Tails and moments estimates for some types of chaos *

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Abstract

Let X_i be a sequence of independent symmetric real random variables with logarithmically concave tails. We consider a variable $X = \sum_{i \neq j} a_{i,j} X_i X_j$, where $a_{i,j}$ are real numbers. We derive approximate formulas for the tails and moments of X and its decoupled version, which are exact up to some universal constants.

Definitions and notation. Let X_i, X_j' be two independent sequences of independent symmetric random variables with logarithmically concave tails, i.e. the functions $N_i, N_j' : [0, \infty) \to [0, \infty]$ defined by the formulas

$$N_i(t) = -\ln P(|X_i| \geq t)$$

and

$$N_j'(t) = -\ln P(|X_j'| \geq t)$$

are convex. Since it is only a matter of normalization we may and will assume that for all i and j

$$\inf\{t : N_i(t) \geq 1\} = \inf\{t : N_j'(t) \geq 1\} = 1. \tag{1}$$

Let us define the functions $\hat{N}_i$ by the formula

$$\hat{N}_i(t) = \begin{cases} t^2 & \text{for } |t| \leq 1 \\ N_i(|t|) & \text{for } |t| > 1. \end{cases}$$

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For sequences (a_i) of real numbers and $p > 0$ we put

$$\|(a_i)\|_{\mathcal{N},p} = \sup\{\sum a_i b_i : \sum \hat{N}_i(b_i) \le p\}$$

and

$$\|(a_i)\|_p = (\sum a_i^p)^{1/p}.$$

In a similar way we define $\hat{N}'_j$ and $\|(a_j)\|_{\mathcal{N}',p}$.

For matrices $(a_{i,j})$ and $p > 0$ we define

$$\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} = \sup\{\sum a_{i,j} b_i c_j : \sum \hat{N}_i(b_i) \le p, \sum \hat{N}'_j(c_j) \le p\}.$$

We denote by (ε_i) the Bernoulli sequence, i.e. a sequence of i.i.d. symmetric r.v. taking on values ± 1. A sequence of independent standard $\mathcal{N}(0,1)$ Gaussian random variables will be denoted by (g_i) and the canonical Gaussian measure on $\mathbb{R}^n$ by γ_n.

For a random variable X and $p > 0$ we write

$$\|X\|_p = (E|X|^p)^{1/p}.$$

We will also use the notation $a \sim_C b$ to denote that $C^{-1}a \le b \le Ca$.

In this paper we will prove the following theorem

Theorem 1 *Let $(a_{i,j})$ be a square summable matrix and $X = \sum a_{i,j} X_i X'_j$. Then for each $p \ge 1$*

$$\|X\|_p \sim_C \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(A_i)\|_{\mathcal{N},p} + \|(B_j)\|_{\mathcal{N}',p},$$

where $A_i = (\sum_j a_{i,j}^2)^{1/2}$, $B_j = (\sum_i a_{i,j}^2)^{1/2}$ and C is a universal constant.

We postpone the proof of Theorem 1 till the end of this article and now present some corollaries and examples.

Corollary 1 *Let $(a_{i,j})$ be a square summable matrix, such that $a_{i,i} = 0$ and $a_{i,j} = a_{j,i}$ for all i, j. Then for each $p \ge 1$*

$$\|\sum a_{i,j} X_i X_j\|_p \sim_{\tilde{C}} \|(a_{i,j})\|_{\mathcal{N},\mathcal{N},p} + \|(A_i)\|_{\mathcal{N},p},$$

where $A_i = (\sum_j a_{i,j}^2)^{1/2}$ and $\tilde{C}$ is a universal constant.

Proof. Let X_i' be an independent copy of X_i, then by the result of de la Peña, Montgomery-Smith (cf [2]) about decoupling chaos we have for $p \geq 1$

$$\|\sum a_{i,j} X_i X_j\|_p \sim_K \|\sum a_{i,j} X_i X_j'\|_p$$

with some universal constant K. Hence Corollary 1 is an immediate concequence of Theorem 1 if we notice that $A_i = B_i$ by the symmetry of the matrix $(a_{i,j})$.

Corollary 2 *There exist universal constants* $0 < c < C < \infty$ *such that under the assumptions of Corollary 1, for each* $t \geq 1$

$$P(|\sum a_{i,j} X_i X_j| \geq C(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N},t} + \|(A_i)\|_{\mathcal{N},t})) \leq e^{-t}$$

and

$$P(|\sum a_{i,j} X_i X_j| \geq c(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N},t} + \|(A_i)\|_{\mathcal{N},t})) \geq \min(c, e^{-t}).$$

Proof. The first inequality follows from Corollary 1 and Chebyshev's inequality. To get the second inequality we first use Corollary 1 and Proposition 1 from below to get

$$\|\sum a_{i,j} X_i X_j\|_{2p} \leq 4\tilde{C}^2 \|\sum a_{i,j} X_i X_j\|_p \text{ for } p \geq 1.$$

The inequality now may be obtained by Corollary 1 and the Paley-Zygmund inequality as in [3].

