SOME ESTIMATES OF NORMS OF RANDOM MATRICES

RAFAŁ LATAŁA

Abstract. We show that for any random matrix (X_{ij}) with independent mean zero entries

$$\mathbf{E}\|(X_{ij})\| \le C\Big(\max_i \sqrt{\sum_j \mathbf{E}X_{ij}^2} + \max_j \sqrt{\sum_i \mathbf{E}X_{ij}^2} + \sqrt[4]{\sum_{ij} \mathbf{E}X_{ij}^4}\Big),$$

where C is some universal constant.

1. Introduction

In this paper we consider the Euclidean operator norm

$$\|(a_{ij})\| = \|(a_{ij})\|_{l^2\to l^2} := \sup\{\sum_{i,j} a_{ij}x_iy_j : \sum x_i^2 \le 1, \sum y_j^2 \le 1\}$$

of random matrices whose entries are independent random variables. The case of $n\times m$ matrices with iid entries is quite well understood - the operator norm is (under some mild assumptions on the underlying ditribution) of order $\max(\sqrt{n}, \sqrt{m})$ (cf.[1, 6, 7] for the Gaussian case and [2, 8] for general iid entries).

Recently Y. Seginer [4] showed that for any $n \times m$ random matrix $(X_{ij})_{i\le n, j\le m}$ with iid mean zero entries

$$\mathbf{E}\|(X_{ij})\| \le C\Big(\mathbf{E}\max_{i\le n} \sqrt{\sum_{j\le m} X_{ij}^2} + \mathbf{E}\max_{j\le m} \sqrt{\sum_{i\le n} X_{ij}^2}\Big),$$

where C is an universal constant.

However if we do not assume the identical distribution (or at least identical variances) of the entries very little seems to be known about the average operator norm of random matrices, even in the Gaussian case. In the paper we present some general estimates for norms of random matrices with independent mean zero entries, assuming only the existence of the 4th moments. In the iid case our estimates are of the right order $\max(\sqrt{n}, \sqrt{m})$.

By (g_{ij}) we denote the iid sequence of standard $\mathcal{N}(0, 1)$ normal r.v.'s. We use the letter C for universal constants that may change from line to line.

2. Main Results

The purpose of the paper is to prove the following

2000 *Mathematics Subject Classification.* Primary 15A52, Secondary 60G15.

Key words and phrases. Random matrices, operator norm.

Research partially supported by KBN grant #2 PO3A 027 22.

Theorem 1. *For any finite matrix (a_{ij}) the following estimate holds*

$$\mathbf{E}\|(a_{ij}g_{ij})\| \le C\Big(\max_i \sqrt{\sum_j a_{ij}^2} + \max_j \sqrt{\sum_i a_{ij}^2} + \sqrt[4]{\sum_{ij} a_{ij}^4}\Big). \tag{1}$$

The next theorem is formally stronger, but it is almost immediate consequence of Theorem 1

Theorem 2. *For any finite matrix of independent mean zero r.v.'s X_{ij} we have*

$$\mathbf{E}\|(X_{ij})\| \le C\Big(\max_i \sqrt{\sum_j \mathbf{E}X_{ij}^2} + \max_j \sqrt{\sum_i \mathbf{E}X_{ij}^2} + \sqrt[4]{\sum_{ij} \mathbf{E}X_{ij}^4}\Big). \tag{2}$$

Remark 1. In the case when (X_{ij}) is an $n \times m$ matrix with iid mean zero entries such that $\mathbf{E}|X_{ij}|^4 < \infty$ Theorem 2 gives the right order estimate $\mathbf{E}\|(X_{ij})\| \le C(\mathbf{E}X_{ij}^2, \mathbf{E}X_{ij}^4)\max(\sqrt{n}, \sqrt{m})$.

Remark 2. The presence of fourth moments in Theorem 2 is not surprising. The result of Silverstein [5] states that if $X_{i,j}$ are iid mean zero random variables such that the norms of random matrices $n^{-1/2}\|(X_{i,j})_{1\le i,j\le n}\|$ are stochastically bounded then $\mathbf{E}|X_{i,j}|^p < \infty$ for any $p < 4$.

