Banach-Mazur distances and projections on random subgaussian polytopes

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Abstract

We consider polytopes in $\mathbb{R}^n$ that are generated by N vectors in $\mathbb{R}^n$ whose coordinates are independent subgaussian random variables. (A particular case of such polytopes are symmetric random ± 1 polytopes generated by N independent vertices of the unit cube.) We show that for a random pair of such polytopes the Banach-Mazur distance between them is essentially of a maximal order n. This result is an analogue of well-known Gluskin's result for spherical vectors. We also study the norms of projections on such polytopes and prove an analogue of Gluskin's and Szarek's results on basis constants. The proofs are based on a version of "small ball" estimates for linear images of random subgaussian vectors.

1 Introduction

The structure of ± 1 polytopes, that is, the convex hulls of subsets of the combinatorial cube $\{-1, 1\}^n$, is a much studied subject in several areas of combinatorics (see e.g., [Z]). In the asymptotic geometric analysis the study of geometric properties of *random* ± 1 polytopes has been initiated by Giannopoulos and Hartzoulaki [GiH] who investigated such properties as volumes, mean widths, inradii and other related geometric parameters. These

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results were further developed by Litvak, Pajor, Rudelson and Tomczak-Jaegermann in [LPRT] to symmetric random polytopes spanned by vectors whose coordinates are independent subgaussian random variables with variance ≥ 1 and uniformly bounded subgaussian constants; and who derived these results from their lower bounds for smallest singular values of rectangular random matrices with independent entries satisfying some moment conditions. We refer the reader to papers [MePR] and [GGi] for further developments and generalizations.

Many results mentioned above suggest that random ± 1 polytopes typically display an extremal or "almost" extremal behavior of many geometric parameters of interest. For polytopes spanned by independent Gaussian (or spherical) vectors in $\mathbb{R}^n$, studied in asymptotic geometric analysis, a similar phenomenon is well known since a major result by Gluskin [Gl2] on the diameter of Minkowski compactum, followed by independent results by Gluskin and Szarek, [Gl3] and [S], on the basis constant. Many other papers continued this direction; the reader may consult the survey [MT2] and references therein for details. However, similar results for ± 1, or more generally, subgaussian, polytopes, although resting on a similar approach, are far from being straightforward and require more sophisticated tools.

Geometric results of this paper are variants of the theorems from [Gl2], [Gl3], [S] for the class of symmetric polytopes spanned by random vectors with independent subgaussian coordinates with a uniform control of parameters. This class includes random ± 1 polytopes. In particular, in Theorem 2.3 we establish a lower bound for the Banach–Mazur distance between a "typical" pair of such polytopes in $\mathbb{R}^n$, of the form cn, where c depends on the numbers of vectors generating each polytope. In the case when the number of vectors for both polytopes is proportional to n, for majority of pairs of such polytopes the Banach–Mazur distance is of maximal possible order, *i.e.,* constant times the dimension.

Another geometric result, Theorem 2.4, provides a lower estimate for the minimal norm of a proportional projection for a "typical" subgaussian polytope B (considered as an operator acting on a Banach space generated by B). As above, if the number of vectors defining B is polynomial in the dimension, the discussed quantity is (up to a logarithmic factor) of the maximal possible order, *i.e.,* square root of the dimension.

The paper is organized as follows:

In Section 2 we introduce basic definitions and provide some known important facts concerning random vectors with subgaussian coordinates. Also the main results of the paper are formulated.

Section 3 contains the proof of the main probabilistic result. Namely we prove in Theorem 2.5 an exponential upper bound for the probability of the event that an image by a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ of a random vector ξ with independent subgaussian coordinates, belongs to the Euclidean ball with a specific radius. This is done in terms of the Hilbert–Schmidt and operator norms of A and the subgaussian constant of ξ.

In the next section we use a recent result from [LPRTV1] to derive from Theorem 2.5 a general upper estimate of the probability that the image by a linear operator of a subgaussian vector belongs to a suitable multiple of a symmetric convex body $K \subset \mathbb{R}^n$.

In Section 5 we prove Theorem 2.3 on the Banach–Mazur distance of a "typical" pair of two random polytopes (that may have different numbers of vertices). The main new ingredient in the proof is the estimate from the previous section.

In the last section we prove the result on norms of projections. Here, in addition to the estimate from Section 4, to overcome difficulties arising from the lack of rotation invariance of subgaussian vectors, we use a recent result from [ST] on decoupling weakly depending events.

2 Preliminaries and main results

2.1 General notation We shall use standard notations. In particular, by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^n$ while $e_1, \dots, e_n$ and B_2^n stand for the canonical basis and the Euclidean unit ball in $\mathbb{R}^n$, respectively.

For a subset $A \subset \mathbb{R}^n$ by conv A we denote the convex hull of A and the symmetric convex hull of A is denoted by absconv $A = \mathrm{conv}\,(A \cup -A)$. By a symmetric convex body $B \subset \mathbb{R}^n$ we mean a centrally symmetric compact subset of $\mathbb{R}^n$ with nonempty interior, *i.e.* B is a convex body satisfying $B = -B$. Often, we shall identify such a symmetric convex body B with the n-dimensional Banach space $(\mathbb{R}^n, \|\cdot\|_B)$ for which B is the unit ball. Consequently, for a pair of symmetric convex bodies $B_1, B_2 \subset \mathbb{R}^n$ and an operator $T \in L(\mathbb{R}^n)$ by $\|T\|_{B_1 \to B_2}$ we shall denote the norm of T considered as an operator acting from $(\mathbb{R}^n, \|\cdot\|_{B_1})$ to $(\mathbb{R}^n, \|\cdot\|_{B_2})$. Moreover,

$$\|T\|_{\mathrm{HS}} = (\sum_{i=1}^{n} |Te_i|^2)^{1/2} \quad \text{and} \quad \|T\|_{op} = \|T\|_{B_2^n \to B_2^n}.$$

Let $B_1, B_2 \subset \mathbb{R}^n$ be a pair of symmetric convex bodies. Recall that the Banach–Mazur distance $d(B_1, B_2)$ is defined by

$$d(B_1, B_2) = \inf\{\|T\|_{B_1 \to B_2} \|T^{-1}\|_{B_2 \to B_1}\}$$

where infimum is taken over all invertible operators $T \in \mathbb{R}^n$.

The volume of a body $B \subset \mathbb{R}^n$ is denoted by $|B|$. For a pair of subsets $B_1, B_2 \subset \mathbb{R}^n$ by $N(B_1, B_2)$ we denote the covering number of B_1 by B_2, *i.e.,* the minimal number of translates of B_2 covering B_1.

$$N(B_1, B_2) := \min\left\{k \mid \exists x_1, \dots x_k \text{ such that } B_1 \subset \bigcup_{i=1}^{k} x_i + B_2\right\}.$$

2.2 Subgaussian random variables and random vectors

By g, g_i, $i \geq 1$, we denote independent $\mathcal{N}(0,1)$ Gaussian random variables. By $\mathbb{P}(\cdot)$ we denote the probability of an event, and by $\mathbb{E}$, the expectation.

For a random variable ξ and $p > 0$ we put $\|\xi\|_p := (\mathbb{E}|\xi|^p)^{1/p}$. A random variable ξ is called *subgaussian* if there exists a constant $\beta < \infty$ such that

$$\|\xi\|_{2k} \leq \beta \|g\|_{2k} \text{ for } k = 1, 2, \dots. \tag{2.1}$$

We shall refer to the infimum over all β satisfying (2.1) as the subgaussian constant of ξ. Let us mention, although we shall not use it directly here, that an equivalent definition is often given in terms of the ψ_2-norm. Denoting by ψ_2 the Orlicz function $\psi_2(x) = \exp(x^2) - 1$, ξ is subgaussian if and only if

$$\|\xi\|_{\psi_2} := \inf\{t > 0 \mid \mathbb{E}\, \psi_2(\xi/t) \leq 1\} < \infty. \tag{2.2}$$

Denoting by $\tilde{\beta}$ the subgaussian constant of ξ, a direct calculation shows the following folklore (and not optimal) estimate

$$\tilde{\beta} \leq \|\xi\|_{\psi_2} \leq \tilde{\beta}\|g\|_{\psi_2} = \tilde{\beta}\sqrt{8/3}.$$

The lower estimate follows since $\mathbb{E}\psi_2(X) \geq \mathbb{E}X^{2k}/k!$ and $\mathbb{E}g^{2k} = (2k-1)!!$ for $k = 1, 2, \dots$. The upper one is using the fact that $\mathbb{E}\exp(tg^2) = (1-2t)^{-1/2}$ for $t < 1/2$.

Apart from Gaussian random variables the prime example of subgaussian variables are Bernoulli random variables, taking values ± 1 with $\mathbb{P}(\xi = 1) =$

$\mathbb{P}(\xi = -1) = 1/2$. Recall that in this case both variation and the subgaussian constant are equal to 1.

In this paper we will work with random vectors in $\mathbb{R}^n$ of the form $\xi = (\xi_1, \ldots, \xi_n)$, where ξ_i are independent subgaussian random variables, and we will refer to such vectors as subgaussian vectors. We will require that $\mathrm{Var}(\xi_i) \geq 1$ and subgaussian constants are at most β. Note that under these assumptions we have $\mathbb{E}\xi_i^2 \geq \mathrm{Var}(\xi_i) \geq 1 = \mathbb{E}g^2$, hence $\beta \geq 1$.

Fact 2.1 *Let $\xi = (\xi_1, \ldots, \xi_n)$ be as above. Then for any $u \geq 0$,*

$$\mathbb{P}(|\xi| \geq u\sqrt{n}) \leq \exp\big(n\,(\ln 2 - u^2/(3\beta^2))\big). \tag{2.3}$$

In particular, $\mathbb{P}\big(|\xi| \geq 3\beta\sqrt{n}\big) \leq e^{-2n}$.