By simple calculations we may easily derive from Corollary 1 the following two examples of interest.

Example 1. If a matrix $(a_{i,j})$ satisfies the assumptions of Corollary 1 then for some universal constant K and any $p \geq 1$ we have

$$\|\sum a_{i,j} g_i g_j\|_p \sim_K p\|(a_{i,j})\|_{l_2 \to l_2} + \sqrt{p}\|(a_{i,j})\|_{HS},$$

where

$$\|(a_{i,j})\|_{l_2 \to l_2} = \sup\{\sum a_{i,j} b_i c_j : \|(b_i)\|_2, \|(c_j)\|_2 \leq 1\}$$

and

$$\|(a_{i,j})\|_{HS} = (\sum a_{i,j}^2)^{1/2}.$$

Example 2. Under the assumptions of Corollary 1 we have

$$\|\sum a_{i,j}\varepsilon_i\varepsilon_j\|_p \sim_K \sup\{\sum a_{i,j}b_ic_j : \|(b_i)\|_2, \|(c_j)\|_2 \le p, |b_i|, |c_j| \le 1\}$$
$$+\sum_{i\le p} A_i^* + \sqrt{p}(\sum_{i>p}(A_i^*)^2)^{1/2},$$

where A_i^* denotes a nondecreasing rearrangement of the sequence A_i and K is a universal constant.

Remark. Example 1 may be also derived in a simpler way. Using the invariance of Gaussian r.v. under orthogonal transformations, it is enough to prove that for any sequence (d_i) of real numbers we have

$$\|\sum d_i g_i g_i^{'}\|_p \sim_{K_1} p\|(d_i)\|_\infty + \sqrt{p}\|(d_i)\|_2.$$

This easily follows from the results of [3] (see Theorem 2 below).

The following theorem was established in a slightly less general setting by Gluskin and Kwapień in [3] and in full generality in [5].

Theorem 2 *There exists a universal constant $C_1 < \infty$ such that for any square summable sequence (a_i) and $p \ge 1$ we have*

$$\|\sum a_i X_i\|_p \sim_{C_1} \|(a_i)\|_{\mathcal{N},p}. \tag{2}$$

In particular for any $p, q \ge 1$ there exists a constant $C_{p,q}$, which depends only on p and q such that

$$\|\sum a_i X_i\|_p \le C_{p,q}\|\sum a_i X_i\|_q. \tag{3}$$

Remark. The inequality (3) may be also obtained by hypercontractive methods or direct calculations.

We will also use the following theorem of M. Talagrand (see [8] and [6] for a simpler proof with better constants).

Theorem 3 *Let λ be the measure on $\mathbb{R}$ with the density $\frac{1}{2}e^{-|x|}$ and λ^n be the product measure $\otimes_{i=1}^n \lambda$ on $\mathbb{R}^n$. Then for any Borel subset A of $\mathbb{R}^n$ with $\lambda^n(A) > 0$ and any $s > 0$ we have*

$$\lambda^n(A + V_s) \ge 1 - \lambda^n(A)^{-1}e^{-s},$$

where

$$V_s = \{x \in \mathbb{R}^n : \sum \min(|x_i|, x_i^2) \le 36s\}.$$

In the next part of the paper we will need some additional definitions. We will say that a measure μ on $I\!\!R$ is symmetric unimodal if it has a density with respect to the Lebesgue measure, which is symmetric and nonincreasing on $[0,\infty)$. A nonnegative Borel measure μ on $I\!\!R^n$ will be called logconcave if

$$\mu(tA+(1-t)B) \geq \mu^t(A)\mu^{1-t}(B)$$

for any nonempty Borel sets A, B in $I\!\!R^n$ and $t \in (0,1)$. A real random variable will be called symmetric unimodal (logconcave) if its distribution is symmetric unimodal (logconcave).

By the results of Borell [1] products of logconcave measures are logconcave and nondegenerate measures on $I\!\!R$ are logconcave if and only if they have logconcave densities with respect to the Lebesgue measure. In particular any symmetric nondegenerate logconcave real r.v. has logconcave tails and is symmetric unimodal.

Proposition 1 *The following inequalities are satisfied*

$$\|(a_i)\|_{\mathcal{N},\lambda p} \leq \lambda \|(a_i)\|_{\mathcal{N},p} \text{ for } \lambda \geq 1, p > 0 \tag{4}$$

$$\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',\lambda p} \leq \lambda^2 \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} \text{ for } \lambda \geq 1, p > 0, \tag{5}$$

and

$$\sqrt{p}(\sum_{i>p}(a_i^*)^2)^{1/2} \leq \|(a_i)\|_{\mathcal{N},p} \leq pa_1^* + \sqrt{p}(\sum_i (a_i^*)^2)^{1/2}, \tag{6}$$

where (a_i^*) *is a nonincreasing rearrangement of the sequence* $(|a_i|)$.