Proof of Theorem 2. Let $\tilde{X}_{ij}$ be an independent copy of r.v.'s X_{ij} and ε_{ij} be a sequence of independent symmetric Bernoulli r.v.'s (i.e. $\mathbf{P}(\varepsilon_{ij} = \pm 1) = 1/2$), independent of all other r.v.'s. Then by Jensen's inequality

$$\mathbf{E}\|(X_{ij})\| \le \mathbf{E}\|(X_{ij} - \tilde{X}_{ij})\| = \mathbf{E}\|(\varepsilon_{ij}(X_{ij} - \tilde{X}_{ij}))\| \le 2\mathbf{E}\|(\varepsilon_{ij}X_{ij})\|.$$

Thus it is enough to show (2) in the case of symmetric r.v.'s (X_{ij}). We have

$$\mathbf{E}\|(a_{ij}g_{ij})\| = \mathbf{E}\|(a_{ij}\varepsilon_{ij}|g_{ij}|)\| \ge \mathbf{E}_\varepsilon\|(a_{ij}\varepsilon_{ij}\mathbf{E}|g_{ij}|)\| = \sqrt{\frac{2}{\pi}}\mathbf{E}\|(a_{ij}\varepsilon_{ij})\|.$$

Therefore Theorem 1 implies (2) for $X_{ij} = a_{ij}\varepsilon_{ij}$. By conditioning we get

$$\mathbf{E}\|(X_{ij})\|^2 = \mathbf{E}_X\mathbf{E}_\varepsilon\|(\varepsilon_{ij}X_{ij})\|^2 \le C\mathbf{E}\Big(\max_i \sum_j X_{ij}^2 + \max_j \sum_i X_{ij}^2 + \sqrt{\sum_{ij} X_{ij}^4}\Big).$$

Obviously

$$\mathbf{E}\sqrt{\sum_{ij} X_{ij}^4} \le \sqrt{\sum_{ij} \mathbf{E}X_{ij}^4}.$$

To estimate the other terms we notice first that

$$\mathbf{E}\max_i \sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2) \le \mathbf{E}\sqrt{\sum_i \big(\sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2)\big)^2}$$

$$\le \sqrt{\sum_i \mathbf{E}\big(\sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2)\big)^2} = \sqrt{\sum_{ij} \mathrm{Var}(X_{ij}^2)} \le \sqrt{\sum_{ij} EX_{ij}^4}.$$

Hence

$$\mathbf{E}\max_i \sum_j X_{ij}^2 \le \mathbf{E}\max_i \sum_j (X_{ij}^2 - \mathbf{E}X_{ij}^2) + \max_i \sum_j \mathbf{E}X_{ij}^2 \le \sqrt{\sum_{ij} EX_{ij}^4} + \max_i \sum_j \mathbf{E}X_{ij}^2.$$

In similar way we show

$$\mathbf{E}\max_j \sum_i X_{ij}^2 \le \sqrt{\sum_{ij} EX_{ij}^4} + \max_j \sum_i \mathbf{E}X_{ij}^2.\square$$

3. Proof of Theorem 1

Lemma 1 (Gaussian concentration)**.** *Suppose that g_n are iid $\mathcal{N}(0,1)$ r.v.'s and*

$$G = \sup_i (l_i + \sum_n k_{i,n} g_n).$$

Then

$$\mathbf{P}\big(G \ge (EG^2)^{1/2} + wt\big) \le \mathbf{P}(G \ge EG + wt) \le e^{-t^2/2},$$

where

$$w = \sup_i \sqrt{\sum_n k_{i,n}^2}.$$

Proof. We may treat G as a w–Lipschitz function of variables g_n and the statement is an immediate consequence of concentration of Gaussian measures (cf. [3, Chapter 5]).$\square$

Lemma 2. *Suppose that r.v. Y_x, A_x and numbers $b_x \ge 0$ satisfy the inequality*

$$\mathbf{P}(Y_x \ge A_x + tb_x) \le e^{-t} \text{ for all } t > 0, x \in U.$$

Then for all $u \ge -\log \#U$

$$\mathbf{E}\max_{x\in U}\big[Y_x - A_x - b_x(u + \log \#U)\big] \le e^{-u}\max_{x\in U} b_x. \tag{3}$$

