

# THE LIL FOR $U$ -STATISTICS IN HILBERT SPACES

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We give necessary and sufficient conditions for the (bounded) law of the iterated logarithm for  $U$ -statistics in Hilbert spaces. As a tool we also develop moment and tail estimates for canonical Hilbert-space valued  $U$ -statistics of arbitrary order, which are of independent interest.

**1. Introduction.** In the last two decades we have witnessed a rapid development in the asymptotic theory of  $U$ -statistics, boosted by the introduction of the so called 'decoupling' techniques (see [5–7]), which allow to treat  $U$ -statistics conditionally as sums of independent random variables. This approach yielded better understanding of  $U$ -statistics versions of the classical limit theorems of probability. Necessary and sufficient conditions were found for the strong law of large numbers [17], the central limit theorem [10, 19] and the law of the iterated logarithm [2, 11]. Also some sharp exponential inequalities for canonical  $U$ -statistics have been found [1, 8, 14]. Analysis of the aforementioned results shows an interesting phenomenon. Namely, the natural counterparts of the necessary and sufficient conditions for sums of i.i.d. random variables ( $U$ -statistics of degree 1), remain sufficient for  $U$ -statistics of arbitrary degree, but with an exception for the CLT, they cease to be necessary. The correct conditions turn out to be much more involved and are expressed for instance in terms of convergence of some series (LLN) or as growth conditions for some functions (LIL).

A natural problem is an extension of the above results to the infinite-dimensional setting. There has been some progress in this direction, and partial answers have been found, usually under the assumption on the geometrical structure of the space in which the values of a  $U$ -statistic are taken. In general however the picture is far from being complete and the necessary and sufficient conditions are known only in the case of the CLT for Hilbert space valued  $U$ -statistics (see [5, 10] for the proof of sufficiency in type 2 spaces and necessity in cotype 2 spaces respectively).

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In this article we generalize to separable Hilbert spaces the results from [2] on necessary and sufficient conditions for the LIL for real valued  $U$ -statistics. The conditions are expressed only in terms of the  $U$ -statistic kernel and the distribution of the underlying i.i.d. sequence and can be also considered a generalization of results from [13], where the LIL for i.i.d. sums in Hilbert spaces was characterized. We consider only the bounded version of the LIL and do not give the exact value of the lim sup nor determine the limiting set. Except for the classical case of sums of i.i.d. random variables, the problem of finding the lim sup is at the moment open even in the one dimensional case (see [3, 5, 15] for some partial results) and the problem of the geometry of the limiting set and the compact LIL is solved only under suboptimal integrability conditions [3].

The organization of the paper is as follows. First, in Section 3 we prove sharp exponential inequalities for canonical  $U$ -statistics, which generalize the results of [1, 8] for the real-valued case. Then, after recalling some basic facts about the LIL we give necessary and sufficient condition for the LIL for decoupled, canonical  $U$ -statistics (Theorem 2). The quite involved proof is given in the two subsequent sections. Finally we conclude with our main result (Theorem 4), which gives a characterization of the LIL for undecoupled  $U$ -statistics and follows quite easily from Theorem 2 and the one dimensional result.

**2. Notation.** For an integer  $d$ , let  $(X_i)_{i \in \mathbb{N}}$ ,  $(X_i^{(k)})_{i \in \mathbb{N}, 1 \leq k \leq d}$  be independent random variables with values in a Polish space  $\Sigma$ , equipped with the Borel  $\sigma$ -field  $\mathcal{F}$ . Let also  $(\varepsilon_i)_{i \in \mathbb{N}}$ ,  $(\varepsilon_i^{(k)})_{i \in \mathbb{N}, 1 \leq k \leq d}$  be independent Rademacher variables, independent of  $(X_i)_{i \in \mathbb{N}}$ ,  $(X_i^{(k)})_{i \in \mathbb{N}, 1 \leq k \leq d}$ .

Consider moreover measurable functions  $h_i: \Sigma^d \rightarrow H$ , where  $(H, |\cdot|)$  is a separable Hilbert space (we will denote both the norm in  $H$  and the absolute value of a real number by  $|\cdot|$ , the context will however prevent ambiguity).

To shorten the notation, we will use the following convention. For  $\mathbf{i} = (i_1, \dots, i_d) \in \{1, \dots, n\}^d$  we will write  $\mathbf{X}_{\mathbf{i}}$  (resp.  $\mathbf{X}_{\mathbf{i}}^{\text{dec}}$ ) for  $(X_{i_1}, \dots, X_{i_d})$ , (resp.  $(X_{i_1}^{(1)}, \dots, X_{i_d}^{(d)})$ ) and  $\varepsilon_{\mathbf{i}}$  (resp.  $\varepsilon_{\mathbf{i}}^{\text{dec}}$ ) for the product  $\varepsilon_{i_1} \cdot \dots \cdot \varepsilon_{i_d}$  (resp.  $\varepsilon_{i_1}^{(1)} \cdot \dots \cdot \varepsilon_{i_d}^{(d)}$ ), the notation being thus slightly inconsistent, which however should not lead to a misunderstanding. The  $U$ -statistics will therefore be

denoted

$$\begin{aligned} \sum_{\mathbf{i} \in I_n^d} h_{\mathbf{i}}(\mathbf{X}_{\mathbf{i}}) & \quad (\text{an undecoupled } U\text{-statistic}) \\ \sum_{|\mathbf{i}| \leq n} h_{\mathbf{i}}(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) & \quad (\text{a decoupled } U\text{-statistic}) \\ \sum_{\mathbf{i} \in I_n^d} \epsilon_{\mathbf{i}} h_{\mathbf{i}}(\mathbf{X}_{\mathbf{i}}) & \quad (\text{an undecoupled randomized } U\text{-statistic}) \\ \sum_{|\mathbf{i}| \leq n} \epsilon_{\mathbf{i}}^{\text{dec}} h_{\mathbf{i}}(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) & \quad (\text{a decoupled randomized } U\text{-statistic}), \end{aligned}$$

where

$$\begin{aligned} |\mathbf{i}| &= \max_{k=1, \dots, d} i_k, \\ I_n^d &= \{\mathbf{i}: |\mathbf{i}| \leq n, i_j \neq i_k \text{ for } j \neq k\}. \end{aligned}$$

Since in this notation  $\{1, \dots, d\} = I_d^1$  we will write

$$I_d = \{1, 2, \dots, d\}.$$

Throughout the article we will write  $L_d, L$  to denote constants depending only on  $d$  and universal constants respectively. In all those cases the values of a constant may differ at each occurrence.

For  $I \subseteq I_d$ , we will write  $\mathbb{E}_I$  to denote integration with respect to variables  $(X_i^{(j)})_{i \in \mathbb{N}, j \in I}$ .

**3. Moment inequalities for  $U$ -statistics in Hilbert space .** In this section we will present sharp moment and tail inequalities for Hilbert space valued  $U$ -statistics, which in the sequel will constitute an important ingredient in the analysis of the LIL. These estimates are a natural generalization of inequalities for real valued  $U$ -statistics presented in [1].

Let us first introduce some definitions.

**DEFINITION 1.** *For a nonempty, finite set  $I$  let  $\mathcal{P}_I$  be the family consisting of all partitions  $\mathcal{J} = \{J_1, \dots, J_k\}$  of  $I$  into nonempty, pairwise disjoint subsets. Let us also define for  $\mathcal{J}$  as above  $\text{deg}(\mathcal{J}) = k$ . Additionally let  $\mathcal{P}_{\emptyset} = \{\emptyset\}$  with  $\text{deg}(\emptyset) = 0$ .*

DEFINITION 2. For a nonempty set  $I \subseteq I_d$  consider  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_I$ . For an array  $(h_i)_{i \in I_n^d}$  of  $H$ -valued kernels and fixed value of  $i_{I^c}$ , define

$$\|(h_i)_{i_I}\|_{\mathcal{J}} = \sup \left\{ \left| \sum_{i_I} \mathbb{E}_I [h_i(\mathbf{X}_i^{\text{dec}}) \prod_{j=1}^{\deg(\mathcal{J})} f_{i_{J_j}}^{(j)}(\mathbf{X}_{i_{J_j}}^{\text{dec}})] \right| : f_{i_{J_j}}^{(j)} : \Sigma^{J_j} \rightarrow \mathbb{R} \right. \\ \left. \mathbb{E} \sum_{i_{J_j}} |f_{i_{J_j}}^{(j)}(\mathbf{X}_{i_{J_j}}^{\text{dec}})|^2 \leq 1 \text{ for } j = 1, \dots, \deg(\mathcal{J}) \right\}.$$

Let moreover  $\|(h_i)_{i_{\emptyset}}\|_{\emptyset} = |h_i|$ .

*Remark.* It is worth mentioning that for  $I = I_d$ ,  $\|\cdot\|_{\mathcal{J}}$  is a deterministic norm, whereas for  $I \subsetneq I_d$  it is a random variable, depending on  $\mathbf{X}_{i_{I^c}}^{\text{dec}}$ .

Quantities given by the above definition suffice to obtain precise moment estimates for real valued  $U$ -statistics. However, to bound the moments of  $U$ -statistics with values in general Hilbert spaces, we will need to introduce one more definition.

DEFINITION 3. For nonempty sets  $K \subseteq I \subseteq I_d$  consider  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I \setminus K}$ . For an array  $(h_i)_{i \in I_n^d}$  of  $H$ -valued kernels and fixed value of  $i_{I^c}$ , define

$$\|(h_i)_{i_I}\|_{K, \mathcal{J}} = \sup \left\{ \left| \sum_{i_I} \mathbb{E}_I [\langle h_i(\mathbf{X}_i^{\text{dec}}), g_{i_K}(\mathbf{X}_{i_K}^{\text{dec}}) \rangle \prod_{j=1}^{\deg(\mathcal{J})} f_{i_{J_j}}^{(j)}(\mathbf{X}_{i_{J_j}}^{\text{dec}})] \right| : \right. \\ \left. f_{i_{J_j}}^{(j)} : \Sigma^{J_j} \rightarrow \mathbb{R}, g_{i_K} : \Sigma^K \rightarrow H, \mathbb{E} \sum_{i_K} |g_{i_K}(\mathbf{X}_{i_K}^{\text{dec}})|^2 \leq 1 \right. \\ \left. \mathbb{E} \sum_{i_{J_j}} |f_{i_{J_j}}^{(j)}(\mathbf{X}_{i_{J_j}}^{\text{dec}})|^2 \leq 1 \text{ for } j = 1, \dots, \deg(\mathcal{J}) \right\}.$$

*Remark.* One can see that the only difference between the above definition and Definition 2 is that the latter distinguishes one set of coordinates and allows functions corresponding to this set to take values in  $H$ . Moreover, since the norm in  $H$  satisfies  $|\cdot| = \sup_{|\phi| \leq 1} \langle \phi, \cdot \rangle$ , we can treat Definition 2 as a counterpart of Definition 3 for  $K = \emptyset$ . We will use this convention to simplify the statements of the subsequent theorems. Thus, from now on, we will write

$$\|\cdot\|_{\emptyset, \mathcal{J}} := \|\cdot\|_{\mathcal{J}}.$$

*Example.* For  $d = 2$  and  $I = \{1, 2\}$ , the above definition gives

$$\begin{aligned}
\|(h_{ij}(X_i, Y_j))_{i,j}\|_{\emptyset, \{\{1,2\}\}} &= \sup \left\{ \left| \mathbb{E} \sum_{ij} h_{ij}(X_i, Y_j) f_{ij}(X_i, Y_j) \right| : \right. \\
&\quad \left. \mathbb{E} \sum_{ij} f(X_i, Y_j)^2 \leq 1 \right\} \\
&= \sup_{\phi \in H, |\phi| \leq 1} \sqrt{\mathbb{E} \sum_{ij} \langle \phi, h_{ij}(X_i, Y_j) \rangle^2}, \\
\|(h_{ij}(X_i, Y_j))_{i,j}\|_{\emptyset, \{\{1\}, \{2\}\}} &= \sup \left\{ \left| \mathbb{E} \sum_{ij} h_{ij}(X_i, Y_j) f_i(X_i) g_j(Y_j) \right| : \right. \\
&\quad \left. \sum_i \mathbb{E} f(X_i)^2, \sum_j \mathbb{E} g(Y_j)^2 \leq 1 \right\}, \\
\|(h_{ij}(X_i, Y_j))_{i,j}\|_{\{1\}, \{\{2\}\}} &= \sup \left\{ \mathbb{E} \sum_{ij} \langle f_i(X_i), h_{ij}(X_i, Y_j) \rangle g_j(Y_j) : \right. \\
&\quad \left. \mathbb{E} \sum_i |f(X_i)|^2, \mathbb{E} \sum_j g(Y_j)^2 \leq 1 \right\}, \\
\|(h_{ij}(X_i, Y_j))_{i,j}\|_{\{1,2\}, \emptyset} &= \sup \left\{ \mathbb{E} \sum_{ij} \langle f_{ij}(X_i, Y_j), h_{ij}(X_i, Y_j) \rangle : \right. \\
&\quad \left. \mathbb{E} \sum_{ij} |f(X_i, Y_j)|^2 \leq 1 \right\} \\
&= \sqrt{\sum_{i,j} \mathbb{E} |h_{ij}(X_i, Y_j)|^2}.
\end{aligned}$$

We can now present the main result of this section.

**THEOREM 1.** *For any array of  $H$ -valued, completely degenerate kernels  $(h_i)_i$  and any  $p \geq 2$ , we have*

$$\mathbb{E} \left| \sum_i h(\mathbf{X}_i^{\text{dec}}) \right|^p \leq L_d^p \left( \sum_{K \subseteq I \subseteq I_d} \sum_{\mathcal{J} \in \mathcal{P}_{I \setminus K}} p^{p(\#I^c + \deg \mathcal{J}/2)} \mathbb{E}_{I^c} \max_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p \right).$$

The proof of the above theorem proceeds along the lines of arguments presented in [1, 8]. In particular we will need the following moment estimates for suprema of empirical processes [8].

**LEMMA 1** ([8, Proposition 3.1], see also [4, Theorem 12]). *Let  $X_1, \dots, X_n$  be independent random variables with values in  $(\Sigma, \mathcal{F})$  and  $\mathcal{T}$  be a countable class of measurable real functions on  $\Sigma$ , such that for all  $f \in \mathcal{T}$  and  $i \in$*

$I_n$ ,  $\mathbb{E}f(X_i) = 0$  and  $\mathbb{E}f(X_i)^2 < \infty$ . Consider the random variable  $S := \sup_{f \in \mathcal{T}} |\sum_i f(X_i)|$ . Then for all  $p \geq 1$ ,

$$\mathbb{E}S^p \leq L^p \left[ (\mathbb{E}S)^p + p^{p/2} \sigma^p + p^p \mathbb{E} \max_i \sup_{f \in \mathcal{T}} |f(X_i)|^p \right],$$

where

$$\sigma^2 = \sup_{f \in \mathcal{T}} \sum_i \mathbb{E}f(X_i)^2.$$

We will also need the following technical lemma.

LEMMA 2 (Lemma 5 in [1]). For  $\alpha > 0$  and arbitrary nonnegative kernels  $g_i: \Sigma^d \rightarrow \mathbb{R}_+$  and  $p > 1$  we have

$$p^{\alpha p} \sum_i \mathbb{E}g_i^p \leq L_d^p p^{\alpha d} \left[ p^{\alpha p} \mathbb{E} \max_i g_i^p + \sum_{I \subsetneq \{1, \dots, d\}} p^{\#I p} \mathbb{E}_I \max_{i_I} \left( \sum_{i_{I^c}} \mathbb{E}_{I^c} g_i \right)^p \right].$$

Before stating the next lemma, let us introduce some more definitions, concerning  $\mathcal{J}$ -norms of deterministic matrices

DEFINITION 4. Let  $(a_i)_{i \in I_n^d}$  be a  $d$ -indexed array of real numbers. For  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I_d}$  define

$$\|(a_i)_i\|_{\mathcal{J}} = \sup \left\{ \sum_i a_i x_{i_{J_1}}^{(1)} \cdots x_{i_{J_k}}^{(k)} : \sum_{i_{J_1}} (x_{i_{J_1}}^{(1)})^2 \leq 1, \dots, \sum_{i_{J_k}} (x_{i_{J_k}}^{(k)})^2 \leq 1 \right\}.$$

We will also need

DEFINITION 5. For  $i \in \mathbb{N}^{d-1} \times I_n$  let  $a_i: \Sigma \rightarrow \mathbb{R}$  be measurable functions and  $Z_1, \dots, Z_n$  be independent random variables with values in  $\Sigma$ . For a partition  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I_d}$  ( $d \in J_1$ ), let us define

$$\|(a_i(Z_{i_d}))_i\|_{\mathcal{J}} = \sup \left\{ \sqrt{\sum_{i_{J_1}} \mathbb{E} \left( \sum_{i_{I_d \setminus J_1}} a_i(Z_{i_d}) x_{i_{J_2}}^{(2)} \cdots x_{i_{J_k}}^{(k)} \right)^2} : \sum_{i_{J_2}} (x_{i_{J_2}}^{(2)})^2 \leq 1, \dots, \sum_{i_{J_k}} (x_{i_{J_k}}^{(k)})^2 \leq 1 \right\}.$$

*Remark.* All the definitions of norms presented so far, seem quite similar and indeed they can be all interpreted as injective tensor-product norms on proper spaces. We have decided to introduce them separately by explicit formulas, because this form appears in our applications.

