r} \le r$ for any $t \in T$. Therefore for $s \ge 2u$ we obtain

$$P(\sup_{t \in T} X_t \le a + s) \ge 1 - (P(\sup_{t \in T} X_t \le a))^{-1} e^{-s/144}. \tag{19}$$

Let M be the median of $\sup_{t \in T} X_t$. Then choosing suitable a in (19) we get for $s \ge 2u$

$$P(\sup_{t \in T} X_t < M - s) \le 2e^{-s/144} \tag{20}$$

and

$$P(\sup_{t \in T} X_t > M + s) \le 2e^{-s/144}. \tag{21}$$

By (21) we get

$$E \sup_{t \in T} X_t \le M + 2u + 2 \int_{2u}^{\infty} e^{-s/144} ds \le M + 290u. \tag{22}$$

This gives

$$P(\sup_{t \in T} X_t \le E \sup_{t \in T} X_t - s) \le P(\sup_{t \in T} X_t \le M + 290u - s). \tag{23}$$

The similar reasoning shows that

$$P(\sup_{t \in T} X_t \ge E \sup_{t \in T} X_t + s) \le P(\sup_{t \in T} X_t \ge M - 290u + s). \tag{24}$$

Theorem follows by (20), (21), (23) and (24).

Corollary 3 *There are two universal constants $0 < \rho, K < \infty$ such that if (17) is satisfied then for any $\ln N \geq u \geq 1$, $t^1, \ldots, t^N \in l^2$ and nonempty sets $T_1, \ldots, T_N \subset l^2$ satisfying*

$$\forall_{k \neq l} \quad t^k - t^l \notin B_{\mathcal{N}}(u)$$

and

$$\forall_k \; T_k \subset \rho B_{\mathcal{N}}(u),$$

the following inequality holds for $T = \bigcup (t^k + T_k)$

$$E \sup_{t \in T} X_t \geq \frac{1}{K} \min(u, \ln N) + \min_k E \sup_{t \in T_k} X_t.$$

Proof. Without lost of generality we may assume that $u \leq \ln N \leq 2u$. Let $a_k = E \sup_{t \in T_k} X_t$, then

$$\sup_{t \in T} X_t \geq \max_k X_{t^k} + \min_k \sup_{t \in T_k} X_t$$

and

$$E \min_k \sup_{t \in T_k} X_t \geq \min_k a_k - E \max_k (a_k - \sup_{t \in T_k} X_t).$$

By Theorem 2 taking $r = K_4 \rho \ln N$ integrating by parts we obtain

$$E \max_k (a_k - \sup_{t \in T_k} X_t) \leq r + 2N \int_r^\infty e^{-s/(\rho K_4)} ds \leq r + 2\rho K_4 \leq 4\rho K_4 u.$$

By Theorem 1

$$E \max_k X_{t^k} \geq K_3^{-1} \min(u, \ln N) = K_3^{-1} u.$$

Hence

$$E \sup_{t \in T} X_t \geq K_3^{-1} u + \min_k a_k - 4\rho K_4 u$$

and Corollary follows if we take $\rho = (8K_4 K_3)^{-1}$ and $K = 2K_3$.

Before formulating our main theorem we will need some more definitions and few technical results.

Definitions and notation 2. Let $r > 1$ be a fixed number. For any set $A \subset l^2, j \in Z$ we define

$$D_{j,r}(A) = \inf\{u > 0 : \exists_{s \in l^2} \; A \subset s + r^{-j} B_{\mathcal{N}}(u)\}$$

and

$$i_r(A) = \max\{i \in Z : D_{i,r}(A) \leq 1\}.$$

Let $T \subset l^2, i \in Z$ and $(\mathcal{A}_j)_{j>i}$ be an increasing family of finite partitions of T. For $t \in T, j > i$ we denote by $A_j(t)$ the unique $A \in \mathcal{A}_j$ such that $t \in A$. We define

$$\theta_{i,r}(T) = \inf \sup_{t \in T} \sum_{j>i} r^{-j}(D_j(A_j(t)) - \ln \mu(A_j(t))),$$

where the infimum is taken over all increasing sequences of partitions $(\mathcal{A}_j)_{j>i}$ of T and the probability measures μ on T for which all the sets $A \in \mathcal{A}_j$ are measurable. We set

$$\theta_{\mathcal{N},r}(T) = \theta_{i_r(T),r}$$

and we will write $D_j(A)$, $i(A)$, $\theta_i(T)$, $\theta_{\mathcal{N}}(T)$ instead of $D_{j,2}(A)$, $i_2(A)$, $\theta_{i,2}(T)$, $\theta_{\mathcal{N},2}(T)$.

