HADAMARD PRODUCTS AND MOMENTS OF RANDOM VECTORS

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ABSTRACT. We derive sharp comparison inequalities between weak and strong moments of random vectors in arbitrary finite dimensional Banach space. As an application, we show that the p-summing constant of any finite dimensional Banach space is upper bounded, up to a universal constant, by the p-summing constant of the Hilbert space of the same dimension. We also apply our result to the concentration of measure theory for log-concave random vectors in Euclidean spaces.

Keywords. Moments of random vectors, Hadamard products, p-summing operators, log-concave measures, concentration of measure.

1. INTRODUCTION

The study of moments of random variables is an essential issue of probability theory, one of the reasons being the fact that tail estimates for random variables are related to bounds for their moments via the Markov inequality. In probabilistic convex geometry and concentration of measure theory one is often interested in bounding pth moments of random vectors. To be more precise, the pth *strong moment* of a random vector X in $\mathbb{R}^n$ with respect to a given norm structure $(\mathbb{R}^n, \|\cdot\|)$ is defined as $M_p(X) = (\mathbb{E}\|X\|^p)^{1/p}$. Another related quantity is the so-called *weak pth moment* defined as $\sigma_p(X) = \sup_{\|t\|_*\le 1}(\mathbb{E}|\langle t, X\rangle|^p)^{1/p}$, where $\|\cdot\|_*$ denotes the dual norm. Weak moments are usually much easier to compute or estimate, and so comparison inequalities between weak and strong moments are of interest in convex geometry, see e.g. [6].

Clearly the strong moment always dominates the weak moment. In this article we derive a sharp (up to a universal constant) reverse bound.

Theorem 1. *For any n-dimensional random vector X and any nonempty set T in $\mathbb{R}^n$ we have*

$$\left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^p\right)^{1/p} \le 2\sqrt{e}\sqrt{\frac{n+p}{p}}\sup_{t\in T}(\mathbb{E}|\langle t, X\rangle|^p)^{1/p} \qquad \text{for } p\ge 2. \tag{1}$$

In particular, for any normed space $(\mathbb{R}^n, \|\cdot\|)$ we have

$$(\mathbb{E}\|X\|^p)^{1/p} \le 2\sqrt{e}\sqrt{\frac{n+p}{p}}\sup_{\|t\|_*\le 1}(\mathbb{E}|\langle t, X\rangle|^p)^{1/p} \qquad \text{for } p\ge 2.$$

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To see that (1) is optimal up to a universal constant it suffices to take any rotationally invariant random vector X and T to be a centered Euclidean ball. In this case $M_p(X)/\sigma_p(X) = (\mathbb{E}|U_1|^p)^{-1/p} \sim \sqrt{\frac{n+p}{p}}$, where U_1 is the first coordinate of a random vector uniformly distributed on the unit sphere in $\mathbb{R}^n$.

Obtaining upper bounds for strong moments in terms of weak moments turns out to be very challenging. As an example let us mention the Paouris inequality $M_p(X) \leq C(M_1(X) + \sigma_p(X))$ valid for the standard Euclidean norm and arbitrary log-concave random vector X in $\mathbb{R}^n$, see [19] and [1] (see also [15] for an extension of this result to a larger class of norms). Here and in the sequel C denotes an absolute constant, whose value may change at each occurrence. Usually, to derive such bounds one applies the concentration of measure theory [17] or the chaining method [23]. What is crucial in these reasonings is the regularity of the random vector X and/or the special form of the norm. Our proof uses a totally different linear algebra method of Hadamard powers inspired by the proof of the so-called Welch bound (see [24]) given in [7].

We now use our result to derive bounds on p-summing norms of operators between Banach spaces. The theory of absolutely summing operators is an important part of the modern Banach space theory and found numerous powerful applications in harmonic analysis, approximation theory, probability theory and operator theory [8]. Recall that a linear operator Φ between Banach spaces F_1 and F_2 is p-summing if there exists a constant $\alpha < \infty$, such that for all $x_1, \ldots x_m \in F_1$ one has

$$\left(\sum_{i=1}^{m} \|\Phi x_i\|^p\right)^{1/p} \leq \alpha \sup_{x^* \in F_1^*, \|x^*\| \leq 1} \left(\sum_{i=1}^{m} |x^*(x_i)|^p\right)^{1/p}.$$