The next lemma is crucial for obtaining moment inequalities for canonical real-valued  $U$ -statistics of order greater than 2. In the context of  $U$ -statistics in Hilbert spaces we will need it already for  $d = 2$ .

LEMMA 3 (Theorem 5 in [1]). *Let  $Z_1, \dots, Z_n$  be independent random variables with values in  $(\Sigma, \mathcal{F})$ . For  $i \in \mathbb{N}^{d-1} \times I_n$  let  $a_i: \Sigma \rightarrow \mathbb{R}$  be measurable functions, such that  $\mathbb{E}_Z a_i(Z_{i_d}) = 0$ . Then, for all  $p \geq 2$  we have*

$$\begin{aligned} \mathbb{E} \left\| \left( \sum_{i_d} a_i(Z_{i_d}) \right)_{i_{I_{d-1}}} \right\| &\leq L_d \sum_{\mathcal{J} \in \mathcal{P}_{I_d}} p^{(1+\deg(\mathcal{J})-d)/2} \|(a_i(Z_{i_d}))_{i_{\mathcal{J}}}\| \\ &\quad + L_d \sum_{\mathcal{J} \in \mathcal{P}_{I_{d-1}}} p^{1+(1+\deg(\mathcal{J})-d)/2} \sqrt{\mathbb{E} \max_{i_d} \|(a_i(Z_{i_d}))_{i_{I_{d-1}}}\|_{\mathcal{J}}^2}, \end{aligned}$$

where  $\|\cdot\|$  denotes the norm of a  $(d-1)$ -indexed matrix, regarded as a  $(d-1)$ -linear operator on  $(\ell_2)^{d-1}$  (thus the  $\|\cdot\|_{\{1\}\dots\{d-1\}}$ -norm in our notation).

To prove Theorem 1, we will need to adapt the above lemma to be able to bound the  $(K, \mathcal{J})$ -norms of sums of independent kernels.

DEFINITION 6. *We define a partial order  $\prec$  on  $\mathcal{P}_I$  as*

$$\mathcal{I} \prec \mathcal{J}$$

*if and only if for all  $I \in \mathcal{I}$ , there exists  $J \in \mathcal{J}$ , such that  $I \subseteq J$ .*

LEMMA 4. *Assume that  $\sum_i \mathbb{E} |h_i(\mathbf{X}_i^{\text{dec}})|^2 < \infty$ . Then for any  $K \subseteq I_{d-1}$  and  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I_{d-1} \setminus K}$  and all  $p \geq 2$ ,*

$$\begin{aligned} (1) \quad &\mathbb{E}_d \left\| \left( \sum_{i_d} h_i(\mathbf{X}_i^{\text{dec}}) \right)_{i_{I_{d-1}}} \right\|_{K, \mathcal{J}} \\ &\leq L_d \left( \sum_{\substack{K \subseteq L \subseteq I_d, \mathcal{K} \in \mathcal{P}_{I_d \setminus L}: \\ \mathcal{J} \cup \{K, \{d\}\} \prec \mathcal{K} \cup \{L\}}} p^{(\deg \mathcal{K} - \deg \mathcal{J})/2} \|(h_i)_{i_{I_d}}\|_{L, \mathcal{K}} \right. \\ &\quad \left. + \sum_{\substack{K \subseteq L \subseteq I_{d-1}, \mathcal{K} \in \mathcal{P}_{I_{d-1} \setminus L}: \\ \mathcal{J} \cup \{K\} \prec \mathcal{K} \cup \{L\}}} p^{1+(\deg \mathcal{K} - \deg \mathcal{J})/2} \sqrt{\mathbb{E}_d \max_{i_d} \|(h_i)_{i_{I_{d-1}}}\|_{L, \mathcal{K}}^2} \right). \end{aligned}$$

*Remark.* In the above lemma we slightly abuse the notation, by identifying for  $K = \emptyset$  the partition  $\{\emptyset\} \cup \mathcal{J}$  with  $\mathcal{J}$ .

Given Lemma 3, the proof of Lemma 4 is not complicated, the main idea is just a change of basis, however due to complicated notation it is quite difficult to write it directly. We find it more convenient to write the proof in terms of tensor products of Hilbert spaces.

Let us begin with a classical fact.

LEMMA 5. *Let  $H$  be a separable Hilbert space and  $X$  a  $\Sigma$ -valued random variable. Then  $H \otimes L^2(X) \simeq L^2(X, H)$ , where  $L^2(X, H)$  is the space of square integrable random variables of the form  $f(X)$ ,  $f: \Sigma \rightarrow H$ -measurable. With the above identification, for  $h \in H$ ,  $f(X) \in L^2(X)$ , we have  $h \otimes f(X) = hf(X) \in L^2(X, H)$ .*

PROOF OF LEMMA 4. To avoid problems with notation, which would lengthen an intuitively easy proof, we will omit some technical details, related to obvious identification of some tensor product of Hilbert spaces (in the spirit of Lemma 5). Similarly, when considering linear functionals on a space, which can be written as a tensor product in several ways, we will switch to the most convenient notation, without further explanations.

Let

$$H_0 = H \otimes \left[ \otimes_{l \in K} \left( \oplus_{i=1}^n L^2(X_i^{(l)}) \right) \right] \simeq \oplus_{|i_K| \leq n} L^2(\mathbf{X}_{i_K}^{\text{dec}}, H)$$

and, for  $j = 1, \dots, k$ ,

$$H_i = \otimes_{l \in J_j} \left( \oplus_{i=1}^n L^2(X_i^{(l)}) \right) \simeq \oplus_{|i_{J_j}| \leq n} L^2(\mathbf{X}_{i_{J_j}}^{\text{dec}}).$$

In the case  $K = \emptyset$ , we have (using the common convention for empty products)  $H_0 \simeq H$ .

For  $i_d = 1, \dots, n$  and fixed value of  $X_{i_d}^{(d)}$ , let  $A_{i_d}$  be a linear functional on  $\tilde{H} = \oplus_{|i_{I_{d-1}}| \leq n} L^2(\mathbf{X}_{i_{I_{d-1}}}^{\text{dec}}, H) \simeq \otimes_{j=0}^k H_k$ , given by  $(h_i(\mathbf{X}_i^{\text{dec}}))_{|i_{I_{d-1}}| \leq n} \in \tilde{H}$ , with the formula

$$\begin{aligned} A_{i_d}((g_{i_{I_{d-1}}}(\mathbf{X}_{i_{I_{d-1}}}^{\text{dec}}))_{i_{I_{d-1}}}) &= \langle (g_{i_{I_{d-1}}}(\mathbf{X}_{i_{I_{d-1}}}^{\text{dec}}))_{i_{I_{d-1}}}, (h_i(\mathbf{X}_i^{\text{dec}}))_{i_{I_{d-1}}} \rangle_{\tilde{H}} \\ &= \sum_{|i_{I_{d-1}}| \leq n} \mathbb{E}_{\{1, \dots, d-1\}} \langle g_{i_{I_{d-1}}}(\mathbf{X}_{i_{I_{d-1}}}^{\text{dec}}), h_i(\mathbf{X}_i^{\text{dec}}) \rangle_H. \end{aligned}$$

As functions of  $X_{i_d}^{(d)}$ ,  $A_{i_d} = A_{i_d}(X_{i_d}^{(d)})$  are independent random linear functionals. Thus they determine also random  $(k+1)$ -linear functionals on  $\oplus_{j=0}^k H_k$ , given by

$$(h_0, h_1, \dots, h_k) \mapsto A_{i_d}(h_0 \otimes h_1 \otimes \dots \otimes h_k).$$



If we denote by  $\|\cdot\|$  the norm of a  $(k+1)$ -linear functional, the left hand-side of (1), can be written as

$$\mathbb{E} \left\| \sum_{i_d=1}^n A_{i_d}(X_{i_d}^{(d)}) \right\|.$$

Moreover, denoting by  $\|A_{i_d}\|_{HS}$  the norm of  $A_{i_d}$  seen as a linear operator on  $\otimes_{j=0}^k H_j$  (by analogy with the Hilbert-Schmidt norm of a matrix), we have

$$\sum_{i_d=1}^n \mathbb{E} \|A_{i_d}(X_{i_d}^{(d)})\|_{HS}^2 = \|(h_i)_i\|_{L_d, \emptyset}^2 < \infty,$$

so the sequence  $A_{i_d}(X_{i_d}^{(d)})$ , determines a linear functional  $A$  on  $\tilde{H} \otimes [\oplus_{i_d=1}^n L^2(X_{i_d}^{(d)})] \simeq \oplus_{|i| \leq n} L^2(\mathbf{X}_i^{\text{dec}}, H) \simeq \oplus_{i_d=1}^n L^2(X_{i_d}^{(d)}, \tilde{H})$ , given by the formula

$$A(g_1(X_1^{(d)}), \dots, g_n(X_n^{(d)})) = \sum_{i_d=1}^n \mathbb{E}[A_{i_d}(X_{i_d}^{(d)})(g_{i_d}(X_{i_d}^{(d)}))].$$

It is easily seen, that if we interpret the domain of this functional as  $\oplus_{|i| \leq n} L^2(\mathbf{X}_i^{\text{dec}}, H)$ , then it corresponds to the multimatrix  $(h_i(\mathbf{X}_i^{\text{dec}}))_i$ .

Let us now introduce the following notation, consistent with the definition of  $\|\cdot\|_{\mathcal{I}}$ . If  $T$  is a linear functional on  $\otimes_{j=0}^m E_j$  for some Hilbert spaces  $E_j$ , and  $\mathcal{I} = \{L_1, \dots, L_r\} \in \mathcal{P}_{I_m \cup \{0\}}$ , then let  $\|T\|_{\mathcal{I}}$  denote the norm of  $T$  as a  $r$ -linear functional on  $\oplus_{i=1}^r [\otimes_{j \in L_i} E_j]$ , given by

$$(e_1, \dots, e_r) \mapsto T(e_1 \otimes \dots \otimes e_r).$$

Now, denoting  $H_{k+1} = \oplus_{i_d=1}^n L^2(X_{i_d}^{(d)})$ , we can apply the above definition to  $\tilde{H} \otimes [\oplus_{i_d=1}^n L^2(X_{i_d}^{(d)})] \simeq \otimes_{j=0}^{k+1} H_j$  and use Lemma 3 to obtain

$$\begin{aligned} \mathbb{E} \left\| \sum_{i_d=1}^n A_{i_d}(X_{i_d}^{(d)}) \right\| &\leq L_d \sum_{\mathcal{I} \in \mathcal{P}_{I_{k+1} \cup \{0\}}} p^{(1+\deg(\mathcal{I})-(k+2))/2} \|A\|_{\mathcal{I}} \\ (2) \quad &+ L_d \sum_{\mathcal{I} \in \mathcal{P}_{I_k \cup \{0\}}} p^{1+(1+\deg(\mathcal{I})-(k+2))/2} \sqrt{\mathbb{E} \max_{i_d} \|A_{i_d}(X_{i_d}^{(d)})\|_{\mathcal{I}}^2}. \end{aligned}$$

This inequality is just the statement of the Lemma, which follows from „associativity” of the tensor product and its „distributivity” with respect to the simple sum of Hilbert spaces. Indeed, denoting  $J_{k+1} = \{d\}$ , we have for  $0 \notin L_i$  and  $U = \bigcup_{j \in L_i} J_j$ ,

$$\otimes_{j \in L_i} H_j \simeq \otimes_{j \in L_i} \otimes_{l \in J_j} (\oplus_{s=1}^n L^2(X_s^{(l)})) \simeq \otimes_{l \in U} (\oplus_{s=1}^n L^2(X_s^{(l)})) \simeq \oplus_{|i_U| \leq n} L^2(\mathbf{X}_{i_U}^{\text{dec}}).$$

Similarly, if  $0 \in L_i$ ,

$$\begin{aligned} \otimes_{j \in L_i} H_j &\simeq [\oplus_{|i_K| \leq n} L^2(\mathbf{X}_{i_K}^{\text{dec}}, H)] \times [\otimes_{0 \neq j \in L_i} \otimes_{l \in J_j} (\oplus_{s=1}^n L^2(X_s^{(l)}))] \\ &\simeq \oplus_{|i_U| \leq n} L^2(\mathbf{X}_{i_U}^{\text{dec}}, H), \end{aligned}$$

where  $U = (\bigcup_{0 \neq j \in L_i} J_j) \cup K$ . Using the fact that for fixed  $X_{i_d}^{(d)}$ ,  $A_{i_d}$  corresponds to the multimatrix  $(h_i(\mathbf{X}_i^{\text{dec}}))_{|i_{I_{d-1}}| \leq n}$ , and  $A$  corresponds to  $(h_i(\mathbf{X}_i^{\text{dec}}))_{|i| \leq n}$ , we can see, that each summand  $\|\cdot\|_{\mathcal{I}}$  on the right hand side of (2) is equal to some summand  $\|\cdot\|_{L, \mathcal{K}}$  on the right hand side of (1). Informally speaking and abusing slightly the notation (in the case  $K = \emptyset$ ), we „merge” the elements of the partition  $\{\{d\}, J_1, \dots, J_k, K\}$  or  $\{J_1, \dots, J_k, K\}$  in a way described by the partition  $\mathcal{I}$ , thus obtaining the partition  $\{L\} \cup \mathcal{K}$ , where  $L$  is the set corresponding in the new partition to the set  $L_i \in \mathcal{I}$ , containing 0 (in particular, if  $K = \emptyset$  and  $\{0\} \in \mathcal{I}$ , then  $L = \emptyset$ ). Let us also notice, that  $\deg(\mathcal{I}) = \deg(\mathcal{K}) + 1$ , hence

$$1 + \deg(\mathcal{I}) - (k + 2) = \deg(\mathcal{K}) - \deg(\mathcal{J}),$$

which shows, that also the powers of  $p$  on the right hand sides of (1) and (2) are the same, completing the proof.  $\square$

PROOF OF THEOREM 1. For  $d = 1$ , the theorem is an obvious consequence of Lemma 1. Indeed, since  $|\cdot| = \sup_{|\phi| \leq 1} |\phi(\cdot)|$ , and we can restrict the supremum to a countable set of functionals, we have

$$\begin{aligned} \mathbb{E} \left| \sum_i h_i(X_i) \right|^p &\leq L^p \left( (\mathbb{E} \left| \sum_i h_i(X_i) \right|)^p + p^{p/2} \sup_{|\phi| \leq 1} \left( \sum_i \mathbb{E} \langle \phi, h_i(X_i) \rangle \right)^{p/2} \right) \\ &\quad + p^p \mathbb{E} \max_i |h_i(X_i)|^p. \end{aligned}$$

But  $\mathbb{E} \left| \sum_i h_i(X_i) \right| \leq \sqrt{\mathbb{E} \left| \sum_i h_i(X_i) \right|^2} = \sqrt{\sum_i \mathbb{E} |h_i(X_i)|^2} = \|(h_i)_i\|_{\{1\}, \emptyset}$  and we also have  $\sup_{|\phi| \leq 1} \left( \sum_i \mathbb{E} \langle \phi, h_i(X_i) \rangle \right)^{1/2} = \|(h_i)_i\|_{\emptyset, \{1\}}$  and  $\max_i |h_i(X_i)| = \max_i \|h_i\|_{\emptyset, \emptyset}$ .