Proof Indeed, for an arbitrary $s > 0$ and $1 \leq j \leq n$ we have

$$\mathbb{E}\exp\left(\frac{\xi_j^2}{s^2}\right) = \sum_{k=0}^{\infty} \frac{1}{k! \cdot s^{2k}} \mathbb{E}\xi_j^{2k} \leq \sum_{k=0}^{\infty} \frac{\beta^{2k}\,\mathbb{E}g^{2k}}{k! \cdot s^{2k}} = \mathbb{E}\exp\left(\frac{(\beta g)^2}{s^2}\right),$$

and this last quantity is less than or equal to 2 for e.g., $s = \sqrt{3}\beta$. For this choice of s,

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{n} \xi_j^2 \geq u^2 n\right) &\leq \mathbb{E}\exp\left(\frac{1}{s^2}\left(\sum_{j=1}^{n} \xi_j^2 - u^2 n\right)\right)\\ &\leq \exp\left(-\frac{u^2 n}{s^2}\right)\prod_{j=1}^{n}\mathbb{E}\exp\left(\frac{\xi_j^2}{s^2}\right) \leq \exp\left(-\frac{u^2 n}{3\,\beta^2}\right)\cdot 2^n,\end{aligned}$$

which is the desired result. $\square$

For $N \geq n$ consider N random vectors $\xi^j = (\xi_{1,j}, \ldots, \xi_{n,j}) \in \mathbb{R}^n$ ($j = 1, \ldots, N$), where $\xi_{i,j}$ are independent subgaussian random variables with $\mathrm{Var}(\xi_{i,j}) \geq 1$ and subgaussian constants at most β. Denote the matrix $[\xi_{i,j}]_{i \leq n, j \leq N}$ by Γ.

Fact 2.2 *Let $N \geq 2n$. In the above notation, consider Γ as an operator $\Gamma : \mathbb{R}^N \to \mathbb{R}^n$. Then there exist $\mu_1, \mu_2 > 0$ depending on β only such that*

(i) $\mathbb{P}\Big(\exists\, x \in \mathbb{R}^n \text{ s.t. } |\Gamma^* x| < \mu_1\sqrt{N}|x|\Big) \leq \exp(-\mu_2 N)$.

(ii) $\mathbb{P}\Big(\mu_1 B_2^n \not\subset \operatorname{absconv}\{\xi^j\}_{j\le N}\Big) \le \exp(-\mu_2 N)$.

Remark Replacing μ_1 and μ_2 by $\min(\mu_1, \mu_2 1)$ we may assume that both μ_1 and μ_2 less than 1.

Proof (i) follows from the main result in [LPRT] on singular values of rectangular matrices (Theorem 3.1 and Remark 4 following it).

(ii) Since $\xi^j = \Gamma e_j$ for $j = 1, \dots, N$, condition $\mu_1 B_2^n \not\subset \operatorname{absconv}\{\Gamma e_j\}_{j\le N}$ immediately implies, by separation theorem, that $|\Gamma^* x| < \mu_1\sqrt{N}$ for some $x \in \mathbb{R}^n$ with $|x| = 1$, and the estimate follows from (i). $\square$

Remark The condition $N \ge 2n$ has been chosen only to avoid inessential constants in further arguments. Indeed, it suffices to use that (i) and (ii) hold for $N \ge \lambda_0 n$ where $\lambda_0 > 1$ is *some* absolute constant. This fact in turn is a consequence of a much easier variant of the main result in [LPRT] (see Remarks 3 and 4 after Theorem 3.1 in [LPRT]).

2.3 Main results

The main subject of this paper are random polytopes in $\mathbb{R}^n$ determined by subgaussian vectors. So such a polytope is of the form

$$B := \operatorname{absconv}\{\xi^j\}_{j\le N}, \tag{2.4}$$

where $N \ge n$ and $\xi^1, \dots, \xi^N$ are random subgaussian vectors as above. In the case when ξ^j's are Bernoulli vectors (i.e. $\xi_{i,j}$ are independent Bernoulli random variables), the body B is a so called random symmetric ±1-polytope, that means the symmetric convex hull of N independently chosen vertices of the unit cube in $\mathbb{R}^n$.

For the moment fix N and M. Let B and $\tilde{B}$ be two such polytopes generated by independent random variables $\{\xi_{i,j}\}_{j\le N}$ and $\{\tilde{\xi}_{i,j}\}_{j\le M}$, respectively. The first theorem says that with a large probability, such a pair $(B, \tilde{B})$ has the Banach–Mazur distance of order n. More precisely,

Theorem 2.3 *There exist $\mu_3 > 2$, $\mu_4 > 1$ and $0 < \mu_5 < 1$, $\mu' > 0$ depending on β only such that the following holds. Suppose $N, M > \mu_3 n$ satisfy*

$$\mu_4 \ln\ln(M/n) \le N/n \le \exp\exp\big((1/\mu_4)M/n\big). \tag{2.5}$$

Let B and $\tilde{B}$ be defined as in (2.4) by independent subgaussian vectors $\{\xi^j\}_{j\le N}$ and $\{\tilde{\xi}^j\}_{j\le M}$, respectively. Then, with probability larger than or equal to $1-2(M+N+3)\exp(-\mu_5 n)$, we have

$$d(B,\tilde{B}) \ge (1/4)\,\alpha\, n,$$

with $\alpha = \big(c_1 \ln(N/n)\ln(M/n)\big)^{-\mu'}$, where $c_1>0$ is a universal constant.

The theorem above has its roots in the ground breaking result of Gluskin [Gl2] which provided the first random construction of pairs of symmetric convex bodies in $\mathbb{R}^n$ with Banach-Mazur distance of the order of cn. For more recent development in this direction we refer the reader to [MT1] where a more general situation in a rotation invariant setting is studied.

Another geometric property of a random polytope which we are going to study is the relation between projection of the polytope onto a linear subspace E and its section by E. The theorem below states that a typical body defined as in (2.4) has the property that for every projection P onto E with $n/4 \le \dim E \le 3n/4$, the image $P(B)$ of the polytope sticks out of the multiple of the section $\alpha(B\cap E)$ for α as large as $\sqrt{n}$ (up to a logarithmic factor).

Theorem 2.4 *There exist $\mu_6>1$ and $\mu'>0$ depending on β only such that the following holds. Suppose that $N \ge \mu_6 n^2 \ln n$. Let B be defined as in (2.4) by independent subgaussian vectors. Then, with probability larger than or equal to $1-(N+1)\exp(-2n)$, we have, for every projection $P:\mathbb{R}^n \to \mathbb{R}^n$ with $n/4 \le \mathrm{rank}\ P \le 3n/4$,*

$$\|P\|_{B\to B} \ge \alpha\sqrt{n},$$

with $\alpha = \big(c_2 \ln(N/n)\big)^{-\mu'}$, where $c_2>0$ is a universal constant.

Theorem 2.4 is modelled on results of Gluskin [Gl3] and Szarek [S] who considered some version of symmetric random polytopes generated by spherical and gaussian points. For more details see also [MT2].

A new ingredient in proofs of the both theorems above is the following estimate for subgassian vectors which seems to be of independent interest.

Theorem 2.5 *Let A be a non-zero $n \times n$ matrix and $\xi = (\xi_1, \ldots, \xi_n)$, where ξ_i are independent subgaussian random variables with $\mathrm{Var}(\xi_i) \geq 1$ and subgaussian constants at most β. Then for any $y \in \mathbb{R}^n$ we have*

$$\mathbb{P}\big(|A\xi - y| \leq \|A\|_{\mathrm{HS}}/2\big) \leq 2\exp\Big(- \frac{c_0}{\beta^4}\Big(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\Big)^2\Big), \tag{2.6}$$

where $c_0 > 0$ is a universal constant.

Remark Note that a standard calculation shows that in the situation of the Theorem, for $A = [a_{ij}]$ one has

$$\mathbb{E}|A\xi - y|^2 \geq \mathbb{E}|A(\xi - \mathbb{E}\xi)|^2 = \sum_{ij} a_{ij}^2 \mathrm{Var}(\xi_j) \geq \|A\|_{\mathrm{HS}}^2.$$

Remark It will be convenient for a further argument to assume that $c_0 \leq 1/2$. This is of course possible to do, replacing c_0 by $\min(c_0, 1/2)$.

For Gaussian variables an easy well-known estimate analogous to Theorem 2.5 may be given in terms of $\det A$ (see *e.g.,* [MT2]). The following strengthening is far less elementary and uses a recent result on small balls probability, [LaO]. As it might be of independent interest, we include it here (with the sketch of the proof at the end of Section 3), although it is not used in the present paper.

Proposition 2.6 *Let A be a non-zero $n \times n$ matrix and $\theta = (g_1, \ldots, g_n)$, where g_i are independent $\mathcal{N}(0,1)$ random variables. Then there exist universal constants $\alpha_0 \in (0,1)$ and $\kappa > 0$ such that for any $y \in \mathbb{R}^n$ and any $\alpha \in (0, \alpha_0)$ we have*

$$\mathbb{P}\big(|A\theta - y| \leq \alpha \|A\|_{\mathrm{HS}}\big) \leq \exp\Big(\kappa \log\alpha \Big(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\Big)^2\Big). \tag{2.7}$$

Remark In the theorem above one can make the probability in (2.7) as small as one wishes by choosing sufficiently small $\alpha \in (0, \alpha_0)$, which is not the case for the subgaussian vectors in (2.6).

Remark It follows directly from (2.6) and the definition of covering number that for for an arbitrary convex body $B \subset \mathbb{R}^n$ one has

$$\mathbb{P}\big(A\xi \in B\big) \leq 2\,N(B, \tfrac{1}{2}\|A\|_{\mathrm{HS}}\, B_2^n) \exp\Big(- \frac{c_0}{\beta^4}\Big(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\Big)^2\Big),$$

while in the Gaussian case, (2.7) yields the estimate

$$\mathbb{P}\big(A\theta \in B\big) \le N(B, \alpha\|A\|_{\mathrm{HS}}\, B_2^n)\, \exp\left(\kappa \log \alpha \left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2\right),$$

for an arbitrary $\alpha \in (0, \alpha_0)$.