In particular

$$\mathbf{E}\max_{x\in U}\big[Y_x - A_x - b_x \log \#U\big] \le \max_{x\in U} b_x.$$

Proof. Let $Z_x = Y_x - A_x - b_x(u + \log \#U)$, then

$$\mathbf{E}\max_{x\in U} Z_x \le \int_0^\infty \mathbf{P}(\max_{x\in U} Z_x \ge s)ds \le \sum_{x\in U}\int_0^\infty \mathbf{P}\big(Y_x \ge A_x + b_x(u + \log \#U + \frac{s}{b_x})\big)ds$$

$$\le \sum_{x\in U} e^{-u}\frac{1}{\#U}\int_0^\infty e^{-s/b_x}ds = e^{-u}\frac{1}{\#U}\sum_{x\in U} b_x \le e^{-u}\max_{x\in U} b_x$$

and (3) follows. $\square$

Lemma 3. *Suppose that $g_{i,j}$ are iid $\mathcal{N}(0,1)$ r.v., U is a finite subset of l_2 and for each $x \in U$ there are given sets $I(x)$ and r.v.'s $(L_j(x))_j$ independent of $(g_{ij})_{i\in I(x),j}$. Then for any $c > 0$*

$$\mathbf{E}\max_{x\in U}\Big[\sum_j (L_j(x) + \sum_{i\in I(x)} a_{ij}g_{ij}x_i)^2 - (1+c)\sum_j \big(L_j(x)^2 + \sum_{i\in I(x)} a_{ij}^2x_i^2\big)$$
$$-2(1+c^{-1})\max_j \sum_{i\in I(x)} a_{ij}^2 x_i^2 \log \#U\Big]$$
$$\le 2(1+c^{-1})\max_{x\in U}\max_j \sum_{i\in I(x)} a_{ij}^2 x_i^2. \tag{4}$$

Proof. Let

$$G_x = \sqrt{\sum_j \big(L_j(x) + \sum_{i\in I(x)} a_{ij} g_{ij} x_i\big)^2} = \sup_{\|y\|_2\le 1} \sum_j \big(L_j(x) + \sum_{i\in I(x)} a_{ij} g_{ij} x_i\big) y_j.$$

Then

$$\mathbf{E}_g G_x^2 = \sum_j \big(L_j(x)^2 + \sum_{i\in I(x)} a_{ij}^2 x_i^2\big),$$

where $\mathbf{E}_g$ denotes the expectation with respect to $(g_{ij})_{i\in I(x)}$. If we put

$$w(x) = \sup_{\|y\|_2\le 1} \sqrt{\sum_j \sum_{i\in I(x)} a_{ij}^2 x_i^2 y_j^2} = \max_j \sqrt{\sum_{i\in I(x)} a_{ij}^2 x_i^2},$$

then since

$$(\sqrt{\mathbf{E}_g G_x^2} + tw(x))^2 \le (1+c)\mathbf{E}_g G_x^2 + (1+c^{-1}) t^2 w^2(x)$$

we get by Lemma 1

$$\mathbf{P}\big(G_x^2 \ge (1+c)\mathbf{E}_g G_x^2 + (1+c^{-1}) t^2 w^2(x)\big)$$
$$\le \mathbf{P}\big(G_x \ge \sqrt{\mathbf{E}_g G_x^2} + tw(x)\big) \le e^{-t^2/2}.$$

Therefore Lemma 3 follows by Lemma 2 applied to

$$Y_x = G_x^2, A_x = (1+c)\sum_j \big(L_j(x)^2 + \sum_{i\in I(x)} a_{ij}^2 x_i^2\big)$$

and

$$b_x = 2(1+c^{-1}) w^2(x).\square$$

Before stating the next lemma let us introduce some notation

$$D_2 = \{x \in l_2 : \|x\|_2 \le 1, \forall_i \exists_{k\in\mathbf{Z}}\ x_i^2 = 2^{-k}\},$$

$$D_2^n = \{x \in l_2^{2^n} : \|x\|_2 \le 1, \forall_i \exists_{k\in\mathbf{Z}}\ x_i^2 = 2^{-k}\}.$$

