On the limit set in the law of the iterated logarithm for U-statistics of order two

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Abstract. We find the cluster set in the Law of the Iterated Logarithm for U-statistics of order 2 in some interesting special cases. The lim sup is an unusual function of the quantities that determine the Bounded LIL.

1. Introduction and Notation.

In [GKLZ] necessary and sufficient conditions were obtained for the law of the iterated logarithm for canonical U-statistics of order 2 to hold. Here we continue the investigation of the LIL for U-statistics of order 2 by describing the cluster (or limit) set for the examples in [GZ], which helped motivate [GKLZ]. Namely, let $X_1, X_2, \ldots$ denote a sequence of iid r.v.'s with values in some measurable space $(S, \mathcal{S})$. In the general case for a measurable kernel h on S^2 we define symmetrized U-statistics by the formula

$$U_n = \sum_{1 \leq i < j \leq n} \varepsilon_i \varepsilon_j h(X_i, X_j),$$

where (ε_i) is a Rademacher sequence (i.e. a sequence of independent symmetric ± 1 valued r.v's) independent of (X_i). In our case we will assume that each X_i has a uniform distribution on $[0, 1]$ and

$$h(x, y) = \sum_{k=1}^{\infty} a_k h_k(x) h_k(y), \tag{1}$$

where

$$h_k(x) = I_{A_k}(x) \text{ and } A_k = (2^{-k}, 2^{-k+1}], k = 1, 2, \ldots.$$

We also assume that $0 \leq a_k \leq k^{-1/2} 2^k$ (this assumption seems not to be necessary but makes the calculations easier).

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The (nas) conditions for the (bounded) LIL for $\{U_n\}$ that were obtained in [GKLZ] imply that the conditions for the LIL in our case be in terms of

$$A = \sup\{\mathbf{E}h(X_1, X_2)f(X_1)f(X_2) : \mathbf{E}f^2(X_i) \le 1\} = \sup_k |a_k 2^{-k}| \tag{2}$$

and

$$B = \limsup_{u\to\infty} \frac{\mathbf{E}(h^2(X_1, X_2) \wedge u)}{L_2 u}. \tag{3}$$

However, what is not so clear is the form of the function of A and B that determines the lim sup. It turns out that the lim sup is

$$C = \phi(A, B) = \begin{cases} A + \frac{B^2}{4A} & \text{if } B \le 2A \\ B & \text{if } B \ge 2A. \end{cases}$$

In the sequel letters like K, K_1, etc., will denote universal constants that may change from line to line, but do not depend on any parameters. To simplify the notation we define $Lx = \log(x \vee e)$ and $L_2 x = LLx$. We also write $\log_2$ for the logarithm to the base 2.

Now, a few comments about the organization of the paper. After presenting in Section 2 some known results, we present in Section 3 a few results for general U-statistics. Finally, in Section 4 we concentrate on the types of kernels of the form (1) that are the main focus of this paper.

2. Preliminary results.

In this section we gather a few inequalities proven elsewhere that we will use in the sequel.

Lemma 1. *(*[KW]*, Theorem 6.2.1) There exists a universal constant K such that for any $t > 0$ and any sequence of real numbers $(a_{ij})_{1\le i<j\le n}$ we have*

$$\mathbf{P}(\max_{1\le k\le n} |\sum_{1\le i<j\le k} a_{ij}\varepsilon_i\varepsilon_j| > t) \le K\mathbf{P}(|\sum_{1\le i<j\le n} a_{ij}\varepsilon_i\varepsilon_j| > t).$$

Lemma 2 (Bernstein inequality)**.**
([de la P,G] *Lemma 4.1.9 and Remark 4.1.10,* [D] *Th. 1.3.2) If Z_i are independent r.v's with $\mathbf{E}Z_i = 0$, $\|Z_i\|_\infty \le a$ and $b^2 = \sum \mathbf{E}Z_i^2$, then for all $t \ge 0$*

$$\mathbf{P}(|\sum_i Z_i| \ge t) \le 2\exp(-\frac{t^2}{2b^2 + \frac{2}{3}at}).$$

Lemma 3 (Kolmogorov's converse exponential inequality)**.**
([S] *Th. 5.2.2) For any $\gamma > 0$ there exist numbers $\tilde{K}(\gamma) < \infty$ and $\tilde{\varepsilon}(\gamma) > 0$ such that if Z_i are independent r.v's with $\mathbf{E}Z_i = 0$, $\|Z_i\|_\infty \le a$, $b^2 = \sum \mathbf{E}Z_i^2$ satisfying $t \ge \tilde{K}(\gamma)b$ and $ta \le \tilde{\varepsilon}(\gamma)b^2$ for some $t > 0$, then*

$$\mathbf{P}(\sum_i Z_i \ge t) \ge \exp(-\frac{(1+\gamma)t^2}{2b^2}).$$

We will, however, prefer to use the following simple corollary of Kolmogorov's converse exponential inequality (one may take below $\varepsilon(\gamma) = (1+\gamma)\tilde{\varepsilon}(\gamma)^2/2$ and $K(\gamma) = \exp((1+\gamma)\tilde{K}(\gamma)^2/2)$.)

Corollary 1. *For any $\gamma > 0$ there exist numbers $K(\gamma) < \infty$ and $\varepsilon(\gamma) > 0$ such that if Z_i are independent r.v's with $\mathbf{E}Z_i = 0$, $\|Z_i\|_\infty \le a$, $b^2 = \sum \mathbf{E}Z_i^2$, then for all $t > 0$*

$$\mathbf{P}(\sum_i Z_i \ge t) \ge \frac{1}{K(\gamma)} \exp(-\frac{(1+\gamma)t^2}{2b^2}) - \exp(-\varepsilon(\gamma)\frac{b^2}{a^2}).$$

Lemma 4. *([GLZ], Corollary 3.4) There exists a universal constant $K < \infty$ such that for all $t > 0$*

$$\mathbf{P}(|U_n| \ge t) \le K \exp\Big[-\frac{1}{K}\min\Big(\frac{t^2}{n^2 Eh^2}, \frac{t}{n\|h\|_{L_2\to L_2}}, \frac{t^{2/3}}{[n(\|E_Y h^2\|_\infty + \|E_X h^2\|_\infty)]^{1/3}}, \frac{t^{1/2}}{\|h\|_\infty^{1/2}}\Big)\Big].$$

3. Technical Lemmas. General Kernels.

In this section we present few technical lemmas that do not require additional assumptions on the form of the kernel h.