3 Operators acting on subgaussian vectors; Euclidean estimates

In this section we shall prove Theorem 2.5. The argument is based on two lemmas.

Lemma 3.1 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be a sequence of independent symmetric subgaussian random variables satisfying (2.1) with constant β and $B = (b_{ij})$ be a symmetric matrix with zero diagonal. Then for any $t > 1$,*

$$\mathbb{P}\Big(\Big|\sum_{i<j} b_{ij}\xi_i\xi_j\Big| \ge C\beta^2(\sqrt{t}\|B\|_{\mathrm{HS}} + t\|B\|_{\mathrm{op}})\Big) \le e^{-t},$$

where C is a universal constant.

Proof By (2.1) and by the symmetry of ξ_i we immediately get $\|a + b\xi_i\|_{2k} \le \|a + b\beta g_i\|_{2k}$ for any real numbers a, b and a positive integer k. Hence easy induction shows that

$$\Big\|\sum_{i<j} b_{ij}\xi_i\xi_j\Big\|_{2k} \le \Big\|\sum_{i<j} b_{ij}\beta g_i \beta g_j\Big\|_{2k} = \beta^2\Big\|\sum_{i<j} b_{ij} g_i g_j\Big\|_{2k}.$$

Using Hanson-Wright estimate (cf. [HaW]) we get

$$\|\sum_{i<j} b_{ij}\xi_i\xi_j\|_{2k} \le C'\beta^2(\sqrt{2k}\|B\|_{\mathrm{HS}} + 2k\|B\|_{\mathrm{op}}).$$

with some universal constant C'. Take $k := \lceil t/2 \rceil$, then by Chebyshev's inequality

$$\mathbb{P}\Big(\Big|\sum_{i<j} b_{ij}\xi_i\xi_j\Big| \ge e\, C'\beta^2(\sqrt{2k}\|B\|_{\mathrm{HS}} + 2k\|B\|_{\mathrm{op}})\Big) \le e^{-2k} \le e^{-t}.$$

The statement easily follows, since $k \le t$. $\square$

Lemma 3.2 *Let $\xi_1, \xi_2, \ldots, \xi_n$ be a sequence of independent random variables with finite fourth moments. Then for any nonnegative coefficients b_i and $t > 0$,*

$$\mathbb{P}\Big(\sum_{i=1}^{n} b_i(\mathbb{E}\xi_i^2 - \xi_i^2) > \big(2t\sum_{i=1}^{n} b_i^2 \mathbb{E}\xi_i^4\big)^{1/2}\Big) \le e^{-t}.$$

Proof We may obviously assume that $\sum_{i=1}^n b_i^2\mathbb{E}\xi_i^4 > 0$. For $x \ge 0$, we have $e^{-x} = (e^x)^{-1} \le (1 + x + x^2/2)^{-1} \le 1 - x + x^2/2$. Thus for $\lambda \ge 0$,

$$\mathbb{E}\exp(-\lambda\xi_i^2) \le 1 - \lambda\mathbb{E}\xi_i^2 + \frac{1}{2}\lambda^2\mathbb{E}\xi_i^4 \le \exp(-\lambda\mathbb{E}\xi_i^2 + \frac{1}{2}\lambda^2\mathbb{E}\xi_i^4).$$

Letting $S = \sum_{i=1}^n b_i(\mathbb{E}\xi_i^2 - \xi_i^2)$, we get $\mathbb{E}\exp(\lambda S) \le \exp(\frac{1}{2}\lambda^2\sum_{i=1}^n b_i^2\mathbb{E}\xi_i^4)$, and for any $u \ge 0$,

$$\mathbb{P}(S \ge u) \le \inf_{\lambda\ge 0}\mathbb{E}\exp(\lambda S - \lambda u) \le \exp\Big(-\frac{u^2}{2\sum_{i=1}^n b_i^2\mathbb{E}\xi_i^4}\Big).$$

□

Proof of Theorem 2.5 Let $\xi' = (\xi_1', \ldots, \xi_n')$ denote the independent copy of ξ and set $\theta = (\theta_1, \ldots, \theta_n) := \xi - \xi'$. Variables θ_i are independent symmetric with subgaussian constants at most 2β. Put

$$p := \mathbb{P}(|A\xi - y| \le \|A\|_{\mathrm{HS}}/2).$$

Then

$$\begin{aligned} p^2 &= \mathbb{P}\big(|A\xi - y| \le \|A\|_{\mathrm{HS}}/2,\ |A\xi' - y| \le \|A\|_{\mathrm{HS}}/2\big) \\ &\le \mathbb{P}\big(|A\theta| \le \|A\|_{\mathrm{HS}}\big). \end{aligned}$$

Let $B = AA^T = (b_{ij})$. Then $b_{ii} = \sum_j a_{ij}^2 \ge 0$ and

$$|A\theta|^2 = \langle B\theta, \theta\rangle = \sum_{ij} b_{ij}\theta_i\theta_j = \sum_i b_{ii}\theta_i^2 + 2\sum_{i<j} b_{ij}\theta_i\theta_j.$$

Notice that $\mathrm{Var}(\theta_i) = 2\mathrm{Var}(\xi_i) \ge 2$, so that $\sum_i b_{ii}\,\mathbb{E}\theta_i^2 \ge 2\mathrm{tr}(B) = 2\|A\|_{\mathrm{HS}}^2$. Thus

$$\begin{aligned} p^2 &\le \mathbb{P}\big(|A\theta|^2 \le \|A\|_{\mathrm{HS}}^2\big) \\ &\le \mathbb{P}\big(2\sum_{i<j} b_{ij}\theta_i\theta_j + \sum_{ii} b_{ii}(\theta_i^2 - \mathbb{E}\theta_i^2) \le -\|A\|_{\mathrm{HS}}^2\big) \qquad (3.1) \\ &\le \mathbb{P}\big(|\sum_{i<j} b_{ij}\theta_i\theta_j| \ge \|A\|_{\mathrm{HS}}^2/3\big) + \mathbb{P}\big(\sum_i b_{ii}(\theta_i^2 - \mathbb{E}\theta_i^2) \le -\|A\|_{\mathrm{HS}}^2/3\big). \end{aligned}$$

Notice that we have $\|(b_{ij}\delta_{i\neq j})\|_{\mathrm{HS}} \le \|B\|_{\mathrm{HS}} = \|AA^T\|_{\mathrm{HS}} \le \|A\|_{\mathrm{op}}\|A\|_{\mathrm{HS}}$ and $\|(b_{ij}\delta_{i\neq j})\|_{\mathrm{op}} \le \|B\|_{\mathrm{op}} + \|(b_{ij}\delta_{i=j})\|_{\mathrm{op}} \le 2\|B\|_{\mathrm{op}} \le 2\|A\|_{\mathrm{op}}^2$. Hence by Lemma 3.1 we get

$$\mathbb{P}\Big(\Big|\sum_{i<j} b_{ij}\theta_i\theta_j\Big| \ge \|A\|_{\mathrm{HS}}^2/3\Big) \le 2\exp\Big(-(C''\beta^4)^{-1}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2\Big). \tag{3.2}$$

We have $\|\theta_i\|_4 \le 2\|\xi_i\|_4 \le 2\beta\|g_i\|_4$, thus

$$\sum_{i=1}^n b_{ii}^2\mathbb{E}\theta_i^4 \le 48\beta^4\sum_{i=1}^n b_{ii}^2 \le 48\beta^4\|B\|_{\mathrm{HS}}^2 \le 48\beta^4\|A\|_{\mathrm{op}}^2\|A\|_{\mathrm{HS}}^2.$$

Therefore, by Lemma 3.2,

$$\mathbb{P}\Big(\sum_i b_{ii}(\mathbb{E}\theta_i^2 - \theta_i^2) \ge \|A\|_{\mathrm{HS}}^2/3\Big) \le \exp\Big(-(C'''\beta^4)^{-1}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2\Big). \tag{3.3}$$

Thus by (3.1)-(3.3),

$$p^2 \le 4\exp\Big(-(\max(C'',C''')\beta^4)^{-1}\left(\frac{\|A\|_{\mathrm{HS}}}{\|A\|_{\mathrm{op}}}\right)^2\Big),$$

which completes the proof. $\square$

Remark Lemma 3.2 is somewhat special in that we assume that coefficients are nonnegative. In the general case one has for any sequence of independent random variables ξ_i with subgaussian constant at most β and $t > 1$,

$$\mathbb{P}\Big(\Big|\sum_{i=1}^n a_i(\mathbb{E}\xi_i^2 - \xi_i^2)\Big| > C\beta^2(t\|(a_i)\|_\infty + \sqrt{t}|(a_i)|)\Big) \le e^{-t}. \tag{3.4}$$

We provide a sketch of the proof of the inequality (3.4) for the sake of completeness. Let $(\tilde{\xi_i})$ be an independent copy of (ξ_i). We have by Jensen's inequality for $p \ge 1$,

$$\big\|\sum_i a_i(\xi_i^2 - \mathbb{E}(\xi_i^2))\big\|_p \le \big\|\sum_i a_i(\xi_i^2 - \tilde{\xi}_i^2)\big\|_p.$$

Variables $\xi_i^2 - \tilde{\xi}_i^2$ are independent, symmetric and for $k \ge 1$, $\|\xi_i^2 - \tilde{\xi}_i^2\|_{2k} \le 2\beta^2\|g_i^2\|_{2k} \le 4\beta^2\|\eta_i\|_{2k}$ where η_i are i.i.d. symmetric exponential r.v.'s with variance 1. Therefore, for positive integer k,

$$\big\|\sum_i a_i(\xi_i^2 - \tilde{\xi}_i^2)\big\|_{2k} \le 4\beta^2\big\|\sum_i a_i\eta_i\big\|_{2k} \le C_1\beta^2(k\|(a_i)\|_\infty + \sqrt{k}\|(a_i)\|_2),$$

where the last inequality follows by Gluskin-Kwapień estimate [GK]. Hence by Chebyshev's inequality

$$\mathbb{P}\Big(\Big|\sum_i a_i(\xi_i^2 - \mathbb{E}(\xi_i^2))\Big| \geq eC_1\beta^2(k\|(a_i)\|_\infty + \sqrt{k}\|(a_i)\|_2)\Big) \leq e^{-2k}$$

and the assertion easily follows.