With a little abuse of notation we will treat $l_2^{2^n}$ as a subset of l_2 and D_2^n as a subset of D_2. For $x \in D_2$, $k \in \mathbf{Z}_+$ let

$$A_k(x) = \{i : i \ge 2^k, x_i^2 \ge 2^{-k} \text{ or } 2^l \le i < 2^{l+1}, x_i^2 \ge 2^{l-2k}, l = 0,1,\ldots,k-1\},$$

$$\pi_k(x) = (\pi_{k,i}(x))_i, \text{ where } \pi_{k,i}(x) = x_i I_{A_k(x)}(i).$$

Finally let

$$\Pi_k^n = \{\pi_k(x) : x \in D_2^n\}.$$

Lemma 4. *For any $k, n \in \mathbf{Z}_+$ we have*

$$\log \#\Pi_k^n \le \begin{cases} C2^k(1+n-k) & \text{for } k \le n \\ C2^n(1+k-n) & \text{for } k \ge n \end{cases}.$$

Proof. Let

$U_l^n = \{x \in l_2^{2^n} : \forall_i \, x_i^2 = 2^{-l} I_{x_i \neq 0}\}$ and $V_l^n = \{x \in l_2^{2^n} : \forall_i \exists_{m=0,1,\dots,l} \, x_i^2 = 2^{-m} I_{x_i \neq 0}\}$. Notice that if $x \in U_l^n$ then $\#\{i : x_i \neq 0\} \leq 2^l$, so for $l \leq n$

$$\#U_l^n \leq \sum_{k=0}^{2^l} \binom{2^n}{k} 2^k \leq 1 + \sum_{k=1}^{2^l} (\frac{2e2^n}{k})^k \leq 1 + 2^l (2e2^{n-l})^{2^l} \leq \exp(C2^l(1+n-l)),$$

where the second inequality follows by the classical estimate $\binom{n}{k} \leq \frac{n^k}{k!} \leq (\frac{en}{k})^k$ and the third one by the monotonicity of the function $x \to a^x x^{-x}$ on $(0, a/e]$. For $l \geq n$ we get

$$\#U_l^n \leq \sum_{k=0}^{2^n} \binom{2^n}{k} 2^k = 3^{2^n}.$$

Now notice that for any $a, b > 0$ there exist finite constants $C_{a,b}$ and $C'_{a,b}$ depending only on a and b such that

$$\sum_{l=-\infty}^{m} 2^{al}(1 + b(n-l)) \leq C_{a,b} 2^{am}(1 + b(n-m)) \text{ for } m \leq n \tag{5}$$

and

$$\sum_{l=m}^{\infty} 2^{-al}(1 + b(l-n)) \leq C'_{a,b} 2^{-am}(1 + b(m-n)) \text{ for } m \geq n. \tag{6}$$

Indeed, observe that for $l \leq m \leq n$, $1 + b(n-l) \leq (1 + b(m-l))(1 + b(n-m))$, so

$$\sum_{l=-\infty}^{m} 2^{al}(1 + b(n-l)) \leq 2^{am}(1 + b(n-m)) \sum_{l=-\infty}^{m} 2^{a(l-m)}(1 + b(m-l))$$

$$= 2^{am}(1 + b(n-m)) \sum_{k=0}^{\infty} 2^{-ak}(1 + bk)$$

and (5) immediately follows. The proof of (6) is entirely similar.

Thus if we use (5) with $a = b = 1$ and the previously obtained estimates of $\#U_l^n$ we get that for any n, k

$$\log \#V_k^n \leq \sum_{l=0}^{k} \log \#U_l^n \leq \begin{cases} C2^k(1+n-k) & \text{for } k \leq n \\ C2^n(1+k-n) & \text{for } k \geq n \end{cases}.$$

Now, for $k \leq n$ we have

$$\#\Pi_k^n \leq \#V_k^n \prod_{l=0}^{k-1} \#V_{2k-l}^l,$$

so by (5) with $a = 1, b = 2$ we obtain

$$\log \#\Pi_k^n \leq C\big[2^k(1+n-k) + \sum_{l=0}^{k-1} 2^l(1+2k-2l)\big] \leq C2^k(1+n-k).$$