The smallest constant α in the above inequality is called the p-summing norm of Φ and will be denoted by $\pi_p(\Phi)$. For a Banach space F by the p-summing constant $\pi_p(F)$ of F we mean the p-summing constant of the identity map of F. It is well known that $\pi_p(F) < \infty$ if and only if F is finite dimensional. Moreover $\pi_2(F) = \sqrt{\dim F}$ (see [10, Theorem 16.12.3]). The p-summing constants of certain finite dimensional spaces were computed by Gordon in [11]. In particular he showed that

$$\pi_p(\ell_2^n) = (\mathbb{E}|U_1|^p)^{-1/p} \sim \sqrt{\frac{n+p}{p}}.$$

An immediate consequence of our main result is that, up to a universal constant, Hilbert spaces have the largest p-summing constant among all Banach spaces of fixed dimension.

Corollary 2. *For any finite dimensional Banach space F and $p \geq 2$ we have*

$$\pi_p(F) \leq 2\sqrt{e}\sqrt{\frac{\mathrm{dim} F + p}{p}} \leq C\pi_p(\ell_2^{\mathrm{dim} F}).$$

Indeed, it suffices to apply Theorem 1 for random vectors uniformly distributed on finite subsets of F and T the unit ball in F^*. We ask the following question.

Question. Is it true that for any finite dimensional Banach space F and $p \geq 2$ we have $\pi_p(F) \leq \pi_p(\ell_2^{\mathrm{dim} F})$? Equivalently, is it true that the best constant in Theorem 1 is equal to $(\mathbb{E}|U_1|^p)^{-1/p}$?

Using the ideal properties of p-summing operators (see [8]) we get a bound for p-summing constants of finite rank operators.

Corollary 3. *Let Φ be a finite rank linear operator between Banach spaces F_1 and F_2. Then the p-summing constant of Φ satisfies*

$$\pi_p(\Phi) \leq 2\sqrt{e}\sqrt{\frac{\mathrm{rk}(\Phi)+p}{p}}\|\Phi\|.$$

For the proof it suffices to consider the decomposition $\Phi = i \circ I \circ \tilde{\Phi}$ where $\tilde{\Phi}$ is Φ considered as an operator between F_1 and $\Phi(F_1)$, I is the identity map on $\Phi(F_1)$ and i is the embedding of $\Phi(F_1)$ into F_2. The ideal property gives $\pi_p(\Phi) \leq \|i\|\pi_p(I)\|\tilde{\Phi}\|$, which combined with Corollary 2 gives the desired bound.

Inequality (1) has been conjectured (with a universal constant in place of $2\sqrt{e}$) in the language of the so-called $\mathcal{Z}_p$ bodies by the first named author in [13] (see Problem 1 therein), where certain special cases have been studied. Inequality (1) arose as a result of investigating optimal concentration of measure inequalities. The discussion of this application will be given is Section 3. Section 2 is devoted to the proof of Theorem 1.

2. Proof of the main result

We shall need the following lemma, which can be found in [2], Lemma 9.2 (see also [20] for a version for Gram matrices).

Lemma 4. *Suppose $A = (a_{ij})$ is a $k \times l$ matrix of rank at most n. Let m be a positive integer. Then the Hadamard power $A^{\circ m} = (a_{ij}^m)$ has rank at most $\binom{n+m-1}{m}$.*

Proof. Since the space spanned by the column vectors of A has dimension at most n, there exist vectors $v^{(1)}, \ldots, v^{(n)}$ in $\mathbb{R}^k$ such that every column $a = (a_1, \ldots, a_k)$ of A can be written as a linear combination of these vectors, that is $a = \sum_{s=1}^n v^{(s)}\lambda_s$ for some real numbers λ_s. Restricting this equality to the ith coordinate gives $a_i = \sum_{s=1}^n v_i^{(s)}\lambda_s$. If we now raise this equality to the mth power, we obtain

$$a_i^m = \sum_{s_1,s_2,\ldots,s_m=1}^{n} v_i^{(s_1)}v_i^{(s_2)} \cdot \ldots \cdot v_i^{(s_m)}\lambda_{s_1}\lambda_{s_2} \cdot \ldots \cdot \lambda_{s_m}.$$