We say that a functions N is of moderate growth with constant C if

$$\forall_{t \geq 1}\ N(2t) \leq CN(t).$$

For a subset $A \subset l^2$ we define

$$\overline{\text{conv}}(A) = \{\sum \lambda_i v_i : v_i \in A, \lambda_i \geq 0, \sum \lambda_i = 1\}.$$

Given two quantities U, V we say that U dominates V with a constant K if $V \leq KU$, and that U is equivalent to V up to a constant K if U dominates V and V dominates U with constant K.

Lemma 2 *For any $r > 1$ we have*

$$\theta_{\mathcal{N},r} \leq (1+r)\theta_{\mathcal{N},r^2}.$$

Proof. Let μ be a probability measure on T and $(\mathcal{A}_j)_{j>i}$ be an increasing family of finite μ-measurable partitions of T. We define partition $(\tilde{\mathcal{A}}_j)_{j>2i}$ by $\tilde{\mathcal{A}}_{2j-1} = \tilde{\mathcal{A}}_{2j} = \mathcal{A}_j$. Since for any $A \subset l^2$,$D_{2j-1,r}(A) \leq D_{j,r^2}(A)$ and $D_{2j,r}(A) = D_{j,r^2}(A)$ we have

$$\sup_{t \in T} \sum_{j>2i} r^{-j}(D_{j,r}(\tilde{A}_j(t)) - \ln \mu(\tilde{A}_j(t))) \leq$$

$$(1+r) \sup_{t \in T} \sum_{j>i} r^{-2j}(D_{j,r^2}(A_j(t)) - \ln \mu(A_j(t))).$$

Hence $\theta_{2i,r}(T) \leq (1+r)\theta_{i,r^2}(T)$ and Lemma follows since $i_r(T) \geq 2i_{r^2}(T)$.

Lemma 3 *Let ρ be as in Corollary 3 and N_i be of moderate growth with constant C. Then there exists $r_0 = r_0(C)$ depending only on C such that*

$$\forall_{r \geq r_0} \forall_{u>0} \ B_{\mathcal{N}}(ru) \subset \frac{\rho}{4} r B_{\mathcal{N}}(u).$$

Proof. One can easily check that for any t

$$\hat{N}_i(2t) \leq 4C\hat{N}_i(t). \tag{25}$$

Let a positive integer n satisfy $2^n \rho \geq 4$, we put $r_0 = (4C)^n$. Let $r \geq r_0$, $x \in B_{\mathcal{N}}(ru)$ and y_i be a sequence such that $\sum \hat{N}_i(y_i) \leq u$. Then by (25) $\sum \hat{N}_i(2^n y_i) \leq (4C)^n u \leq ru$ and therefore $\sum x_i y_i \leq 2^{-n} ru \leq \rho ru/4$. Hence $\|x\|_{\mathcal{N},u} \leq \rho ru/4$ and $x \in \rho r/4 B_{\mathcal{N}}(u)$.

Lemma 4 *Let Y_i be a sequence of independent, symmetric random varaibles such that*

$$\ln P(|Y_i| \geq t) = -\tilde{N}_i(t),$$

where $\tilde{N}_i(t) = t$ for $t \in [0,1]$ and $\tilde{N}_i(t) = N_i(t)$ otherwise. Then functions $\tilde{N}_i$ are convex and for any $u > 0$, $t \in l^2$ and $T \subset l^2$ we have

$$P(|\sum t_i Y_i| \geq u) \leq 2P(|X_t| \geq u) \tag{26}$$

and

$$(1+e)^{-1} E \sup_{t \in T} X_t \leq E \sup_{t \in T} \sum t_i Y_i \leq E \sup_{t \in T} X_t. \tag{27}$$