Simirarily for $k \geq n$

$$\#\Pi_k^n \leq \prod_{l=0}^{n} \#V_{2k-l}^l$$

and therefore

$$\log \#\Pi_k^n \le C\sum_{l=0}^{n} 2^l(1+2k-2l) \le C2^n(1+k-n).\Box$$

Lemma 5. *For any $x \in l^2$ with $\|x\|_2 \le 1$ we can find $y \in D_2$ such that $\|x-y\|_2 \le 3/10$. Therefore for any linear operator T on l^2 we have*

$$\sup_{\|x\|\le 1}\|Tx\| \le \frac{10}{7}\sup_{x\in D_2}\|Tx\|.$$

Proof. To prove the first part of the statement it is enough to show

$$\forall_{a\in[-1,1]}\ \exists_b\ b^2 = 2^{-k}I_{b\neq 0}, b^2 \le a^2, |a-b| \le \frac{3}{10}|a|.$$

If $a = 0$ we take $b = 0$, so we may assume $2^{-k} \le a^2 \le 2^{-k+1}$ and put $b = \mathrm{sgn}(a)2^{-k/2}$, then

$$\frac{|a-b|}{|a|} = (1-\frac{|b|}{|a|}) \le (1-\frac{1}{\sqrt{2}}) \le \frac{3}{10}.$$

Therefore any $x \in l^2$ with $\|x\|_2 \le 1$ can be represented as

$$x = \sum_{i\ge 0}(\frac{3}{10})^i y_i, \text{ where } y_0, y_1, \ldots \in D_2$$

and the last part of the lemma immediately follows. $\Box$

Now we are ready to formulate the main technical lemma needed for the proof of Theorem 1.

Lemma 6. *Let $\Delta = \max_j(\sum_{i<2^n} a_{ij}^4)^{1/4}$ then*

$$\mathbf{E}\sup_{x\in D_2}\Big[\sum_j(\sum_{i<2^n} a_{ij}g_{ij}x_i)^2 - C_1\sum_{ij}a_{ij}^2x_i^2 - C_1\Delta^2 2^{\frac{n}{2}}\sum_{l=1}^{n}2^{\frac{l-n}{8}}\sqrt{\sum_{2^{l-1}\le i<2^l}x_i^2}\Big] \le C_1\Delta^2 2^{\frac{n}{4}}.$$

Proof. For $x \in D_2^n$, $l, n = 1, 2, \ldots$ let us define

$$A_l^n(x) = \{i < 2^n : i \in A_l(x)\},$$

$$B_0^n(x) = A_0^n(x) = \{i < 2^n \colon x_i^2 = 1\}, b_0(x) = 2^{\frac{n}{4}}\log\#\Pi_0^n \max_j \sum_{i\in B_0^n(x)} a_{ij}^2x_i^2,$$

$$B_l^n(x) = A_l^n(x)\setminus A_{l-1}^n(x) = \big\{i < 2^n : i \ge 2^{l-1}, x_i^2 = 2^{-l}$$
$$\text{or } 2^{m-1} \le i < 2^m, x_i^2 \in \{2^{m-2l}, 2^{m-1-2l}\}, m = 1, 2, \ldots, l\big\},$$

$$c_l(x) = \sqrt{\sum_{2^{l-1}\le i<2^l}x_i^2},\ b_l(x) = 2^{\frac{|n-l|}{4}}\log\#\Pi_l^n\max_j\sum_{i\in B_l^n(x)}a_{ij}^2x_i^2$$

and

$$d_k = \prod_{l=0}^{k}(1+2^{-\frac{|n-l|}{4}}).$$

We first prove by induction on $k = 0, 1, \dots$ that

$$\mathbf{E} \sup_{x \in D_2^n} \Big[\sum_j \big(\sum_{i \in A_k^n(x)} a_{ij} g_{ij} x_i \big)^2 - d_k \sum_{i \in A_k^n(x), j} a_{ij}^2 x_i^2 - 4 d_k \sum_{l=0}^{k} b_l(x) \Big]$$
$$\leq 4 d_k \sum_{l=0}^{k} 2^{\frac{|n-l|}{4}} \sup_{x \in D_2^n} \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2. \tag{7}$$

For $k = 0$ inequality (7) follows by Lemma 3 applied to $U = \Pi_0^n, L_j(x) = 0, I(x) = B_0^n(x)$ and $c = 2^{-n/4}$.