Thus, every column a^m of $A^{\circ m}$ can be written as a linear combination of the vectors $(v_i^{(s_1)}v_i^{(s_2)} \cdot \ldots \cdot v_i^{(s_m)})_{i=1,\ldots,k}$. Since these vectors are invariant under permuting the numbers s_i, we can assume that $1 \leq s_1 \leq s_2 \leq \ldots \leq s_m \leq n$. The number of such sequences is precisely $\binom{n+m-1}{m}$. □

Corollary 5. *Let k, l, m and n be positive integers. For any vectors $t_1, \ldots, t_k$ and $x_1, \ldots, x_l$ in $\mathbb{R}^n$ there exist vectors $\tilde{t_1}, \ldots, \tilde{t_k}$ and $\tilde{x}_1, \ldots, \tilde{x}_l$ in $\mathbb{R}^N$ with $N = \binom{n+m-1}{m}$ such that $\langle t_i, x_j\rangle^m = \langle \tilde{t_i}, \tilde{x_j}\rangle$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$.*

Proof. The $k \times l$ matrix $A = (\langle t_i, x_j\rangle)$ is of the form $A = TX$ where T is a $k \times n$ matrix whose ith row is the vector t_i and X is a $n \times l$ matrix whose jth column is the vector x_j. Thus A is a matrix of a composition of two linear maps $\mathbb{R}^l \to \mathbb{R}^n$ and $\mathbb{R}^n \to \mathbb{R}^k$ and thus has rank at most n. According to Lemma 4 the rank of $A^{\circ m} = (\langle t_i, x_j\rangle^m)$ is at most N. From the rank factorization theorem the matrix $A^{\circ m}$ can be written as a product $\tilde{T}\tilde{X}$, where $\tilde{T}$ is a $k \times N$ matrix and $\tilde{X}$ is a $N \times l$ matrix. It suffices to take $\tilde{t_i}$ to be ith row of $\tilde{T}$ and $\tilde{x_j}$ to be the jth column of $\tilde{X}$. $\square$

We now consider the case $p = 2$ of Theorem 1.

Lemma 6. *For any n-dimensional random vector X and any nonempty set T in $\mathbb{R}^n$ we have*

$$\mathbb{E} \sup_{t \in T} |\langle t, X\rangle|^2 \leq \ n \ \sup_{t\in T} \mathbb{E}|\langle t, X\rangle|^2.$$

Proof. By an approximation argument without loss of generality we may assume that X is bounded and has a nondegenerate covariance matrix C. We can also assume that X is symmetric (if not multiply X by an independent symmetric ± 1 random variable). Let $\alpha := \sup_{t\in T} \mathbb{E}|\langle t, X\rangle|^2 = \sup_{t\in T}\langle Ct, t\rangle$. Then

$$\begin{aligned}\mathbb{E} \sup_{t \in T} |\langle t, X\rangle|^2 &\leq \mathbb{E} \sup\{|\langle s, X\rangle|^2 \colon \langle Cs, s\rangle \leq \alpha\} \\ &= \mathbb{E} \sup\{|\langle C^{1/2}s, C^{-1/2}X\rangle|^2 \colon |C^{1/2}s|^2 \leq \alpha\} = \alpha \mathbb{E}|C^{-1/2}X|^2 = \alpha n.\end{aligned}$$

$\square$

The crucial case of Theorem 1 is the case of p being an even integer.

Proposition 7. *Suppose m is a positive integer. Then for any n-dimensional random vector X and any nonempty set T in $\mathbb{R}^n$ we have*

$$\mathbb{E} \sup_{t \in T} |\langle t, X\rangle|^{2m} \leq \ \binom{n+m-1}{m} \ \sup_{t\in T} \mathbb{E}|\langle t, X\rangle|^{2m}.$$

Proof. By an easy approximation argument we can assume that $T = \{t_1, \ldots, t_k\}$ is a finite subset of $\mathbb{R}^n$ and X is uniformly distributed on a finite number of points $x_1, \ldots, x_l$ in $\mathbb{R}^n$. In this case the above inequality reads

$$\sum_{j=1}^{l} \sup_{1\leq i\leq k} |\left\langle t_i, x_j\right\rangle|^{2m} \leq \binom{n+m-1}{m} \sup_{1\leq i\leq k} \sum_{j=1}^{l} |\left\langle t_i, x_j\right\rangle|^{2m}. \tag{2}$$