Lemma 5. *We have*

$$\mathbf{E}\exp\big(\lambda(\sum_{i=1}^n \varepsilon_i)^2\big) \le \frac{1}{\sqrt{1-2\lambda n}} \text{ for all } 0 \le \lambda < \frac{1}{2n}. \tag{4}$$

Moreover, for each $\gamma > 0$, there exist positive numbers $K(\gamma)$ and $\delta(\gamma)$ such that for any n

$$\mathbf{P}(\sum_{i=1}^n \varepsilon_i \ge t\sqrt{n}) \ge \frac{1}{K(\gamma)} \exp(-\frac{(1+\gamma)t^2}{2}) - \exp(-\delta(\gamma)n). \tag{5}$$

Proof. Notice that for any t

$$\mathbf{E}\exp(t\sum_{i=1}^n \varepsilon_i) = (\frac{1}{2}e^t + \frac{1}{2}e^{-t})^n \le e^{\frac{nt^2}{2}}.$$

So if g is $\mathcal{N}(0,1)$ r.v. independent of ε_i, then

$$\mathbf{E}\exp\big(\lambda(\sum_{i=1}^n \varepsilon_i)^2\big) = \mathbf{E}_\varepsilon \mathbf{E}_g e^{\sqrt{2\lambda}(\sum_{i=1}^n \varepsilon_i)g}$$

$$= \mathbf{E}_g E_\varepsilon e^{\sqrt{2\lambda}g\sum_{i=1}^n \varepsilon_i} \le \mathbf{E}e^{n\lambda g^2} = \frac{1}{\sqrt{1-2\lambda n}}.$$

Inequality (5) is an immediate consequence of Kolmogorov's converse exponential inequality (Corollary 1). □

Lemma 6. *Suppose that $a_{ij}^{(n)}$ is a tripley indexed sequence of numbers such that*

$$\limsup_{n\to\infty} |\sum_{i,j=1}^{n} a_{ij}^{(n)} \varepsilon_i \varepsilon_j| \le C \ a.s.$$

Then,

$$\limsup_{n\to\infty} |\sum_{i=1}^{n} a_{ii}^{(n)}| \le C.$$

Proof. Let $t > C$, then $I(\sum_{i,j=1}^{n} a_{ij}^{(n)} \varepsilon_i \varepsilon_j \ge t) \to 0$ a.s. so in particular

$$\mathbf{P}(\sum_{i,j=1}^{n} a_{ij}^{(n)} \varepsilon_i \varepsilon_j \ge t) \to 0.$$

However

$$\mathbf{P}(\sum_{i,j=1}^{n} a_{ij}^{(n)} \varepsilon_i \varepsilon_j \ge \sum_{i=1}^{n} a_{ii}^{(n)}) = \mathbf{P}(\sum_{1\le i\ne j\le n} a_{ij}^{(n)} \varepsilon_i \varepsilon_j \ge 0) \ge \frac{1}{K}$$

for some universal K ([de la P,G] Proposition 3.3.7 combined with Theorem 3.2.2). This implies $\sum_{i=1}^{n} a_{ii}^{(n)} \le t$ for large enough n, so $\limsup_{n\to\infty} \sum_{i=1}^{n} a_{ii}^{(n)} \le C$. In a similar way we prove that $\limsup_{n\to\infty}(-\sum_{i=1}^{n} a_{ii}^{(n)}) \le C$. □

Lemma 7. *a)If $C < \infty$ is a number such that*

$$\forall_{\varepsilon>0} \exists_{K,N} \forall_{n\ge N}\ \mathbf{P}(|U_n| \ge C(1+\varepsilon) n L_2 n) \le \frac{K}{\log n (L_2 n)^{1+\varepsilon}}, \tag{6}$$

then

$$\limsup_{n\to\infty} \frac{|U_n|}{n L_2 n} \le C \ a.s.$$

b)If $C < \infty$ is a number such that

$$\forall_{\varepsilon>0, n_0} \exists_{K, N>n_0} \forall_{N\le n\le N^2}\ \mathbf{P}(|U_n| \ge C(1+\varepsilon) n L_2 n) \ge \frac{1}{K \log n}, \tag{7}$$

then

$$\limsup_{n\to\infty} \frac{|U_n|}{n L_2 n} \ge C \ a.s.$$

Proof. We start with the proof of part a). Let $\alpha > 1$, in this part of the proof we will denote $U_a = U_{\lfloor a \rfloor}$ for all $a \ge 0$. Let $\varepsilon > 0$ and K, N be given by formula (6). Let us choose k_0 such that $\alpha^{k_0} \ge N$. Then, we have for all $t > 0$

$$\mathbf{P}(\max_{n\ge \alpha^{k_0}} \frac{|U_n|}{n L_2 n} \ge t) \le \sum_{k=k_0}^{\infty} \mathbf{P}(\max_{\alpha^k \le n \le \alpha^{k+1}} \frac{|U_n|}{n L_2 n} \ge t)$$

$$\le \sum_{k=k_0}^{\infty} \mathbf{P}(\max_{1\le n\le \alpha^{k+1}} |U_n| \ge t\alpha^k L_2(\alpha^k)) \le \sum_{k=k_0}^{\infty} K \mathbf{P}(|U_{\alpha^{k+1}}| \ge t\alpha^k L_2(\alpha^k)),$$

where in the last line we used the maximal inequality (Lemma 1). Since for large enough k we have $L_2(\alpha^k) \geq \alpha^{-1}L_2(\alpha^{k+1})$ we get that for sufficiently large k_0

$$\mathbf{P}(\max_{n\geq\alpha^{k_0}} \frac{|U_n|}{nL_2n} \geq C\alpha^2(1+\varepsilon)) \leq \sum_{k=k_0}^{\infty} K\mathbf{P}(|U_{\alpha^{k+1}}| \geq C(1+\varepsilon)\alpha^{k+1}L_2(\alpha^{k+1}))$$

$$\leq \sum_{k=k_0}^{\infty} \frac{K}{\log\lfloor\alpha^{k+1}\rfloor(L_2\lfloor\alpha^{k+1}\rfloor)^{1+\varepsilon}}.$$

This implies that

$$\lim_{k\to\infty} \mathbf{P}\big(\max_{n\geq\alpha^k} \frac{|U_n|}{nL_2n} \geq C\alpha^2(1+\varepsilon)\big) = 0,$$

so $\limsup_{n\to\infty} \frac{|U_n|}{nL_2n} \leq C\alpha^2(1+\varepsilon)$ a.s. and part a) follows, when $\alpha \to 1^+$ and $\varepsilon \to 0^+$.