To show the inductive step let us denote

$$S_k(x) = \sum_j \big(\sum_{i \in A_k^n(x)} a_{ij} g_{ij} x_i \big)^2 - d_k \sum_{i \in A_k^n(x), j} a_{ij}^2 x_i^2 - 4 d_k \sum_{l=0}^{k} b_l(x).$$

Then

$$\begin{aligned} & S_k(x) - (1 + 2^{-\frac{|n-k|}{4}}) S_{k-1}(x) \\ & \quad \leq \sum_j \big(\sum_{i \in A_k^n(x)} a_{ij} g_{ij} x_i \big)^2 - (1 + 2^{-\frac{|n-k|}{4}}) \sum_j \big(\sum_{i \in A_{k-1}^n(x)} a_{ij} g_{ij} x_i \big)^2 \\ & \quad -(1 + 2^{-\frac{|n-k|}{4}}) \sum_j \sum_{i \in B_k^n(x)} a_{ij}^2 x_i^2 - 2(1 + 2^{\frac{|n-k|}{4}}) \log \# \Pi_k^n \max_j \sum_{i \in B_k^n(x)} a_{ij}^2 x_i^2. \end{aligned}$$

Lemma 3 applied to

$$U = \Pi_k^n, L_j(x) = \sum_{i \in A_{k-1}^n(x)} a_{ij} g_{ij} x_i, I(x) = B_k^n(x), c = 2^{-\frac{|n-k|}{4}}$$

implies

$$\mathbf{E} \sup_{x \in D_2^n} \big[S_k(x) - (1 + 2^{-\frac{|n-k|}{4}}) S_{k-1}(x) \big] \leq 2(1 + 2^{\frac{|n-k|}{4}}) \sup_{x \in D_2^n} \max_j \sum_{i \in B_k^n(x)} a_{ij}^2 x_i^2.$$

The induction step from $k - 1$ to k easily follows, since

$$\mathbf{E} \sup_{x \in D_2^n} S_k(x) \leq \mathbf{E} \sup_{x \in D_2^n} \big[S_k(x) - (1 + 2^{-\frac{|n-k|}{4}}) S_{k-1}(x) \big] + (1 + 2^{-\frac{|n-k|}{4}}) \mathbf{E} \sup_{x \in D_2^n} S_{k-1}(x).$$

When we take $k \to \infty$ in (7) we get

$$\mathbf{E} \sup_{x \in D_2^n} \Big[\sum_j \big(\sum_{i < 2^n} a_{ij} g_{ij} x_i \big)^2 - d_\infty \sum_{i,j} a_{ij}^2 x_i^2 - 4 d_\infty \sum_{l=0}^{\infty} b_l(x) \Big]$$
$$\leq 4 d_\infty \sum_{l \geq 0} 2^{\frac{|n-l|}{4}} \sup_{x \in D_2^n} \max_j \sum_{i \in B_l^n(x)} a_{ij}^2 x_i^2, \tag{8}$$

where

$$d_\infty = \lim_{k \to \infty} d_k = \prod_{l \geq 0} (1 + 2^{-\frac{|n-l|}{4}}) \leq C.$$

Now notice that

$$\max_j \sum_{i < 2^n} a_{ij}^2 y_i^2 \leq \max_j \sqrt{\sum_{i < 2^n} a_{ij}^4} \sqrt{\sum_i y_i^4} \leq \Delta^2 \|y\|_2 \|y\|_\infty. \tag{9}$$

Thus

$$4\sum_{l\geq 0}2^{\frac{\lfloor n-l\rfloor}{4}}\sup_{x\in D_2^n}\max_j\sum_{i\in B_l^n(x)}a_{ij}^2x_i^2\leq 4\Delta^2\Big(\sum_{l=0}^{n}2^{\frac{n-l}{4}}2^{-\frac{l}{2}}+\sum_{l\geq n+1}2^{\frac{l-n}{4}}2^{\frac{n-2l}{2}}\Big)$$
$$\leq C\Delta^2 2^{\frac{n}{4}}.$$