Remark The assumption about independence of coordinates of random subgaussian vector ξ is important. Let $G \sim \mathcal{N}(0, I_n)$ and let θ be a random variable independent of G with $\mathbb{P}(\theta = \sqrt{2}) = \mathbb{P}(\theta = 0) = 1/2$. Put $\xi := \theta G$, then $\|\langle y, \xi\rangle\|_{2k} \leq \sqrt{2}|y|\|g\|_{2k}$ and $\mathrm{Var}(\langle y, \xi\rangle) \geq |y|^2$ for any $y \in \mathbb{R}^n$. However $A\xi$ has an atom with mass at least $1/2$ at zero for any matrix A and estimate given in Theorem 2.5 does not hold in this case.

Proof of Proposition 2.6 By Anderson's inequality (cf. e.g., [LeTa, p.73]) it suffices to consider the case $y = 0$. Let T be a countable dense subset of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and let $(G_t)_{t\in T}$ be a centered Gaussian process defined by $G_t = \langle A\theta, t\rangle$. Then one can easily check that $\sigma = \sup_{t\in T}(\mathbb{E}G_t^2)^{1/2} = \|A\|_{\mathrm{op}}$ whereas $(\mathbb{E}|A\theta|^2)^{1/2} = \|A\|_{\mathrm{HS}}$. It is a standard fact that a median of a supremum of a centered Gaussian process is equal to its L_2-norm, up to a universal factor. That is, there exist universal constants $a, b > 0$ such that $M = \mathrm{Med}\,(\sup_{t\in T}|G_t|) = \mathrm{Med}\,(|A\theta|)$ satisfies $a\|A\|_{\mathrm{HS}} \leq M \leq b\|A\|_{\mathrm{HS}}$. Hence the assertion immediately follows from Theorem 4 in [LaO]. $\Box$

4 Operators acting on subgaussian vectors; general estimates

In this section we will get estimates of a similar type as in Theorem 2.5 for probability that $A\xi$ belongs to a general convex and symmetric set K rather than to a Euclidean ball. In the case of polytopes given by Gaussian vectors, studied in depth by many authors cited in the Introduction, and especially in the case of Gaussian projections of arbitrary bodies, this probability was naturally expressed in terms of the volume of K. In the present subgaussian case, a natural approach (introduced in a similar context in [LPRTV1, LPRTV2] and developed in [LPT]) is to cover K by N Euclidean balls of an appropriate radius and apply Theorem 2.5 combined with an upper bound for the covering number N. Although one could estimate N in general by, for example, Sudakov's inequality, the use of volumes often gives slightly better bounds. We therefore follow this approach in our estimates.

Unfortunately a simple bound for N by the volume ratio argument is too weak in the present situation since we may not be able to make the probability in the proof of Theorem 2.5 as small as would be required; hence we need to use a more sophisticated covering argument given by the following proposition from [LPRTV1, LPRTV2].

For $K \subset \mathbb{R}^n$ we let $V_K = \big(|K|/|B_2^n|\big)^{1/n}$.

Proposition 4.1 *For every $0 < r \le 1/e$, every symmetric convex body $K \subset \mathbb{R}^n$ satisfying $B_2^n \subset K$, and every $0 < \tilde\eta \le \ln(4\pi V_K)/\ln(1/r)$ one has*

$$N\left(\tilde\alpha K \cap B_2^n, rB_2^n\right) \le 2^{\tilde\eta n} \quad \textit{for} \quad \tilde\alpha = (4\pi V_K)^{(c_1/\tilde\eta)\ln r},$$

where $c_1 > 0$ is a universal constant.

Theorem 4.2 *Let A be a non-zero $n \times n$ matrix and $\xi = (\xi_1, \dots, \xi_n)$, where ξ_i are independent subgaussian random variables with $\mathrm{Var}(\xi_i) \ge 1$ and subgaussian constants at most β. Let $K \subset \mathbb{R}^n$ be a symmetric convex body satisfying $B_2^n \subset K$. Then*

$$\mathbb{P}\big(A\xi \in \alpha \|A\|_{\mathrm{op}}\sqrt{n}\, K\big) \le 3\exp\big(-c_0\, a(A)^2/(2\beta^4)\big),$$

where $a(A) := \|A\|_{\mathrm{HS}}/\|A\|_{\mathrm{op}}$,

$$\alpha = 3\beta\, (4\pi V_K)^{(c_2/\eta)\ln(a(A)/(6\beta\sqrt{n}))}$$

and $\eta := a(A)^2/(n\beta^4)$, where c_0 is the constant from Theorem 2.5, and $c_2 > 0$ is a universal constant.

It follows from the proof that one can take $c_2 = 2c_1 \ln 2/c_0$, where c_1 is the constant from Proposition 4.1.

Proof Set $\Omega_1 = \big(|\xi| \le 3\beta\sqrt{n}\big)$ so that by Fact 2.1 the probability of the complement satisfies $\mathbb{P}(\Omega_1^c) \le e^{-2n}$. We shall show that

$$\mathbb{P}\big(A\xi \in \alpha\|A\|_{\mathrm{op}}\sqrt{n}\, K \cap 3\beta\|A\|_{\mathrm{op}}\sqrt{n}B_2^n\big) \le 2e^{-c_0\, a(A)^2/(2\beta^4)}. \tag{4.1}$$

Since $|A\xi| \le \|A\|_{\mathrm{op}}|\xi| \le \|A\|_{\mathrm{op}} \cdot 3\beta\sqrt{n}$ on Ω_1, then

$$\big(A\xi \in \alpha\|A\|_{\mathrm{op}}\sqrt{n}\, K\big) \subset \big(A\xi \in \alpha\|A\|_{\mathrm{op}}\sqrt{n}\, K \cap 3\beta\|A\|_{\mathrm{op}}\sqrt{n}B_2^n\big) \cup \Omega_1^c,$$

which will immediately conclude the desired estimate, by (4.1) and the fact that $c_0\, a(A)^2/(2\beta^4) \le 2n$ (note that $c_0 \le 1$, $a(A) \le \sqrt{n}$ and $\beta \ge 1$).

Consider the covering

$$\begin{aligned} N := {} & N\big(\alpha \|A\|_{\rm op}\sqrt{n}\, K \cap 3\beta\|A\|_{\rm op}\sqrt{n} B_2^n, (\|A\|_{\rm HS}/2)B_2^n\big) \\ = {} & N\big(\frac{\alpha}{3\beta}\, K \cap B_2^n, \frac{a(A)}{6\beta\sqrt{n}} B_2^n\big), \end{aligned}$$

with the equality obtained by rescaling all sets involved by dividing them by $3\beta\|A\|_{\rm op}\sqrt{n}$. We use Proposition 4.1 with $r = a(A)/(6\beta\sqrt{n})$ and $\tilde{\eta} = (c_0/(2\ln 2))a(A)^2/(n\beta^4)$. In particular, $\tilde{\eta} = (18c_0/\ln 2)r^2/\beta^2$ and it is easy to check that if c_0 is sufficiently small (since $\sup_{0\le r\le 1/6} r^2 \ln(1/r) = (\ln 6)/36$, $c_0 \le 1/2$ works), then $\tilde{\eta} \le 1/\ln(1/r)$, and hence r and $\tilde{\eta}$ remain in the appropriate ranges. This yields α satisfying

$$\alpha/3\beta = \tilde{\alpha} = (4\pi V_K)^{(c_1/\tilde{\eta})\ln r} = (4\pi V_K)^{(c_2/\eta)\ln(a(A)/(6\beta\sqrt{n}))},$$

where $\eta = a(A)^2/(n\beta^4)$ and $c_2 = 2c_1 \ln 2/c_0$. Most importantly, we get the estimate $N \le 2^{\tilde{\eta} n} = \exp\big(c_0 a(A)^2/(2\beta^4)\big)$.

Let $y_1, \dots, y_N$ be such that

$$\alpha\|A\|_{\rm op}\sqrt{n}\, K \cap 3\beta\|A\|_{\rm op}\sqrt{n} B_2^n \subset \bigcup_{i=1}^{N} (y_i + (\|A\|_{\rm HS}/2)B_2^n).$$

By Theorem 2.5 for every $1 \le i \le N$ we have

$$\mathbb{P}\Big(A\xi \in y_i + (\|A\|_{\rm HS}/2)B_2^n\Big) \le 2\exp\big(-c_0 a(A)^2/\beta^4\big).$$

Thus

$$\begin{aligned} \mathbb{P}\Big(A\xi \in \alpha\|A\|_{\rm op}\sqrt{n}\, K \cap 3\beta\|A\|_{\rm op}\sqrt{n} B_2^n\Big) & \le N \cdot 2e^{-c_0 a(A)^2/\beta^4} \\ & \le 2e^{-c_0 a(A)^2/(2\beta^4)}, \end{aligned}$$

which concludes the proof of (4.1). $\square$

The results of the next two Sections are based on a probabilistic estimate which easily follows from a special case of Theorem 4.2. To fix the setting, let $M \ge n$, let $u_1, \dots, u_M \in \mathbb{R}^n$ with $|u_i| \le \sqrt{n}$ for $1 \le i \le M$, and set

$$B_0 := \text{absconv}\,\{u_i\}_{i\le M}. \tag{4.2}$$

Recall that every operator $T : \mathbb{R}^n \to \mathbb{R}^n$ can be written in the form

$$Tx = \sum_{l=1}^{n} s_l(T)\langle x, v_l\rangle \tilde{v}_l \quad \text{for every } x \in \mathbb{R}^n,$$

where $\{v_l\}_{l=1}^n$ and $\{\tilde{v}_l\}_{l=1}^n$ are orthogonal bases and $s_1(T) \geq s_2(T) \geq \ldots \geq s_n(T) \geq 0$ are uniquely determined by T. The sequence $\{s_l(T)\}_{l=1}^n$ is called the sequence of s-numbers of T.