Proof. Inequalities (4) and (5) follow easily from the observation that $\hat{N}_i(tx) \leq t\hat{N}_i(x)$ for any $t \in [0,1]$ and real number x. To prove (6) let us fix a sequence (b_i) such that $\sum_i \hat{N}_i(b_i) \leq p$ and let $J = \{i : b_i \geq 1\}$. Then since $\hat{N}_i(x) \geq x$ for $x \geq 1$ we have $\sum_{i\in J} a_i b_i \leq pa_1^*$ and since $\hat{N}_i(x) = x^2$ for $|x| \leq 1$ we get $\sum_{i\notin J} a_i b_i \leq \sqrt{p}(\sum_i (a_i^*)^2)^{1/2}$.

To prove the other inequality in (6) let $k = \lfloor p \rfloor + 1$, $A = (k(a_k^*)^2 + \sum_{i>k}(a_i^*)^2)^{1/2}$, $b_i = \operatorname{sgn}(a_i)\sqrt{p}a_k^*/A$ for $|a_i| \geq a_k^*$ and $b_i = \sqrt{p}a_i/A$ for $|a_i| \leq a_k^*$. Then $|b_i| \leq 1$, $\sum \hat{N}_i(b_i) = \sum b_i^2 = p$ and

$$\sum a_i b_i \geq \sqrt{p}A \geq \sqrt{p}(\sum_{i>p}(a_i^*)^2)^{1/2}.$$

Proposition 2 *For any random variable X_i with logconcave tails normalized as in (1) we have*

$$\frac{1}{2} < 1 - e^{-1} \leq E|X_i| \leq 1 \tag{7}$$

and

$$\frac{1}{2} < 2 - 4e^{-1} \leq E|X_i|^2 \leq 2. \tag{8}$$

Proof. By our normalization property (1) and the convexity of N_i we get $0 \leq N_i(t) \leq t$ for $t \in [0,1]$ and $N_i(t) \geq \max(0, k(t-1)+1)$ for some $k \geq 1$ and all $t \geq 0$. Proposition easily follows by integration by parts.

Lemma 1 *Let $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$ be symmetric probabilistic logconcave measures on $I\!R$ such that*

$$\forall_i \ \forall_{t>0} \ \mu_i([-t,t]) \leq \nu_i([-t,t]), \tag{9}$$

$\mu = \mu_1 \otimes \ldots \otimes \mu_n$ and $\nu = \nu_1 \otimes \ldots \otimes \nu_n$. Then for any convex symmetric Borel set K in $I\!R^n$ we have

$$\mu(K) \leq \nu(K).$$

Proof. It is enough to prove that for any symmetric logconcave measure on $I\!R^{n-1}$ and convex symmetric set K we have

$$\mu_1 \otimes \mu(K) \leq \nu_1 \otimes \mu(K).$$

Let for $t \in I\!R$, $K_t = \{x \in I\!R^{n-1} : (t,x) \in K\}$ and $f(t) = \mu(K_t)$. By the convexity of K we have for any $\lambda \in (0,1)$ and $s,t \in I\!R$ such that $K_t, K_s \neq \emptyset$

$$\lambda K_t + (1-\lambda) K_s \subset K_{\lambda t + (1-\lambda)s}.$$

Therefore f is logconcave on $I\!R$ and since it is also symmetric, it is nonincreasing on $[0,\infty)$. Hence approximating f by $\sum a_j I_{[-t_j,t_j]}$ we obtain from (9)

$$\mu_1 \otimes \mu(K) = \int_R f(t) d\mu_1(t) \leq \int_R f(t) d\nu_1(t) = \nu_1 \otimes \mu(K).$$

Lemma 2 *For all $t > 0$ the following inequality holds*

$$\gamma_1([-t,t]) = \frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-x^2/2}dx \geq e^{-2/t^2}.$$

Proof. Since for any $x > 0$, $e^{-x} + e^{-1/x} \leq \frac{1}{1+x} + \frac{1}{1+x^{-1}} = 1$, Lemma 1 follows from well known (and easy to check) estimate $\gamma_1([-t,t]) \geq 1 - e^{-t^2/2}$.

Lemma 3 *For any matrix $(a_{i,j})$ and $C \geq 2\sum_{i,j} a_{i,j}^2$ we have*

$$\gamma_n(\max_j |\sum_{i=1}^n a_{i,j}x_i| \leq 1, \sum_j |\sum_{i=1}^n a_{i,j}x_i|^2 \leq C) \geq \frac{1}{2}e^{-2\sum_{i,j} a_{i,j}^2}.$$

Proof. From the result of Khatri [4] and Sidak [7] we have

$$\gamma_n(\max_j |\sum_{i=1}^n a_{i,j}x_i| \leq 1, \sum_j |\sum_{i=1}^n a_{i,j}x_i|^2 \leq C)$$

$$\geq (\prod_j \gamma_n(|\sum_{i=1}^n a_{i,j}x_i| \leq 1))\gamma_n(\sum_j |\sum_{i=1}^n a_{i,j}x_i|^2 \leq C). \tag{10}$$

By Lemma 2 we have

$$\gamma_n(|\sum_{i=1}^n a_{i,j}x_i| \leq 1) = \gamma_1([-(\sum_{i=1}^n a_{i,j}^2)^{-1/2}, (\sum_{i=1}^n a_{i,j}^2)^{-1/2}]) \geq e^{-2\sum_{i=1}^n a_{i,j}^2}. \tag{11}$$

Since $E\sum_j |\sum_{i=1}^n a_{i,j}g_i|^2 = \sum_{i,j} a_{i,j}^2$, from Chebyshev's inequality we obtain

$$\gamma_n(\sum_j |\sum_{i=1}^n a_{i,j}x_i|^2 \leq C) = 1 - P(\sum_j |\sum_{i=1}^n a_{i,j}g_i|^2 > C) \geq \frac{1}{2}. \tag{12}$$

Lemma 3 follows from (10), (11) and (12).