Using the inequality (9) we also get

$$\max_j\sum_{i\in B_l^n(x)}a_{ij}^2x_i^2\leq\begin{cases}\Delta^2\big(\sum_{m=1}^{l}2^{\frac{m-2l}{2}}c_m(x)+2^{-\frac{l}{2}}\sum_{m=l+1}^{n}c_m(x)\big) & \text{for } l\leq n\\ \Delta^2\sum_{m=1}^{n}2^{\frac{m-2l}{2}}c_m(x) & \text{for } l>n\end{cases}.$$

Therefore by Lemma 4

$$\sum_{l\geq 0}b_l(x)=\sum_{l\geq 0}2^{\frac{\lfloor n-l\rfloor}{4}}\log\#\Pi_l^n\max_j\sum_{i\in B_l^n(x)}a_{ij}^2x_i^2\leq C\Delta^2\sum_{m=1}^{n}\alpha_m c_m(x),$$

where

$$\alpha_m=\sum_{l=0}^{m-1}2^{\frac{n-l}{4}}2^l(1+n-l)2^{-\frac{l}{2}}+\sum_{l=m}^{n}2^{\frac{n-l}{4}}2^l(1+n-l)2^{\frac{m-2l}{2}}+\sum_{l\geq n+1}2^{\frac{l-n}{4}}2^n(1+l-n)2^{\frac{m-2l}{2}}.$$

To estimate the first and third term we use respectively (5) with $a=\frac{1}{4}$ and $b=1$ and (6) with $a=\frac{1}{4}$ and $b=1$. To bound the second term we notice that $1+n-l\leq 1+n-m$ for $m\leq l\leq n$. Hence we get

$$\alpha_m\leq C2^{\frac{n+m}{4}}(1+n-m)\leq C2^{\frac{n}{2}}2^{\frac{m-n}{8}}\sup_{x\geq 0}((1+x)2^{-\frac{x}{8}})\leq C2^{\frac{n}{2}}2^{\frac{m-n}{8}}.$$

Therefore

$$\sum_{l\geq 0}b_l(x)\leq C\Delta^2 2^{\frac{n}{2}}\sum_{l=1}^{n}2^{\frac{l-n}{8}}c_l(x)$$

and Lemma follows by the inequality (8).□

Proof of Theorem 1. We may and will assume that

$$\sum_i a_{i,j_1}^4\geq\sum_i a_{i,j_2}^4 \text{ for } j_1<j_2$$

and

$$\sum_j a_{i_1,j}^4\geq\sum_j a_{i_2,j}^4 \text{ for } i_1<i_2.$$

Notice first that

$$\sup_{\|x\|_2,\|y\|_2\leq 1}\sum_{ij}a_{ij}g_{ij}x_iy_j\leq\sup_{\|x\|_2,\|y\|_2\leq 1}\sum_{i\leq j}a_{ij}g_{ij}x_iy_j+\sup_{\|x\|_2,\|y\|_2\leq 1}\sum_{j<i}a_{ij}g_{ij}x_iy_j$$
$$=\sup_{\|x\|_2\leq 1}\sqrt{\sum_j(\sum_{i\leq j}a_{ij}g_{ij}x_i)^2}+\sup_{\|y\|_2\leq 1}\sqrt{\sum_i(\sum_{j<i}a_{ij}g_{ij}y_j)^2}.$$

So to prove Theorem 1 it is enough to show

$$E\sup_{\|x\|_2\leq 1}\sum_j(\sum_{i\leq j}a_{ij}g_{ij}x_i)^2\leq C\Big(\max_i\sum_j a_{ij}^2+\sqrt{\sum_{ij}a_{ij}^4}\Big) \tag{10}$$

and

$$E\sup_{\|y\|_2\le 1}\sum_i(\sum_{j<i}a_{ij}g_{ij}y_j)^2 \le C\Big(\max_j\sum_i a_{ij}^2+\sqrt{\sum_{ij}a_{ij}^4}\Big). \tag{11}$$

We will prove only (10), since (11) is basically the same inequality. By Lemma 5 it is enough to show

$$E\sup_{x\in D_2}\sum_j(\sum_{i\le j}a_{ij}g_{ij}x_i)^2 \le C\Big(\max_i\sum_j a_{ij}^2+\sqrt{\sum_{ij}a_{ij}^4}\Big). \tag{12}$$