From Corollary 5 there exist vectors $\tilde{t_1}, \ldots, \tilde{t_k}$ and $\tilde{x_1}, \ldots, \tilde{x_l}$ in $\mathbb{R}^N$ with $N = \binom{n+m-1}{m}$ such that $\langle t_i, x_j\rangle^m = \left\langle \tilde{t_i}, \tilde{x_j}\right\rangle$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$. From Lemma 6 used to the set $T' = \{\tilde{t_1}, \ldots, \tilde{t_k}\} \subset \mathbb{R}^N$ and a random variable X' uniformly distributed in $\{\tilde{x_1}, \ldots, \tilde{x_l}\} \subset \mathbb{R}^N$ we have

$$\sum_{j=1}^{l} \sup_{1\leq i\leq k} |\left\langle \tilde{t_i}, \tilde{x_j}\right\rangle|^{2} \leq N \sup_{1\leq i\leq k} \sum_{j=1}^{l} |\left\langle \tilde{t_i}, \tilde{x_j}\right\rangle|^{2}.$$

This is precisely (2). $\square$

Our next lemma shows that the best constant $C_{n,p}$ in the inequality (1) is a monotone function of p.

Lemma 8. *Let $p > 0$ and let $C_{n,p}$ be the best constant such that for any n-dimensional random vector X and any nonempty set T in $\mathbb{R}^n$ we have*

$$\left(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^p\right)^{1/p} \leq\ C_{n,p}\ \sup_{t\in T}\left(\mathbb{E}|\langle t, X\rangle|^p\right)^{1/p}. \tag{3}$$

Then the function $p \to C_{n,p}$ is non-increasing.

Proof. Suppose $0 < p < q$ and let μ_X be the law of X. By rescaling X one can assume that $\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^{q-p} = 1$. This allows us to define a new random vector Y whose law μ_Y is given by

$$\mu_Y(A) = \int_A \sup_{t\in T}|\langle t, x\rangle|^{q-p}\mathrm{d}\mu_X(x).$$

Thus, by Hölder's inequality

$$\begin{aligned}\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^q &= \mathbb{E}\sup_{t\in T}|\langle t, Y\rangle|^p \leq C_{n,p}^p\sup_{t\in T}\mathbb{E}|\langle t, Y\rangle|^p \\ &= C_{n,p}^p\sup_{t\in T}\mathbb{E}\left[|\langle t, X\rangle|^p\sup_{s\in T}|\langle s, X\rangle|^{q-p}\right] \\ &\leq C_{n,p}^p\sup_{t\in T}\left(\mathbb{E}|\langle t, X\rangle|^q\right)^{\frac{p}{q}}\left(\mathbb{E}\sup_{s\in T}|\langle s, X\rangle|^q\right)^{\frac{q-p}{q}}.\end{aligned}$$

Rearranging gives the inequality

$$(\mathbb{E}\sup_{t\in T}|\langle t, X\rangle|^q)^{1/q} \leq C_{n,p}\sup_{t\in T}(\mathbb{E}|\langle t, X\rangle|^q)^{1/q}$$

and thus $C_{n,q} \leq C_{n,p}$. □

We are now ready to give a proof of the main result.

Proof of Theorem 1. Let m be a positive integer such that $2m \leq p < 2m+2$. By Lemma 8 and Proposition 7 we get that the best constant $C_{n,p}$ in (3) satisfies

$$C_{n,p}^2 \leq C_{n,2m}^2 \leq \binom{n+m-1}{m}^{1/m} \leq \frac{e(n+m-1)}{m} \leq e\frac{n+p/2}{p/4} \leq 4e\frac{n+p}{p}.$$

□

3. Optimal concentration of measure

Let us notice that by homogeneity one can always assume that the supremum on the right hand side of (1) is one. Then by enlarging the set T we may assume that T is the set of all vectors t satisfying $\mathbb{E}|\langle t, X\rangle|^p \leq 1$. Thus, inequality (1) may be equivalently stated as

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p\right)^{1/p} \leq 2\sqrt{e}\sqrt{\frac{n+p}{p}}, \tag{4}$$

where

$$\|s\|_{\mathcal{Z}_p(X)} := \sup\{|\langle t, s\rangle|:\ \mathbb{E}|\langle t, X\rangle|^p \leq 1\}.$$

This has been conjectured (with a universal constant in place of $2\sqrt{e}$) by the first named author in [13] (see Problem 1 therein), where the result in the case of unconditional random vectors was obtained. Thus Theorem 1 positively resolves this conjecture.