We will now proceed by induction with respect to  $d$ . Assume that the theorem is true for all integers smaller than  $d \geq 2$  and denote  $\tilde{I}^c = I^c \setminus \{d\}$  for  $I \subseteq I_d$ . Then, applying it for fixed  $X_{i_d}^{(d)}$  to the array of functions  $(\sum_{i_d} h_i(x_1, \dots, x_{d-1}, X_{i_d}^{(d)}))_{i_{I_{d-1}}}$ , we get by the Fubini theorem

$$\begin{aligned} \mathbb{E} \left| \sum_i h_i(\mathbf{X}_i^{\text{dec}}) \right|^p &\leq L_{d-1}^p \left( \sum_{K \subseteq I \subseteq I_{d-1}} \sum_{\mathcal{J} \in \mathcal{P}_{I \setminus K}} p^{p(\#\tilde{I}^c + \deg \mathcal{J}/2)} \sum_{i_{\tilde{I}^c}} \mathbb{E}_{I^c} \left\| \left( \sum_{i_d} h_i \right)_{i_I} \right\|_{K, \mathcal{J}}^p \right), \end{aligned}$$

where we have replaced the maxima in  $i_{I^c}$  by sums (we can afford this apparent loss, since we will be able to fix it with Lemma 2). Now, from Lemma 1 (applied to  $\mathbb{E}_d$ ) it follows that

$$\begin{aligned} \mathbb{E}_d \left\| \sum_{i_d} h_i \right\|_{K, \mathcal{J}}^p &\leq L^p \left( (\mathbb{E}_d \left\| \sum_{i_d} h_i \right\|_{K, \mathcal{J}})^p + p^{p/2} \|(h_i)_{i_{I \cup \{d}\}}\|_{K, \mathcal{J} \cup \{\{d\}\}}^p \right. \\ &\quad \left. + p^p \sum_{i_d} \mathbb{E}_d \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p \right). \end{aligned}$$

Since  $\tilde{I}^c = (I \cup \{d\})^c$ ,  $\deg \mathcal{J} \cup \{\{d\}\} = \deg \mathcal{J} + 1$  and  $\#I^c = \#\tilde{I}^c + 1$ , combining the above inequalities gives

$$\begin{aligned} \mathbb{E} \left| \sum_i h_i(\mathbf{X}_i^{\text{dec}}) \right|^p &\leq L_d^p \left( \sum_{K \subseteq I \subseteq I_d} \sum_{\mathcal{J} \in \mathcal{P}_{I \setminus K}} p^{p(\#I^c + \deg \mathcal{J}/2)} \mathbb{E}_{I^c} \sum_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p \right. \\ &\quad \left. + \sum_{K \subseteq I \subseteq I_{d-1}} \sum_{\mathcal{J} \in \mathcal{P}_{I \setminus K}} p^{p(\#\tilde{I}^c + \deg \mathcal{J}/2)} \sum_{i_{\tilde{I}^c}} \mathbb{E}_{\tilde{I}^c} (\mathbb{E}_d \left\| \sum_{i_d} h_i \right\|_{K, \mathcal{J}})^p \right). \end{aligned}$$

By applying Lemma 4 to the second sum on the right hand side, we get

$$(3) \quad \mathbb{E} \left| \sum_i h_i(\mathbf{X}_i^{\text{dec}}) \right|^p \leq L_d^p \left( \sum_{K \subseteq I \subseteq I_d} \sum_{\mathcal{J} \in \mathcal{P}_{I \setminus K}} p^{p(\#I^c + \deg \mathcal{J}/2)} \mathbb{E}_{I^c} \sum_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p \right).$$

We can now finish the proof using Lemma 2. We apply it to  $\mathbb{E}_{I^c}$  for  $I \neq I_d$ , with  $\#I^c$  instead of  $d$  and  $p/2$  instead of  $p$  (for  $p = 2$  the theorem is trivial, so we can assume that  $p > 2$ ) and  $\alpha = 2\#I^c + \deg \mathcal{J} + \#I^c$ . Using the fact that  $(p/2)^{\alpha \#I^c} \leq L_d^p$  and  $\mathbb{E} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^2 \leq \sum_{i_I} \mathbb{E}_I |h_i|^2$ , we get

$$\begin{aligned} \mathbb{E}_{I^c} \sum_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p &\leq p^{-\alpha p/2} \tilde{L}_d^p \left( p^{\alpha p/2} \mathbb{E}_{I^c} \max_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p \right. \\ &\quad \left. + \max_{J \subsetneq I^c} p^{\#J p/2} \mathbb{E}_J \max_{i_J} \left( \sum_{i_{I^c \setminus J}} \mathbb{E}_{I^c \setminus J} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^2 \right)^{p/2} \right) \\ &\leq \tilde{L}_d^p \left( \mathbb{E}_{I^c} \max_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p \right. \\ &\quad \left. + p^{-(\#I^c + \deg \mathcal{J}/2)p} \max_{J \subseteq I^c} \mathbb{E}_J \max_{i_J} \left( \sum_{i_{J^c}} \mathbb{E}_{J^c} |h(\mathbf{X}_i^{\text{dec}})|^2 \right)^{p/2} \right) \\ &= \tilde{L}_d^p \left( \mathbb{E}_{I^c} \max_{i_{I^c}} \|(h_i)_{i_I}\|_{K, \mathcal{J}}^p + p^{-(\#I^c + \deg \mathcal{J}/2)p} \max_{J \subseteq I^c} \mathbb{E}_J \max_{i_J} \|(h_i)_{i_{J^c}}\|_{J^c, \emptyset}^p \right), \end{aligned}$$

which allows us to replace the sums in  $i_{I^c}$  on the right-hand side of (3) by the corresponding maxima, proving the inequality in question.  $\square$

Theorem 1 gives a precise estimate for moments of canonical Hilbert space valued  $U$ -statistics. In the sequel however we will need a weaker estimate, using the  $\|\cdot\|_{K,\mathcal{J}}$  norms only for  $I = I_d$  and specialized to the case  $h_i = h$ . Before we formulate a proper corollary, let us introduce

**DEFINITION 7.** *Let  $h: \Sigma^d \rightarrow H$  be a canonical kernel. Let moreover  $X_1, X_2, \dots, X_d$  be i.i.d random variables with values in  $\Sigma$ . Denote  $X = (X_1, \dots, X_d)$  and for  $J \subseteq I_d$ ,  $X_J = (X_j)_{j \in J}$ . For  $K \subseteq I \subseteq I_d$  and  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I \setminus K}$ , we define*

$$\|h\|_{K,\mathcal{J}} = \sup \left\{ \mathbb{E}_I \langle h(X), g(X_K) \rangle \prod_{j=1}^k f_j(X_{J_j}) : g: \Sigma^{\#K} \rightarrow H, \mathbb{E} |g(X_K)|^2 \leq 1, \right. \\ \left. f_j: \Sigma^{\#J_j} \rightarrow \mathbb{R}, \mathbb{E} f_j(X_{J_j})^2 \leq 1, j = 1, \dots, k \right\}.$$

In other words  $\|h\|_{K,\mathcal{J}}$  is the  $\|\cdot\|_{K,\mathcal{J}}$  of an array  $(h_i)_{|i|=1}$ , with  $h_{(1,\dots,1)} = h$ .

*Remark.* For  $I = I_d$ ,  $\|h\|_{K,\mathcal{J}}$  is a norm, whereas for  $I \subsetneq I_d$ , it is a random variable, depending on  $X_{I^c}$ .

It is also easy to see that if all the variables  $X_i^{(j)}$  are i.i.d. and for all  $|i| \leq n$  we have  $h_i = h$ , then for any fixed value of  $i_{I^c}$ ,

$$\|(h_i)_{|i| \leq n}\|_{K,\mathcal{J}} = \|h\|_{K,\mathcal{J}} n^{\#I/2},$$

where  $\|h\|_{K,\mathcal{J}}$  is defined with respect to any i.i.d. sequence  $X_1, \dots, X_d$  of the form  $X_j = X_{i_j}^{(j)}$  for  $j \in I^c$ .

We also have  $\|h\|_{K,\mathcal{J}} \leq \sqrt{\mathbb{E}_I |h(X)|^2}$ , which together with the above observations allows us to derive the following

**COROLLARY 1.** *For all  $p \geq 2$ , we have*

$$\mathbb{E} \left| \sum_i h(\mathbf{X}_i^{\text{dec}}) \right|^p \leq L_d^p \left( \sum_{K \subseteq I_d} \sum_{\mathcal{J} \in \mathcal{P}_{I_d \setminus K}} p^{p \deg \mathcal{J}/2} n^{dp/2} \|h\|_{K,\mathcal{J}}^p \right. \\ \left. + \sum_{I \subsetneq I_d} p^{p(d+\#I^c)/2} n^{\#Ip/2} \mathbb{E}_{I^c} \max_{i_{I^c}} (\mathbb{E}_I |h(\mathbf{X}_i^{\text{dec}})|^2)^{p/2} \right)$$

The Chebyshev inequality gives the following corollary for bounded kernels

COROLLARY 2. *If  $h$  is bounded, then for all  $t \geq 0$ ,*

$$\begin{aligned} \mathbb{P}\left(\left|\sum_{\mathbf{i}} h(\mathbf{X}_{\mathbf{i}}^{\text{dec}})\right| \geq L_d(n^{d/2}(\mathbb{E}|h|^2)^{1/2} + t)\right) \leq \\ L_d \exp\left[-\frac{1}{L_d}\left(\min_{K \subsetneq I_d, \mathcal{J} \in \mathcal{P}_{I_d \setminus K}} \left(\frac{t}{n^{d/2}\|h\|_{K, \mathcal{J}}}\right)^{2/\deg(\mathcal{J})}\right) \wedge \right. \\ \left. \wedge \left(\min_{I \subsetneq I_d} \left(\frac{t}{n^{\#I/2}\|(\mathbb{E}_I|h|^2)^{1/2}\|_{\infty}}\right)^{2/(d+\#I^c)}\right)\right]. \end{aligned}$$

Before we formulate the version of exponential inequalities that will be useful for the analysis of the LIL, let us recall the classical definition of Hoeffding projections.

DEFINITION 8. *For an integrable kernel  $h: \Sigma^d \rightarrow H$ , define  $\pi_d h: \Sigma^k \rightarrow \mathbb{R}$  with the formula*

$$\pi_d h(x_1, \dots, x_k) = (\delta_{x_1} - \mathbf{P}) \times (\delta_{x_2} - \mathbf{P}) \times \dots \times (\delta_{x_d} - \mathbf{P})h,$$

where  $\mathbf{P}$  is the law of  $X_1$ .

*Remark.* It is easy to see that  $\pi_k h$  is canonical. Moreover  $\pi_d h = h$  iff  $h$  is canonical.

The following Lemma was proven for  $H = \mathbb{R}$  in [2] (Lemma 1). The proof given there works for an arbitrary Banach space.

LEMMA 6. *Consider an arbitrary family of integrable kernels  $h_i: \Sigma^d \rightarrow H$ ,  $|\mathbf{i}| \leq n$ . For any  $p \geq 1$  we have*

$$\left\| \sum_{|\mathbf{i}| \leq n} \pi_d h_{\mathbf{i}}(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) \right\|_p \leq 2^d \left\| \sum_{|\mathbf{i}| \leq n} \epsilon_{\mathbf{i}}^{\text{dec}} h_{\mathbf{i}}(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) \right\|_p.$$

In the sequel we will use exponential inequalities to  $U$ -statistics generated by  $\pi_d h$ , where  $h$  will be a non-necessarily canonical kernel of order  $d$ . Since the kernel  $\tilde{h}((\varepsilon_1, X_1), \dots, (\varepsilon_d, X_d)) = \varepsilon_1 \cdots \varepsilon_d h(X_1, \dots, X_d)$ , where  $\varepsilon_i$ 's are i.i.d. Rademacher variables independent of  $X_i$ 's is always canonical, Corollary 1, Lemma 6 and the Chebyshev inequality give us also the following corollary (note that  $\|\tilde{h}\|_{K, \mathcal{J}} = \|h\|_{K, \mathcal{J}}$ )

COROLLARY 3. *If  $h$  is bounded, then for all  $p \geq 0$ ,*

$$\mathbb{P}\left(\left|\sum_i \pi_d h(\mathbf{X}_i^{\text{dec}})\right| \geq L_d(n^{d/2}(\mathbb{E}|h|^2)^{1/2} + t)\right) \leq$$

$$L_d \exp\left[-\frac{1}{L_d} \left(\min_{K \subsetneq I_d, \mathcal{J} \in \mathcal{P}_{I_d \setminus K}} \left(\frac{t}{n^{d/2} \|h\|_{K, \mathcal{J}}}\right)^{2/\deg(\mathcal{J})}\right) \wedge\right.$$

$$\left.\wedge \left(\min_{I \subsetneq I_d} \left(\frac{t}{n^{\#I/2} \|(\mathbb{E}_I |h|^2)^{1/2}\|_\infty}\right)^{2/(d+\#I^c)}\right)\right].$$

**4. The equivalence of several LIL statements.** In this section we will recall general results on the correspondence of various statements of the LIL. We will state them without proofs, since all of them have been proven in [9] and [2] in the real case and the proofs can be directly transferred to the Hilbert space case, with some simple modifications that we will indicate.

Before we proceed, let us introduce the assumptions and notation common for the remaining part of the article.

- We assume that  $(X_i)_{i \in \mathbb{N}}$ ,  $(X_i^{(k)})_{i \in \mathbb{N}, 1 \leq k \leq d}$  are i.i.d. and  $h: \Sigma^d \rightarrow H$  is a measurable function.
- Recall that  $(\varepsilon_i)_{i \in \mathbb{N}}$ ,  $(\varepsilon_i^{(k)})_{i \in \mathbb{N}, 1 \leq k \leq d}$  are independent Rademacher variables, independent of  $(X_i)_{i \in \mathbb{N}}$ ,  $(X_i^{(k)})_{i \in \mathbb{N}, 1 \leq k \leq d}$ .
- To avoid technical problems with small values of  $h$  let us also define  $\text{LL}x = \log \log(x \vee e^e)$ .
- We will also occasionally write  $X$  for  $(X_1, \dots, X_d)$  and for  $I \subseteq I_d$ ,  $X_I = (X_i)_{i \in I}$ . Sometimes we will write simply  $h$  instead of  $h(X)$ .
- We will use the letter  $K$  to denote constants depending only on the function  $h$ .

We will need the following simple fact

LEMMA 7. *If  $\mathbb{E}|h|^2/(\text{LL}|h|)^d = K < \infty$  then  $\mathbb{E}(|h|^2 \wedge u) \leq L(\log \log u)^d$  with  $L$  depending only on  $K$  and  $d$ .*

The next lemma comes from [9]. It is proven there for  $H = \mathbb{R}$  but the argument is valid also for general Banach spaces.

LEMMA 8. *Let  $h: \Sigma^d \rightarrow H$  be a symmetric function. There exist constants  $L_d$ , such that if*

$$(4) \quad \limsup_{n \rightarrow \infty} \frac{1}{(n \log \log n)^{d/2}} \left| \sum_{i \in I_n^d} h(\mathbf{X}_i) \right| < C \text{ a.s.,}$$

then

$$(5) \quad \sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}})\right| \geq D 2^{nd/2} \log^{d/2} n\right) < \infty$$

for  $D = L_d C$ .

LEMMA 9. *For a symmetric function  $h: \Sigma^d \rightarrow H$ , the LIL (4) is equivalent to the decoupled LIL*

$$(6) \quad \limsup_{n \rightarrow \infty} \frac{1}{(n \log \log n)^{d/2}} \left| \sum_{i \in I_n^d} h(\mathbf{X}_i^{\text{dec}}) \right| < D \text{ a.s.},$$

meaning that (4) implies (6) with  $D = L_d C$ , and conversely (6) implies (4) with  $C = L_d D$ .

PROOF. This is Lemma 8 in [2]. The proof is the same as there, one needs only to replace  $l^\infty$  with  $l^\infty(H)$  – the space of bounded  $H$ -valued sequences.  $\square$

The next lemma also comes from [2] (Lemma 9). Although stated for real kernels, its proof relies on an inductive argument with a stronger, Banach-valued hypothesis.

LEMMA 10. *There exists a universal constant  $L < \infty$ , such that for any kernel  $h: \Sigma^d \rightarrow H$  we have*

$$\mathbb{P}\left(\max_{|j| \leq n} \left| \sum_{i: i_k \leq j_k, k=1 \dots d} h(\mathbf{X}_i^{\text{dec}}) \right| \geq t\right) \leq L^d \mathbb{P}\left(\left| \sum_{|i| \leq n} h(\mathbf{X}_i^{\text{dec}}) \right| \geq t/L^d\right).$$

COROLLARY 4. *Consider a kernel  $h: \Sigma^d \rightarrow H$  and  $\alpha > 0$ . If*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \sum_{|i| \leq 2^n} h(\mathbf{X}_i^{\text{dec}}) \right| \geq C 2^{n\alpha} \log^\alpha n\right) < \infty,$$

then

$$\limsup_{n \rightarrow \infty} \frac{1}{(n \log \log n)^\alpha} \left| \sum_{|i| \leq 2^n} h(\mathbf{X}_i^{\text{dec}}) \right| \leq L_{d,\alpha} C \text{ a.s.}$$

PROOF. Given Lemma 10, the proof is the same as the one for real kernels, presented in [2] (Corollary 1 therein).  $\square$

The next lemma shows that the contribution to a decoupled U-statistic from the 'diagonal', i.e. from the sum over multiindices  $\mathbf{i} \notin I_n^d$  is negligible. The proof given in [2] (Lemma 10) is still valid, since the only part which cannot be directly transferred to the Banach space setting is the estimate of variance of canonical U-statistics, which is the same in the real and general Hilbert space case.

LEMMA 11. *If  $h: \Sigma^d \rightarrow H$  is canonical and satisfies*

$$\mathbb{E}(|h|^2 \wedge u) = \mathcal{O}((\log \log u)^\beta),$$

for some  $\beta$ , then

$$(7) \quad \limsup_{n \rightarrow \infty} \frac{1}{(n \log \log n)^{d/2}} \left| \sum_{\substack{|\mathbf{i}| \leq n \\ \exists_{j \neq k} i_j = i_k}} h(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) \right| = 0 \text{ a.s.}$$

COROLLARY 5. *The randomized decoupled LIL*

$$(8) \quad \limsup_{n \rightarrow \infty} \frac{1}{(n \log \log n)^{d/2}} \left| \sum_{|\mathbf{i}| \leq n} \epsilon_{\mathbf{i}}^{\text{dec}} h(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) \right| < C$$

is equivalent to (5), meaning then if (8) holds then so does (5) with  $D = L_d C$  and (5) implies (8) with  $C = L_d D$ .

The proof is the same as for the real-valued case, given in [2] (Corollary 2), one only needs to replace  $h^2$  by  $|h|^2$  and use the formula for the second moments in Hilbert spaces.

COROLLARY 6. *For a symmetric, canonical kernel  $h: \Sigma^d \rightarrow H$ , the LIL (4) is equivalent to the decoupled LIL 'with diagonal'*

$$(9) \quad \limsup_{n \rightarrow \infty} \frac{1}{(n \log \log n)^{d/2}} \left| \sum_{|\mathbf{i}| \leq n} h(\mathbf{X}_{\mathbf{i}}^{\text{dec}}) \right| < D$$

again meaning that there are constants  $L_d$  such that if (4) holds for some  $D$  then so does (9) for  $D = L_d C$ , and conversely, (9) implies (4) for  $C = L_d D$ .