Corollary 4.3 *Let $\xi = (\xi_1, \ldots, \xi_n)$, where ξ_i are independent subgaussian random variables with $\mathrm{Var}(\xi_i) \geq 1$ and subgaussian constants at most β. Let $1 \leq m \leq n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an operator with the m'th s-number $s_m(T) \geq 1$. Then*

$$\mathbb{P}\Big(T\xi \in \alpha\sqrt{n}\, B_0\Big) \leq 3e^{-\mu m},$$

where $\alpha = 3\beta\big(c_3 \ln(1 + M/n)\big)^{-\mu'}$, $c_3 > 0$ is a universal constant, $\mu > 0$ depends on β and $\mu' > 0$ depends on β and $\lambda := n/m$.

Proof Let $T = \sum_{l=1}^n s_l(T)\langle\, \cdot, v_l\,\rangle\, \tilde{v}_l$ be a polar decomposition of T. Set $E \subset \mathbb{R}^n$ to be a subspace spanned by $\{v_1, \ldots, v_m\}$ and let $F := T(E)$. Let P_E and P_F denote the orthogonal projections onto E and F respectively, let $\tilde{T} : E \to F$ be the restriction of T to E and let $S = \tilde{T}^{-1} : F \to E$. Thus $P_F T = \tilde{T} P_E$, and consequently, the event $T\xi \in \alpha\sqrt{n}\, B_0$ is contained in $P_E \xi \in \alpha\sqrt{n}\, S P_F(B_0)$.

Note that by the assumption on T we get that $\|S\|_{\mathrm{op}} \leq 1$. So $SP_F(B_0) = \mathrm{absconv}\,\{SP_F u_i\}_{i \leq M}$ and $|SP_F u_i| \leq \sqrt{n}$ for all $i \leq M$. Finally, let

$$B_0' = \mathrm{absconv}\,\{SP_F u_1, \ldots, SP_F u_M, \sqrt{n} e_1, \ldots \sqrt{n} e_n\},$$

where $e_1, \ldots, e_n$ is the standard unit vector basis in $\mathbb{R}^n$. We shall apply Theorem 4.2 for $A = P_E$ and $K = B_0'$. Note that $B_0' \supset B_2^n$ and by a well known estimate (*cf.,* [CP], see also [Gl1]), $V_{B_0'} \leq c'\sqrt{\ln(1 + M/n)}$, where $c' > 0$ is a universal constant. Also, $a(P_E) = \sqrt{m}$. This yields (setting $\lambda := n/m$),

$$\alpha = 3\beta\big(c_3 \ln(1 + M/n)\big)^{-c_4 \beta^4 \lambda \ln(36\beta^2\lambda)},$$

where $c_3, c_4 > 0$ are universal constants. Since $SP_F(B_0) \subset B_0'$, by Theorem 4.2 we infer that

$$\mathbb{P}\big(P_E\, \xi \in \alpha\sqrt{n}\, SP_F(B_0)\big) \leq \mathbb{P}\big(P_E\, \xi \in \alpha\sqrt{n}\, B_0'\big) \leq 3e^{-c_0 m/(2\beta^4)}.$$

This concludes the proof by letting $\mu := c_0/(2\beta^4)$ and $\mu' := c_4\beta^4\lambda\ln(36\beta^2\lambda)$. $\square$

Remark An alternative direct proof of Corollary 4.3 could be done following the lines of the argument in Theorem 4.2, but replacing the estimate of Proposition 4.1 by Carl's estimates ([C]) of the entropy numbers of operators from a body of the form B_0 (or B_0') to ℓ_2^n.

5 Banach–Mazur distances for subgaussian polytopes

For $j \geq 1$ let $\xi^j = (\xi_{1,j}, \ldots, \xi_{n,j}), \tilde{\xi}^j = (\tilde{\xi}_{1,j}, \ldots, \tilde{\xi}_{n,j}) \in \mathbb{R}^n$ where $\xi_{i,j}, \tilde{\xi}_{i,j}$ are independent subgaussian random variables with variances at least 1 and subgaussian constants at most β, and let Ω, $\tilde{\Omega}$ denote the underlying probability spaces. Fix $N, M > 2n$ and recall that we consider polytopes $B = B(\omega) := \mathrm{absconv}\,\{\xi^j\}_{j\leq N}$ and $\tilde{B} = B(\tilde{\omega}) := \mathrm{absconv}\,\{\tilde{\xi}^j\}_{j\leq M}$.

We first define "good" subsets $\Omega_0 \subset \Omega$ and $\tilde{\Omega}_0 \subset \tilde{\Omega}$. Set

$$\Omega_0 = \Big(|\xi^i| \leq 3\beta\sqrt{n} \text{ for } 1 \leq i \leq N \text{ and } \mu_1 B_2^n \subset \mathrm{absconv}\,\{\xi^j\}_{j\leq 2n}\Big). \tag{5.1}$$

The set $\tilde{\Omega}_0 \subset \tilde{\Omega}$ is defined fully analogously for random vectors $\tilde{\xi}^j$ and replacing N by M.

Thus fix $\tilde{\omega}_0 \in \tilde{\Omega}_0$ and note that the polytope

$$B_0 := (1/(3\beta))B(\tilde{\omega}_0) = \mathrm{absconv}\,\{\tilde{\xi}^j(\tilde{\omega}_0)/(3\beta)\}_{j\leq M}$$

satisfies (4.2), with $u_i = \tilde{\xi}_i(\tilde{\omega}_0)/(3\beta)$ for $1 \leq i \leq M$.

The argument in the proof of Theorem 2.3 splits into three steps which will be addressed separately. Let us emphasize that the roles played by random variables ξ^j for $j \leq 2n$ and $j > 2n$ are completely different. In particular, the latter variables are used in Step I below to obtain probabilistic estimates while the first $2n$ of the ξ^j's are used to control cardinality of nets in the space of operators considered in Step II below.

It is therefore convenient to consider Ω as $\Omega = \Omega' \times \Omega''$, where Ω' corresponds to the variables $\{\xi^j\}_{j\leq 2n}$ and Ω'' corresponds to $\{\xi^j\}_{2n<j\leq N}$. Furthermore, represent Ω_0 as $\Omega_0' \times \Omega_0''$ where $\Omega_0' \subset \Omega'$ and $\Omega_0'' \subset \Omega''$ are defined in a natural way. We will also denote by $\mathbb{P}, \mathbb{P}''$ and $\tilde{\mathbb{P}}$ underlying probability measures defined on Ω, Ω'' and $\tilde{\Omega}$, respectively.

Step I. Estimate for a fixed operator. With the notation above, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an operator with the $[n/2]$'th s-number $s_{[n/2]}(T) \geq 1$. Then

$$\mathbb{P}''\Big(\omega'' \,|\, T\xi^j \in \alpha\sqrt{n}\, B_0 \text{ for } 2n < j \leq N\Big) \leq 3^{N-2n} \exp\big(-\mu n(N-2n)\big), \tag{5.2}$$

where $\alpha = 3\beta\big(c_3 \ln(M/n)\big)^{-\mu'}$, $c_3 > 0$ is a universal constant and $\mu, \mu' > 0$ depend on β. Due to the independence of the ξ^j's this is an immediate consequence of Corollary 4.3. The value of α will be fixed till the end of this section.

The argument in the next two steps will be done for a fixed $\omega_0' \in \Omega_0'$. Set $\xi_1^j = \xi^j(\omega_0')$ for $1 \leq j \leq 2n$ and notice that

$$\big|\mathrm{absconv}\,\{\xi_1^j\}_{j\leq 2n}\big| \geq \mu_1^n \cdot |B_2^n|. \tag{5.3}$$

Step II. δ-net in the set of operators. Fix $\omega_0' \in \Omega_0'$. By (5.3) we get

$$\mu_1^n \cdot |B_2^n| \leq \big|\mathrm{absconv}\,\{\xi_1^j\}_{j\leq 2n}\big| \leq \binom{2n}{n} \max_{|\sigma|=n} \big|\mathrm{absconv}\,\{\xi_1^j\}_{j\in\sigma}\big|.$$

Hence there is $\sigma_1 \subset \{1,\ldots,2n\}$ with $|\sigma_1| = n$ such that $\big|\mathrm{absconv}\,\{\xi_1^j\}_{j\in\sigma_1}\big| \geq \bar{\mu}^n |B_2^n|$, where $\bar{\mu} = \mu_1/4 > 0$ depends on β only.

Consider the following set of operators $T : \mathbb{R}^n \to \mathbb{R}^n$

$$\mathcal{A} := \{T \,|\, T\xi_1^j \in \alpha\sqrt{n} B_0 \text{ for } j \in \sigma_1\}.$$

We shall control the cardinality of a minimal net in a smaller set $\mathcal{A}_0$ of operators which will be still sufficient for our purposes. Let

$$\mathcal{A}_0 := \{T \,|\, T\xi_1^j \in \alpha\sqrt{n} B_0 \text{ for } j \leq 2n \text{ and } s_{[n/2]}(T) = 1\} \subset \mathcal{A}.$$

Note that $B_0 \supset \mu_1 B_2^n$ (as $\tilde{\omega}_0 \in \tilde{\Omega}_0$) and observe that the set of operators $\mathcal{A}$ is of the form that allows to use Proposition 5.3 of [MT1] to estimate cardinality of nets in its subsets. Namely, using this proposition (with $B := \alpha\sqrt{n} B_0$ and $\mathcal{E} := \mu_1 \alpha\sqrt{n} B_2^n$) we get, that for every $0 < \delta' \leq 1/(3\beta\sqrt{n}) \leq 1/\max|\xi_1^j|$ there exists a δ'-net $\mathcal{N} \subset \mathcal{A}_0$ in the operator norm in $L(B_2^n, \mu_1\alpha\sqrt{n} B_2^n)$, from B_2^n to $\mu_1\alpha\sqrt{n} B_2^n$, of cardinality

$$\begin{aligned} |\mathcal{N}| &\leq \left(\frac{C}{\delta'}\right)^{n^2} \left(\frac{|B|\,|B_1^n|}{|\mathcal{E}|\,|\,\mathrm{absconv}\,\{\xi_1^j\}_{j\in\sigma_1}|}\right)^n \\ &\leq \left(\frac{C}{\delta'}\right)^{n^2} \left(\frac{|B_0|\,|B_1^n|}{|\mu_1 B_2^n|\,|\,\mathrm{absconv}\,\{\xi_1^j\}_{j\in\sigma_1}|}\right)^n \leq \left(\frac{\bar{\mu}'\sqrt{\ln(1+M/n)}}{\mu_1 \delta' \sqrt{n}}\right)^{n^2}, \end{aligned}$$

where C is a universal constant and $\bar{\mu}'$ depends on β. For the last inequality note that the estimate from [CP] and the definition of B_0 yield

$$|B_0|/|B_2^n| \le (c'\sqrt{\ln(1+M/n)})^n,$$

while by the choice of σ_1 one has

$$|B_1^n|/|\,\mathrm{absconv}\,\{\xi_1^j\}_{j\in\sigma_1}| \le (\mu''/\sqrt{n})^n,$$

where c' is a numerical constant and μ'' depends only on β.