Lemma 4 *Let $Y_1, \ldots, Y_n$ be symmetric unimodal real r.v.'s and $d_i = EY_i^2$. Then for any matrix $(b_{i,j})$ we have*

$$P(\max_j |\sum_{i=1}^n b_{i,j}Y_i| \leq 1, \sum_j |\sum_{i=1}^n b_{i,j}Y_i|^2 \leq 1 + 4\sum d_i b_{i,j}^2) \geq \frac{1}{4}e^{-4\sum_{i,j} d_i b_{i,j}^2}.$$

Proof. Let Y_i have the distribution μ_i with the density f_i and $\mu = \otimes_{i=1}^n \mu_i$. Since f_i are nonnegative, symmetric, nonincreasing on $[0,\infty)$ and μ_i are probability measures, there exist probability measures $m_1, \ldots, m_n$ on $\mathbb{R}$ such that for each i

$$f_i(x) = \int_0^\infty \frac{1}{2t} I_{[-t,t]}(x) dm_i(t).$$

We also have

$$\int_0^\infty t^2 dm_i(t) = 3\int_0^\infty \int_R x^2 \frac{1}{2t} I_{[-t,t]}(x) dx dm_i(t) = 3\int_R x^2 f_i(x) dx = 3d_i.$$

For any Borel set A in $\mathbb{R}^n$ we have

$$\mu(A) = \int_{R^n} \nu^n_{t_1,\ldots,t_n}(A) dm_1(t_1)\ldots dm_n(t_n), \tag{13}$$

where $\nu^n_{t_1,\ldots,t_n}$ denotes a uniform probability measure on $[-t_1,t_1] \times \ldots \times [-t_n,t_n]$. We will also write ν^n_t instead of $\nu^n_{t,\ldots,t}$.

From Lemma 1 it immediately follows that for any convex symmetric set K in $\mathbb{R}^n$, $\nu^n_{\sqrt{\pi/2}}(K) \geq \gamma_n(K)$. Hence by Lemma 3

$$\nu^n_{t_1,\ldots,t_n}(x \in \mathbb{R}^n : \max_j |\sum_{i=1}^n b_{i,j}x_i| \leq 1, \sum_j |\sum_{i=1}^n b_{i,j}x_i|^2 \leq C)$$

$$= \nu^n_{\sqrt{\pi/2}}(x \in \mathbb{R}^n : \max_j |\sum_{i=1}^n b_{i,j}t_i\sqrt{\frac{2}{\pi}}x_i| \leq 1, \sum_j |\sum_{i=1}^n b_{i,j}t_i\sqrt{\frac{2}{\pi}}x_i|^2 \leq C)$$

$$\geq \frac{1}{2} e^{-\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2} I_{\{\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2 \leq C\}}. \tag{14}$$

Since the function e^{-x} is convex we obtain

$$\int_{R^n} e^{-\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2} dm_1(t_1)\ldots dm_n(t_n) \geq \exp(-\int_{R^n} \frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2 dm_1(t_1)\ldots dm_n(t_n))$$

$$= \exp(-\frac{12}{\pi}\sum_{i,j} d_i b_{i,j}^2).$$

Using the above and the obvious estimate

$$\int_{R^n} e^{-\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2} I_{\{\frac{4}{\pi}\sum_{i,j} t_i^2 b_{i,j}^2 > C\}} dm_1(t_1)\ldots dm_n(t_n) \leq e^{-C}$$

we obtain by (13) and (14)

$$\mu(x \in I\!R^n : \max_j |\sum_{i=1}^n b_{i,j} x_i| \le 1, \sum_j |\sum_{i=1}^n b_{i,j} x_i|^2 \le 1 + 4 \sum d_i b_{i,j}^2)$$

$$\ge \frac{1}{2} e^{-4 \sum_{i,j} d_i b_{i,j}^2} - \frac{1}{2} e^{-1-4 \sum_{i,j} d_i b_{i,j}^2} \ge \frac{1}{4} e^{-4 \sum_{i,j} d_i b_{i,j}^2}.$$

Lemma 5 *Let $Y_1, \ldots, Y_n$ be symmetric unimodal r.v.'s such that $EY_i^2 \le 4$. Then for any $p > 0$*

$$P(\|(\sum_{i=1}^n a_{i,j} Y_i)\|_{\mathcal{N}',p} \le 10 \|(B_j)\|_{\mathcal{N}',p}) \ge \frac{1}{4} e^{-8p}.$$