PROOF. The proof is the same as in the real case (see [2], Corollary 3). Although the integrability of the kernel guaranteed by the LIL is worse in the Hilbert space case, it still allows one to use Lemma 11.  $\square$



**5. The canonical decoupled case.** Before we formulate the necessary and sufficient conditions for the bounded LIL in Hilbert spaces, we need

DEFINITION 9. For a canonical kernel  $h: \Sigma^d \rightarrow H$ ,  $K \subseteq I_d$ ,  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I_d \setminus K}$  and  $u > 0$  we define

$$\|h\|_{K, \mathcal{J}, u} = \sup \left\{ \mathbb{E} \langle h(X), g(X_K) \rangle \prod_{i=1}^k f_i(X_{J_i}) : g: \Sigma^K \rightarrow H, \right. \\ \left. f_i: \Sigma^{J_i} \rightarrow \mathbb{R}, \|g\|_2, \|f_i\|_2 \leq 1, \|g\|_\infty, \|f_i\|_\infty \leq u \right\},$$

where for  $K = \emptyset$  by  $g(X_K)$  we mean an element  $g \in H$ , and  $\|g\|_2$  denotes just the norm of  $g$  in  $H$  (alternatively we may think of  $g$  as of a random variable measurable with respect to  $\sigma((X_i)_{i \in \emptyset})$ , hence constant). Thus the condition on  $g$  becomes in this case just  $|g| \leq 1$ .

Example. For  $d = 2$ , the above definition reads as

$$\begin{aligned} \|h(X_1, X_2)\|_{\emptyset, \{\{1,2\}\}, u} &= \sup \left\{ |\mathbb{E} h(X_1, X_2) f(X_1, X_2)| : \right. \\ &\quad \left. \mathbb{E} f(X_1, X_2)^2 \leq 1, \|f\|_\infty \leq u \right\}, \\ \|h(X_1, X_2)\|_{\emptyset, \{\{1\}\{2\}\}, u} &= \sup \left\{ |\mathbb{E} h(X_1, X_2) f(X_1) g(X_2)| : \right. \\ &\quad \left. \mathbb{E} f(X_1)^2, \mathbb{E} g(X_2)^2 \leq 1 \right. \\ &\quad \left. \|f\|_\infty, \|g\|_\infty \leq u \right\}, \\ \|h(X_1, X_2)\|_{\{1\}, \{\{2\}\}, u} &= \sup \left\{ \mathbb{E} \langle f(X_1), h(X_1, X_2) \rangle g(X_2) : \right. \\ &\quad \left. \mathbb{E} |f(X_1)|^2, \mathbb{E} g(X_2)^2 \leq 1 \right. \\ &\quad \left. \|f\|_\infty, \|g\|_\infty \leq u \right\} \\ \|h(X_1, X_2)\|_{\{1,2\}, \emptyset, u} &= \sup \left\{ \mathbb{E} \langle f(X_1, X_2), h(X_1, X_2) \rangle : \right. \\ &\quad \left. \mathbb{E} |f(X_1, X_2)|^2 \leq 1, \|f\|_\infty \leq u \right\}. \end{aligned}$$

THEOREM 2. Let  $h$  be a canonical  $H$ -valued symmetric kernel in  $d$  variables. Then the decoupled LIL

$$(10) \quad \limsup_{n \rightarrow \infty} \frac{1}{n^{d/2} (\log \log n)^{d/2}} \left| \sum_{|i| \leq n} h(\mathbf{X}_i^{\text{dec}}) \right| < C$$

holds if and only if

$$(11) \quad \mathbb{E} \frac{|h|^2}{(\text{LL}|h|)^d} < \infty$$

and for all  $K \subseteq I_d, \mathcal{J} \in \mathcal{P}_{I_d \setminus K}$

$$(12) \quad \limsup_{u \rightarrow \infty} \frac{1}{(\log \log u)^{(d - \deg \mathcal{J})/2}} \|h\|_{K, \mathcal{J}, u} < D.$$

More precisely, if (10) holds for some  $C$  then (12) is satisfied for  $D = L_d C$  and conversely, (11) and (12) implies (10) with  $C = L_d D$ .

*Remark.* Using Lemma 7 one can easily check that the condition (12) with  $D < \infty$  for  $I = I_d$  is implied by (11).

**6. Necessity.** The proof is a refinement of ideas from [16], used to study random matrix approximations of the operator norm of kernel integral operators.

LEMMA 12. *If  $a, t > 0$  and  $h$  is a nonnegative  $d$ -dimensional kernel such that  $N^d \mathbb{E}h(X) \geq ta$  and  $\|\mathbb{E}_I h(X)\|_\infty \leq N^{-\#I} a$  for all  $\emptyset \subseteq I \subsetneq \{1, \dots, d\}$ , then*

$$\forall \lambda \in (0, 1) \quad \mathbb{P}\left(\sum_{|i| \leq N} h(\mathbf{X}_i^{\text{dec}}) \geq \lambda ta\right) \geq (1 - \lambda)^2 \frac{t}{t + 2^d - 1} \geq (1 - \lambda)^2 2^{-d} \min(1, t).$$

PROOF. We have

$$\begin{aligned} \mathbb{E}\left(\sum_{|i| \leq N} h(\mathbf{X}_i^{\text{dec}})\right)^2 &= \sum_{|i| \leq N} \sum_{|j| \leq N} \mathbb{E}h(\mathbf{X}_i^{\text{dec}})h(\mathbf{X}_j^{\text{dec}}) \\ &= \sum_{I \subseteq I_d} \sum_{|i| \leq N} \sum_{\substack{|j| \leq N: \\ \{k: i_k = j_k\} = I}} \mathbb{E}h(\mathbf{X}_i^{\text{dec}})h(\mathbf{X}_j^{\text{dec}}) \\ &= \sum_{I \subseteq I_d} \sum_{|i| \leq N} \sum_{\substack{|j| \leq N: \\ \{k: i_k = j_k\} = I}} \mathbb{E}[h(\mathbf{X}_i^{\text{dec}})\mathbb{E}_{I^c} h(\mathbf{X}_j^{\text{dec}})] \\ &\leq N^{2d}(\mathbb{E}h(X))^2 + \sum_{\emptyset \neq I \subseteq I_d} N^{d + \#I^c} \mathbb{E}h(X) \|\mathbb{E}_{I^c} h(X)\|_\infty \\ &\leq N^{2d}(\mathbb{E}h(X))^2 + (2^d - 1)N^d a \mathbb{E}h(X) \\ &\leq N^{2d}(\mathbb{E}h(X))^2 + (2^d - 1)t^{-1}N^{2d}(\mathbb{E}h(X))^2 \\ &\leq \frac{t + 2^d - 1}{t} \left(\mathbb{E} \sum_{|i| \leq n} h(\mathbf{X}_i^{\text{dec}})\right)^2. \end{aligned}$$

The lemma follows now from the Paley-Zygmund inequality (see e.g. [5], Corollary 3.3.2.), which says that for an arbitrary nonnegative random variable  $S$ ,

$$\mathbb{P}(S \geq \lambda S) \geq (1 - \lambda)^2 \frac{(\mathbb{E}S)^2}{\mathbb{E}S^2}.$$

□

COROLLARY 7. *Let  $A \subseteq \Sigma^d$  be a measurable set, such that*

$$\forall_{\emptyset \subsetneq I \subsetneq \{1, \dots, d\}} \forall_{x_{I^c} \in \Sigma^{I^c}} \mathbb{P}_I((x_{I^c}, X_I) \in A) \leq N^{-\#I}.$$

*Then*

$$\mathbb{P}(\exists_{|i| \leq N} \mathbf{X}_i^{\text{dec}} \in A) \geq 2^{-d} \min(N^d \mathbb{P}(X \in A), 1).$$

PROOF. We apply Lemma 12 with  $h = I_A$ ,  $a = 1$ ,  $t = N^d \mathbb{P}(X \in A)$  and  $\lambda \rightarrow 0+$ . □

LEMMA 13. *Suppose that  $Z_j$  are nonnegative r.v.'s,  $p > 0$  and  $a_j \in \mathbb{R}$  are such that  $\mathbb{P}(Z_j \geq a_j) \geq p$  for all  $j$ . Then*

$$\mathbb{P}\left(\sum_j Z_j \geq p \sum_j a_j / 2\right) \geq p/2.$$

PROOF. Let  $\alpha := \mathbb{P}(\sum_j Z_j \geq p \sum_j a_j / 2)$ , then

$$p \sum_j a_j \leq \mathbb{E}\left(\sum_j \min(Z_j, a_j)\right) \leq \alpha \sum_j a_j + p \sum_j a_j / 2.$$

□

THEOREM 3. *Let  $Y$  be a r.v. independent of  $X_i^{(j)}$ . Suppose that for each  $n$ ,  $a_n \in \mathbb{R}$ ,  $h_n$  is a  $d + 1$ -dimensional nonnegative kernel such that*

$$\sum_n \mathbb{P}\left(\sum_{|i| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n\right) < \infty.$$

*Let  $p > 0$ , then there exists a constant  $C_d(p)$  depending only on  $p$  and  $d$  such that the sets*

$$A_n := \{x \in S^d : \forall_{n \leq m \leq 2^{d-1}n} \mathbb{P}_Y(h_m(x, Y) \geq C_d(p) 2^{d(n-m)} a_m) \geq p\},$$

*satisfy  $\sum 2^{dn} \mathbb{P}(X \in A_n) < \infty$ .*

PROOF. We will show by induction on  $d$ , that the assertion holds with  $C_1(p) := 1$ ,  $C_2(p) := 12/p$  and

$$C_d(p) := 12p^{-1} \max_{1 \leq l \leq d-1} C_{d-l}(2^{-l-4}p/3) \text{ for } d = 3, 4, \dots$$

For  $d = 1$  we have

$$\begin{aligned}
\frac{1}{2} \min(2^n \mathbb{P}(X \in A_n), 1) &\leq \mathbb{P}(\exists_{|i| \leq 2^n} \mathbf{X}_i^{\text{dec}} \in A_n) \\
&= \mathbb{P}(\exists_{|i| \leq 2^n} \mathbb{P}_Y(h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n) \geq p) \\
&\leq \mathbb{P}(\mathbb{P}_Y(\sum_{|i| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n) \geq p) \\
&\leq p^{-1} \mathbb{P}(\sum_{|i| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n).
\end{aligned}$$

Before investigating the case  $d > 1$  let us define

$$\tilde{A}_n := A_n \setminus \bigcup_{m > n} A_m.$$

The sets  $\tilde{A}_n$  are pairwise disjoint and obviously  $\tilde{A}_n \subset A_n$ . Notice that since  $C_d(p) \geq 1$ ,

$$\mathbb{P}(X \in A_n) \leq \mathbb{P}(\mathbb{P}_Y(h_n(X, Y) \geq a_n)) \leq p^{-1} \mathbb{P}(\sum_{|i| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n).$$

Hence  $\sum_n \mathbb{P}(X \in A_n) < \infty$ , so  $\mathbb{P}(X \in \limsup A_n) = 0$ . But if  $x \notin \limsup A_n$ , then  $\sum_n 2^{nd} I_{A_n}(x) \leq \sum_n 2^{nd+1} I_{\tilde{A}_n}(x)$ . So it is enough to show that  $\sum 2^{dn} \mathbb{P}(X \in \tilde{A}_n) < \infty$ .

*Induction step.* Suppose that the statement holds for all  $d' < d$ , we will show it for  $d$ . First we will inductively construct sets

$$\tilde{A}_n = A_n^0 \supset A_n^1 \supset \dots \supset A_n^{d-1}$$

such that for  $1 \leq l \leq d-1$ ,

$$(13) \quad \forall_{\emptyset \subsetneq I \subsetneq \{1, \dots, d-1\}, \#I \leq l} \forall_{x_{I^c}} \mathbb{P}_I((x_{I^c}, X_I) \in A_n^l) \leq 2^{-n\#I}$$

and

$$(14) \quad \sum_n 2^{nd} \mathbb{P}(X \in A_n^{l-1} \setminus A_n^l) < \infty.$$

Suppose that  $1 \leq l \leq d-1$  and the set  $A_n^{l-1}$  was already defined. Let  $I \subset \{1, \dots, d\}$  be such that  $\#I = l$  and let  $j \in I$ . Notice that

$$\mathbb{P}_I((x_{I^c}, X_I) \in A_n^{l-1}) = \mathbb{E}_j \mathbb{P}_{I \setminus \{j\}}((x_{I^c}, X_j, X_{I \setminus \{j\}}) \in A_n^{l-1}) \leq 2^{-n(l-1)}$$

by the property (13) of the set  $A_n^{l-1}$ . Let us define for  $n(l-1)+1 \leq k \leq nl$ ,

$$B_{n,k}^I := \{x_{I^c} : \mathbb{P}_I((x_{I^c}, X_I) \in A_n^{l-1}) \in (2^{-k}, 2^{-k+1}]\}$$

and

$$B_n^I := \bigcup_{k=n(l-1)+1}^{nl} B_{n,k}^I = \{x_{I^c} : \mathbb{P}_I((x_{I^c}, X_I) \in A_n^{l-1}) > 2^{-nl}\}.$$

We have

$$\begin{aligned} \sum_n 2^{dn} \mathbb{P}(X \in A_n^{l-1}, X_{I^c} \in B_n^I) &\leq 2 \sum_n \sum_{k=n(l-1)+1}^{nl} 2^{dn-k} \mathbb{P}(X_{I^c} \in B_n^I) \\ &= 2 \mathbb{E} k_1^I(X_{I^c}), \end{aligned}$$

where

$$k_1^I(x_{I^c}) := \sum_n \sum_{k=n(l-1)+1}^{nl} 2^{dn-k} I_{B_{n,k}^I}(x_{I^c}).$$

Let  $m \geq 1$  and

$$C_m^I := \{x_{I^c} : 2^{(m+1)(d-l)} > k_1(x_{I^c}) \geq 2^{m(d-l)}\}.$$

Notice that for  $n > m$  and  $k \leq nl$ ,  $2^{dn-k} \geq 2^{(d-l)(m+1)}$ , moreover

$$\sum_{n < m/2} \sum_{k=n(l-1)+1}^{nl} 2^{dn-k} \leq \sum_{n < m/2} 2^{(d-l+1)n} \leq \frac{4}{3} 2^{(d-l+1)(m-1)/2} \leq \frac{2}{3} 2^{(d-l)m}.$$

Hence

$$(15) \quad x_{I^c} \in C_m^I \Rightarrow \sum_{m/2 \leq n \leq m} \sum_{k=n(l-1)+1}^{nl} 2^{dn-k} I_{B_{n,k}^I}(x_{I^c}) \geq \frac{1}{3} 2^{(d-l)m}.$$

Let  $m \leq r \leq 2^{d-2}m$ , if  $m/2 \leq n \leq m$ , then since  $A_n^{l-1} \subset A_n$  we have for all  $x \in S^d$ ,

$$\mathbb{P}_Y(h_r(x, Y) \geq C_d(p)2^{d(n-r)}a_r I_{A_n^{l-1}}(x)) \geq p,$$

therefore, since  $A_n^{l-1} \subset \tilde{A}_n$  are pairwise disjoint,

$$\mathbb{P}_Y\left(h_r(x, Y) \geq C_d(p)2^{-dr}a_r \sum_{m/2 \leq n \leq m} 2^{dn} I_{A_n^{l-1}}(x)\right) \geq p.$$

Hence, by Lemma 13,

$$(16) \quad \mathbb{P}_Y\left(\sum_{|i_I| \leq 2^r} h_r(x_{I^c}, \mathbf{X}_{i_I}^{\text{dec}}, Y) \geq \frac{p}{2} C_d(p)2^{-dr}a_r \sum_{|i_I| \leq 2^r} k_{2, x_{I^c}}(\mathbf{X}_{i_I}^{\text{dec}})\right) \geq \frac{p}{2},$$

where

$$k_{2, x_{I^c}}(x_I) := \sum_{m/2 \leq n \leq m} 2^{dn} I_{A_n^{l-1}}(x_{I^c}, x_I).$$

We have  $\|k_{2, x_{I^c}}\|_\infty \leq 2^{dm}$  and for  $\emptyset \neq J \subsetneq I$ , by the property (13) of  $A_n^{l-1}$ ,

$$\begin{aligned} \mathbb{E}_J k_{2, x_{I^c}}(x_{I \setminus J}, X_J) &= \sum_{m/2 \leq n \leq m} 2^{dn} \mathbb{P}_J\left((x_{I^c}, x_{I \setminus J}, X_J) \in A_n^{l-1}\right) \\ &\leq \sum_{m/2 \leq n \leq m} 2^{(d-\#J)n} \leq 2^{(d-\#J)m+1}. \end{aligned}$$

Moreover for  $x_{I^c} \in C_m^I$ , by the definition of  $B_{n,k}^I$  and (15),

$$\begin{aligned} \mathbb{E} k_{2, x_{I^c}}(X_I) &\geq \sum_{m/2 \leq n \leq m} \sum_{k=n(l-1)+1}^{nl} 2^{dn} \mathbb{P}_I((x_{I^c}, X_I) \in A_n^{l-1}) I_{B_{n,k}^I}(x_{I^c}) \\ &\geq \sum_{m/2 \leq n \leq m} \sum_{k=n(l-1)+1}^{nl} 2^{dn-k} I_{B_{n,k}^I}(x_{I^c}) \geq \frac{1}{3} 2^{(d-l)m}. \end{aligned}$$