To prove part b) suppose that

$$\limsup_{n\to\infty} \frac{|U_n|}{nL_2n} \leq C_1 < C \text{ a.s.}$$

(By the 0-1 Law we know that the lim sup is constant a.s.). Let $m > 1$ be an integer (to be chosen later) and $\tilde{\varepsilon}_i$ be another Rademacher sequence independent of ε_i and X_i. Since for any choice of signs $\eta_i = \pm 1$ the sequence $\eta_i\varepsilon_i$ has the same distribution as ε_i we get that

$$\limsup_{n\to\infty} \frac{|\sum_{k,l=1}^n \tilde{\varepsilon}_k\tilde{\varepsilon}_l \sum_{m^{k-1}\leq i<m^k, m^{l-1}\leq j<m^l, i<j} \varepsilon_i\varepsilon_j h(X_i,X_j)|}{m^nL_2(m^n)} \leq C_1 \text{ a.s..}$$

So

$$\mathbf{P}_{\varepsilon,X}\Big(\limsup_{n\to\infty} \frac{|\sum_{k,l=1}^n \tilde{\varepsilon}_k\tilde{\varepsilon}_l \sum\limits_{m^{k-1}\leq i<m^k, m^{l-1}\leq j<m^l, i<j} \varepsilon_i\varepsilon_j h(X_i,X_j)|}{m^nL_2(m^n)} \leq C_1 \ \tilde{\varepsilon}\text{-a.s.}\Big) = 1.$$

However by Lemma 6 it implies

$$\mathbf{P}\Big(\limsup_{n\to\infty} \frac{|\sum_{k=1}^n \sum_{m^{k-1}\leq i<j<m^k} \varepsilon_i\varepsilon_j h(X_i,X_j)|}{m^nL_2(m^n)} \leq C_1\Big) = 1.$$

Let $1/2 > \delta > 0$ to be chosen later and $C_1 < C_2 < C$, then

$$\mathbf{P}\Big(\max_{n\geq n_0} \frac{|\sum_{k=1}^n \sum_{m^{k-1}\leq i<j<m^k} \varepsilon_i\varepsilon_j h(X_i,X_j)|}{m^nL_2(m^n)} > C_2\Big) < \delta$$

for sufficiently large n_0. Notice that if $|s_n| \leq C_2m^nL_2(m^n)$ for $n \geq n_0$, then $|s_n - s_{n-1}| \leq C_2(m^n + m^{n-1})L_2(m^n)$ for $n > n_0$. Therefore

$$\mathbf{P}\Big(\max_{n>n_0} \frac{|\sum_{m^{n-1}\leq i<j<m^n} \varepsilon_i\varepsilon_j h(X_i,X_j)|}{m^nL_2(m^n)} > C_2(1+\frac{1}{m})\Big) < \delta.$$

Thus by the independence (since $\mathbf{P}(\bigcup A_i) \geq 1/2 \sum \mathbf{P}(A_i)$ if A_i are independent and $\mathbf{P}(\bigcup A_i) \leq 1/2$)

$$\sum_{n>n_0} \mathbf{P}\Big(|\sum_{1\leq i<j\leq m^n-m^{n-1}} \varepsilon_i\varepsilon_j h(X_i,X_j)| \geq C_2 m^n(1+\frac{1}{m})L_2(m^n)\Big)$$
$$= \sum_{n>n_0} \mathbf{P}\Big(|\sum_{m^{n-1}\leq i<j<m^n} \varepsilon_i\varepsilon_j h(X_i,X_j)| \geq C_2 m^n(1+\frac{1}{m})L_2(m^n)\Big) < 2\delta.$$

Now choose m and increase n_0, if necessary, in such a way that

$$C_2 m^n(1+\frac{1}{m})L_2(m^n) \leq C(1+\varepsilon)(m^n-m^{n-1})L_2(m^n-m^{n-1})$$

for $n > n_0$. By our assumption (7) we can find $N > m^{n_0}$ such that

$$\mathbf{P}\big(|U_{m^n-m^{n-1}}| \geq C(1+\varepsilon)(m^n-m^{n-1})L_2(m^n-m^{n-1})\big)$$
$$\geq \frac{1}{K\log(m^n-m^{n-1})} \geq \frac{1}{Kn\log m}$$

for all n such that $N \leq m^n - m^{n-1} \leq N^2$. However

$$\sum_{n:N\leq m^n-m^{n-1}\leq N^2} \frac{1}{Kn\log m} \gtrsim \frac{\log 2}{K\log m} > 2\delta$$

if we choose δ small enough. □

The next Lemma shows why the LIL-limit depends on two quantities in a very non-obvious way.

Lemma 8. *Suppose that S_1, S_2 are independent r.v's, $A, B > 0$ and*

$$C = \begin{cases} A + \frac{B^2}{4A} & \text{if } B \leq 2A \\ B & \text{if } B \geq 2A \end{cases}$$

a) If for some $K \geq 1$ and $\varepsilon > 0$

$$\mathbf{P}(S_1 \geq sAn) \geq \frac{1}{K}e^{-s(1+\varepsilon)} - \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0$$

and

$$\mathbf{P}(S_2 \geq sBn\sqrt{L_2 n}) \geq \frac{1}{K}e^{-s^2(1+\varepsilon)^2} - \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0,$$

then for sufficiently large n

$$\mathbf{P}(S_1+S_2 \geq (1+\varepsilon)^{-1}CnL_2 n) \geq \frac{1}{K^2}\frac{1}{\log n} - \frac{2}{(\log n)^{1+\varepsilon}}.$$

b) On the other hand if for some $K, \varepsilon > 0$

$$\mathbf{P}(S_1 \geq sAn) \leq Ke^{-\frac{s}{1+\varepsilon}} + \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0$$

and

$$\mathbf{P}(S_2 \geq sBn\sqrt{L_2 n}) \leq Ke^{-\frac{s^2}{(1+\varepsilon)^2}} + \frac{1}{(\log n)^{1+\varepsilon}} \text{ for all } s \geq 0,$$

then

$$\mathbf{P}(S_1 + S_2 \geq (1+\varepsilon)^3 CnL_2 n) \leq (\frac{1}{\varepsilon} + 1)\frac{(K+2)^2}{(\log n)^{1+\varepsilon}}.$$