For an arbitrary $0 < \delta < \mu_1\alpha/(3\beta)$, considering $\delta' = \delta/(\mu_1\alpha\sqrt{n})$, the above δ'-net $\mathcal{N} \subset \mathcal{A}_0$ becomes a δ-net in the operator norm in $L(B_2^n, B_2^n)$, of cardinality

$$|\mathcal{N}| \le \left(\frac{\bar{\mu}'\,\alpha\sqrt{\ln(1+M/n)}}{\delta}\right)^{n^2}.$$

Step III. Approximation argument. Fix $\omega_0' \in \Omega_0'$. Let $0 < \delta < \mu_1\alpha/(3\beta)$, and let $\mathcal{N} \subset \mathcal{A}_0$ be the δ-net constructed at the end of Step II. Let $T \in \mathcal{A}_0$ and let $T_0 \in \mathcal{N}$ be such that $\|T - T_0\|_{\mathrm{op}} \le \delta$.

Consider $\omega'' \in \Omega_0''$ such that for some $2n < j \le N$,

$$T_0\xi^j \notin \alpha\sqrt{n}B_0,$$

which is equivalent to $\|T_0\xi^j\|_{B_0} > \alpha\sqrt{n}$. Since $|\xi^j| \le 3\beta\sqrt{n}$ and $\mu_1 B_2^n \subset B_0$, we get

$$\begin{aligned}\|T\xi^j\|_{B_0} &\ge \|T_0\xi^j\|_{B_0} - \|(T-T_0)\xi^j\|_{B_0} > \alpha\sqrt{n} - \|(T-T_0)\xi^j\|_{\mu_1 B_2^n}\\ &\ge \alpha\sqrt{n} - \delta|\xi^j|/\mu_1 \ge (\alpha - 3\beta\delta/\mu_1)\sqrt{n}.\end{aligned}$$

Choosing $\delta = \mu_1\alpha/(6\beta)$ we obtain $\|T\xi^j\|_{B_0} > (\alpha/2)\sqrt{n}$.

This leads to an estimate for probability in Ω_0 valid in a wide range of N and M.

Lemma 5.1 *There exist $\mu_3 > 2$, $\mu_4 > 1$ and $0 < \tilde{\mu} < 1$ depending on β only such that the following holds. Suppose $N, M > \mu_3 n$ satisfy*

$$N/n \ge \mu_4 \ln\ln(M/n). \tag{5.4}$$

Set $\alpha = 3\beta\big(c_3\ln(M/n)\big)^{-\mu'}$, as in Step I. Then

$$\begin{aligned}\mathbb{P}\big(\omega \in \Omega_0 \mid \exists\, T : \mathbb{R}^n \to \mathbb{R}^n \text{ with } & s_{[n/2]}(T) = 1 \text{ and}\\ &\|T\xi^j\|_{B_0} \le (\alpha/2)\sqrt{n} \text{ for } 1 \le j \le N\big) \le 3^N \exp(-\tilde{\mu} n\, N).\end{aligned}$$

Proof For every fixed $\omega_0' \in \Omega_0'$ consider a subset of Ω_0'' defined by

$$\Theta_{\omega_0'} = \big\{\omega'' \in \Omega_0'' \mid \exists\, T : \mathbb{R}^n \to \mathbb{R}^n \text{ with } s_{[n/2]}(T) = 1 \text{ and } \\ \|T\xi^j\|_{B_0} \le (\alpha/2)\sqrt{n} \text{ for } 1 \le j \le N\big\}$$

and note that the condition on T automatically implies that $T \in \mathcal{A}_0$. Thus, by the argument above,

$$\begin{aligned}\Theta_{\omega_0'} &\subset \big\{\omega'' \in \Omega_0'' \mid \exists\, T \in \mathcal{A}_0 \text{ with } \|T\xi^j\|_{B_0} \le (\alpha/2)\sqrt{n} \text{ for } 2n < j \le N\big\} \\ &\subset \big\{\omega'' \in \Omega_0'' \mid \exists\, T_0 \in \mathcal{N} \text{ with } \|T_0\xi^j\|_{B_0} \le \alpha\sqrt{n} \text{ for } 2n < j \le N\big\}.\end{aligned}$$

By Step II and Step I, $\mathbb{P}''(\Theta_{\omega_0'})$ is less than or equal to

$$\begin{aligned}&\big(6\bar{\mu}'\beta\mu_1^{-1}\sqrt{\ln(1+M/n)}\big)^{n^2} \max_{T\in\mathcal{N}} \mathbb{P}''\big(\omega'' \mid \|T\xi^j\|_{B_0} \le \alpha\sqrt{n} \text{ for } 2n < j \le N\big) \\ &\qquad\qquad \le \big(\bar{\mu}''\sqrt{\ln(1+M/n)}\big)^{n^2} 3^{N-2n} \exp(-\mu n(N-2n)),\end{aligned}$$

where $\bar{\mu}'', \mu$ depend on β only. The assumption on N and M ensures, with an appropriate choice of μ_3, μ_4 and $\tilde{\mu}$, that the last quantity is less than or equal to $3^N \exp(-\tilde{\mu}n\, N)$. The proof is completed by integrating $\mathbb{P}''(\Theta_{\omega_0'})$ with respect to ω_0'. □

Proof of Theorem 2.3. If $\exp(\mu_5 n) \le 6$ then $1-(M+N+4)\exp(-\mu_5 n) < 0$ and the statement is obvious. So (decreasing the value of μ_5 if necessary) we may assume that $n \ge n_0$, where n_0 depends on β.

Recall that $\tilde{\Omega}_0$ is the subset of $\tilde{\Omega}$ defined as in (5.1) for subgaussian variables $\{\tilde{\xi}^j\}_{j\le M}$. In particular, by Facts 2.1 and 2.2

$$\tilde{p}_0 := \tilde{\mathbb{P}}(\tilde{\Omega}_0) \ge (1-e^{-2n})^M - e^{-\mu_2 2n} \ge 1-(M+1)e^{-\mu_2' n}.$$

Similarly $p_0 := \mathbb{P}(\Omega_0) \ge 1-(N+1)e^{-\mu_2' n}$. As observed at the beginning of this Section, for each $\tilde{\omega} \in \tilde{\Omega}_0$ the polytope

$$B_0 = (1/(3\beta))B(\tilde{\omega}) = (1/(3\beta))\,\text{absconv}\,\{\tilde{\xi}^j\}_{j\le M}$$

satisfies the condition (4.2).

Note that $\|T\|_{K\to B(\tilde{\omega})} = (1/(3\beta))\|T\|_{K\to B_0}$, for any convex body $K \subset \mathbb{R}^n$. Since N and M in particular satisfy (5.4), by Lemma 5.1 we get, for every

$\tilde{\omega} \in \tilde{\Omega}_0$,

$$\begin{aligned}\mathbb{P}\big(\omega \in \Omega_0 \mid \|T\|_{B(\omega)\to B(\tilde{\omega})} \geq (\alpha/(6\beta))\sqrt{n} \ \text{ for every } T : \mathbb{R}^n \to \mathbb{R}^n \\ \text{with } \ s_{n/2}(T) = 1\big) &\geq p_0 - 3^N \exp(-\tilde{\mu} n\, N) \\ &\geq 1 - (N+2)\exp(-\tilde{\mu}' n),\end{aligned}$$

where $\tilde{\mu}, \tilde{\mu}'$ depend on β only. By integrating with respect to $\tilde{\omega} \in \tilde{\Omega}_0$ we conclude

$$\begin{aligned}(\mathbb{P} \times \tilde{\mathbb{P}})\big((\omega, \tilde{\omega}) \in \Omega_0 \times \tilde{\Omega}_0 \mid \|T\|_{B(\omega)\to B(\tilde{\omega})} \geq (\alpha/(6\beta))\sqrt{n} \\ \text{for every } T : \mathbb{R}^n \to \mathbb{R}^n \text{ with } s_{[n/2]}(T) = 1\big) \\ \geq \big(1 - (N+2)\exp(-\tilde{\mu}' n)\big)\, \tilde{p}_0 \geq 1 - (M+N+3)\exp(-\mu_2' n).\end{aligned}$$

Denote the subset above by Θ.