Therefore by Lemma 12 (with  $l$  instead of  $d$  and  $a = 2^{(d-l)m+rl+1}$ ,  $t = 1/6$ ,  $N = 2^r$ ,  $\lambda = 1/2$ ), for  $m \leq r \leq 2^{d-2}m$ ,

$$\mathbb{P}\left(\sum_{|i_I| \leq 2^r} k_{2, x_{I^c}}(\mathbf{X}_{i_I}^{\text{dec}}) \geq \frac{1}{6} 2^{(d-l)m+rl}\right) \geq \frac{1}{3} 2^{-l-3}.$$

Combining the above estimate with (16) we get (for  $x_{I^c} \in C_m^I$  and  $m \leq r \leq 2^{d-2}m$ ),

$$\mathbb{P}_{I,Y} \left( \sum_{|I| \leq 2^r} h_r(x_{I^c}, \mathbf{X}_{i_I}^{\text{dec}}, Y) \geq \frac{p}{12} C_d(p) 2^{(d-l)(m-r)} a_r \right) \geq \frac{1}{3} 2^{-l-4} p.$$

Let us define  $\tilde{Y} := ((X_i^{(j)})_{j \in I}, Y)$  and  $\tilde{h}_n(x_{I^c}, \tilde{Y}) := \sum_{|I| \leq 2^n} h_n(x_{I^c}, \mathbf{X}_{i_I}^{\text{dec}}, Y)$ . Then

$$\sum_n \mathbb{P} \left( \sum_{|I^c| \leq 2^n} \tilde{h}_n(\mathbf{X}_{i_{I^c}}^{\text{dec}}, \tilde{Y}) \geq a_n \right) = \sum_n \mathbb{P} \left( \sum_{|I| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n \right) < \infty.$$

Moreover (since  $C_d(p) \geq 12p^{-1}C_{d-l}(2^{-l-4}p/3)$ ),

$$C_m^I \subseteq \left\{ \forall_{m \leq r \leq 2^{d-2}m} \mathbb{P}_{\tilde{Y}}(\tilde{h}_r(x_{I^c}, \tilde{Y}) \geq C_{d-l}(2^{-l-4}p/3) 2^{(d-l)(m-r)} a_r) \geq 2^{-l-4}p/3 \right\}.$$

Hence by the induction assumption,

$$\sum_m 2^{(d-l)m} \mathbb{P}(X_{I^c} \in C_m^I) < \infty,$$

so  $\mathbb{E}k_1^I(X_{I^c}) < \infty$  and thus

$$(17) \quad \forall_{\#I=l} \sum_n 2^{dn} \mathbb{P}(X \in A_n^{l-1}, X_{I^c} \in B_n^I) < \infty.$$

We set

$$A_n^l := \{x \in A_n^{l-1} : x_{I^c} \notin B_n^I \text{ for all } I \subset \{1, \dots, d\}, \#I = l\}.$$

The set  $A_n^l$  satisfies the condition (13) by the definition of  $B_n^I$  and the property (13) for  $A_n^{l-1}$ . The condition (14) follows by (17).

Notice that the set  $A_n^{d-1}$  satisfies the assumptions of Corollary 7 with  $N = 2^n$ , therefore if  $C_d(p) \geq 1$ ,

$$\begin{aligned} 2^{-d} \min(1, 2^{nd} \mathbb{P}(X \in A_n^{d-1})) &\leq \mathbb{P}(\exists_{|i| \leq 2^n} \mathbf{X}_i^{\text{dec}} \in A_n^{d-1}) \leq \mathbb{P}(\exists_{|i| \leq 2^n} \mathbf{X}_i^{\text{dec}} \in \tilde{A}_n) \\ &\leq \mathbb{P}(\exists_{|i| \leq 2^n} \mathbb{P}_Y(h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq C_d(p)a_n) \geq p) \\ &\leq \mathbb{P}(\mathbb{P}_Y \left( \sum_{|i| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n \right) \geq p) \\ &\leq p^{-1} \mathbb{P} \left( \sum_{|i| \leq 2^n} h_n(\mathbf{X}_i^{\text{dec}}, Y) \geq a_n \right). \end{aligned}$$

Therefore  $\sum_n 2^{nd} \mathbb{P}(X \in A_n^{d-1}) < \infty$ , so by (14) we get

$$\sum_n 2^{nd} \mathbb{P}(X \in \tilde{A}_n) = \sum_n 2^{nd} \left( \sum_{l=1}^{d-1} \mathbb{P}(X \in A_n^{l-1} \setminus A_n^l) + \mathbb{P}(X \in A_n^{d-1}) \right) < \infty.$$

□

COROLLARY 8. *If*

$$\sum_n \mathbb{P} \left( \sum_{|i| \leq 2^n} h^2(\mathbf{X}_i^{\text{dec}}) \geq \varepsilon 2^{nd} (\log n)^\alpha \right) < \infty$$

for some  $\varepsilon > 0$  and  $\alpha \in \mathbb{R}$ ,

then  $\mathbb{E} \frac{h^2}{(\text{LL}|h|)^\alpha} < \infty$ .

PROOF. We apply Theorem 3 with  $h_n = h^2$  and  $a_n = \varepsilon 2^{nd} \log^d n$  in the degenerate case when  $Y$  is deterministic. It is easy to notice that  $h^2 \geq \tilde{C}_d(p, \varepsilon) 2^{dn} \log^d n$  implies that

$$\forall_{n \leq m \leq 2^{d-1}n} h^2 \geq C_d(p) 2^{d(n-m)} a_m.$$

□

To prove the necessity part of Theorem 2 we will also need the following Lemmas

LEMMA 14 ([2], Lemma 12). *Let  $g: \Sigma^d \rightarrow \mathbb{R}$  be a square integrable function. Then*

$$\text{Var} \left( \sum_{|i| \leq n} g(\mathbf{X}_i^{\text{dec}}) \right) \leq (2^d - 1) n^{2d-1} \mathbb{E} g(X)^2.$$

LEMMA 15 ([2], Lemma 5). *If  $\mathbb{E}(|h|^2 \wedge u) = \mathcal{O}((\log \log u)^\beta)$  then*

$$\mathbb{E}|h| \mathbf{1}_{\{|h| \geq s\}} = \mathcal{O} \left( \frac{(\log \log s)^\beta}{s} \right).$$

LEMMA 16. *Let  $(a_i)_{i \in I_n^d}$  be a  $d$ -indexed array of vectors from a Hilbert space  $H$ . Consider a random variable*

$$S := \left| \sum_{|i| \leq n} a_i \prod_{k=1}^d \varepsilon_{i_k}^{(k)} \right| = \left| \sum_{|i| \leq n} a_i \varepsilon_i^{\text{dec}} \right|.$$



For any set  $K \subseteq I_d$  and a partition  $\mathcal{J} = \{J_1, \dots, J_m\} \in \mathcal{P}_{I_d \setminus K}$  let us define

$$\|(a_i)\|_{K, \mathcal{J}, p}^* := \sup \left\{ \left| \sum_{|i| \leq n} \langle a_i, \alpha_{i_K}^{(0)} \rangle \prod_{k=1}^m \alpha_{i_{J_k}}^{(k)} \right| : \sum_{i_K} |\alpha_{i_K}^{(0)}|^2 \leq 1, \sum_{i_{J_k}} (\alpha_{i_{J_k}}^{(k)})^2 \leq p, \right. \\ \left. \forall_{i_{\max J_k} \in I_n} \sum_{i_{\diamond J_k}} (\alpha_{i_{J_k}}^{(k)})^2 \leq 1, k = 1, \dots, m \right\},$$

where  $\diamond J = J \setminus \{\max J\}$  (here  $\sum_{i_{\emptyset}} a_i = a_i$ ).

Then, for all  $p \geq 1$ ,

$$\|S\|_p \geq \frac{1}{L_d} \sum_{K \subseteq I_d, \mathcal{J} \in \mathcal{P}_{I_d \setminus K}} \|(a_i)\|_{K, \mathcal{J}, p}^*.$$

In particular for some constant  $c_d$

$$\mathbb{P}(S \geq c_d \sum_{K \subseteq I_d, \mathcal{J} \in \mathcal{P}_{I_d \setminus K}} \|(a_i)\|_{K, \mathcal{J}, p}^*) \geq c_d \wedge e^{-p}.$$

*Remark.* For  $K = \emptyset$ , we define

$$\|(a_i)\|_{\emptyset, \mathcal{J}, p}^* := \sup \left\{ \left| \sum_{|i| \leq n} a_i \prod_{k=1}^m \alpha_{i_{J_k}}^{(k)} \right| : (\alpha_{i_{J_k}}^{(k)})_{i_{J_k}} \in \mathbb{R}^{(I_n^{\# J_k})}, \sum_{i_{J_k}} (\alpha_{i_{J_k}}^{(k)})^2 \leq p, \right. \\ \left. \forall_{i_{\max J_k} \in I_n} \sum_{i_{\diamond J_k}} (\alpha_{i_{J_k}}^{(k)})^2 \leq 1, k = 1, \dots, m \right\},$$

It is also easy to see that for a  $d$ -indexed matrix,  $\|(a_i)_i\|_{I_d, \{\emptyset\}, p} = \sqrt{\sum_i |a_i|^2} = \|S\|_2$  and thus does not depend on  $p$ . Since it will not be important in the applications, we keep a uniform notation with the subscript  $p$ .

*Examples.* For  $d = 1$ , we have

$$\|(a_i)_{i \leq n}\|_{\emptyset, \{\{1\}\}, p}^* = \sup \left\{ \left| \sum_{i=1}^n a_i \alpha_i \right| : \sum_{i=1}^n \alpha_i^2 \leq p, |\alpha_i| \leq 1, i = 1, \dots, n \right\},$$

$$\|(a_i)_{i \leq n}\|_{\{1\}, \emptyset, p}^* = \sup \left\{ \sum \langle a_i, \alpha_i \rangle : \sum |\alpha_i|^2 \leq 1 \right\} = \sqrt{\sum_{i=1}^n |a_i|^2},$$

whereas for  $d = 2$ , we get

$$\begin{aligned} \|(a_{ij})_{i,j \leq n}\|_{\emptyset, \{\{1\}, \{2\}\}, p}^* &= \sup \left\{ \left| \sum_{i,j=1}^n a_{ij} \alpha_i \beta_j \right| : \sum_{i=1}^n \alpha_i^2 \leq p, \sum_{j=1}^n \beta_j^2 \leq p, \right. \\ &\quad \left. \forall_{i \in I_n} |\alpha_i| \leq 1, \forall_{j \in I_n} |\beta_j| \leq 1 \right\}, \\ \|(a_{ij})_{i,j \leq n}\|_{\emptyset, \{I_2\}, p}^* &= \sup \left\{ \left| \sum_{i,j=1}^n a_{ij} \alpha_{ij} \right| : \sum_{i,j=1}^n \alpha_{ij}^2 \leq p, \forall_{j \in I_n} \sum_{i=1}^n \alpha_{ij}^2 \leq 1 \right\}, \\ \|(a_{ij})_{i,j \leq n}\|_{\{1\}, \{\{2\}\}, p}^* &= \sup \left\{ \left| \sum_{i,j=1}^n \langle a_{ij}, \alpha_i \rangle \beta_j \right| : \sum_{i=1}^n |\alpha_i|^2 \leq 1, \right. \\ &\quad \left. \sum_{j=1}^n \beta_j^2 \leq p, \forall_{j \in I_n} |\beta_j| \leq 1 \right\}, \\ \|(a_{ij})_{i,j \leq n}\|_{I_2, \emptyset, p}^* &= \sup \left\{ \left| \sum_{i,j=1}^n \langle a_{ij}, \alpha_{ij} \rangle \right| : \sum_{i,j=1}^n \alpha_{ij}^2 \leq 1 \right\} = \sqrt{\sum_{ij} |a_{ij}|^2}. \end{aligned}$$

PROOF OF LEMMA 16. We will combine the classical hypercontractivity property of Rademacher chaoses (see e.g. [5], p. 110-116) with Lemma 3 in [2], which says that for  $H = \mathbb{R}$  we have

$$(18) \quad \|S\|_p \geq \frac{1}{L_d} \sum_{\mathcal{J} \in \mathcal{P}_{I_d}} \|(a_i)\|_{\emptyset, \mathcal{J}, p}.$$

Since  $\|(a_i)\|_{I_d, \{\emptyset\}, p} = \sqrt{\sum_i |a_i|^2} = \|S\|_2$ , the inequality  $\|S\|_p \geq L^{-1} \|(a_i)\|_{I_d, \{\emptyset\}, p}$  is just Jensen's inequality ( $p \geq 2$ ) or the aforesaid hypercontractivity of Rademacher chaos ( $p \in (1, 2)$ ). On the other hand, for  $K \neq I_d$  and  $\mathcal{J} \in$

$\mathcal{P}_{I_d \setminus K}$ , we have

$$\begin{aligned}
\|S\|_p &= \left( \mathbb{E}_{I_d \setminus K} \mathbb{E}_K \left| \sum_{i_K} \prod_{k \in K} \varepsilon_{i_k}^{(k)} \sum_{i_{I_d \setminus K}} a_i \prod_{k \notin K} \varepsilon_{i_k}^{(k)} \right|^p \right)^{1/p} \\
&\geq \frac{1}{L \# K} \left( \mathbb{E}_{I_d \setminus K} \left( \sum_{i_K} \left| \sum_{i_{I_d \setminus K}} a_i \prod_{k \notin K} \varepsilon_{i_k}^{(k)} \right|^2 \right)^{p/2} \right)^{1/p} \\
&= \frac{1}{L \# K} \left( \mathbb{E}_{I_d \setminus K} \sup_{\sum_{i_K} |\alpha_{i_K}^{(0)}|^2 \leq 1} \left| \sum_{i_{I_d \setminus K}} \sum_{i_K} \langle \alpha_{i_K}^{(0)}, a_i \rangle \prod_{k \notin K} \varepsilon_{i_k}^{(k)} \right|^p \right)^{1/p} \\
&\geq \frac{1}{L \# K} \left( \sup_{\sum_{i_K} |\alpha_{i_K}^{(0)}|^2 \leq 1} \mathbb{E}_{I_d \setminus K} \left| \sum_{i_{I_d \setminus K}} \sum_{i_K} \langle \alpha_{i_K}^{(0)}, a_i \rangle \prod_{k \notin K} \varepsilon_{i_k}^{(k)} \right|^p \right)^{1/p} \\
&\geq \frac{1}{L \# K L_{d-\#K}} \sup_{\sum_{i_K} |\alpha_{i_K}^{(0)}|^2 \leq 1} \left\| \left( \sum_{i_K} \langle \alpha_{i_K}^{(0)}, a_i \rangle \right)_{i_{I_d \setminus K}} \right\|_{\emptyset, \mathcal{J}, p} \\
&= \frac{1}{L \# K L_{d-\#K}} \|(a_i)\|_{K, \mathcal{J}, p},
\end{aligned}$$

where the first inequality follows from hypercontractivity applied conditionally on  $(\varepsilon_i^{(k)})_{k \notin K, i \in I_n}$ , the second is Jensen's inequality and the third is (18) applied for a chaos of order  $d - \#K$ .

The tail estimate follows from moment estimates by the Paley-Zygmund inequality and the inequality  $\|(a_i)\|_{K, \mathcal{J}, tp} \leq t^{\deg \mathcal{J}} \|(a_i)\|_{K, \mathcal{J}, p}$  for  $t \geq 1$  just like in [12, 18].  $\square$

**PROOF OF NECESSITY.** First we will prove the integrability condition (11). Let us notice that by classical hypercontractive estimates for Rademacher chaoses and the Paley-Zygmund inequality (or by Lemma 16), we have

$$\mathbb{P}_\varepsilon \left( \left| \sum_{|i| \leq 2^n} \varepsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}}) \right| \geq c_d \sqrt{\sum_{|i| \leq 2^n} h(\mathbf{X}_i^{\text{dec}})^2} \right) \geq c_d$$

for some constant  $c_d > 0$ . By the Fubini theorem it gives

$$\mathbb{P}_\varepsilon \left( \left| \sum_{|i| \leq 2^n} \varepsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}}) \right| \geq D 2^{nd/2} \log^{d/2} n \right) \geq c_d \mathbb{P} \left( \sum_{|i| \leq 2^n} h(\mathbf{X}_i^{\text{dec}})^2 \geq D^2 c_d^{-2} 2^{nd} \log^d n \right),$$

which together with Lemma 8 yields

$$\sum_n \mathbb{P} \left( \sum_{|i| \leq 2^n} h(\mathbf{X}_i^{\text{dec}})^2 \geq D^2 c_d^{-2} 2^{nd} \log^d n \right) < \infty.$$

The integrability condition (11) follows now from Corollary 8.

Before we proceed to the proof of (12), let us notice that (11) and Lemma 7 imply that

$$(19) \quad \mathbb{E}(|h|^2 \wedge u) \leq K(\log \log u)^d$$

for  $n$  large enough. The proof of (12) can be now obtained by adapting the argument for the real valued case.