Proof. For the first part of the statement it is enough to notice that in the case when $B \geq 2A$ we get for sufficiently large n

$$\mathbf{P}(S_1 + S_2 \geq (1+\varepsilon)^{-1} CnL_2 n) \geq \mathbf{P}(S_1 \geq 0)\mathbf{P}(S_2 \geq (1+\varepsilon)^{-1} BnL_2 n)$$

$$\geq (\frac{1}{K} - \frac{1}{(\log n)^{1+\varepsilon}})(\frac{1}{K}e^{-L_2 n} - \frac{1}{(\log n)^{1+\varepsilon}}) \geq \frac{1}{K^2 \log n} - \frac{2}{(\log n)^{1+\varepsilon}}.$$

In the case when $B \leq 2A$ we have for large enough n

$$\begin{aligned} &\mathbf{P}(S_1 + S_2 \geq (1+\varepsilon)^{-1} CnL_2 n) \\ &\geq \mathbf{P}(S_1 \geq (1+\varepsilon)^{-1}(A - \tfrac{B^2}{4A})nL_2 n)\mathbf{P}(S_2 \geq (1+\varepsilon)^{-1}\tfrac{B^2}{2A}nL_2 n) \\ &\geq (\tfrac{1}{K}\exp(-(1-\tfrac{B^2}{4A^2})L_2 n) - \tfrac{1}{(\log n)^{1+\varepsilon}})(\tfrac{1}{K}\exp(-\tfrac{B^2}{4A^2}L_2 n) - \tfrac{1}{(\log n)^{1+\varepsilon}}) \\ &\geq \tfrac{1}{K^2 \log n} - \tfrac{2}{(\log n)^{1+\varepsilon}}. \end{aligned}$$

To prove part b) first notice that for all $x \in [0, C]$

$$\frac{x}{A} + \frac{(C-x)^2}{B^2} \geq 1.$$

Hence, for such x

$$\begin{aligned} &\mathbf{P}(S_1 \geq (1+\varepsilon)^2 xnL_2 n, S_2 \geq (1+\varepsilon)^2 (C-x)nL_2 n) \\ &\leq \Big(K\exp\big(-(1+\varepsilon)\frac{x}{A}L_2 n\big) + \frac{1}{(\log n)^{1+\varepsilon}}\Big)\cdot \\ &\Big(K\exp\big(-(1+\varepsilon)\frac{(C-x)^2}{B^2}L_2 n\big) + \frac{1}{(\log n)^{1+\varepsilon}}\Big) \leq \frac{(K+1)^2}{(\log n)^{1+\varepsilon}}. \end{aligned}$$

Moreover,

$$\mathbf{P}(S_1 \leq 0, S_1 + S_2 \geq (1+\varepsilon)^2 CnL_2 n) \leq \mathbf{P}(S_2 \geq (1+\varepsilon)^2 CnL_2 n)$$

$$\leq K\exp\big(-(1+\varepsilon)\frac{C^2}{B^2}L_2 n\big) + \frac{1}{(\log n)^{1+\varepsilon}} \leq \frac{K+1}{(\log n)^{1+\varepsilon}}$$

and

$$\mathbf{P}(S_1 \geq (1+\varepsilon)^2 CnL_2 n) \leq \frac{K+1}{(\log n)^{1+\varepsilon}}.$$

Let $k_0 = \lfloor \varepsilon^{-1} \rfloor$. Then,

$$\mathbf{P}(S_1 + S_2 \geq (1+\varepsilon)^3 C n L_2 n)$$
$$\leq \mathbf{P}(S_1 \leq 0, \frac{S_1 + S_2}{(1+\varepsilon)^2 n L_2 n} \geq C) + \mathbf{P}(\frac{S_1}{(1+\varepsilon)^2 n L_2 n} \geq C)$$
$$+ \sum_{k=0}^{k_0} \mathbf{P}(\frac{S_1}{(1+\varepsilon)^2 n L_2 n} \in [k\varepsilon C, (k+1)\varepsilon C), \frac{S_2}{(1+\varepsilon)^2 n L_2 n} \geq C - k\varepsilon C)$$
$$\leq \frac{2K+2}{(\log n)^{1+\varepsilon}} + (k_0+1)\frac{(K+1)^2}{(\log n)^{1+\varepsilon}} \leq (\frac{1}{\varepsilon}+1)\frac{(K+2)^2}{(\log n)^{1+\varepsilon}}.$$

□

4. Special Kernels

From this point on we will assume that our kernel is of the form (1). We consider the following (undecoupled) U-statistics Let

$$\tilde{U}_n = \sum_{k=1}^{\infty} a_k \sum_{1 \leq i < j \leq N_k} \varepsilon_i^k \varepsilon_j^k = \sum_{k=1}^{\infty} \frac{a_k}{2}\Big((\sum_{i=1}^{N_k} \varepsilon_i^k)^2 - N_k\Big),$$

where

$$N_k = \#\{1 \leq i \leq n : X_i \in (2^{-k}, 2^{-k+1}]\}, k = 1, 2 \ldots .$$

Notice that

$$\mathcal{L}\big(U_n | \sigma(X_1, X_2, \ldots)\big) = \mathcal{L}\big(\tilde{U}_n | \sigma(X_1, X_2, \ldots)\big),$$

so U_n and $\tilde{U}_n$ have the same distribution.