Proof. The convexity of $\tilde{N}_i$ follows from convexity of N_i and normalization property (1). Since $\tilde{N}_i \geq N_i$, we obtain (26) and the second inequality in (27) by the contraction principle (see [3], Corollary 1.2.1). Moreover we may represent Y_i as $X_i - Z_i$, where Z_i are independent random variables with $|Z_i| \leq 1$. So again by the contraction principle and (13)

$$E \sup_{t \in T} \sum t_i Z_i \leq E \sup_{t \in T} \sum t_i \varepsilon_i \leq e E \sup_{t \in T} \sum t_i Y_i.$$

Therefore

$$E \sup_{t \in T} X_t \leq E \sup_{t \in T} \sum t_i Y_i + E \sup_{t \in T} \sum t_i Z_i \leq (1+e) E \sup_{t \in T} \sum t_i Y_i.$$

Theorem 3 *Let N_i be of moderate growth with the same constant C. Then for any nonempty set $T \subset l^2$ the following four quantities are equivalent (up to constants, which depend only on C)*

$$E \sup_{t\in T} X_t \tag{28}$$

$$\theta_{\mathcal{N}}(T) \tag{29}$$

$$\inf\{M : \forall_{n\geq 1}\exists_{t_n\in l^2}\ \|t_n\|_{\mathcal{N},log(n+1)} \leq M,\ T-T\subset \overline{\mathrm{conv}}\{\pm t_n : n\geq 1\}\} \tag{30}$$

$$\begin{aligned}\inf\{M : \forall_{n\geq 1}\exists_{t_n\in l^2}\ \|t_n\|_2 \leq M,\ P(|X_{t_n}|\geq M)\leq (n+1)^{-3},\\ T-T\subset \overline{\mathrm{conv}}\{\pm t_n : n\geq 1\}\}.\end{aligned} \tag{31}$$

Proof. (30) dominates (31). Since $\|x\|_{\mathcal{N},\lambda u} \leq \lambda \|x\|_{\mathcal{N},u}$ for $\lambda \geq 1$ we have by Lemma 1

$$P(|X_{t_n}| \geq 3K_1\|t_n\|_{\mathcal{N},\ln(n+1)}) \leq e^{-3\ln(n+1)} = (n+1)^{-3}.$$

We conclude by the observation that $\|x\|_{\mathcal{N},u} \geq \min(\sqrt{u},1)\|x\|_2$.

(31) dominates (28). Let Y_i be as in Lemma 4 and $Y_t = \sum t_i Y_i$ for $t\in l^2$. For any $t \in T-T$ we have $|Y_t| \leq \sup_{n\geq 1}|Y_{t_n}|$ a.e. Hence for any $t_0 \in T$

$$E\sup_{t\in T} Y_t = E\sup_{t\in T}(Y_t - Y_{t_0}) = E\sup_{t\in T} Y_{t-t_0} \leq E\sup_{t\in T-T} Y_t \leq E\sup_{n\geq 1}|Y_{t_n}|.$$

By (31) and Lemma 4 we obtain

$$P(\sup_{n\geq 1}|Y_{t_n}| \geq M) \leq 2\sum P(|X_{t_n}|\geq M) \leq 2\sum (n+1)^{-3} \leq 1/2.$$

So using (22) we have

$$E\sup_{t\in T} Y_t \leq E\sup_{n\geq 1}|Y_{t_n}| \leq M + 290\sup_{n\geq 1}\|t_n\|_{\mathcal{N},1} \leq 291M.$$

Lemma 4 completes this part of the proof.

(28) dominates (29). By Lemma 2 it is enough to show that (28) dominates $\theta_{\mathcal{N},r}$ for $r = 2^{2^n}$ with n sufficiently large. Using Lemma 4, we may assume that (17) is satisfied. We proceed as in [12], pp. 317-324, using Corollary 3 and Lemma 3 instead of Lemma 5.3 and 5.4.