Proof. For $p \le 1$, $\|(a_j)\|_{\mathcal{N}',p} = \sqrt{p} \|(a_j)\|_2$ and the lemma follows easily from Chebyshev's inequality. So we will assume that $p > 1$. Without loss of generality we may also assume that $B_1 \ge B_2 \ge \ldots$ and $\|(B_j)\|_{\mathcal{N}',p} = p$. Let

$$b_{i,j} = \begin{cases} a_{i,j}/B_j & \text{for } j \le p \\ a_{i,j} & \text{for } j > p \end{cases}$$

and $d_i = EY_i^2/4$. Then by (6) we get

$$\sum_{i,j} d_i b_{i,j}^2 \le \sum_{i,j} b_{i,j}^2 = \lfloor p \rfloor + \sum_{j>p} B_j^2 \le p + p^{-1} \|(B_j)\|_{\mathcal{N}',p}^2 \le 2p.$$

Moreover if $\max_j |\sum_i b_{i,j} y_i| \le 1$ and $\sum_j |\sum_i b_{i,j} y_i|^2 \le 1 + 4 \sum_{i,j} d_i b_{i,j}^2$ then by (6)

$$\|(\sum_{i=1}^n a_{i,j} y_i)\|_{\mathcal{N}',p} \le \|(\sum_{i=1}^n a_{i,j} y_i)_{j \le p}\|_{\mathcal{N}',p} + p \max_{j>p} |\sum_i a_{i,j} y_i| + \sqrt{p} (\sum_{j>p} |\sum_i a_{i,j} y_i|^2)^{1/2}$$

$$\le \|(B_j)\|_{\mathcal{N}',p} + p + \sqrt{p(8p+1)} \le 5p.$$

Hence by Lemma 4

$$P(\|(\sum_{i=1}^n a_{i,j} Y_i/2)\|_{\mathcal{N}',p} \le 5 \|(B_j)\|_{\mathcal{N}',p}) \ge \frac{1}{4} \exp(-4 \sum_{i,j} d_i b_{i,j}^2) \ge \frac{1}{4} e^{-8p}.$$

Corollary 3 *There exists* $C_3 < \infty$ *such that for any matrix* $(a_{i,j})$ *and* $p > 0$

$$E\|(\sum_{i=1}^{n} a_{i,j}X_i)\|_{\mathcal{N}',p} \leq C_3(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(B_j)\|_{\mathcal{N}',p}).$$

Proof. For $p < 1$ Corollary follows easily by (8), so we will asssume that $p \geq 1$. Suppose first that all X_i's are also unimodal. Then $P(|X_i| < t) \geq tP(|X_i| \leq 1) = t(1 - e^{-1})$ for $t \in [0, 1]$. So for all $t > 0$

$$N_i(t) = -\ln P(|X_i| \geq t) \geq (1 - e^{-1})t \geq \frac{t}{2}. \tag{15}$$

Let F_i be an odd function, whose restriction to $I\!R^+$ is the inverse of N_i. Then X_i has the distribution $F_i(\lambda)$, where λ is the same symmetric exponential measure as in Theorem 3. Let

$$A = \{x \in I\!R^n : \|(\sum_{i=1}^{n} a_{i,j}F_i(x_i))\|_{\mathcal{N}',p} \leq 10\|(B_j)\|_{\mathcal{N}',p}\}.$$

Then by Lemma 5 and (8)

$$\lambda^n(A) = P(\|(\sum_{i=1}^{n} a_{i,j}X_i)\|_{\mathcal{N}',p} \leq 10\|(B_j)\|_{\mathcal{N}',p}) \geq \frac{1}{4}e^{-8p}.$$

Let $s > 0$, $x = y + z$ with $y \in A$ and $z \in V_s$. Let $\Delta_i = F_i(x_i) - F_i(y_i)$, then by (15) and since N_i is convex we get $\Delta_i \leq 2\min(F_i(|z_i|), |z_i|)$. So for $36s \geq p$ we get

$$\|(\sum_{i=1}^{n} a_{i,j}\Delta_i)\|_{\mathcal{N}',p} \leq 2\sup\{\|(\sum_{i=1}^{n} a_{i,j}b_i)\|_{\mathcal{N}',p} : \sum \hat{N}_i(b_i) \leq 36s\}$$

$$\leq \frac{72s}{p}\sup\{\|(\sum_{i=1}^{n} a_{i,j}b_i)\|_{\mathcal{N}',p} : \sum \hat{N}_i(b_i) \leq p\} = \frac{72s}{p}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}.$$

Hence for $36s \geq p$

$$\|(\sum_{i=1}^{n} a_{i,j}F_i(x_i))\|_{\mathcal{N}',p} \leq 10\|(B_j)\|_{\mathcal{N}',p} + \frac{72s}{p}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}.$$