Lemma 9. *We have for all $\delta > 0$*

$$\mathbf{P}\big(\exists_{k \leq m} |N_k - n2^{-k}| \geq \delta n 2^{-k}\big) \leq \frac{2^{m+1}}{\delta^2 n}.$$

Proof. Notice that

$$\mathbf{P}(\exists_{k \leq m} |N_k - n2^{-k}| \geq \delta n 2^{-k}) \leq \sum_{k=1}^{m} \mathbf{P}(|N_k - \mathbf{E}N_k| \geq \delta n 2^{-k})$$
$$\leq \sum_{k=1}^{m} \frac{2^{2k}}{\delta^2 n^2} \mathrm{Var}(N_k) \leq \frac{1}{\delta^2 n} \sum_{k=1}^{m} 2^k \leq \frac{2^{m+1}}{\delta^2 n}.$$

□

Lemma 10. *Suppose that $s > 0$ and $|n_k - n2^{-k}| \leq \varepsilon n 2^{-k-1}$ for $k = 1, \ldots, m$. Let $\alpha = \max\{2^{-k}|a_k| : 1 \leq k \leq m\}$, then*

$$\mathbf{P}\Big(\Big|\sum_{k=1}^{m} \frac{a_k}{2}\big((\sum_{i=1}^{n_k} \varepsilon_i^k)^2 - n_k\big)\Big| \geq \alpha s n\Big) \leq (\frac{2e(1+\varepsilon)}{\varepsilon})^{m/2} e^{-\frac{s}{1+\varepsilon}} \tag{8}$$

On the other hand, if $\alpha_1 = \max\{2^{-k}a_k : 1 \le k \le m\} > 0$*, then*

$$\mathbf{P}\big(\sum_{k=1}^{m} \frac{a_k}{2}\big((\sum_{i=1}^{n_k} \varepsilon_i^k)^2 - n_k\big) \ge \alpha_1 sn\big) \ge \frac{1}{K(\varepsilon)} e^{-(1+\varepsilon)s} - \exp(-\delta(\varepsilon)2^{-m}n), \qquad (9)$$

and if $\alpha_2 = \max\{-2^{-k}a_k : 1 \le k \le m\}$*, then*

$$\mathbf{P}\big(-\sum_{k=1}^{m} \frac{a_k}{2}\big((\sum_{i=1}^{n_k} \varepsilon_i^k)^2 - n_k\big) \ge \alpha_2 sn\big) \ge \frac{1}{K(\varepsilon)} e^{-(1+\varepsilon)s} - \exp(-\delta(\varepsilon)2^{-m}n), \quad (10)$$

where $K(\varepsilon)$ *and* $\delta(\varepsilon)$ *depend only on* ε*.*

Proof. Let $S = \sum_{i=1}^{m} \frac{|a_k|}{2}(\sum_{i=1}^{n_k} \varepsilon_i^k)^2$, then by (4) we have

$$\mathbf{E}e^{\lambda S} \le \prod_{i=1}^{m} \frac{1}{\sqrt{1 - \lambda |a_k| n_k}}.$$

But by our assumptions $|a_k|n_k \le (1 + \frac{\varepsilon}{2})\alpha n$, so

$$\mathbf{E}\exp(\frac{1}{\alpha n(1+\varepsilon)} S) \le (1 - \frac{1+\frac{\varepsilon}{2}}{1+\varepsilon})^{-m/2} = (\frac{2(1+\varepsilon)}{\varepsilon})^{m/2}.$$

Notice that

$$\big|\sum_{k=1}^{m} \frac{a_k}{2}\big((\sum_{i=1}^{n_k} \varepsilon_i^k)^2 - n_k\big)\big| \le S + \frac{1}{2}\sum_{k=1}^{m} |a_k| n_k \le S + \frac{1}{2}(1+\varepsilon)\alpha n m,$$

so (8) immediately follows, since

$$\mathbf{P}\Big(\big|\sum_{k=1}^{m} \frac{a_k}{2}\big((\sum_{i=1}^{n_k} \varepsilon_i^k)^2 - n_k\big)\big| \ge \alpha sn\Big) \le \mathbf{P}(S \ge \alpha n(s - \frac{1}{2}(1+\varepsilon)m)).$$

To get (9) let k_0 be such that $a_{k_0} = \alpha_1 2^{k_0}$, then

$$\mathbf{P}\big(\sum_{k=1}^{m} \frac{a_k}{2}\big((\sum_{i=1}^{n_k} \varepsilon_i^k)^2 - n_k\big) \ge \alpha_1 sn\big)$$

$$\ge \mathbf{P}(\frac{a_{k_0}}{2}(\sum_{i=1}^{n_{k_0}} \varepsilon_i)^2 \ge \alpha_1 sn)\mathbf{P}(\sum_{k \ne k_0} a_k \sum_{1 \le i < j \le n_k} \varepsilon_i^k \varepsilon_j^k \ge 0)$$

$$\ge \frac{1}{K}\mathbf{P}((\sum_{i=1}^{n_{k_0}} \varepsilon_i)^2 \ge 2^{-k_0+1} sn),$$

where in the last inequality we used the same properties of Rademacher chaoses as in the proof of Lemma 6 (see [de la P,G], Proposition 3.3.7). Thus (9) follows by (5). The proof of (10) is similar. □

Lemma 11. *Suppose that $0 < \delta < 1, k_1 \geq 1$, $|n_k - n2^{-k}| \leq \delta n 2^{-k}$ for $k_1 \leq k \leq k_2$ and*

$$a = \sup\{|a_k|2^{-k} : k_1 \leq k \leq k_2\}, b^2 = \sum_{k=k_1}^{k_2} a_k^2 2^{-2k}.$$

Then, for any $s > 0$ and $t > 0$, we have

$$\mathbf{P}\Big(\frac{1}{2}\sum_{k=k_1}^{k_2} a_k\big((\sum_{i=1}^{n_k}\varepsilon_i^k)^2 - n_k\big) \geq t + 4k_2 nae^{-s/8}\Big)$$
$$\leq \exp(-\frac{t^2}{(1+\delta)^2 n^2 b^2(1+50e^{-s/8}) + 2tsan}) + 2k_2 e^{-s/4} \tag{11}$$

and

$$\mathbf{P}\Big(\frac{1}{2}\sum_{k=k_1}^{k_2} a_k\big((\sum_{i=1}^{n_k}\varepsilon_i^k)^2 - n_k\big) \geq t - 4k_2 nae^{-s/8}\Big)$$
$$\geq \frac{1}{K(\delta)}\exp(-\frac{(1+\delta)t^2}{(1-\delta)^2 n^2 b^2(1-50e^{-s/8}) - 2^{k_2+1}nb^2})$$
$$-\exp(-\frac{\varepsilon(\delta)b^2[(1-\delta)^2(1-50e^{-s/8}) - n^{-1}2^{k_2+1}]}{s^2a^2}) - 2k_2e^{-s/4}, \tag{12}$$

where positive constants $K(\delta)$ and $\varepsilon(\delta)$ depend only on δ.