Since  $\lim_{n \rightarrow \infty} \sum_{k=n}^{2n} \frac{1}{k} = \log 2$ , (5) implies that there exists  $N_0$ , such that for all  $N > N_0$ , there exists  $N \leq n \leq 2N$ , satisfying

$$(20) \quad \mathbb{P}\left(\left| \sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}}) \right| > L_d C 2^{nd/2} \log^{d/2} n\right) < \frac{1}{10n}.$$

Let us thus fix  $N > N_0$  and consider  $n$  as above. Let  $K \subseteq I_d$ ,  $\mathcal{J} = \{J_1, \dots, J_k\} \in \mathcal{P}_{I_d \setminus K}$ . Let us also fix functions  $g: \Sigma^{\#K} \rightarrow H$ ,  $f_j: \Sigma^{\#J_j} \rightarrow \mathbb{R}$ ,  $j = 1, \dots, k$ , such that

$$\begin{aligned} \|g(X_k)\|_2 &\leq 1, \|g(X_K)\|_\infty \leq 2^{n/(2k+3)}, \\ \|f_j(X_{J_j})\|_2 &\leq 1, \|f_j(X_{J_j})\|_\infty \leq 2^{n/(2k+3)}. \end{aligned}$$

The Chebyshev inequality gives

$$(21) \quad \mathbb{P}\left(\sum_{|i_{J_j}| \leq 2^n} f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}})^2 \log n \leq 10 \cdot 2^d 2^{\#J_j n} \log n\right) \geq 1 - \frac{1}{10 \cdot 2^d}.$$

Similarly, if  $K \neq \emptyset$ ,

$$(22) \quad \mathbb{P}\left(\sum_{|i_K| \leq 2^n} |g(\mathbf{X}_{i_K}^{\text{dec}})|^2 \leq 10 \cdot 2^d 2^{\#K n}\right) \geq 1 - \frac{1}{10 \cdot 2^d}$$

and for  $K = \emptyset$ ,  $|g| \leq 1$  (recall that for  $K = \emptyset$ , the function  $g$  is constant).

Moreover for  $j = 1, \dots, k$  and sufficiently large  $N$ ,

$$\begin{aligned} \sum_{|i_{\circ J_j}| \leq 2^n} \frac{1}{2^{n\#J_j}} f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}})^2 \cdot \log n &\leq \frac{2^{n\#J_j} 2^{n/(2k+3)} \log n}{2^{n\#J_j}} \\ &\leq \frac{2^{2n/(2k+3)} \log n}{2^n} \leq 1. \end{aligned}$$

Without loss of generality we may assume that the sequences  $(X_i^{(j)})_{i,j}$  and  $(\varepsilon_i^{(j)})_{i,j}$  are defined as coordinates of a product probability space. If for

each  $j = 1, \dots, k$  we denote the set from (21) by  $A_k$ , and the set from (22) by  $A_0$ , we have  $\mathbb{P}(\bigcap_{j=0}^k A_k) \geq 0.9$ . Recall now Lemma 16. On  $\bigcap_{j=0}^k A_k$  we can estimate the  $\|\cdot\|_{K, \mathcal{J}, \log n}^*$  norms of the matrix  $(h(\mathbf{X}_i^{\text{dec}}))_{|i| \leq 2^n}$  by using the test sequences

$$\alpha_{i_{J_j}}^{(j)} = \frac{f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \sqrt{\log n}}{10^{1/2} 2^{d/2} 2^{n \# J_j / 2}}$$

for  $j = 1, \dots, k$  and

$$\alpha_{i_K}^{(0)} = \frac{g(\mathbf{X}_{i_K}^{\text{dec}})}{10^{1/2} 2^{d/2} 2^{n \# K / 2}}.$$

Therefore with probability at least 0.9 we have

(23)

$$\begin{aligned} & \| (h(\mathbf{X}_i^{\text{dec}}))_{|i| \leq 2^n} \|_{K, \mathcal{J}, \log n}^* \\ & \geq \frac{(\log n)^{k/2}}{2^{d(k+1)/2} 10^{(k+1)/2} 2^{(\#K + \sum_j \#J_j)n/2}} \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h(\mathbf{X}_i^{\text{dec}}) \rangle \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \right| \\ & = \frac{(\log n)^{k/2}}{2^{d(k+1)/2} 10^{(k+1)/2} 2^{dn/2}} \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h(\mathbf{X}_i^{\text{dec}}) \rangle \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \right|. \end{aligned}$$

Our aim is now to further bound from below the right hand side of the above inequality, to have, via Lemma 16, control from below on the conditional tail probability of  $\sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}})$ , given the sample  $(X_i^{(j)})$ .

From now on let us assume that

$$(24) \quad |\mathbb{E} \langle g(X_K), h(X) \rangle \prod_{j=1}^k f_j(X_{J_j})| > 1.$$

The Markov inequality, (19) and Lemma 15 give

$$\begin{aligned} \mathbb{P} \left( \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_K), h(\mathbf{X}_i^{\text{dec}}) \rangle \mathbf{1}_{\{|h(\mathbf{X}_i^{\text{dec}})| > 2^n\}} \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \right| \geq \frac{2^{nd} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j|}{4} \right) \\ \leq 4 \frac{2^{nd} (\|g\|_\infty \prod_{j=1}^k \|f_j\|_\infty) \cdot \mathbb{E} |h| \mathbf{1}_{\{|h| > 2^n\}}}{2^{nd} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j|} \leq 4 2^{n(k+1)/(2k+3)} \mathbb{E} |h| \mathbf{1}_{\{|h| > 2^n\}} \end{aligned}$$

(25)

$$\leq 4K \frac{(\log n)^d}{2^{\frac{n(k+2)}{2k+3}}}.$$

Let now  $h_n = h \mathbf{1}_{\{|h| \leq 2^n\}}$ . By the Chebyshev inequality, Lemma 14 and (19)

$$\begin{aligned}
(26) \quad & \mathbb{P} \left( \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h_n(\mathbf{X}_i^{\text{dec}}) \rangle \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) - 2^{nd} \mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j \right| \right. \\
& \geq \frac{2^{nd}}{5} |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j| \Big) \\
& \leq 25 \frac{\text{Var}(\sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h_n(\mathbf{X}_i^{\text{dec}}) \rangle \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}))}{2^{2nd} |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j|^2} \\
& \leq 25 \frac{(2^d - 1) 2^{n(2d-1)}}{2^{2nd} |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j|^2} \mathbb{E} |\langle g, h_n \rangle \prod_{j=1}^k f_j|^2 \\
& \leq 25 (2^d - 1) \frac{2^{2n(k+1)/(2k+3)} \mathbb{E} |h_n|^2}{2^n |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j|^2} \\
& \leq 25 K (2^d - 1) \frac{\log^d n}{2^{n/(2k+3)} |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j|^2}.
\end{aligned}$$

Let us also notice that for large  $n$ , by (19), Lemma 15 and (24)

$$\begin{aligned}
(27) \quad & |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^k f_j| \geq |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| - |\mathbb{E} \langle g, h \rangle \mathbf{1}_{\{|h| > 2^n\}} \prod_{j=1}^k f_j| \\
& \geq |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| - 2^{n(k+1)/(2k+3)} K \frac{(\log n)^d}{2^n} \geq \frac{5}{8} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| \geq \frac{5}{8}.
\end{aligned}$$

Inequalities (25), (26) and (27) imply, that for large  $n$  with probability at

least 0.9 we have

$$\begin{aligned}
& \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h(\mathbf{X}_i^{\text{dec}}) \rangle \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \right| \\
& \geq \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h_n(\mathbf{X}_i^{\text{dec}}) \rangle \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \right| \\
& \quad - \left| \sum_{|i| \leq 2^n} \langle g(\mathbf{X}_{i_K}^{\text{dec}}), h(\mathbf{X}_i^{\text{dec}}) \rangle \mathbf{1}_{\{|h(\mathbf{X}_i^{\text{dec}})| > 2^n\}} \prod_{j=1}^k f_j(\mathbf{X}_{i_{J_j}}^{\text{dec}}) \right| \\
& \geq 2^{nd} \left( \frac{4}{5} |\mathbb{E} \langle g, h_n \rangle \prod_{j=1}^n f_j| - \frac{1}{4} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| \right) \\
& \geq 2^{nd} \left( \frac{4}{5} \cdot \frac{5}{8} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^n f_j| - \frac{1}{4} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| \right) \geq \frac{2^{nd}}{4} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j|.
\end{aligned}$$

Together with (23) this yields that for large  $n$  with probability at least 0.8,

$$\|(h_i)_{|i| \leq 2^n}\|_{K, \mathcal{J}, \log n}^* \geq \frac{2^{nd/2} \log^{k/2} n}{4 \cdot 2^{d(k+1)/2} 10^{(k+1)/2}} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j|.$$

Thus, by Lemma 16, for large  $n$

$$\mathbb{P} \left( \left| \sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}}) \right| \geq c_d \frac{2^{nd/2} \log^{k/2} n}{4 \cdot 2^{d(k+1)/2} 10^{(k+1)/2}} |\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| \right) \geq \frac{8}{10n},$$

which together with (20) gives

$$|\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| \leq L_d C \frac{4 \cdot 2^{d(k+1)/2} 10^{(k+1)/2}}{c_d} \log^{(d-k)/2} n.$$

In particular for sufficiently large  $N$ , for arbitrary functions  $g: \Sigma^{\#K} \rightarrow H$ ,  $f_j: \Sigma^{\#J_j} \rightarrow \mathbb{R}$ ,  $j = 1, \dots, k$ , such that

$$\begin{aligned}
& \|g(\mathbf{X}_K)\|_\infty, \|f_j(X_{J_j})\|_2 \leq 1, \\
& \|g(\mathbf{X}_K)\|_2, \|f_j(X_{J_j})\|_\infty \leq 2^{N/(2k+3)}
\end{aligned}$$

we have

$$|\mathbb{E} \langle g, h \rangle \prod_{j=1}^k f_j| \leq L_d C \frac{4 \cdot 2^{d(k+1)/2} 10^{(k+1)/2}}{c_d} \log^{(d-k)/2} n \leq \tilde{L}_d C \log^{(d-k)/2} N,$$

which clearly implies (12).  $\square$

### 7. Sufficiency.

LEMMA 17. *Let  $H = H(X_1, \dots, X_d)$  be a nonnegative random variable, such that  $\mathbb{E}H^2 < \infty$ . Then for  $I \subseteq I_d$ ,  $I \neq \emptyset, I_d$ ,*

$$\sum_{l=0}^{\infty} \sum_{n=1}^{\infty} 2^{l+\#I^c n} \mathbb{P}_{I^c}(\mathbb{E}_I H^2 \geq 2^{2l+\#I^c n}) < \infty.$$

PROOF.

$$\begin{aligned} \sum_l \sum_n 2^{l+\#I^c n} \mathbb{P}_{I^c}(\mathbb{E}_I H^2 \geq 2^{2l+\#I^c n}) &= \sum_l 2^l \mathbb{E}_{I^c} \left[ \sum_n 2^{\#I^c n} \mathbf{1}_{\{\mathbb{E}_I |H|^2 \geq 2^{2l+\#I^c n}\}} \right] \\ &\leq \sum_l 2^{1-l} \mathbb{E}_{I^c} \mathbb{E}_I H^2 \leq 4 \mathbb{E}H^2 < \infty. \end{aligned}$$

□

LEMMA 18. *Let  $\mathbf{X} = (X_1, \dots, X_d)$  and  $\tilde{\mathbf{X}}(I) = ((X_i)_{i \in I}, (X_i^{(1)})_{i \in I^c})$ . Denote  $H = |h|/(\mathbb{L}\mathbb{L}|h|)^{d/2}$ . If  $\mathbb{E}|H|^2 < \infty$  and  $h_n = h \mathbf{1}_{A_n}$ , where*

$$A_n \subseteq \{x: |h(x)|^2 \leq 2^{nd} \log^d n \text{ and } \forall_{I \neq \emptyset, I_d} \mathbb{E}_I H^2 \leq 2^{\#I^c n}\},$$

then for  $I \subseteq I_d$ ,  $I \neq \emptyset$ , we have

$$\sum_n \frac{2^{-n\#I}}{\log^{2d} n} \mathbb{E}[|h_n(X)|^2 |h_n(\tilde{\mathbf{X}}(I))|^2] < \infty.$$

PROOF. a)  $I = I_d$

$$\begin{aligned} \sum_n \frac{\mathbb{E}|h_n|^4}{2^{nd} \log^{2d} n} &\leq \mathbb{E}|h|^4 \sum_n \frac{1}{2^{nd} \log^{2d} n} \mathbf{1}_{\{|h|^2 \leq 2^{nd} \log^d n\}} \\ &\leq L_d \mathbb{E}|h|^4 \frac{1}{|h|^2 (\mathbb{L}\mathbb{L}|h|)^d} < \infty. \end{aligned}$$

b)  $I \neq I_d, \emptyset$ . Let us denote by  $\mathbb{E}_I, \mathbb{E}_{I^c}, \tilde{\mathbb{E}}_{I^c}$  respectively, the expectation with respect to  $(X_i)_{i \in I}, (X_i)_{i \in I^c}$  and  $(X_i^{(1)})_{i \in I^c}$ . Let also  $\tilde{h}, \tilde{h}_n$  stand for  $h(\tilde{\mathbf{X}}(I)), h_n(\tilde{\mathbf{X}}(I))$  respectively. Then



$$\begin{aligned}
& \sum_n \frac{\mathbb{E}(|h_n|^2 \cdot |\tilde{h}_n|^2)}{2^{n\#I} \log^{2d} n} \leq 2 \sum_n \frac{\mathbb{E}(|h_n|^2 \cdot |\tilde{h}_n|^2 \mathbf{1}_{\{|h| \leq |\tilde{h}|\}})}{2^{n\#I} \log^{2d} n} \\
& \leq 2\mathbb{E}\left(|h|^2 |\tilde{h}|^2 \mathbf{1}_{\{|h| \leq |\tilde{h}|\}}\right) \\
& \times \sum_n \frac{1}{2^{n\#I} \log^{2d} n} \mathbf{1}_{\{\mathbb{E}_{I^c} |h|^2 \mathbf{1}_{\{|h|^2 \leq 2^{2nd}\}} \leq L_d 2^{\#I n} \log^d n, |\tilde{h}|^2 \leq 2^{2nd}\}} \\
& \leq 2\mathbb{E}\left(|h|^2 |\tilde{h}|^2 \mathbf{1}_{\{|h| \leq |\tilde{h}|\}}\right) \\
& \times \sum_n \frac{1}{2^{n\#I} \log^{2d} n} \mathbf{1}_{\{\mathbb{E}_{I^c} |h|^2 \mathbf{1}_{\{|h|^2 \leq |\tilde{h}|^2\}} \leq L_d 2^{\#I n} \log^d n, |\tilde{h}|^2 \leq 2^{2nd}\}} \\
& \leq \tilde{L}_d \mathbb{E}\left(|h|^2 |\tilde{h}|^2 \mathbf{1}_{\{|h| \leq |\tilde{h}|\}} \frac{1}{(\mathbb{E}_{I^c} |h|^2 \mathbf{1}_{\{|h|^2 \leq |\tilde{h}|^2\}})(\mathbb{L}\mathbb{L}|\tilde{h}|)^d}\right) \\
& = \tilde{L}_d \mathbb{E}_I \tilde{\mathbb{E}}_{I^c} \left[ |\tilde{h}|^2 \mathbb{E}_{I^c} \left( |h|^2 \mathbf{1}_{\{|h| \leq |\tilde{h}|\}} \frac{1}{(\mathbb{E}_{I^c} |h|^2 \mathbf{1}_{\{|h|^2 \leq |\tilde{h}|^2\}})(\mathbb{L}\mathbb{L}|\tilde{h}|)^d} \right) \right] \\
& \leq \tilde{L}_d \mathbb{E} \frac{|\tilde{h}|^2}{(\mathbb{L}\mathbb{L}|\tilde{h}|)^d} < \infty,
\end{aligned}$$

where to obtain the second inequality, we used the fact that

$$\begin{aligned}
& \mathbb{E}_{I^c} |h|^2 \mathbf{1}_{\{|h|^2 \leq 2^{2nd}, \mathbb{E}_{I^c} H^2 \leq 2^{\#I n}\}} \\
& \leq \mathbb{E}_{I^c} \frac{|h|^2}{(\mathbb{L}\mathbb{L}|h|)^d} (\log \log 2^{2nd})^d \mathbf{1}_{\{\mathbb{E}_{I^c} H^2 \leq 2^{\#I n}\}} \\
& \leq L_d \mathbb{E}_{I^c} H^2 \mathbf{1}_{\{\mathbb{E}_{I^c} H^2 \leq 2^{\#I n}\}} \log^d n \leq L_d 2^{\#I n} \log^d n.
\end{aligned}$$

□

LEMMA 19. Consider a square integrable, nonnegative random variable  $Y$ . Let  $Y_n = Y \mathbf{1}_{B_n}$ , with  $B_n = \bigcup_{k \in K(n)} C_k$ , where  $C_0, C_1, C_2, \dots$  are pairwise disjoint subsets of  $\Omega$  and

$$K(n) = \{k \leq n : \mathbb{E}(Y^2 \mathbf{1}_{C_k}) \leq 2^{k-n}\}.$$

Then

$$\sum_n (\mathbb{E} Y_n^2)^2 < \infty$$

PROOF. Let us first notice that by the Schwarz inequality, we have

$$\begin{aligned} \left( \sum_{k \in K(n)} \mathbb{E}(Y^2 \mathbf{1}_{C_k}) \right)^2 &= \left( \sum_{k \in K(n)} 2^{(n-k)/2} 2^{(k-n)/2} \mathbb{E}(Y^2 \mathbf{1}_{C_k}) \right)^2 \\ &\leq \sum_{k \in K(n)} [2^{n-k} (\mathbb{E}(Y^2 \mathbf{1}_{C_k}))^2] \sum_{k \leq n} 2^{k-n} \\ &= 2 \sum_{k \in K(n)} [2^{n-k} (\mathbb{E}(Y^2 \mathbf{1}_{C_k}))^2]. \end{aligned}$$

Thus

$$\begin{aligned} \sum_n (\mathbb{E}Y_n^2)^2 &\leq \sum_n 2 \sum_{k \in K(n)} [2^{n-k} (\mathbb{E}(Y^2 \mathbf{1}_{C_k}))^2] \\ &\leq 2 \sum_{k: \mathbb{E}(Y^2 \mathbf{1}_{C_k}) > 0} (\mathbb{E}(Y^2 \mathbf{1}_{C_k}))^2 \sum_{n: k \in K(n)} 2^{n-k} \\ &\leq 4 \sum_{k: \mathbb{E}(Y^2 \mathbf{1}_{C_k}) > 0} (\mathbb{E}(Y^2 \mathbf{1}_{C_k}))^2 \max_{n: k \in K(n)} 2^{n-k} \\ &\leq 4 \sum_{k: \mathbb{E}(Y^2 \mathbf{1}_{C_k}) > 0} (\mathbb{E}(Y^2 \mathbf{1}_{C_k}))^2 \frac{1}{\mathbb{E}(Y^2 \mathbf{1}_{C_k})} \\ &\leq 4 \sum_k \mathbb{E}(Y^2 \mathbf{1}_{C_k}) = 4\mathbb{E}Y^2 < \infty. \end{aligned}$$

□

PROOF OF SUFFICIENCY. The proof consists of several truncation arguments. The first part of it follows the proofs presented in [11] and [2] for the real-valued case. Then some modifications are required, reflecting the diminished integrability condition in the Hilbert space case. At each step we will show that

$$(28) \quad \sum_{n=1}^{\infty} \mathbb{P}\left( \left| \sum_{|i| \leq 2^n} \pi_d h_n(\mathbf{X}_i^{\text{dec}}) \right| \geq C 2^{nd/2} \log^{d/2} n \right) < \infty,$$

with  $h_n = h \mathbf{1}_{A_n}$  for some sequence of sets  $A_n$ . In the whole proof we keep the notation  $H = |h| / (\mathbb{L}L|h|)^{d/2}$ .