Now observe that the role of $B(\omega)$ and $B(\tilde{\omega})$ can be interchanged and that a version of condition (5.4) for N and M replacing each other is satisfied as well. Therefore, if we define $\tilde{\Theta}$ in a analogous way as Θ above then, clearly,

$$(\mathbb{P} \times \tilde{\mathbb{P}})(\tilde{\Theta}) \geq 1 - (M+N+3)\exp(-\mu_2' n).$$

Putting these estimates together we get

$$(\mathbb{P} \times \tilde{\mathbb{P}})(\Theta \cap \tilde{\Theta}) \geq 1 - 2(M+N+3)\exp(-\mu_2' n),$$

where $\mu_2' > 0$ depends on β only. The proof is completed by observing that for every $(\omega, \tilde{\omega}) \in \Theta \cap \tilde{\Theta}$ (using α as defined in Lemma 5.1),

$$d(B(\omega), B(\tilde{\omega})) \geq (1/4)\big(c_3^2 \ln(N/n)\ln(M/n)\big)^{-\mu'} n,$$

where $c_3, \mu' > 0$ are as in Step I. Indeed, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible operator. By multiplying T by a suitable constant we may assume that both $s_{[n/2]}(T)$ and $s_{[n/2]}(T^{-1})$ are bigger than or equal to 1. Thus, by the definition of Θ,

$$\|T\|_{B(\omega)\to B(\tilde{\omega})} \geq (1/2)\big(c_3 \ln(M/n)\big)^{-\mu'}\sqrt{n},$$

and, by the definition of $\tilde{\Theta}$,

$$\|T^{-1}\|_{B(\tilde{\omega})\to B(\omega)} \geq (1/2)\big(c_3 \ln(N/n)\big)^{-\mu'}\sqrt{n}.$$

Passing to the infimum over all T we get the required bound for $d(B(\omega), B(\tilde{\omega}))$.
□

6 Mixing operators

Recall that an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $(m,1)$-mixing, for $m \le n/2$, if there exists a subspace $E \subset \mathbb{R}^n$ with $\dim E \ge m$ such that

$$|P_{E^\perp} Tx| \ge |x| \quad \text{for every } x \in E.$$

The notion of mixing operators has proved to be a convenient tool in discussions of some geometric properties of finite–dimensional Banach spaces. We shall use the following obvious properties of mixing operators: an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is $(m,1)$-mixing if and only if for every $\lambda \in \mathbb{R}$, the operator $T + \lambda \mathrm{Id}$ is $(m,1)$-mixing. Also, if T is $(m,1)$-mixing then T is $(k,1)$-mixing for every $k \le m$. Finally, let us also recall that every operator $T : \mathbb{R}^n \to \mathbb{R}^n$ of the form $T = 2P$, where P is a (not necessarily orthogonal) projection of rank l, is $(\min\{l, n-l\}, 1)$-mixing.

Theorem 2.4 which we will prove in this Section is an immediate consequence of a more general result below, that provides lower estimates of norms of operators acting on $B(\omega)$ in terms of their mixing properties.

Theorem 6.1 *Let $\rho \in (0, 1/2)$. There exists $\mu_6' > 1$ depending on β and ρ such that the following holds. Let $N \ge \mu_6' n^2 \ln(en)$ and let B be defined as in (2.4) by independent subgaussian vectors with variance ≥ 1 and subgaussian constants at most β. Then on a set of probability larger than or equal to $1 - (N+1)\exp(-2n)$ body $B(\omega)$ has the property that*

$$\|T\|_{B(\omega) \to B(\omega)} \ge \alpha \sqrt{n} \tag{6.1}$$

for every $(m,1)$-mixing operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and every $\rho n \le m \le n/2$. Here $\alpha = \big(c_3' \ln(N/n)\big)^{-\mu'}$, where $c_3' > 0$ is a universal constant and $\mu' > 0$ depends on ρ and β.

Remark If $e^{2n} \le \mu_6' n^2 \ln(en)$ then $1 - (N+1)\exp(-2n) < 0$ and the statement is obvious. So increasing if necessary the value of μ_6' we may assume that $n \ge n_0$, where n_0 depends on β and ρ. Also we may assume that $\mu' \ge 1$ so it is enough to show (6.1) for $\alpha = \big(c_3 \ln(N/n)\big)^{-\mu'}/4$ and then choose $c_3' = 4c_3$.

First note that by the comments above on mixing operators, it is sufficient to prove that (6.1) holds for all $(m,1)$-mixing operators T, for $m := \lceil \rho n \rceil$. By adjusting the parameter ρ if necessary we may also assume that $\lceil \rho n \rceil \le n/2$.

For technical reasons in this section we require somewhat more delicate definitions of the "good" subsets of Ω. Namely we let

$$\Omega_0 = \Big(|\xi^i| \le 3\beta\sqrt{n} \text{ for } 1 \le i \le N \qquad \text{and}$$
$$\mu_1 B_2^n \subset \operatorname{absconv}\{\xi^j\}_{j\le 2n} \cap \operatorname{absconv}\{\xi^j\}_{2n<j\le 4n} \Big), \tag{6.2}$$

where μ_1 is from Fact 2.2.

For $i \le N$ set $B_i = B_i(\omega) = \operatorname{absconv}\{\xi^j\}_{j\ne i}$ and note that by (6.2) for each $i \le N$ and every $\omega \in \Omega_0$

$$\mu_1 B_2^n \subset B_i(\omega). \tag{6.3}$$

The proof of Theorem 6.1 goes along lines parallel to these of the proof of Theorem 2.3. Namely, the first step is to establish a probabilistic estimate for an arbitrary mixing operator. This is the main part of the argument that requires more delicate combinatorial-probabilistic considerations. The next steps are already standard: we find a sufficiently dense net, of well controlled cardinality, in an appropriate class of operators and, finally, by a standard approximation argument we deduce the required inequalities for all mixing operators from similar inequalities for each member of the net.

Step I. Estimate for a fixed operator.

Proposition 6.2 *Let $N > 4n$ and $1 \le m \le n/2$ and set $\rho := m/n$. For every $(m,1)$-mixing operator $T : \mathbb{R}^n \to \mathbb{R}^n$ we have*

$$\mathbb{P}\big(\omega \in \Omega_0 \mid \|T\|_{B(\omega)\to B(\omega)} \le \alpha\sqrt{n}\big) \le \binom{N}{\ell} \left(\mu''\sqrt{n}e^{-\mu m}\right)^{\ell},$$

for $\alpha = \big(c_3 \ln(N/n)\big)^{-\mu'}/2$, where $\ell = \lceil N/(2n+3)\rceil$, $c_3 > 0$ is a universal constant, $\mu, \mu'' > 0$ depend on β and $\mu' > 0$ depends on β and ρ.

Remark For future reference note that for sufficiently large $n \ge n_0(\beta,\rho)$ we have $\binom{N}{\ell}\left(\mu''\sqrt{n}\exp(-\mu m)\right)^{\ell} \le \exp(-\tilde{\mu}N)$ where $\tilde{\mu} > 0$ depends on ρ and β.

Proof Fix n, N, m, T as in the proposition and $\alpha' > 0$ to be specified later. For $1 \le i \le N$ consider the subset of Ω_0

$$\begin{aligned} A_i &:= \big\{\omega \in \Omega_0 \mid T\xi^i \in \alpha' \operatorname{absconv}(\xi^j)_{j\le N}\big\} \\ &= \big\{\omega \in \Omega_0 \mid \exists\, |\lambda| \le 1 \text{ s.t. } T\xi^i - \alpha'\lambda\xi^i \in \alpha'(1-|\lambda|)B_i(\omega)\big\} \\ &\subset \big\{\omega \in \Omega_0 \mid \exists\, |\lambda| \le 1 \text{ s.t. } T\xi^i - \alpha'\lambda\xi^i \in \alpha' B_i(\omega)\big\}. \end{aligned}$$

Let Λ be a $\mu_1/(3\beta\sqrt{n})$-net in the interval $[-1,1]$ with $|\Lambda| \le 6\mu_1^{-1}\beta\sqrt{n}$. Then by (6.3) and the triangle inequality one gets

$$\begin{aligned} A_i \quad &\subset \quad \bigcup_{\lambda\in\Lambda} \big\{\omega \in \Omega_0 \mid T\xi^i - \alpha'\lambda\xi^i \in 2\alpha' B_i(\omega)\big\} \\ &\subset \quad \bigcup_{\lambda\in\Lambda} \big\{\omega \in \Omega_{0,i} \mid T\xi^i - \alpha'\lambda\xi^i \in 2\alpha' B_i(\omega)\big\}, \end{aligned} \tag{6.4}$$

where

$$\Omega_{0,i} := \{\omega \in \Omega \mid \ |\xi^j| \le 3\beta\sqrt{n} \text{ for } j \ne i\} \supset \Omega_0.$$

For $\lambda \in \Lambda$ denote by $\tilde{A}_{i,\lambda}$ the term corresponding to λ in the latter union in (6.4). First consider a fixed $\lambda \in \Lambda$. Set $S = T - \alpha'\lambda \mathrm{Id}_{\mathbb{R}^n}$. Let E be an m-dimensional subspace from the definition of the mixing property for the operator T, that is, such that $|P_{E^\perp}Tx| \ge |x|$ for every $x \in E$ and $\dim E = m$. Then, by the definition of the operator S and a property of mixing operators mentioned in the beginning of this section,

$$|Sx| \ge |P_{E^\perp}Sx| \ge |x| \quad \text{ for every } \quad x \in E. \tag{6.5}$$

Since $\dim E = m$, this yields $s_m(S) \ge 1$, and the probability of $\tilde{A}_{i,\lambda}$ can be tackled using Corollary 4.3.

Additionally, further parts of the argument will also require some conditional probability considerations. Let $\Sigma_{\{i\}^c}$ denote the σ-algebra generated by $\{\xi^j\}_{j\ne i}$. For the moment, let f denote the indicator function of $\tilde{A}_{i,\lambda}$ and let $h = \mathbb{E}(f \mid \Sigma_{\{i\}^c})$ be the conditional expectation of f given $\Sigma_{\{i\}^c}$. We want to get a pointwise estimate for the conditional probability $\mathbb{P}\big(\tilde{A}_{i,\lambda} \,|\, \Sigma_{\{i\}^c}\big)(\omega) := h(\omega)$, for every $\omega \in \Omega$.