Proof. Let

$$S_k = (\sum_{i=1}^{n_k}\varepsilon_i^k)^2 I_{(\sum_{i=1}^{n_k}\varepsilon_i^k)^2 \leq sn_k},$$

then

$$\|a_k(S_k - ES_k)\|_\infty \leq sa_k n_k \leq 2san.$$

Notice that by (4) we have

$$\mathbf{P}(|\sum_{i=1}^{n_k}\varepsilon_i^k| \geq \sqrt{sn_k}) \leq 2e^{-s/4},$$

so

$$|n_k - \mathbf{E}S_k| = \mathbf{E}(\sum_{i=1}^{n_k}\varepsilon_i^k)^2 I_{(\sum_{i=1}^{n_k}\varepsilon_i^k)^2 > sn_k}$$
$$\leq \sqrt{\mathbf{E}(\sum_{i=1}^{n_k}\varepsilon_i^k)^4}\sqrt{\mathbf{P}(|\sum_{i=1}^{n_k}\varepsilon_i^k| \geq \sqrt{sn_k})} \leq 4n_k e^{-s/8}.$$

Therefore

$$\sum_{k=k_1}^{k_2}|a_k(\mathbf{E}S_k - n_k)| \leq 8n\sum_{k=k_1}^{k_2}|a_k|2^{-k}e^{-s/8} \leq 8k_2 nae^{-s/8} \tag{13}$$

and

$$\mathbf{P}\Big(\sum_{k=k_1}^{k_2} a_k(\sum_{i=1}^{n_k}\varepsilon_i^k)^2 \neq \sum_{k=k_1}^{k_2} a_k S_k\Big) \le \sum_{k=k_1}^{k_2} \mathbf{P}\big(S_k \neq (\sum_{i=1}^{n_k}\varepsilon_i^k)^2\big) \le 2k_2 e^{-s/4}. \quad (14)$$

We have

$$|\mathbf{E}(\sum_{i=1}^{n_k}\varepsilon_i^k)^4 - \mathbf{E}S_k^2| = \mathbf{E}(\sum_{i=1}^{n_k}\varepsilon_i^k)^4 I_{(\sum_{i=1}^{n_k}\varepsilon_i^k)^2 > sn_k}$$

$$\le \sqrt{\mathbf{E}(\sum_{i=1}^{n_k}\varepsilon_i^k)^8}\sqrt{\mathbf{P}(|\sum_{i=1}^{n_k}\varepsilon_i^k| \ge \sqrt{sn_k})} \le 80 n_k^2 e^{-s/8}$$

by the Khinchine inequality. Moreover,

$$|(\mathbf{E}S_k)^2 - (\mathbf{E}(\sum_{i=1}^{n_k}\varepsilon_i^k)^2)^2| = |(\mathbf{E}S_k)^2 - n_k^2| = |\mathbf{E}S_k + n_k|\cdot|\mathbf{E}S_k - n_k|$$

$$\le 2n_k \cdot 4n_k e^{-s/8} = 8n_k^2 e^{-s/8},$$

so

$$|\mathrm{Var}(S_k) - \mathrm{Var}((\sum_{i=1}^{n_k}\varepsilon_i^k)^2)| \le 100 n_k^2 e^{-s/8}.$$

Therefore

$$\mathrm{Var}(\frac{1}{2}\sum_{k=k_1}^{k_2} a_k S_k) \le \sum_{k=k_1}^{k_2} a_k^2(\frac{1}{2}n_k(n_k-1) + 25n_k^2 e^{-s/8})$$

$$\le \frac{1}{2}(1+\delta)^2 n^2 b^2 (1+50e^{-s/8})$$

and by the Bernstein inequality (Lemma 2) we have

$$\mathbf{P}\Big(\frac{1}{2}\sum_{k=k_1}^{k_2} a_k(S_k - ES_k)) \ge t\Big) \le \exp\big(-\frac{t^2}{(1+\delta)^2 n^2 b^2(1+50e^{-s/8}) + 2stan}\big). \quad (15)$$

Inequality (11) follows by (13), (14) and (15). To get the other estimate notice that

$$2\mathrm{Var}(\frac{1}{2}\sum_{k=k_1}^{k_2} a_k S_k) \ge \sum_{k=k_1}^{k_2} a_k^2(n_k(n_k-1) - 50n_k^2 e^{-s/8})$$

$$\ge (1-\delta)^2 n^2 b^2(1-50e^{-s/8}) - \sum_{k=k_1}^{k_2} a_k^2 n_k \ge (1-\delta)^2 n^2 b^2 (1-50e^{-s/8}) - 2^{k_2+1} n b^2.$$

So by Kolmogorov's converse exponential inequality (Corollary 1) we get

$$\mathbf{P}\Big(\frac{1}{2}\sum_{k=k_1}^{k_2} a_k(S_k - ES_k)) \ge t\Big)$$
$$\ge \frac{1}{K(\delta)}\exp(-\frac{(1+\delta)t^2}{(1-\delta)^2n^2b^2(1-50e^{-s/8}) - 2^{k_2+1}nb^2}$$
$$- \exp(-\frac{\varepsilon(\delta)b^2[(1-\delta)^2(1-50e^{-s/8}) - n^{-1}2^{k_2+1}]}{s^2a^2}). \qquad (16)$$

Inequality (12) follows by (13), (14) and (16). □

Lemma 12. *Suppose that* $|n_k - n2^{-k}| \le \delta n 2^{-k}$, $|a_k| \le k^{-1/2}2^k$ *for* $k \le k_2$ *and*

$$k_0 = \sqrt{L_2 n}, k_1 = (L_2 n)^{10}, k_2 = \log_2 n - 10L_2 n.$$

Let, moreover,

$$A_n = \sup\{|a_k|2^{-k} : k \le k_0\}, B_n^2 = \frac{1}{L_2 n}\sum_{k=k_1}^{k_2} a_k^2 2^{-2k}$$

and

$$C_n = \begin{cases} A_n + \frac{B_n^2}{4A_n} & \text{if } B_n \le 2A_n \\ B_n & \text{if } B_n \ge 2A_n \end{cases}.$$