So by Theorem 3

$$P(\|(\sum_{i=1}^n a_{i,j}X_i)\|_{\mathcal{N}',p} > 10\|(B_j)\|_{\mathcal{N}',p} + \frac{72s}{p}\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p})$$

$$\leq 1 - \lambda^n(A + V_s) \leq 4e^{8p-s}.$$

Integrating by parts we therefore get for any $s_0 \geq p/36$

$$E\|(\sum_{i=1}^n a_{i,j}X_i)\|_{\mathcal{N}',p} \leq 10\|(B_j)\|_{\mathcal{N}',p} + \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}(\frac{72s_0}{p} + \frac{288}{p}\int_{s_0}^\infty e^{8p-x}dx).$$

Choosing $s_0 = 9p$ we get

$$E\|(\sum_{i=1}^n a_{i,j}X_i)\|_{\mathcal{N}',p} \leq \tilde{C}_3(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(B_j)\|_{\mathcal{N}',p}).$$

Now let X_i be arbitrary r.v.'s with logconcave tails, satisfying the normalization property (1). Let $\tilde{X}_i$ be a r.v. with the density $c_ie^{-N_i(|x|)}$, where $c_i = (\int_R e^{-N_i(|x|)}dx)^{-1}$. Let $\alpha_i = \inf\{t > 0 : P(|t\tilde{X}_i| \geq 1) \geq e^{-1}\}$, $Y_i = \alpha_i X_i$,

$$\tilde{N}_i(t) = -\ln P(|Y_i| \geq t) = -\ln P(|\alpha_i\tilde{X}_i| \geq t)$$

and

$$M_i(t) = \begin{cases} \tilde{N}_i(|t|) & \text{for } |t| > 1 \\ t^2 & \text{for } |t| \leq 1 \end{cases}.$$

Functions $\tilde{N}_i$ are convex, satisfy the normalization property (1) and variables Y_i are unimodal, so by the first part of this proof

$$E\|(\sum_{i=1}^n a_{i,j}Y_i)\|_{\mathcal{N}',p} \leq \tilde{C}_3(\|(a_{i,j})\|_{\tilde{\mathcal{N}},\mathcal{N}',p} + \|(B_j)\|_{\mathcal{N}',p}), \tag{16}$$

where

$$\|(a_{i,j})\|_{\tilde{\mathcal{N}},\mathcal{N}',p} = \sup\{\sum a_{i,j}b_ic_j : \sum M_i(b_i) \leq p, \sum \hat{N}_j'(c_j) \leq p\}.$$

Let us notice that

$$c_i^{-1} \leq 2(1 + \int_1^\infty e^{-x}dx) < 3$$

and

$$c_i^{-1} \geq 2\int_0^1 e^{-x}dx > 1.$$

Hence

$$P(|\tilde{X}_i| \geq t) = 2c_i \int_t^\infty e^{-N_i(x)}dx \leq 2\int_t^\infty e^{-xN_i(t)/t}dx = \frac{2t}{N_i(t)}e^{-N_i(t)},$$

so for $t \geq 2$ we obtain

$$P(|\tilde{X}_i| \geq t) \leq 2e^{-N_i(t)} \leq 2e^{-2N_i(t/2)} \leq e^{-N_i(t/2)}. \tag{17}$$

We also get

$$P(|\tilde{X}_i| \geq t) \geq \frac{2}{3}\int_t^\infty e^{-N_i(x)}dx \geq te^{-N_i(5t/2)}. \tag{18}$$

From (17) and the normalization property (1) we get that $\alpha_i \geq 1/2$. Since $P(|\tilde{X}_i| \leq t) \leq 2c_i t \leq 2t$ we obtain that $\alpha_i \leq 4$. Therefore we get by (18) for $t \geq 5/2$

$$P(|X_i| \geq t) \leq P(|5\tilde{X}_i/2| \geq t) \leq P(|5\alpha_i\tilde{X}_i| \geq t) = P(|5Y_i| \geq t).$$

So by the contraction principle

$$E\|(\sum_{i=1}^n a_{i,j}X_iI_{\{|X_i|\geq 5/2\}})\|_{\mathcal{N}',p} \leq 5E\|(\sum_{i=1}^n a_{i,j}Y_i)\|_{\mathcal{N}',p}.$$

Also by the contraction principle, since by (7) $E|Y_i| \geq 1/2$

$$E\|(\sum_{i=1}^n a_{i,j}X_iI_{\{|X_i|\leq 5/2\}})\|_{\mathcal{N}',p} \leq \frac{5}{2}E\|(\sum_{i=1}^n a_{i,j}\varepsilon_i)\|_{\mathcal{N}',p} \leq 5E\|(\sum_{i=1}^n a_{i,j}Y_i)\|_{\mathcal{N}',p}.$$

Therefore

$$E\|(\sum_{i=1}^n a_{i,j}X_i)\|_{\mathcal{N}',p} \leq 10E\|(\sum_{i=1}^n a_{i,j}Y_i)\|_{\mathcal{N}',p}. \tag{19}$$

Let $t \geq 8$, then by (17) we have

$$P(|Y_i| \geq t) = P(|\alpha_i\tilde{X}_i| \geq t) \leq P(|\tilde{X}_i| \geq t/4) \leq e^{-N_i(t/8)}.$$

Hence $\tilde{N}_i(t) \geq N_i(t/8)$ for $t \geq 8$, so $M_i(t) \geq \hat{N}_i(t/8)$ for all t and

$$\|(a_{i,j})\|_{\tilde{\mathcal{N}},\mathcal{N}',p} \leq 8\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}. \tag{20}$$

(16), (19) and (20) complete the proof.