(29) dominates (30). Let $h(x) = \ln(1/x)$, $i = i(T)$ and choose a probibility measure μ and a sequence of $\mu-$measurable partitions $(\mathcal{A}_{j>i})$ such that

$$\sup_{t\in T}\sum_{j>i} 2^{-j}(D_j(A_j(t)) + h(\mu(A_j(t)))) \le 2\theta_{\mathcal{N}}(T).$$

By definition of $i(T)$ we have $D_{i+1}(T) > 1$. There are two possibilities

1. $\mathcal{A}_{i+1} = \{T\}$, then $2\theta_{\mathcal{N}}(T) \ge 2^{-i-1}D_{i+1}(T) \ge 2^{-i-1}$.
2. $\mathcal{A}_{i+1} \ne \{T\}$, then there exists $A \in \mathcal{A}_{i+1}$ with $\ln(A) \le 1/2$, so $2\theta_{\mathcal{N}}(T) \ge 2^{-i-1}h(\mu(A)) \ge 2^{-i-1}\ln 2$.

So in any case $\theta_{\mathcal{N}}(T) \ge 2^{-i-2}\ln 2$. Let us additionally put $\mathcal{A}_i = \{T\}$, then since $h(ab) \le h(a) + h(b)$ we obtain

$$\sup_{t\in T}\sum_{j\ge i} 2^{-j}(D_j(A_j(t)) + h(2^{i-j}\mu(A_j(t)))) \le 2\theta_{\mathcal{N}}(T) + C_1 2^{-i} \le C_2\theta_{\mathcal{N}}(T), \tag{32}$$

where $C_1 = 1 + \sum_{k=0}^{\infty} k2^{-k}\ln 2$ and $C_2 = 2 + 4C_1/\ln 2$. Let $A \in \mathcal{A}_j, j > i$, then there exists unique $A^{'} \in \mathcal{A}_{j-1}$ such that $A \subset A^{'}$ and we may define

$$a_j(A) = (2C_2\theta_{\mathcal{N}}(T))^{-1}(2^{-j+1}D_{j-1}(A^{'}) + 2^{-j}h(2^{i-j}\mu(A))).$$

From (32) we obtain

$$\forall_{t\in T} \sum_{j>i} a_j(A_j(t)) \le \frac{1}{2}. \tag{33}$$

For any $A \in \mathcal{A}_j, j \ge i$ let us choose $x_A \in A$ and define

$$t(A) = a_j(A)^{-1}(x_A - x_{A'}) \text{ for } A \in \mathcal{A}_j, j > i.$$

Let us notice that

$$t - x_{A_j(t)} \in 22^{-j}B_{\mathcal{N}}(D_j(A_j(t))) \subset 2^{-j+1}D_j(A_j(t))B_2,$$

so for any $t \in T$, by (32), $\|t - x_{A_j(t)}\|_2 \to 0$. Therefore for any $t \in T$

$$t - x_T = t - x_{A_i(t)} = \sum_{j>i} a_j(A_j(t))t(A_j(t)).$$

Hence by (33) we have $\{t - x_T : t \in T\} \subset 2^{-1}\overline{\mathrm{conv}}\{t(A) : A \in \mathcal{A}_j, j > i\}$ and so

$$T - T \subset \overline{\mathrm{conv}}\{\pm t(A) : A \in \mathcal{A}_j, j > i\}.$$

Let $t_n = t(A_n)$ be such rearrangement of the set $\{t(A) : A \in \mathcal{A}_j, j > i\}$ that the sequence $2^{i-j(n)}\mu(A_n)$ decreases. Since

$$\sum_{j>i}\sum_{A\in\mathcal{A}_j} 2^{i-j}\mu(A) = \sum_{j>i} 2^{i-j} = 1,$$

we have $2^{i-j(n)}\mu(A_n) \leq 1/n$ (and $2^{i-j(1)}\mu(A_1) \leq 1/2$), so

$$h(2^{i-j(n)}\mu(A_n)) \geq \frac{1}{2}\ln(n+1). \tag{34}$$

For $A \in \mathcal{A}_j, j > i$, $x_A, x_{A'} \in A'$, so

$$t(A) \in 2a_j(A)^{-1}2^{-j+1}B_{\mathcal{N}}(D_{j-1}(A')) \subset$$

$$8C_2\theta_{\mathcal{N}}(T)(2D_{j-1}(A') + h(2^{i-j}\mu(A)))^{-1}B_{\mathcal{N}}(2D_{j-1}(A') + h(2^{i-j}\mu(A))).$$