Then, for any $\varepsilon > 0$, *there exists* $K(\varepsilon)$ *such that for sufficiently large* n *and sufficiently small* δ *we have*

$$\mathbf{P}\Big(\big|(\sum_{k\le k_0} + \sum_{k=k_1}^{k_2})\frac{a_k}{2}((\sum_{i=1}^{n_k}\varepsilon_i^k)^2 - n_k)\big| \ge (1+\varepsilon)C_n n L_2 n\Big) \le \frac{1}{(\log n)^{1+\varepsilon}}$$

and

$$\mathbf{P}\Big(\big|(\sum_{k\le k_0} + \sum_{k=k_1}^{k_2})\frac{a_k}{2}((\sum_{i=1}^{n_k}\varepsilon_i^k)^2 - n_k)\big| \ge (1-\varepsilon)C_n n L_2 n\Big) \ge \frac{1}{K(\varepsilon)\log n}.$$

Proof. Let

$$S_1 = \sum_{k\le k_0}\frac{a_k}{2}((\sum_{i=1}^{n_k}\varepsilon_i^k)^2 - n_k) \text{ and } S_2 = \sum_{k=k_1}^{k_2}\frac{a_k}{2}((\sum_{i=1}^{n_k}\varepsilon_i^k)^2 - n_k).$$

We will show that for sufficiently small δ and sufficiently large n

$$\mathbf{P}\big(|S_2| \ge un(B_n\sqrt{L_2 n} + 1)\big) \le 2\exp(-\frac{u^2}{(1+\varepsilon/10)^2}) + \frac{1}{(\log n)^2}. \qquad (17)$$

Obviously we may assume $0 < \varepsilon < 1$. It is enough to show that

$$\mathbf{P}\big(\pm S_2 \ge un(B_n\sqrt{L_2 n} + 1)\big) \le \exp(-\frac{u^2}{(1+\varepsilon/10)^2}) + \frac{1}{4(\log n)^2} \qquad (18)$$

for $u \in [1/2, 4\sqrt{L_2 n}]$. Indeed, for $u < 1/2$ the right hand side of (17) is greater than 1 and for $u = 4\sqrt{L_2 n}$ the right hand side of (18) is less than $(2\log n)^{-2}$. Now apply Lemma 11 with $s = 20 L_2 n$, $t = un(B_n\sqrt{L_2 n} + 1/2)$ and $b^2 = \max(B_n^2 L_2 n, 1/4)$ (notice that then $t^2/(n^2 b^2) \geq u^2$ and that part (11) of Lemma 11 holds also under the assumption $b^2 \geq \sum_{k=k_1}^{k_2} a_k^2 2^{-2k}$ - the estimates are monotone in b^2). Since

$$a = \sup\{|a_k| 2^{-k} : k_1 \leq k \leq k_2\} \leq k_1^{-1/2} \leq (L_2 n)^{-5}$$

we have

$$\begin{aligned} 2tsan &\leq 2 \cdot 4\sqrt{L_2 n} \cdot n(B_n\sqrt{L_2 n} + 1) \cdot 20 L_2 n \cdot (L_2 n)^{-5} \cdot n \\ &\leq 160 (L_2 n)^{-3} n^2 (B_n \sqrt{L_2 n} + 1) \leq \delta n^2 b^2 \end{aligned}$$

for sufficiently large n. Also

$$2k_2 e^{-s/4} \leq 2(\log_2 n)(\log n)^{-5} < (4\log n)^{-1},$$

$$4k_2 n a e^{-s/8} \leq 4(\log_2 n) n e^{-s/8} \leq n/4 \leq un/2$$

and $50 e^{-s/8} < \delta$ for sufficiently large n. Now it is enough to choose sufficiently small δ (which will depend on ε). Lemma easily follows by Lemmas 8 and 10. □

Lemma 13. *If $\varepsilon > 0$, $|a_k| \leq k^{-1/2} 2^k$ for all k and*

$$k_0 = \sqrt{L_2 n}, k_1 = (L_2 n)^{10}, k_2 = \log_2 n - 10 L_2 n,$$

then for sufficiently large n

$$\mathbf{P}\Big(\big|(\sum_{k=k_0}^{k_1} + \sum_{k=k_2}^{\infty}) \frac{a_k}{2}((\sum_{i=1}^{N_k} \varepsilon_i^k)^2 - N_k)\big| \geq \varepsilon n L_2 n\Big) \leq \frac{5}{\log n (L_2 n)^{3/2}}$$

Proof. In this proof K denotes a universal constant that may change from line to line. Let us additionally define

$$k_3 = \log_2 n, k_4 = \log_2 n + \frac{1}{4}\log_2 \log n \text{ and } k_5 = \log_2 n + \frac{1}{2}\log_2\log n + \frac{3}{4}\log_2(L_2 n)$$

Notice that

$$\begin{aligned} &\mathbf{P}\big(\sum_{k=k_5}^{\infty} \frac{a_k}{2}((\sum_{i=1}^{N_k} \varepsilon_i^k)^2 - N_k) \neq 0\big) \leq \mathbf{P}(\exists_{k \geq k_5} N_k > 1) \\ &\leq \mathbf{P}(\exists_{i,j \leq n} |X_i|, |X_j| \leq 2^{-k_5+1}) \leq n^2 2^{-2k_5+1} \leq \tfrac{2}{\log n (L_2 n)^{3/2}}. \end{aligned} \tag{19}$$

For $k \leq k_5$ we have $|a_k| \leq k_5^{-1/2} 2^{k_5} \leq K n (L_2 n)^{3/4}$, therefore

$$\begin{aligned} &|\sum_{k=k_4}^{k_5-1} \frac{a_k}{2}((\sum_{i=1}^{N_k} \varepsilon_i^k)^2 - N_k)| \leq K n (L_2 n)^{3/4} (\sum_{i=k_4}^{k_5-1} N_k)^2 \\ &\leq K n (L_2 n)^{3/4} (\#\{i \leq n : |X_i| \leq 2^{-k_4+1}\})^2. \end{aligned}$$

Thus for fixed ε and sufficiently large n

$$\begin{aligned}\mathbf{P}\Big(\Big|\sum_{k=k_4}^{k_5-1}\frac{a_k}{2}\Big((\sum_{i=1}^{N_k}\varepsilon_i^k)^2-N_k\Big)\Big|\geq\frac{\varepsilon}{2}nL_2n\Big)&\\ \leq\mathbf{P}(\#\{i\leq n:|X_i|\geq 2^{-k_4+1}\}\geq(L_2n)^{1/8})&\\ \leq\Big(\frac{en2^{-k_4+1}}{(L_2n)^{1/8}}\Big)^{(L_2n)^{1/8}}\leq\frac{1}{\log n(L_2n)^{3/2}}.&\qquad(20)\end{aligned}$$