The motivation behind inequality (4) was the study of the concentration of measure phenomenon. Let ν be a symmetric exponential measure with parameter 1, i.e. the measure on the real line with the density $\frac{1}{2}e^{-|x|}$. Talagrand [22] showed that the product measure ν^n satisfies the following two-sided concentration inequality

$$\nu^n(A) \geq \frac{1}{2} \implies 1 - \nu^n(A + C\sqrt{p}B_2^n + CpB_1^n) \leq e^{-p}(1 - \nu^n(A)), \quad p > 0.$$

This is a remarkably strong concentration result implying, for example, the celebrated concentration of measure phenomenon for the canonical Gaussian measure γ_n on $\mathbb{R}^n$:

$$\gamma_n(A) \geq \frac{1}{2} \implies 1 - \gamma_n(A + C\sqrt{p}B_2^n) \leq e^{-p}(1 - \gamma_n(A)), \quad p > 0,$$

discovered (in the sharp isoperimetric form) by Sudakov and Tsirelson in [21], and independently by Borell in [4].

It is not hard to check that $\mathcal{Z}_p(\nu^n) \sim \sqrt{p}B_2^n + pB_1^n$ and $\mathcal{Z}_p(\gamma_n) \sim \sqrt{p}B_2^n$ for $p \geq 2$, where for a probability measure μ on $\mathbb{R}^n$ and a random vector X distributed according to μ we set

$$\mathcal{Z}_p(\mu) = \mathcal{Z}_p(X) = \{t \in \mathbb{R}^n \colon \ \|t\|_{\mathcal{Z}_p(X)} \leq 1\}.$$

In the context of convex geometry it is natural to ask if similar inequalities hold for other log-concave measures, namely measures with densities of the form e^{-V}, where $V : \mathbb{R}^n \to (-\infty, \infty]$ is convex. An easy observation from [16] shows that if μ is a symmetric log-concave probability measure and K is a convex set such that for any halfspace A satisfying $\mu(A) \geq \frac{1}{2}$ we have $\mu(A+K) \geq 1 - \frac{1}{2}e^{-p}$, then necesarily $K \supset \frac{1}{C}\mathcal{Z}_p$. This motivates the following definition proposed in [16].

Definition 9. We say that a measure μ satisfies the *optimal concentration inequality with constant* β (CI(β) in short) if for any Borel set A we have

$$\mu(A) \geq \frac{1}{2} \quad \implies \quad 1 - \mu(A + \beta\mathcal{Z}_p(\mu)) \leq e^{-p}(1 - \mu(A)), \qquad p \geq 2.$$

All centered product log-concave measures satisfy the optimal concentration inequality with a universal constant β ([16]). A natural conjecture (discussed in [16, 14]) states that this is true also for nonproduct measures. However, one has to mention that it would imply (see Corollary 3.14. in [16]) the celebrated KLS conjecture (proposed in [12] as a tool for proving efficiency of certain Metropolis type algorithms for computing volumes of convex sets) on the boundedness of the Cheeger constant for isotropic log-concave measures . It was shown in [14] that every log-concave measure on $\mathbb{R}^n$ satisfies $CI(c\sqrt{n})$ with a universal constant c. The following corollary improves upon this bound.

Corollary 10. *Every centered log-concave probability measure on* $\mathbb{R}^n$ *satisfies the optimal concentration inequality with constant* $\beta \leq Cn^{5/12}$.

Proof. We follow the ideas expained after the proof of Proposition 7 in [14], but instead of Eldan's bound on the Cheeger constant [9] we use the recent result of Lee and Vempala [18].

Since the concentration inequality is invariant with respect to linear transformations we may assume that μ is isotropic. Then in particular $\mathcal{Z}_p(\mu) \supset \mathcal{Z}_2(\mu) = B_2^n$.