Let us also fix  $\eta_d \in (0, 1)$ , such that the following implication holds

$$(29) \quad \forall_{n=1,2,\dots} |h|^2 \leq \eta_d^2 2^{nd} \log^d n \implies H^2 \leq 2^{nd}.$$

**Step 1.** Inequality (28) holds for any  $C > 0$  if

$$A_n \subseteq \{x : |h(x)|^2 \geq \eta_d^2 2^{nd} \log^d n\}.$$

We have, by the Chebyshev inequality and the inequality  $\mathbb{E}|\pi_d h_n| \leq 2^d \mathbb{E}|h_n|$  (which follows directly from the definition of  $\pi_d$  or may be considered a trivial case of Lemma 6),

$$\begin{aligned} \sum_n \mathbb{P}(|\sum_{|i| \leq 2^n} \pi_d h_n(\mathbf{X}_i^{\text{dec}})| \geq C 2^{nd/2} \log^{d/2} n) & \\ & \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}|\sum_{|i| \leq 2^n} \pi_d h_n(\mathbf{X}_i^{\text{dec}})|}{C 2^{nd/2} \log^{d/2} n} \\ & \leq 2^d \sum_n \frac{2^{nd} \mathbb{E}|h| \mathbf{1}_{\{|h| > \eta_d 2^{nd/2} \log^{d/2} n\}}}{C 2^{nd/2} \log^{d/2} n} \\ & = 2^d C^{-1} \mathbb{E}\left(|h| \sum_n \frac{2^{nd/2}}{\log^{d/2} n} \mathbf{1}_{\{|h| > \eta_d 2^{nd/2} \log^{d/2} n\}}\right) \\ & \leq L_d C^{-1} \mathbb{E} \frac{|h|^2}{(\text{LL}|h|)^d} < \infty. \end{aligned}$$

**Step 2.** Inequality (28) holds for any  $C > 0$  if

$$A_n \subseteq \{x : |h(x)|^2 \leq \eta_d^2 2^{nd} \log^d n, \exists_{I \neq \emptyset, I_d} \mathbb{E}_I H^2 \geq 2^{\#I^c n}\}.$$

As in the previous step, it is enough to prove that

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}|\sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h_n(\mathbf{X}_i^{\text{dec}})|}{2^{nd/2} \log^{d/2} n} < \infty.$$

The set  $A_n$  can be written as

$$A_n = \bigcup_{I \subseteq I_d, I \neq I_d, \emptyset} A_n(I),$$

where the sets  $A_n(I)$  are pairwise disjoint and

$$A_n(I) \subseteq \{x : |h(x)|^2 \leq 2^{2nd}, \mathbb{E}_I H^2 \geq 2^{\#I^c n}\}.$$

Therefore it suffices to prove that

$$(30) \quad \sum_{n=1}^{\infty} \frac{\mathbb{E}|\sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h(\mathbf{X}_i^{\text{dec}}) \mathbf{1}_{A_n(I)}(\mathbf{X}_i^{\text{dec}})|}{2^{nd/2} \log^{d/2} n} < \infty.$$

Let for  $l \in \mathbb{N}$ ,

$$A_{n,l}(I) := \{x : |h(x)|^2 \leq 2^{2nd}, \\ 2^{2l+2+\#I^c n} > \mathbb{E}_I H^2 \geq 2^{2l+\#I^c n}\} \cap A_n(I).$$

Then  $h_n \mathbf{1}_{A_n(I)} = \sum_{l=0}^{\infty} h_{n,l}$ , where  $h_{n,l} := h_n \mathbf{1}_{A_{n,l}(I)}$  (notice that the sum is actually finite in each point  $x \in \Sigma^d$  as for large  $l$ ,  $x \notin A_{n,l}(I)$ ).

We have

$$\begin{aligned} \mathbb{E} \left| \sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h_{n,l}(\mathbf{X}_i^{\text{dec}}) \right| &\leq \sum_{|i^c| \leq 2^n} \mathbb{E}_{I^c} \mathbb{E}_I \left| \sum_{|i| \leq 2^n} \epsilon_{i_I}^{\text{dec}} h_{n,l}(\mathbf{X}_i^{\text{dec}}) \right| \\ &\leq \sum_{|i^c| \leq 2^n} \mathbb{E}_{I^c} (\mathbb{E}_I \left| \sum_{|i| \leq 2^n} \epsilon_{i_I}^{\text{dec}} h_{n,l}(\mathbf{X}_i^{\text{dec}}) \right|^2)^{1/2} \\ &\leq 2^{(\#I^c + \#I/2)n} \mathbb{E}_{I^c} (\mathbb{E}_I |h_{n,l}|^2)^{1/2} \\ &\leq L_d [2^{(\#I^c + d/2)n + l + 1} \log^{d/2} n] \mathbb{P}_{I^c} (\mathbb{E}_I H^2 \geq 2^{2l + \#I^c n}), \end{aligned}$$

where in the last inequality we used the estimate

$$\begin{aligned} \mathbb{E}_I h_{n,l}^2 &\leq L_d \mathbb{E}_I [(\log n)^d H^2 \mathbf{1}_{\{2^{2l+2+\#I^c n} > \mathbb{E}_I H^2 \geq 2^{2l+\#I^c n}\}}] \\ &\leq L_d 2^{2l+2+\#I^c n} (\log n)^d \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{2l+\#I^c n}\}}. \end{aligned}$$

Therefore to get (30) it is enough to show that

$$\sum_{l=0}^{\infty} \sum_n 2^{l+\#I^c n} \mathbb{P}_{I^c} (\mathbb{E}_I H^2 \geq 2^{2l+\#I^c n}) < \infty.$$

But this is just the statement of Lemma 17.

**Step 3.** Inequality (28) holds for any  $C > 0$  if

$$A_n \subseteq \{x : |h(x)|^2 \leq \eta_d^2 2^{2nd} \log^d n, \forall_{I \neq \emptyset, I_d} \mathbb{E}_I H^2 \leq 2^{\#I^c n}\} \cap \bigcup_{I \subsetneq I_d} B_n^I,$$

with  $B_n^I = \bigcup_{k \in K(I,n)} C_k^I$  and  $C_0^I = \{x : \mathbb{E}_I H^2 \leq 1\}$ ,  $C_k^I = \{x : 2^{\#I^c(k-1)} < \mathbb{E}_I H^2 \leq 2^{\#I^c k}\}$ ,  $k \geq 1$ ,  $K(I,n) = \{k \leq n : \mathbb{E}(H^2 \mathbf{1}_{C_k^I}) \leq 2^{k-n}\}$ .

By Lemma 6 and the Chebyshev inequality, it is enough to show that

$$\sum_n \frac{\mathbb{E} \left| \sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h_n(\mathbf{X}_i^{\text{dec}}) \right|^4}{2^{2nd} \log^{2d} n} < \infty.$$

The Khintchine inequality for Rademacher chaoses gives

$$\begin{aligned}
L_d^{-1} \mathbb{E} \left| \sum_{|i| \leq 2^n} \epsilon_i^{\text{dec}} h_n(\mathbf{X}_i^{\text{dec}}) \right|^4 &\leq \mathbb{E} \left( \sum_{|i| \leq 2^n} |h_n(\mathbf{X}_i^{\text{dec}})|^2 \right)^2 \\
&= \sum_{I \subseteq I_d} \sum_{|i| \leq 2^n} \sum_{\substack{|j| \leq 2^n : \\ \{k : i_k = j_k\} = I}} \mathbb{E} [|h_n(\mathbf{X}_i^{\text{dec}})|^2 |h_n(\mathbf{X}_j^{\text{dec}})|^2] \\
&\leq \sum_{I \subseteq I_d} 2^{nd} 2^{n(d-\#I)} \mathbb{E} [|h_n(\mathbf{X})|^2 \cdot |h_n(\tilde{\mathbf{X}}(I))|^2],
\end{aligned}$$

where  $\mathbf{X} = (X_1, \dots, X_d)$  and  $\tilde{\mathbf{X}}(I) = ((X_i)_{i \in I}, (X_i^{(1)})_{i \in I^c})$ .

To prove the statement of this step it thus suffices to show that for all  $I \subseteq I_d$ ,

$$(31) \quad S(I) := \sum_n \frac{2^{-n\#I}}{\log^{2d} n} \mathbb{E} [|h_n(X)|^2 |h_n(\tilde{X}(I))|^2] < \infty.$$

The case of nonempty  $I$  follows from Lemma 18. It thus remains to consider the case  $I = \emptyset$ . Set  $H_I^2 = \mathbb{E}_I H^2$ . We have

$$\begin{aligned}
S(\emptyset) &= \sum_n \frac{(\mathbb{E} |h_n|^2)^2}{\log^{2d} n} = \sum_n (\mathbb{E} (\frac{|h|^2}{\log^d n} \mathbf{1}_{A_n}))^2 \leq L_d \sum_n (\mathbb{E} (H^2 \mathbf{1}_{A_n}))^2 \\
&\leq L_d \sum_n (\mathbb{E} (H^2 \sum_{I \subsetneq I_d} \mathbf{1}_{B_n^I}))^2 \leq \tilde{L}_d \sum_{I \subsetneq I_d} \sum_n (\mathbb{E} (H^2 \mathbf{1}_{B_n^I}))^2 \\
&= \tilde{L}_d \sum_{I \subsetneq I_d} \sum_n (\mathbb{E} (H_I^2 \mathbf{1}_{B_n^I}))^2 < \infty
\end{aligned}$$

by Lemma 19, applied for  $Y^2 = \mathbb{E}_I H^2$ , since  $\mathbb{E} H_I^2 = \mathbb{E} H^2 < \infty$ .

**Step 4.** Inequality (28) holds for some  $C \leq L_d D$  if

$$A_n = \{x : |h(x)|^2 \leq \eta_d^2 2^{nd} \log^d n, \forall_{I \neq \emptyset, I_d} \mathbb{E}_I H^2 \leq 2^{\#I c n}\} \cap \bigcap_{I \subsetneq I_d} (B_n^I)^c,$$

where  $B_n^I$  is defined as in the previous step.

Let us first estimate  $\|(E_I |h_n|^2)^{1/2}\|_\infty$  for  $I \subsetneq I_d$ . We have

$$\begin{aligned}
\mathbb{E}_I |h_n|^2 &\leq \mathbb{E}_I \left[ |h|^2 \mathbf{1}_{\{|h|^2 \leq \eta_d^2 2^{nd} \log^d n\}} \sum_{k \leq n, k \notin K(I, n)} \mathbf{1}_{C_k^I} \right] \\
&\leq L_d \log^d n \sum_{k \leq n, k \notin K(I, n)} \mathbf{1}_{C_k^I} \mathbb{E}_I H^2.
\end{aligned}$$

The fact that we can restrict the summation to  $k \leq n$  follows directly from the definition of  $A_n$  for  $I \neq \emptyset$  and for  $I = \emptyset$  from (29).

The sets  $C_k^I$  are pairwise disjoint and thus

$$(32) \quad \|\mathbb{E}_I |h_n|^2\|_\infty \leq (L_d \log^d n) \max_{k \leq n, k \notin K(I, n)} 2^{\#I^c k} = L_d 2^{\#I^c k_I(n)} \log^d n,$$

where

$$k_I(n) = \max\{k \leq n : k \notin K(I, n)\}.$$

Therefore for  $C > 0$ ,

$$\begin{aligned} & \sum_n \exp \left[ -\frac{1}{\bar{L}_d} \left( \frac{C 2^{nd/2} \log^{d/2} n}{2^{\#In/2} \|\mathbb{E}_I |h_n|^2\|_\infty^{1/2}} \right)^{2/(d+\#I^c)} \right] \\ & \leq \sum_n \sum_{k \leq n, k \notin K(I, n)} \exp \left[ -\frac{1}{\bar{L}_d} \left( \frac{C 2^{nd/2} \log^{d/2} n}{2^{\#In/2} 2^{\#I^c k/2} \log^{d/2} n} \right)^{2/(d+\#I^c)} \right] \\ & = \sum_k \sum_{n \geq k, k \notin K(I, n)} \exp \left[ -\frac{1}{\bar{L}_d} \left( C 2^{\#I^c(n-k)/2} \right)^{2/(d+\#I^c)} \right]. \end{aligned}$$

Notice that for each  $k$  the inner series is bounded by a geometric series with the ratio smaller than some  $q_{d,C} < 1$  ( $q_{d,C}$  depending only on  $d$  and  $C$ ). Therefore the right hand side of the above inequality is bounded by

$$K \sum_k \sup_{n \geq k, k \notin K(I, n)} \exp \left[ -\frac{1}{\bar{L}_d} \left( C 2^{\#I^c(n-k)/2} \right)^{2/(d+\#I^c)} \right],$$

with the convention  $\sup \emptyset = 0$ . But  $k \notin K(I, n)$  implies that  $2^{\#I^c(n-k)/2} \geq (\mathbb{E}(H^2 \mathbf{1}_{C_k^I}))^{-\#I^c/2}$ . Therefore the above quantity is further bounded by

$$\begin{aligned} K \sum_k \exp \left[ -\frac{1}{\bar{L}_d} \left( C^{-2/\#I^c} \mathbb{E}(H^2 \mathbf{1}_{C_k^I}) \right)^{-\#I^c/(d+\#I^c)} \right] & \leq \bar{L}_d C^{-2/\#I^c} \sum_k \mathbb{E}(H^2 \mathbf{1}_{C_k^I}) \\ & = \bar{L}_d C^{-2/\#I^c} \mathbb{E} H^2 < \infty, \end{aligned}$$

where we used the inequality  $e^x \geq c_d x^\alpha$  for all  $x \geq 0$  and  $0 \leq \alpha \leq 2d$ . We have thus proven that for all  $I \subsetneq I_d$  and  $C, L_d > 0$ ,

$$(33) \quad \sum_{n: A_n \neq \emptyset} \exp \left[ -\frac{1}{\bar{L}_d} \left( \frac{C 2^{nd/2} \log^{d/2} n}{2^{\#In/2} \|\mathbb{E}_I |h_n|^2\|_\infty^{1/2}} \right)^{2/(d+\#I^c)} \right] < \infty.$$

Now we will turn to the estimation of  $\|h_n\|_{J_0, \mathcal{J}}$ . Let us consider  $J_0 \subseteq I_d$ ,  $\mathcal{J} = \{J_1, \dots, J_l\} \in \mathcal{P}_{I_d \setminus J_0}$  and denote as before  $X = (X_1, \dots, X_d)$ ,  $X_I = (X_i)_{i \in I}$ . Recall that

$$\|h_n\|_{J_0, \mathcal{J}} = \sup \left\{ \mathbb{E} \langle h_n(X), f_0(\mathbf{X}_{J_0}) \rangle \prod_{i=1}^l f_i(X_{J_i}) : \mathbb{E} |f_0(\mathbf{X}_{J_0})|^2 \leq 1, \right. \\ \left. \mathbb{E} f_i^2(X_{J_i}) \leq 1, i \geq 1 \right\}.$$

In what follows, to simplify the already quite complicated notation, let us suppress the arguments of all the functions and write just  $h$  instead of  $h(X)$  and  $f_i$  instead of  $f_i(X_{J_i})$ .