Here we used the fact that

$$\mathbf{P}(X\geq k)\leq\binom{n}{k}p^k\leq(\frac{enp}{k})^k \text{ if } X\sim\mathrm{Bin}(n,p).$$

Similarly, for $k\leq k_4$, $|a_k|\leq k_4^{-1/2}2^{k_4}\leq Kn(\log n)^{-1/4}$, so

$$\begin{aligned}\Big|\sum_{k=k_3}^{k_4-1}\frac{a_k}{2}((\sum_{i=1}^{N_k}\varepsilon_i^k)^2-N_k)\Big|&\leq Kn(\log n)^{-1/4}\Big(\sum_{i=k_3}^{k_4-1}N_k\Big)^2\\ &\leq Kn(\log n)^{-1/4}(\#\{i\leq n:|X_i|\leq 2^{-k_3+1}\})^2.\end{aligned}$$

Therefore, for sufficiently large n

$$\begin{aligned}\mathbf{P}\Big(\Big|\sum_{k=k_3}^{k_4-1}\frac{a_k}{2}\Big((\sum_{i=1}^{N_k}\varepsilon_i^k)^2-N_k\Big)\Big|\geq\frac{\varepsilon}{2}nL_2n\Big)&\\ \leq\mathbf{P}(\#\{i\leq n:|X_i|\geq 2^{-k_3+1}\}\geq(\log n)^{1/8})&\qquad(21)\\ \leq\Big(\frac{en2^{-k_3+1}}{(\log n)^{1/8}}\Big)^{(\log n)^{1/8}}\leq\frac{1}{\log n(L_2n)^{3/2}}.&\qquad(22)\end{aligned}$$

Finally

$$\mathcal{L}\Big(\sum_{k=k_0}^{k_1}+\sum_{k=k_2}^{k_3-1}\Big)\frac{a_k}{2}\Big((\sum_{i=1}^{N_k}\varepsilon_i^k)^2-N_k)\Big)=\mathcal{L}\Big(\sum_{i,j=1}^{n}\varepsilon_i\varepsilon_j\tilde{h}(X_i,X_j)\Big),$$

where

$$\tilde{h}(x,y)=\Big(\sum_{k=k_0}^{k_1}+\sum_{k=k_2}^{k_3-1}\Big)a_kh_k(x)h_k(y).$$

Let $A=[k_0,k_1]\cup[k_2,k_3-1]$, notice that

$$\|\tilde{h}\|_{L^2\to L^2}=\max_{k\in A}|a_k2^k|\leq\frac{1}{\sqrt{k_0}}\leq\frac{1}{(L_2n)^{1/4}},$$

$$E\tilde{h}^2=\sum_{k\in A}a_k^22^{2k}\leq\sum_{k\in A}\frac{1}{k}\leq CL_3n,$$

$$\|E_X\tilde{h}^2\|_\infty=\|E_Yh^2\|_\infty=\max_{k\in A}a_k^22^{-k}\leq\max_{k\in A}\frac{2^k}{k}\leq\frac{2^{k_3}}{k_3}\leq\frac{n}{\log_2 n}$$

and

$$\|\tilde{h}\|_\infty = \max_{k\in A}|a_k| \le \frac{2^{k_3}}{\sqrt{k_3}} \le \frac{n}{\sqrt{\log_2 n}}.$$

So by Lemma 4 it easily follows that

$$\mathbf{P}\Big(\Big|\Big(\sum_{k=k_0}^{k_1} + \sum_{k=k_2}^{k_3}\Big)\frac{a_k}{2}\Big((\sum_{i=1}^{N_k}\varepsilon_i^k)^2 - N_k\Big)\Big| \ge \frac{\varepsilon}{2}nL_2n\Big) \le \frac{1}{\log n(L_2n)^{3/2}}. \qquad (23)$$

The lemma follows by (19)–(23). □

Theorem 1. *If* $|a_k| \le \frac{2^k}{\sqrt{k}}$ *and* A *and* B *are given by (2) and (3), then*

$$\limsup_{n\to\infty}\frac{|U_n|}{nL_2n} = \begin{cases} A + \frac{B^2}{4A} & \text{if } B \le 2A \\ B & \text{if } B \ge 2A \end{cases} \quad a.s.$$

Proof. Let A_n, B_n be as in Lemma 12 notice that $\lim_{n\to\infty} A_n = A$ and

$$(L_2n)B_n^2 \le \mathbf{E}(h^2 \wedge n) \le (L_2n)B_n^2 + \Big(\sum_{k\le k_1} + \sum_{k=k_2}^{\log_2 n}\Big)a_k^2 2^{-2k} + n\sum_{k\ge \log_2 n} 2^{-2k}$$

$$\le (L_2n)B_n^2 + CL_3n.$$

Since $L_2n/L_2(n^2) \to 1$ as $n\to\infty$ we get that $\limsup_{n\to\infty} B_n \le B$ and

$$\forall_{\varepsilon>0}\forall_{n_0}\exists_{n\ge n_0}\forall_{N\le n\le N^2} B_n \ge B - \varepsilon.$$

So the theorem follows by Lemmas 7, 9, 12 and 13. □

References

[de la P,G] V. de la Peña and E. Giné, *Decoupling. From dependence to independence. Randomly stopped processes. U-statistics and processes. Martingales and beyond*, Probability and its Applications. Springer-Verlag, New York, 1999.

[D] R. M. Dudley, Uniform central limit theorems. *Cambridge Studies in Advanced Mathematics*, 63. Cambridge University Press, Cambridge, 1999.

[GKLZ] E. Giné, S. Kwapień, R. Latała, J. Zinn The LIL for canonical U-statistics of order 2 *Ann. Probab.* **29** (2001), 502–557.

[GLZ] E. Giné, R. Latała, J. Zinn Exponential and moment inequalities for U-statistics, *High Dimensional Probability II*, 13–38, Progress in Probability 47, Birkhauser, Boston 2000.

[GZ] E. Giné, C.-H. Zhang On the integrability in the LIL for degenerate U-statistics *J. Theoret. Probab.* **9** (1996), 385–412.

[KW] S. Kwapień, W. Woyczyński *Random series and stochastic integrals: single and multiple*, Probability and Its Applications, Birkhauser, Boston 1992.

[S] W. F. Stout, *Almost sure convergence*, Probability and Mathematical Statistics, Vol. 24. Academic Press, New York-London, 1974.

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