Therefore (2) and (34) yields

$$t_n \in 16C_2\theta_{\mathcal{N}}(T)(\ln(n+1))^{-1}B_{\mathcal{N}}(\ln(n+1)),$$

that means that

$$\|t_n\|_{\mathcal{N},\ln(n+1)} \leq 16C_2\theta_{\mathcal{N}}(T).$$

Remark 2. The only part of the proof, where we used the assumptions about the moderate growth was the proof of the domination of (29) over (28).

Finally as an application of Theorem 3 we will prove the following Corollary, which generalize previous result of Krawczyk [2]. Let us recall that a family $\mathcal{A}$ of subsets of I is called a VC class of order $\leq n$ if no subset B of I with cardinality n is shattered by $\mathcal{A}$, i.e. $\{A \cap B : A \in \mathcal{A}\} \neq 2^B$.

Corollary 4 *Let $(X_i)_{i\in I}$ be the same as in Theorem 3 and $\mathcal{A}$ be a VC class of subsets of I of order ν. Then there exists a constant K, which depends only on C and ν such that for any $T \subset l^2$ we have*

$$E\sup_{t\in T}\sup_{A\in\mathcal{A}}|\sum_{i\in A} t_i X_i| \leq KE\sup_{t\in T}|X_t|.$$

In particular this means that for any sequence of vectors v_i in some Banach space such that the sum $\sum v_i X_i$ is a.e. convergent, the following inequality holds

$$E \sup_{A\in\mathcal{A}} \|\sum_{i\in A} v_i X_i\| \leq K E \|\sum_{i\in I} v_i X_i\|.$$

Proof. We may obviously assume that $0 \in T$, class $\mathcal{A}$ is finite and $E\sup_{t\in T} X_t = E\sup_{t\in T}|X_t| = 1$. For simplifying the notation we put $Y_t = \sup_{A\in\mathcal{A}} |\sum_{i\in A} t_i X_i|$ for $t \in l^2$. Let us notice that Y_t is of the form $\sup_{s\in S_t} X_s$ for some set $S_t \subset l^2$. Using Lemma 4 we may assume that functions N_i satisfy (17).

By Theorem 3 we may find constant K_1 (depending only on C) and sequence $t_n \in l^2$ such that $\|t_n\|_{\mathcal{N},\ln(n+1)} \leq K_1$, $E|X_{t_n}| \leq K_1$ and

$$T \subset T - T \subset \overline{\mathrm{conv}}\{\pm t_n : n \geq 1\}.$$

Therefore

$$E \sup_{t\in T} |Y_t| \leq E \sup_n |Y_{t_n}|. \tag{35}$$

As a consequence of [6] Theorem 11.1 and 14.12 we have for some universal constant K_2

$$\forall_{s\in l^2}\ E \sup_{A\in\mathcal{A}} |\sum_{i\in A} s_i \varepsilon_i| \leq K_2\sqrt{v}\sqrt{\sum s_i^2} \leq K_2\sqrt{2\nu} E|\sum_{i\in I} s_i\varepsilon_i|.$$

Hence by Fubini theorem we obtain for all $t \in T$

$$E|Y_t| \leq K_2\sqrt{2\nu}E|X_t| \leq K_1K_2\sqrt{2\nu} = K_3.$$

By Theorem 2 we get for $x \geq K_4K_1$ (K_4 is the same constant as in (18))

$$P(|Y_{t_n}| \geq E|Y_{t_n}| + x) \leq e^{-x\ln(n+1)/K_1K_4}.$$

Hence

$$\forall_{x>1}\ P(\sup_n |Y_{t_n}| \geq K_3 + xK_1K_4) \leq \sum_{n\geq 1}(n+1)^{-x}. \tag{36}$$

Corollary follows by (35), (36) and integration by parts.

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