By Proposition 2.7 in [16] CI(β) may be equivalently stated as

$$\mu(A + \beta \mathcal{Z}_p(\mu)) \geq \min\left\{\frac{1}{2}, e^p \mu(A)\right\}, \qquad p \geq 2.$$

To show the above bound with $\beta = Cn^{5/12}$ we consider two cases.

i) If $2 \leq p \leq n^{1/6}$ then

$$\mu(A + Cn^{5/12}\mathcal{Z}_p(\mu)) \geq \mu(A + Cn^{1/4}pB_2^n) \geq \min\left\{\frac{1}{2}, e^p\mu(A)\right\},$$

where the last inequality follows by the Lee-Vempala [18] $Cn^{1/4}$ bound on the Cheeger constant.

ii) If $p \geq \max\{2, n^{1/6}\}$ then observe first that inequality (4) yields

$$\mu\left(2e^{3/2}\sqrt{\frac{n+p}{p}}\mathcal{Z}_p(\mu)\right) \geq 1 - e^{-p}.$$

Therefore Lemma 9 in [14] gives

$$\mu(A + Cn^{5/12}\mathcal{Z}_p(\mu)) \geq \mu\left(A + 18e^{3/2}\sqrt{\frac{n+p}{p}}\mathcal{Z}_p(\mu)\right) \geq \min\left\{\frac{1}{2}, e^p\mu(A)\right\}.$$

□

In general one cannot reverse bound (4) for $2 \ll p \ll n$. Indeed, let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$ and $\mathbb{P}(X = \pm e_i) = 1/(2n)$ for $1 \leq i \leq n$. Then for $s, t \in \mathbb{R}^n$,

$$\mathbb{E}|\langle t, X\rangle|^p = \frac{1}{n}\sum_{i=1}^n |t_i|^p, \quad \|s\|_{\mathcal{Z}_p(X)} = n^{1/p}\left(\sum_{i=1}^n |s_i|^q\right)^{1/q},$$

where q denotes the Hölder dual to p. Thus for $2 \ll p \ll n$,

$$\left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^p\right)^{1/p} = n^{1/p} \ll \sqrt{\frac{n+p}{p}}.$$

Corollary 6 in [13] states that for unconditional log-concave vectors in $\mathbb{R}^n$ and $2 \leq p \leq n$ we have

$$\frac{1}{C}\sqrt{\frac{n}{p}} \leq \mathbb{E}\|X\|_{\mathcal{Z}_p(X)} \leq \left(\mathbb{E}\|X\|_{\mathcal{Z}_p(X)}^{\sqrt{np}}\right)^{1/\sqrt{np}} \leq C\sqrt{\frac{n}{p}}.$$

We do not know whether such bounds hold without unconditionality assumptions. We are only able to show the following weaker lower bound. Recall that the isotropic constant of a centered logconcave vector X with density g is defined as

$$L_X := (\sup_x g(x))^{1/n}(\det \operatorname{Cov}(X))^{1/(2n)}.$$

It is known that for all log-concave vectors one has $L_X \geq 1/C$. The famous open conjecture, due to Bourgain [5], states that $L_X \leq C$ (see [3, 6] for more details and discussions of known upper bounds).

Proposition 11. *For any centered log-concave n-dimensional random vector with non-degenerate covariance matrix we have*

$$\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} \geq \frac{1}{CL_X}\sqrt{\frac{n}{p}} \quad \text{for } 1 \leq p \leq n,$$

where L_X is the isotropic constant of X.

Proof. Since the assertion is linearly invariant we may and will assume that X is isotropic, i.e. it has the identity covariance matrix. The density of X is then bounded by L_X^n, hence

$$\begin{aligned}\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)} \leq t\sqrt{n/p}\right) &= \mathbb{P}\left(X \in t\sqrt{n/p}\mathcal{Z}_p(X)\right) \\ &\leq L_X^n \mathrm{vol}\left(t\sqrt{n/p}\mathcal{Z}_p(X)\right) \leq (C_1 t L_X)^n,\end{aligned}$$

where the last estimate follows by the Paouris [19] bound on the volume of $\mathcal{Z}_p$-bodies (see also [6, Theorem 5.1.17]).

Thus

$$\mathbb{E}\|X\|_{\mathcal{Z}_p(X)} \geq \frac{1}{2C_1L_X}\sqrt{\frac{n}{p}}\mathbb{P}\left(\|X\|_{\mathcal{Z}_p(X)} > \frac{1}{2C_1L_X}\sqrt{\frac{n}{p}}\right) \geq \frac{1}{4C_1L_X}\sqrt{\frac{n}{p}}.$$

□

In the last years it was showed that various constants related to the n-dimensional log-concave measures (isotropic constant, Cheeger constant, thin-shell constant) are bounded by $Cn^{1/4}$. We think that the same should be true for the CI constant.

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