Proof of Theorem 1. First we estimate $\|X\|_p$ from below. By the Jensen inequality and symmetry of X_i we have

$$\|X\|_p \geq \|\sum_i X_i E|\sum_j a_{i,j} X_j^{'}|\|_p.$$

But by (3) and (8) we get

$$E|\sum_j a_{i,j} X_j^{'}| \geq C_{2,1}^{-1} \|\sum_j a_{i,j} X_j^{'}\|_2 \geq C_{2,1}^{-1} (\sum_j \frac{1}{2} a_{i,j}^2)^{1/2} = \tilde{C}^{-1} A_i.$$

Hence by Theorem 2

$$\|X\|_p \geq \tilde{C}^{-1} \|\sum_i A_i X_i\|_p \geq (\tilde{C} C_1)^{-1} \|(A_i)\|_{\mathcal{N},p}. \tag{21}$$

In a similar way we prove that

$$\|X\|_p \geq (\tilde{C} C_1)^{-1} \|(B_j)\|_{\mathcal{N}',p}. \tag{22}$$

By Theorem 2 we also have that for any (c_j) with $\sum_j \hat{N}_j^{'}(c_j) \leq p$ we have

$$\|X\|_p \geq C_1^{-1} (E|\sum_{i,j} a_{i,j} c_j X_i|^p)^{1/p} \geq C_1^{-2} \|(\sum_j a_{i,j} c_j)\|_{\mathcal{N},p}.$$

Taking the supremum over all such sequences (c_j) we get

$$\|X\|_p \geq C_1^{-2} \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p}. \tag{23}$$

(21), (22) and (23) complete the proof of this part of Theorem 1.

To prove the estimation from above let us first notice that $X_i = Y_i + Z_i$ for some symmetric random variables Y_i and Z_i such that

$$P(|Y_i| \geq t) = e^{-\tilde{N}_i(t)}, \text{ where } \tilde{N}_i(t) = \begin{cases} t & \text{for } t \leq 1 \\ N_i(t) & \text{for } t > 1 \end{cases}$$

and $|Z_i| \leq 1$ a.e., we will also assume that the Y_i are independent and so are the Z_i. In the same way we split $X_i^{'} = Y_i^{'} + Z_i^{'}$. By the contraction principle and since by (7) $E|Y_i|, E|Y_i^{'}| \geq 1/2$,

$$\|\sum a_{i,j} Z_i Y_j^{'}\|_p \leq \|\sum a_{i,j} \varepsilon_i Y_j^{'}\|_p \leq 2\|\sum a_{i,j} Y_i Y_j^{'}\|_p$$

and

$$\|\sum a_{i,j}Z_iZ_j^{'}\|_p \leq \|\sum a_{i,j}\varepsilon_i Z_j^{'}\|_p \leq 2\|\sum a_{i,j}Y_iZ_j^{'}\|_p \leq 4\|\sum a_{i,j}Y_iY_j^{'}\|_p.$$

Hence

$$\|X\|_p \leq \|\sum a_{i,j}Z_iZ_j^{'}\|_p + \|\sum a_{i,j}Z_iY_j^{'}\|_p + \|\sum a_{i,j}Y_iZ_j^{'}\|_p + \|\sum a_{i,j}Y_iY_j^{'}\|_p$$
$$\leq 9\|\sum a_{i,j}Y_iY_j^{'}\|_p.$$

So it is enough to prove that for $p \geq 1$

$$\|\sum a_{i,j}Y_iY_j^{'}\|_p \leq C(\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(A_i)\|_{\mathcal{N},p} + \|(B_j)\|_{\mathcal{N}',p}).$$

To simplify the notation let

$$m_p = \|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} + \|(A_i)\|_{\mathcal{N},p} + \|(B_j)\|_{\mathcal{N}',p}.$$

Then $m_p \geq \|(A_i)\|_{\mathcal{N},1} + \|(B_j)\|_{\mathcal{N}',1} = 2(\sum_{i,j} a_{i,j}^2)^{1/2}$. Since by (8), $EY_i^2, E(Y_j^{'})^2 \leq 2$, we get

$$P(|\sum a_{i,j}Y_iY_j^{'}| \geq 2m_p) \leq P(|\sum a_{i,j}Y_iY_j^{'}|^2 \geq 4E|\sum a_{i,j}Y_iY_j^{'}|^2) \leq \frac{1}{4}. \quad (24)$$

From Corollary 3 we have

$$P(\|(\sum_{i=1}^n a_{i,j}Y_i)\|_{\mathcal{N}',p} \geq 4C_3m_p) \leq \frac{1}{4} \quad (25)$$

and

$$P(\|(\sum_{j=1}^n a_{i,j}Y_j^{'})\|_{\mathcal{N},p} \geq 4C_3m_p) \leq \frac{1}{4}. \quad (26)$$