Let us introduce some additional notation for $n=1,2,\dots$ and $x\in D_2$

$$\Delta_n^4=\sum_i a_{i,2^{n-1}}^4=\max_{j\ge 2^{n-1}}\sum_i a_{ij}^4\ge \max_{j\ge 2^{n-1}}\sum_{i\le j}a_{ij}^4$$

and

$$f_n(x)=\sum_{2^{n-1}\le j<2^n}\sum_{i\le j}a_{ij}^2x_i^2+\Delta_n^2 2^{\frac{n}{2}}\sum_{l\le n}2^{\frac{l-n}{8}}c_l(x),$$

where as in the proof of Lemma 6

$$c_n(x)=\sqrt{\sum_{2^{n-1}\le i<2^n}x_i^2}.$$

Let C_1 be the same constant as in Lemma 6, then

$$\mathbf{E}\sup_{x\in D_2}\big[\sum_j(\sum_{i\le j}a_{ij}g_{ij}x_i)^2-C_1\sum_{n=1}^{\infty}f_n(x)\big]$$

$$\le\sum_{n=1}^{\infty}\mathbf{E}\sup_{x\in D_2}\big[\sum_{2^{n-1}\le j<2^n}(\sum_{i\le j}a_{ij}g_{ij}x_i)^2-C_1f_n(x)\big]\le\sum_{n=1}^{\infty}C_1\Delta_n^2 2^{\frac{n}{4}}. \tag{13}$$

Notice that $2^{n-1}\Delta_n^4\le\sum_{ij}a_{ij}^4$, so

$$\sum_{n\ge 1}\Delta_n^2 2^{\frac{n}{4}}\le C\sqrt{\sum_{ij}a_{ij}^4}. \tag{14}$$

Obviously we have

$$\sup_{x\in D_2}\sum_{i\le j}a_{ij}^2x_i^2\le\max_i\sum_j a_{ij}^2. \tag{15}$$

For any $u,v\in l^2$ we have

$$\sum_{l\le n}u_lv_n2^{\frac{l-n}{8}}=\sum_{i\ge 0}2^{-\frac{i}{8}}\sum_l u_lv_{l+i}\le\sum_{i\ge 0}2^{-\frac{i}{8}}\|u\|_2\|v\|_2\le C\|u\|_2\|v\|_2.$$

Since $\sum_l c_l(x)^2=\|x\|_2^2$ and $\sum_n\Delta_n^4 2^n\le C\sum_{ij}a_{ij}^4$ we get

$$\sum_{l\le j}c_l(x)\Delta_j^2 2^{\frac{j}{2}}2^{\frac{l-j}{8}}\le C\|x\|_2\sqrt{\sum_{ij}a_{ij}^4}. \tag{16}$$

By (15) and (16) we get

$$\sum_{n=1}^{\infty}f_n(x)\le\max_i\sum_j a_{ij}^2+C\sqrt{\sum_{ij}a_{ij}^4}\ \text{ for any } x\in D_2. \tag{17}$$

Inequalities (13), (14) and (17) imply (12) and the proof of Theorem 1 is completed.□

Acknowledgement. The author would like to thank the referee for a careful reading of the manuscript and valuable remarks.

References

[1] S. Chevet, Séries de variables aléatoires gaussiennes à valeurs dans $E\hat{\otimes}_\varepsilon F$. Application aux produits Wiener abstraits, *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, Exp No. 19, École Polytechnique, Palaiseau, 1978

[2] S. Geman, A limit theorem for the norm of random matrices, *Ann. Probab.* 8 (1980), 252–261

[3] M. Ledoux, *The concentration of measure phenomenon*, American Mathematical Society, Providence, RI, 2001.

[4] Y. Seginer, The expected norm of random matrices, *Combin. Probab. Comput.* 9 (2000), 149–166.

[5] J.W. Silverstein, ,*J. Multivariate Anal.* 30 (1989), 307–311.

[6] S. Szarek, Condition numbers of random matrices, *J. Complexity* 7 (1991), 131–149.

[7] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* 62 (1955), 548–564.

[8] Y.Q. Yin, Z.D. Bai and P.R. Krishnaiah, On the limit of the largest eigenvalue of the large sample covariance matrix, *Probab. Theory Relat. Fields* 78 (1988), 509–521.

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warszawa, Poland
E-mail address: rlatala@mimuw.edu.pl