Let us also remark that although  $f_0$  plays special role in the definition of  $\|\cdot\|_{J_0, \mathcal{J}}$ , in what follows the same arguments will apply to all  $f_i$ 's with the obvious use of Schwarz inequality for the scalar product in  $H$ . We will therefore not distinguish the case  $i = 0$  and  $f_i^2$  will denote either the usual power or  $\langle f_0, f_0 \rangle$ , whereas  $\|f_i\|_2$  for  $i = 0$  will be the norm in  $L^2(H, X_{J_0})$ , which may happen to be equal just  $H$  if  $J_0 = \emptyset$ .

Since  $\mathbb{E} |f_i|^2 \leq 1$ ,  $i = 0, \dots, l$ , then for each  $j = 0, \dots, l$  and  $J \subsetneq J_j$  by the Schwarz inequality applied conditionally to  $X_{J_j \setminus J}$

$$\begin{aligned} & \mathbb{E} |\langle h_n, f_0 \rangle \prod_{i=1}^l f_i \mathbf{1}_{\{\mathbb{E}_J f_j^2 > a^2\}}| \\ & \leq \mathbb{E}_{J_j \setminus J} [(\mathbb{E}_{(J_j \setminus J)^c} \prod_{i=0}^l f_i^2)^{1/2} \mathbf{1}_{\{\mathbb{E}_J f_j^2 \geq a^2\}} (\mathbb{E}_{(J_j \setminus J)^c} |h_n|^2)^{1/2}] \\ & \leq \mathbb{E}_{J_j \setminus J} [(\mathbb{E}_J f_j^2)^{1/2} \mathbf{1}_{\{\mathbb{E}_J f_j^2 \geq a^2\}} (\mathbb{E}_{(J_j \setminus J)^c} |h_n|^2)^{1/2}] \\ & \leq L_d 2^{k_{(J_j \setminus J)^c}(n) \#(J_j \setminus J)/2} \log^{d/2} n \mathbb{E}_{J_j \setminus J} [(\mathbb{E}_J f_j^2)^{1/2} \mathbf{1}_{\{\mathbb{E}_J f_j^2 \geq a^2\}}] \\ & \leq L_d [2^{k_{(J_j \setminus J)^c}(n) \#(J_j \setminus J)/2} \log^{d/2} n] a^{-1}, \end{aligned}$$

where the third inequality follows from (32) and the last one from the ele-

mentary fact  $\mathbb{E}|X|\mathbf{1}_{\{|X|\geq a\}} \leq a^{-1}\mathbb{E}|X|^2$ . This way we obtain

(34)

$$\begin{aligned}
& \|\tilde{h}_n\|_{J_0, \mathcal{J}} \\
& \leq \sup\{\mathbb{E}[\langle h_n, f_0 \rangle \prod_{i=1}^l f_i] : \|f_i\|_2 \leq 1, \forall J \subsetneq J_i, \|(\mathbb{E}_J f_i^2)^{1/2}\|_\infty \leq 2^{n\#(J_i \setminus J)/2}\} \\
& + L_d \sum_{i=0}^l \sum_{J \subsetneq J_i} 2^{(k_{(J_i \setminus J)^c}(n) - n)\#(J_i \setminus J)/2} \log^{d/2} n \\
& \leq \sup\{\mathbb{E}[\langle h_n, f_0 \rangle \prod_{i=1}^l f_i] : \|f_i\|_2 \leq 1, \forall J \subsetneq J_i, \|(\mathbb{E}_J f_i^2)^{1/2}\|_\infty \leq 2^{n\#(J_i \setminus J)/2}\} \\
& + L_d \sum_{I \subsetneq I_d} 2^{(k_I(n) - n)\#I^c/2} \log^{d/2} n.
\end{aligned}$$

Let us thus consider arbitrary  $f_i$ ,  $i = 0, \dots, k$  such that  $\|f_i\|_2 \leq 1$ ,  $\|(\mathbb{E}_J f_i^2)^{1/2}\|_\infty \leq 2^{n\#(J_i \setminus J)/2}$  for all  $J \subsetneq J_i$  (note that the latter condition means in particular that  $\|f_i\|_\infty \leq 2^{n\#J_i/2}$ ).

We have by assumption (12) for sufficiently large  $n$ ,

$$|\mathbb{E}[\langle h, f_0 \rangle \prod_{i=1}^k f_i]| \leq \|h\|_{K, \mathcal{J}, 2^{nd/2}} \leq L_d D \log^{(d - \deg \mathcal{J})/2} n.$$

We have also

$$\begin{aligned}
\mathbb{E}[\langle h, f_0 \rangle \mathbf{1}_{\{|h|^2 \geq \eta_d 2^{nd} \log^d n\}} \prod_{i=1}^k f_i] & \leq \mathbb{E}[|h| \mathbf{1}_{\{|h|^2 \geq \eta_d 2^{nd} \log^d n\}}] \prod_{i=0}^k \|f_i\|_\infty \\
& \leq 2^{nd/2} \mathbb{E}[|h| \mathbf{1}_{\{|h|^2 \geq \eta_d 2^{nd} \log^d n\}}] =: \alpha_n.
\end{aligned}$$

Also for  $I \subseteq I_d, I \neq \emptyset, I_d$ , denoting  $\tilde{h}_n = h \mathbf{1}_{\{|h|^2 \leq \eta_d 2^{nd} \log^d n\}}$ , we get

$$\begin{aligned}
& \mathbb{E}[\langle \tilde{h}_n, f_0 \rangle \prod_{i=1}^k f_i \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}}] \\
& \leq \mathbb{E}_{I^c} [(\mathbb{E}_I |\tilde{h}_n|^2)^{1/2} \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}} \prod_{i=0}^k (\mathbb{E}_{J_i \cap I} |f_i|^2)^{1/2}] \\
& \leq [\prod_{i=0}^l 2^{n\#(J_i \cap I^c)/2}] \mathbb{E}_{I^c} [(\mathbb{E}_I |\tilde{h}_n|^2)^{1/2} \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}}] \\
& \leq L_d 2^{n\#I^c/2} \mathbb{E}_{I^c} [(\mathbb{E}_I H^2 \log^d n)^{1/2} \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}}] \\
& \leq L_d [2^{n\#I^c/2} \log^{d/2} n] \mathbb{E}_{I^c} [(\mathbb{E}_I H^2)^{1/2} \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}}] =: \beta_n^I.
\end{aligned}$$



Let us denote  $\bar{h}_n = \tilde{h}_n \prod_{\emptyset \neq I \subsetneq I_d} \mathbf{1}_{\{\mathbb{E}_I H^2 \leq 2^{\#I^c} n\}}$  and  $\gamma_n^I = \mathbb{E}|\bar{h}_n \mathbf{1}_{B_n^I}|^2$ . Combining the three last inequalities we obtain

$$\begin{aligned}
|\mathbb{E}\langle h_n, f_0 \rangle \prod_{i=1}^l f_i| &\leq |\mathbb{E}\langle h, f_0 \rangle \prod_{i=0}^l f_i| + |\mathbb{E}\langle h_n \mathbf{1}_{A_n^c}, f_0 \rangle \prod_{i=1}^l f_i| \\
&\leq L_d D \log^{(d-\deg \mathcal{J})/2} n + \mathbb{E}|\langle h \mathbf{1}_{\{|h|^2 \geq 2^{nd} \log^d n\}}, f_0 \rangle \prod_{i=1}^l f_i| \\
&\quad + \sum_{\emptyset \neq I \subsetneq I_d} \mathbb{E}|\langle \tilde{h}_n \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}}, f_0 \rangle \prod_{i=1}^l f_i| \\
&\quad + \sum_{I \subsetneq I_d} \mathbb{E}|\langle \bar{h}_n \mathbf{1}_{B_n^I}, f_0 \rangle \prod_{i=1}^l f_i| \\
&\leq L_d D \log^{(d-\deg \mathcal{J})/2} n + \alpha_n + \sum_{\emptyset \neq I \subsetneq I_d} \beta_n^I + \sum_{I \subsetneq I_d} \sqrt{\gamma_n^I}.
\end{aligned}$$

Now, combining the above estimate with (34), we obtain

$$\begin{aligned}
(35) \quad \|h_n\|_{J_0, \mathcal{J}} &\leq L_d \sum_{I \subsetneq I_d} 2^{(k_I(n)-n)\#I^c/2} \log^{d/2} n + L_d D \log^{(d-\deg \mathcal{J})/2} n \\
&\quad + \alpha_n + \sum_{\emptyset \neq I \subsetneq I_d} \beta_n^I + \sum_{I \subsetneq I_d} \sqrt{\gamma_n^I}.
\end{aligned}$$

Let us notice that

$$\begin{aligned}
(36) \quad &\sum_n \frac{\alpha_n}{\log^{d/2} n} < \infty, \\
&\forall I \neq \emptyset, I_d \sum_n \frac{\beta_n^I}{\log^{d/2} n} < \infty, \\
&\forall I \neq \emptyset, I_d \sum_n \frac{(\gamma_n^I)^2}{\log^{2d} n} < \infty.
\end{aligned}$$

The first inequality was proved in Step 1. The proof of the second one is straightforward. Indeed, we have

$$\begin{aligned}
\sum_n \frac{\beta_n^I}{\log^{d/2} n} &= L_d \mathbb{E}_{I^c} [(\mathbb{E}_I H^2)^{1/2} \sum_n 2^{n\#I^c/2} \mathbf{1}_{\{\mathbb{E}_I H^2 \geq 2^{n\#I^c}\}}] \\
&\leq \tilde{L}_d \mathbb{E}_{I^c} \mathbb{E}_I H^2 = \tilde{L}_d \mathbb{E} H^2 < \infty.
\end{aligned}$$

The third inequality is implicitly proved in Step 3. Let us however present an explicit argument.

$$\begin{aligned} \sum_n \frac{(\gamma_n^I)^2}{\log^{2d} n} &\leq \sum_n \left( \mathbb{E} \frac{|h|^2 \mathbf{1}_{\{|h| \leq \eta_d 2^{nd/2} \log^{d/2} n\}} \mathbf{1}_{B_n^I}}{\log^d n} \right)^2 \\ &\leq L_d \sum_n (\mathbb{E}_{I^c} \mathbb{E}_I (H^2 \mathbf{1}_{B_n^I}))^2 < \infty \end{aligned}$$

by Lemma 19 applied to the random variable  $\sqrt{\mathbb{E}_I H^2}$ .

We are now in position to finish the proof. Let us notice that we have either  $\mathbb{E}(|h|^2 \mathbf{1}_{\{|h|^2 \leq 2^{2nd}\}}) \leq 1$ , or we can use the function

$$g = \frac{h \mathbf{1}_{\{|h|^2 \leq 2^{2nd}\}}}{(\mathbb{E}(|h|^2 \mathbf{1}_{\{|h|^2 \leq 2^{2nd}\}}))^{1/2}}$$

as a test function in the definition of  $\|h\|_{I_d, \emptyset, 2^{2nd}}$ , obtaining

$$(\mathbb{E}(|h|^2 \mathbf{1}_{\{|h|^2 \leq 2^{2nd}\}}))^{1/2} = \mathbb{E} \langle h, g \rangle \leq \|h\|_{I_d, \emptyset, 2^{2nd}} < D \log^d n$$

for large  $n$ . Combining this estimate with Corollary 3, we can now write

$$\begin{aligned} (37) \quad &\sum_n \mathbb{P} \left( \left| \sum_{|i| \leq 2^n} \pi_d h_n(\mathbf{X}_i^{\text{dec}}) \right| \geq \tilde{L}_d (D + C) 2^{nd/2} \log^{d/2} n \right) \\ &\leq \tilde{L}_d \sum_{J_0 \subsetneq I_d} \sum_{\mathcal{J} \in \mathcal{P}_{I_d} \setminus J_0} \sum_n \exp \left[ - \frac{1}{\tilde{L}_d} \left( \frac{C 2^{nd/2} \log^{d/2} n}{2^{nd/2} \|h_n\|_{J_0, \mathcal{J}}} \right)^{2/\deg \mathcal{J}} \right] \\ &\quad + \tilde{L}_d \sum_{I \subsetneq I_d} \sum_n \exp \left[ - \frac{1}{\tilde{L}_d} \left( \frac{C 2^{nd/2} \log^{d/2} n}{2^{n \# I/2} \|(\mathbb{E}_I |h_n|^2)^{1/2}\|_\infty} \right)^{2/(d + \# I^c)} \right]. \end{aligned}$$

The second series is convergent by (33).

Thus it remains to prove the convergence of the first series. By (35), we have for all  $J_0, \mathcal{J}$

$$\begin{aligned} &\exp \left[ - \frac{1}{\tilde{L}_d} \left( \frac{C \log^{d/2} n}{\|h_n\|_{J_0, \mathcal{J}}} \right)^{2/\deg \mathcal{J}} \right] \\ &\leq \sum_{I \subsetneq I_d} \exp \left[ - \frac{1}{L_d} \left( \frac{C \log^{d/2} n}{2^{(k_I(n) - n) \# I^c / 2} \log^{d/2} n} \right)^{2/\deg \mathcal{J}} \right] \\ &\quad + \exp \left[ - \frac{1}{L_d} \left( \frac{C \log^{d/2} n}{D \log^{(d - \deg \mathcal{J})/2} n} \right)^{2/\deg \mathcal{J}} \right] + \exp \left[ - \frac{1}{L_d} \left( \frac{C \log^{d/2} n}{\alpha_n} \right)^{2/\deg \mathcal{J}} \right] \\ &\quad + \exp \left[ - \frac{1}{L_d} \left( \frac{C \log^{d/2} n}{\sum_{\emptyset \neq I \subsetneq I_d} \beta_n^I} \right)^{2/\deg \mathcal{J}} \right] + \exp \left[ - \frac{1}{L_d} \left( \frac{C \log^{d/2} n}{\sum_{I \subsetneq I_d} \sqrt{\gamma_n^I}} \right)^{2/\deg \mathcal{J}} \right], \end{aligned}$$

(under our permanent convention that the values of  $L_d$  in different equations need not be the same). The series determined by the three last components at the right-hand side are convergent by (36) since  $e^{-x} \leq L_r x^{-r}$  for  $r > 0$ . The series corresponding to the second component is convergent for  $C$  large enough and we can take  $C = L_d D$ . As for the series corresponding to the first term, we have, just as in the proof of (33) for any  $I \subsetneq I_d$ ,

$$\begin{aligned} & \sum_n \exp \left[ - \frac{1}{L_d} \left( \frac{C \log^{d/2} n}{L_d 2^{(k_I(n)-n)\#I^c/2} \log^{d/2} n} \right)^{2/\deg \mathcal{J}} \right] \\ & \leq \sum_k \sum_{n \geq k, k \notin K(I, n)} \exp \left[ - \frac{1}{L_d} \left( C 2^{(n-k)\#I^c/2} \right)^{2/\deg \mathcal{J}} \right] \\ & \leq K \sum_k \sup_{n \geq k, k \notin K(I, n)} \exp \left[ - \frac{1}{L_d} \left( C 2^{(n-k)\#I^c/2} \right)^{2/\deg \mathcal{J}} \right] \\ & \leq \bar{K} \sum_k \mathbb{E}(H^2 \mathbf{1}_{C_k^I}) = \bar{K} \mathbb{E} H^2 < \infty. \end{aligned}$$

We have thus proven the convergence of the series at the left-hand side of (37) with  $C \leq L_d D$ , which ends Step 5.

Now to finish the proof, we just split  $\Sigma^d$  for each  $n$  into four sets, described by steps 1–4 and use the triangle inequality, to show that

$$\sum_n \mathbb{P} \left( \left| \sum_{|i| \leq 2^n} h(\mathbf{X}_i^{\text{dec}}) \right| \geq L_d D 2^{nd/2} \log^{d/2} n \right) < \infty,$$

which proves the sufficiency part of the theorem by Corollary 4.  $\square$

### 8. The undecoupled case.

**THEOREM 4.** *For any function  $h: \Sigma^d \rightarrow H$  and a sequence  $X_1, X_2, \dots$  of i.i.d.,  $\Sigma$ -valued random variables, the LIL (4) holds if and only if  $h$*

$$\mathbb{E} \frac{|h|^2}{(\text{LL}|h|)^d} < \infty,$$

*$h$  is completely degenerate for the law of  $X_1$  and the growth conditions (12) are satisfied.*

*More precisely, if (4) holds, then (12) is satisfied with  $D = L_d C$  and conversely, (12) together with complete degeneration and the integrability condition imply (4) with  $C = L_d D$ .*

PROOF. Sufficiency follows from Corollary 6 and Theorem 2. To prove the necessity assume that (4) holds and observe that from Lemma 8 and Corollary 5,  $h$  satisfies the randomized decoupled LIL (8) and thus, by Theorem 2, (11) holds and the growth conditions (12) on functions  $\|h\|_{K, \mathcal{J}, u}$  are satisfied (note that the  $\|\cdot\|_{\mathcal{J}, u}$  norms of the kernel  $h(X_1, \dots, X_d)$  and  $\varepsilon_1 \cdots \varepsilon_d h(X_1, \dots, X_d)$  are equal). The complete degeneracy of  $\langle \varphi, h \rangle$  for any  $\varphi \in H$  follows from the necessary conditions for real-valued kernels. Since by (11),  $\mathbb{E}_i h$  is well defined in the Bochner sense, we must have  $\mathbb{E}_i h = 0$ .  $\square$

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