Note that $\Omega_{0,i}$ is $\Sigma_{\{i\}^c}$-measurable and $\tilde{A}_{i,\lambda} \subset \Omega_{0,i}$. This implies $h(\omega) = 0$ for $\omega \notin \Omega_{0,i}$. On the other hand, for arbitrary fixed $\omega \in \Omega_{0,i}$, by Corollary 4.3 and the definition of conditional expectation we have,

$$\begin{aligned} h(\omega) &= \mathbb{P}\big(\omega' \mid S\xi^i(\omega') \in 2\alpha' \,\mathrm{absconv}\, \{\xi^j(\omega)\}_{j\ne i}\big) \\ &= \mathbb{P}\big(\omega' \mid S\xi^i(\omega') \in 6\alpha'\beta \,\mathrm{absconv}\, \{\xi^j(\omega)/(3\beta)\}_{j\ne i}\big) \le 3e^{-\mu m}, \end{aligned}$$

therefore,

$$\mathbb{P}\big(\tilde{A}_{i,\lambda} \,|\, \Sigma_{\{i\}^c}\big)(\omega) = h(\omega) \le 3e^{-\mu m} \quad \text{ for every } \omega \in \Omega. \tag{6.6}$$

Here $\alpha' = \big(c_3 \ln((N-1)/n)\big)^{-\mu'} \sqrt{n}/2$, where $c_3 > 0$ is a universal constant, $\mu > 0$ depends on β and $\mu' > 0$ depends on β and ρ. Since λ was an arbitrary member of the net Λ, this yields,

$$\mathbb{P}(\tilde{A}_{i,\lambda} \mid \Sigma_{\{i\}^c}) \le 3e^{-\mu m} \quad \text{for every } \lambda \in \Lambda. \tag{6.7}$$

Setting $\tilde{A}_i = \bigcup_{\lambda\in\Lambda} \tilde{A}_{i,\lambda}$, by (6.7) we get

$$\mathbb{P}\big(\tilde{A}_i | \Sigma_{\{i\}^c}\big) \le \sum_{\lambda\in\Lambda} \mathbb{P}\big(\tilde{A}_{i,\lambda} | \Sigma_{\{i\}^c}\big) \le 18\mu_1^{-1}\beta\sqrt{n}\; e^{-\mu m}. \tag{6.8}$$

Now we are in a position to use a decoupling principle for weakly dependent sets from [ST] which allows us to pass from an estimate of an individual event such as (6.8) to an estimate for the product of the events which is needed for the proposition. To this end, for $1 \le i \le N$ and a subset $D \subset \{1, \dots, N\}$, consider the event

$$\Theta_{i,D} = \bigcup_{\lambda\in\Lambda} \big\{\omega \in \Omega_0 \mid T\xi^i - \alpha'\lambda\xi^i \in 2\alpha' \,\mathrm{absconv}\,(\xi^j)_{j\in D}\big\}.$$

By Caratheodory's theorem, for any D,

$$\Theta_{i,D} \subset \bigcup_{D'\subset D, |D'|\le n+1} \Theta_{i,D'}.$$

For $J \subset \{1, \dots, N\}$ by Σ_J denote the σ-algebra generated by $\{\xi^j\}_{j\in J}$. Clearly, for $I, J \subset \{1, \dots, N\}$ with $I \cap J = \emptyset$, the events $\{\Theta_{j,I} \mid j \in J\}$ are Σ_I-conditionally independent. Finally note that in the notation above, by (6.4) and the definition of $\tilde{A}_i$, for every $i = 1, 2, \dots, N$,

$$A_i \subset \Theta_{i,\{i\}^c} \subset \tilde{A}_i. \tag{6.9}$$

Hence, by (6.8) we get

$$\mathbb{P}\big(\Theta_{i,\{i\}^c} | \Sigma_{\{i\}^c}\big) \le \mathbb{P}\big(\tilde{A}_i | \Sigma_{\{i\}^c}\big) \le 18\mu_1^{-1}\beta\sqrt{n}\; e^{-\mu m}. \tag{6.10}$$

Thus, by Theorem 2 in [ST] combined with (6.9) and (6.10) we have

$$\mathbb{P}\left(\bigcap_{i\le N} A_i\right) \le \mathbb{P}\left(\bigcap_{i\le N} \Theta_{i,\{i\}^c}\right) \le \binom{N}{\ell} \big(18\mu_1^{-1}\beta\sqrt{n}e^{-\mu m}\big)^{\ell}, \tag{6.11}$$

where $\ell = \lceil N/(2n+3) \rceil$.

The proof of the proposition is completed by observing that letting $\alpha := \alpha'/\sqrt{n}$ we get

$$\left\{\omega \in \Omega_0 \mid \|T\|_{B(\omega)\to B(\omega)} \le \alpha\sqrt{n}\right\} = \bigcap_{i\le N} A_i.$$

$\square$

Step II. δ-net in the set of operators. Note that by the definition of Ω_0

$$\mu_1 B_2^n \subset B(\omega) \subset 3\beta\sqrt{n} B_2^n,$$

for every $\omega \in \Omega_0$. Hence for every operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and $\omega \in \Omega_0$

$$\|T\|_{B(\omega)\to B(\omega)} \le \|T : 3\beta\sqrt{n} B_2^n \to \mu_1 B_2^n\| = 3\beta\mu_1^{-1}\sqrt{n}\|T\|_{\mathrm{op}}. \tag{6.12}$$

Similarly

$$\|T\|_{B(\omega)\to B(\omega)} \ge \mu_1(3\beta\sqrt{n})^{-1}\|T\|_{\mathrm{op}}. \tag{6.13}$$

The last inequality yields that if for some $T : \mathbb{R}^n \to \mathbb{R}^n$ we have $\|T\|_{\mathrm{op}} \ge 3\alpha\beta\mu_1^{-1}n$, then $\|T\|_{B(\omega)\to B(\omega)} \ge \alpha\sqrt{n}$. Therefore, in our proof of Theorem 6.1 we may restrict our considerations to operators satisfying $\|T\|_{\mathrm{op}} \le 3\alpha\beta\mu_1^{-1}n$. Let

$$\mathcal{A} := \{T \mid \|T\|_{\mathrm{op}} \le 3\alpha\beta\mu_1^{-1}n \text{ and } T \text{ is } (m,1)\text{-mixing}\},$$

let $\delta = \alpha\mu_1/(6\beta)$ and let $\mathcal{N}$ be a δ–net in $\mathcal{A}$ (in the operator norm), with cardinality

$$|\mathcal{N}| \le (72\beta^2\mu_1^{-2}n)^{n^2}. \tag{6.14}$$

(The existence of such a net follows from the well known "comparison of volumes" argument.)

Step III. Approximation argument. For every $(m,1)$-mixing operator $T : \mathbb{R}^n \to \mathbb{R}^n$ denote by $\Theta_T \subset \Omega_0$ the set considered in Proposition 6.2. Set

$$\Theta = \Omega_0 \setminus \bigcup_{T\in\mathcal{N}} \Theta_T. \tag{6.15}$$

We claim that for every $\omega \in \Theta$ and every $(m,1)$-mixing operator $T : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\|T\|_{B(\omega)\to B(\omega)} \ge (\alpha/2)\sqrt{n}, \tag{6.16}$$

with, recall, $\alpha = \big(c_3 \ln(N/n)\big)^{-\mu'}/2$, where c_3 and μ' are from Proposition 6.2.

Indeed, fix $\omega \in \Theta$ and an $(m,1)$-mixing operator T. By the considerations in Step II, if $T \notin \mathcal{A}$ then (6.16) is satisfied (even with α in place of $\alpha/2$). Therefore we may assume that $T \in \mathcal{A}$. Let $T_0 \in \mathcal{N}$ be such that $\|T - T_0\|_{\rm op} \le \delta$. Since $\omega \in \Theta$ then $\|T_0\|_{B(\omega)\to B(\omega)} \ge \alpha\sqrt{n}$. This implies that there exists ξ^j, for some $1 \le j \le N$, such that

$$\|T_0 \xi^j\|_{B(\omega)} \ge \alpha\sqrt{n}.$$

Note that always $\|\xi^j\|_{B(\omega)} \le 1$. Hence by (6.12) and the choice of δ, we get, for the same j,

$$\begin{aligned}
\|T\|_{B(\omega)\to B(\omega)} &\ge \|T\xi^j\|_{B(\omega)} \\
&\ge \|T_0\xi^j\|_{B(\omega)} - \|(T - T_0)\xi^j\|_{B(\omega)} \\
&\ge \alpha\sqrt{n} - \|T - T_0\|_{B(\omega)\to B(\omega)} \|\xi^j\|_{B(\omega)} \\
&\ge \alpha\sqrt{n} - 3\beta\mu_1^{-1}\sqrt{n}\|T - T_0\|_{\rm op} \\
&\ge (\alpha/2)\sqrt{n}. \qquad (6.17)
\end{aligned}$$

This proves (6.16).

Proof of Theorem 6.1. In view of (6.16), the proof of Theorem 6.1 will be completed once we get the probability estimate for Θ. By (6.2) and Facts 2.1 and 2.2, and by Proposition 6.2 (and the remark afterwards), (6.15) and (6.14), we get

$$\begin{aligned}
\mathbb{P}(\Theta) &\ge \mathbb{P}(\Omega_0) - (72\beta^2\mu_1^{-2}n)^{n^2} \binom{N}{\ell} \left(\mu''\sqrt{n}e^{-\mu m}\right)^{\ell} \\
&\ge 1 - Ne^{-2n} - 2e^{-\mu_2 N} - (72\beta^2\mu_1^{-2}n)^{n^2} e^{-\tilde{\mu}N}.
\end{aligned}$$

The latter expression is larger than or equal to

$$1 - Ne^{-2n} - 3e^{-\tilde{\mu}' N} \ge 1 - (N+1)e^{-2n},$$

provided that $N \ge \tilde{\mu}'' n^2 \ln n$, where $\tilde{\mu}', \tilde{\mu}'' > 0$ depend on ρ and β. $\square$

Proof of Theorem 2.4. The theorem is an immediate consequence of Theorem 6.1 and the remarks following the definition of mixing operators. Namely, for every rank m projection P with $n/4 \le m \le 3n/4$, the operator $2P$ is $(n/4, 1)$-mixing and the conclusion follows from Theorem 6.1, with the constant in the norm estimate divided by 2. $\square$

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