Let $F_i, F_j^{'} : R \to R$ be odd functions, whose restrictions to $\mathbb{R}^+$ are the inverses of $\tilde{N}_i$, $\tilde{N}_j^{'}$ respectively. Then $Y_i, Y_j^{'}$ have distributions $F_i(\lambda)$ and $F_j^{'}(\lambda)$, where λ is the same symmetric exponential measure as in Theorem 3. Let

$$A = \{(x, x^{'}) \in \mathbb{R}^{2n} : |\sum a_{i,j}F_i(x_i)F_j^{'}(x_j^{'})| \leq 2m_p,$$
$$\|(\sum_{i=1}^n a_{i,j}F_i(x_i))\|_{\mathcal{N}',p}, \|(\sum_{j=1}^n a_{i,j}F_j^{'}(x_j^{'}))\|_{\mathcal{N},p} \leq 4C_3m_p\},$$

then by (24), (25) and (26)

$$\lambda^{2n}(A) \geq \frac{1}{4}.$$

Hence by Theorem 3 for $s > 0$

$$\lambda^{2n}(A + V_s) \geq 1 - 4e^{-s}.$$

Let $(x, x^{'}) = (y + z, y^{'} + z^{'})$ with $(y, y^{'}) \in A, (z, z^{'}) \in V_s$. Let $\Delta_i = F_i(x_i) - F_i(y_i)$ and $\Delta_j^{'} = F_j^{'}(x_j^{'}) - F_j^{'}(y_j^{'})$. By the convexity of $\tilde{N}_i$ we have $|\Delta_i| \leq 2F_i(|x_i - y_i|)$, therefore

$$\sum_i \hat{N}_i(\Delta_i/2) \leq \sum_i \hat{N}_i(F_i(|z_i|)) = \sum_i \min(|z_i|, z_i^2) \leq 36s.$$

By the similar reason we have

$$\sum_j \hat{N}_j^{'}(\Delta_j^{'}/2) \leq 36s.$$

Hence

$$|\sum_{i,j} a_{i,j}\Delta_i F_j^{'}(y_j^{'})| \leq 2\sup\{\sum_i(\sum_j a_{i,j}F_j^{'}(y_j^{'}))b_i : \sum_i \hat{N}_i(b_i) \leq 36s\}$$

$$\leq 2\|(\sum_j a_{i,j}F_j^{'}(y_j^{'}))\|_{\mathcal{N},36s}.$$

Therefore by (4) we have

$$|\sum_{i,j} a_{i,j}\Delta_i F_j^{'}(y_j^{'})| \leq \frac{72s}{p}\|(\sum_j a_{i,j}F_j^{'}(y_j^{'}))\|_{\mathcal{N},p} \leq \frac{288s}{p}C_3 m_p \text{ for } 36s \geq p. \tag{27}$$

In a similar way we prove

$$|\sum_{i,j} a_{i,j}F_i(y_i)\Delta_j^{'}| \leq \frac{288s}{p}C_3 m_p \text{ for } 36s \geq p. \tag{28}$$

We also have

$$|\sum_{i,j} a_{i,j}\Delta_i\Delta_j^{'}| \leq 4\sup\{\sum_{i,j} a_{i,j}b_i c_j : \sum_i \hat{N}_i(b_i), \sum_j \hat{N}_j^{'}(c_j) \leq 36s\}$$

$$= 4\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',36s}.$$

So by (5) we get for $36s \geq p$

$$|\sum_{i,j} a_{i,j}\Delta_i\Delta_j^{'}| \leq 4(\frac{36s}{p})^2\|(a_{i,j})\|_{\mathcal{N},\mathcal{N}',p} \leq 4(\frac{36s}{p})^2 m_p. \tag{29}$$

By (27), (28) and (29) and the definition of the set A we get

$$|\sum_{i,j} a_{i,j}F_i(x_i)F_j^{'}(x_j^{'})| \leq (2 + 2\frac{288s}{p}C_3 + 4(\frac{36s}{p})^2)m_p \leq C_4(\frac{s}{p})^2 m_p \text{ for } s \geq p.$$

Therefore for $s \geq p$

$$P(|\sum_{i,j} a_{i,j}Y_iY_j^{'}| > C_4(\frac{s}{p})^2 m_p)$$

$$= \lambda^{2n}((x,x') \in \mathbb{R}^{2n} : |\sum_{i,j} a_{i,j}F_i(x_i)F_j^{'}(x_j^{'})| > C_4(\frac{s}{p})^2 m_p)$$

$$\leq 1 - \lambda^{2n}(A + V_s) \leq 4e^{-s}.$$

Hence integrating by parts

$$E|\sum_{i,j} a_{i,j}Y_iY_j^{'}|^p \leq C_4^p m_p^p(1 + 4\int_1^\infty x^{p-1}e^{-p\sqrt{x}}dx) \leq C^p m_p^p.$$

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