CHEVET-TYPE INEQUALITIES FOR SUBEXPONENTIAL WEIBULL VARIABLES AND ESTIMATES FOR NORMS OF RANDOM MATRICES

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Abstract. We prove two-sided Chevet-type inequalities for independent symmetric Weibull random variables with shape parameter $r \in [1,2]$. We apply them to provide two-sided estimates for operator norms from ℓ_p^n to ℓ_q^m of random matrices $(a_i b_j X_{i,j})_{i\le m, j\le n}$, in the case when $X_{i,j}$'s are iid symmetric Weibull variables with shape parameter $r \in [1,2]$ or when X is an isotropic log-concave unconditional random matrix. We also show how these Chevet-type inequalities imply two-sided bounds for maximal norms from ℓ_p^n to ℓ_q^m of submatrices of X in both Weibull and log-concave settings.

1. Introduction and main results

1.1. **Chevet-type two-sided bounds.** The classical Chevet inequality [13] is a two-sided bound for operator norms of Gaussian random matrices with iid entries. It states that if $g_{i,j}$, g_i, $i,j \ge 1$ are iid standard Gaussian random variables, then for every pair of nonempty bounded sets $S \subset \mathbb{R}^m$, $T \subset \mathbb{R}^n$ we have

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} g_{i,j} s_i t_j \sim \sup_{s\in S} \|s\|_2 \mathbb{E} \sup_{t\in T} \sum_{j=1}^n g_j t_j + \sup_{t\in T} \|t\|_2 \mathbb{E} \sup_{s\in S} \sum_{i=1}^m g_i s_i. \tag{1}$$

Here and in the sequel we write $f \lesssim g$ or $g \gtrsim f$, if $f \le Cg$ for a universal constant C, and $f \sim g$ if $f \lesssim g \lesssim f$. (We write $\lesssim_\alpha$, $\sim_{K,\gamma}$, etc. if the underlying constant depends on the parameters given in the subscripts.) The original motivation for Chevet's result was convergence of Gaussian random sums in tensor spaces, but Chevet-type bounds (i.e., two-sided bounds allowing to compare the expected double supremum of the linear combination of some random variables with some simpler quantities, involving only expectations of a single supremum) are also a useful tool in providing bounds for operator norms of random matrices, as we shall see in the next subsection.

Chevet's inequality was generalised to several settings. A version for iid stable r.v.'s was provided in [14]. Moreover, it was shown in [1, Theorem 3.1] that the upper bound in (1) holds if one replaces – on both sides of (1) – iid Gaussians by iid symmetric exponential r.v.'s. It is however not hard to see (cf. Remark 3 following Theorem 3.1 in [1]) that such a bound cannot be reversed.

The first result of this note is a counterpart of (1) for symmetric Weibull matrices $(X_{i,j})$ with a fixed (shape) parameter $r \in [1,2]$, i.e., symmetric random variables $X_{i,j}$ such that

$$\mathbb{P}(|X_{i,j}| \ge t) = \exp(-t^r) \quad \text{for every } t \ge 0.$$

It is natural to consider Weibull r.v.'s since they interpolate between Gaussian and exponential r.v.'s – the case $r=1$ corresponds to exponential r.v.'s, whereas in the case $r=2$ the r.v.'s $X_{i,j}$ are comparable to Gaussian r.v.'s with variance $1/2$ (see Lemma 19 below). In particular, our result in the case $r=1$ provides two-sided bounds for iid exponential r.v.'s, which is therefore

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a better version of the aforementioned upper bound obtained in [1] (in this note ρ^* denotes the Hölder conjugate of $\rho \in [1,\infty]$, i.e., the unique element of $[1,\infty]$ satisfying $\frac{1}{\rho}+\frac{1}{\rho^*}=1$.).

Theorem 1. *Let $X_{i,j}$, X_i, X_j, $1\le i\le m$, $1\le j\le n$ be iid symmetric Weibull r.v.'s with parameter $r\in[1,2]$. Then for every nonempty bounded sets $S\subset\mathbb{R}^m$ and $T\subset\mathbb{R}^n$ we have*

$$\begin{aligned}\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}X_{i,j}s_it_j &\sim \sup_{s\in S}\|s\|_{r^*}\mathbb{E}\sup_{t\in T}\sum_{j=1}^n X_jt_j+\sup_{t\in T}\|t\|_{r^*}\mathbb{E}\sup_{s\in S}\sum_{i=1}^m X_is_i\\ &\quad+\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}g_{i,j}s_it_j\\ &\sim \sup_{s\in S}\|s\|_{r^*}\mathbb{E}\sup_{t\in T}\sum_{j=1}^n X_jt_j+\sup_{s\in S}\|s\|_2\mathbb{E}\sup_{t\in T}\sum_{j=1}^n g_jt_j\\ &\quad+\sup_{t\in T}\|t\|_{r^*}\mathbb{E}\sup_{s\in S}\sum_{i=1}^m X_is_i+\sup_{t\in T}\|t\|_2\mathbb{E}\sup_{s\in S}\sum_{i=1}^m g_is_i.\end{aligned}$$

This result generalizes to the case of independent ψ_r random variables. There are several equivalent definitions of ψ_r r.v.'s – in this paper we say that a random variable Z is ψ_r with constant σ if

$$\mathbb{P}(|Z|\ge t)\le 2e^{-(t/\sigma)^r}\qquad\text{for every } t\ge 0.$$

One of the reasons to investigate Weibull r.v.'s is that Weibulls with parameter r are extremal in the class of ψ_r random variables, which appear frequently in probability theory, statistics, and their applications, e.g., in convex geometry (see, e.g., [10, 12, 30]). In particular, Theorem 1 and a standard estimate (see Lemma 7 below) yield the following result (observe that we do not assume that the r.v.'s $Y_{i,j}$ are identically distributed).

Corollary 2. *Let $X_1,X_2,\ldots$ be iid symmetric Weibull r.v.'s with parameter $r\in[1,2]$, and let $Y_{i,j}$, $1\le i\le m$, $1\le j\le n$ be independent centered ψ_r random variables with constant σ. Then for every bounded nonempty sets $S\subset\mathbb{R}^m$ and $T\subset\mathbb{R}^n$ we have*

$$\begin{aligned}\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\le m,j\le n}Y_{i,j}s_it_j \lesssim \sigma\Big(&\sup_{s\in S}\|s\|_{r^*}\mathbb{E}\sup_{t\in T}\sum_{j=1}^n X_jt_j+\sup_{s\in S}\|s\|_2\mathbb{E}\sup_{t\in T}\sum_{j=1}^n g_jt_j\\ &+\sup_{t\in T}\|t\|_{r^*}\mathbb{E}\sup_{s\in S}\sum_{i=1}^m X_is_i+\sup_{t\in T}\|t\|_2\mathbb{E}\sup_{s\in S}\sum_{i=1}^m g_is_i\Big).\end{aligned}$$

Let us now focus on the case $r=1$. Random vectors with independent symmetric exponential coordinates are extremal in the class of unconditional isotropic log-concave random vectors (cf., [8, 19]). Recall that we call a random vector Z in $\mathbb{R}^k$ log-concave, if for any compact nonempty sets $K,L\subset\mathbb{R}^k$ and $\lambda\in[0,1]$ we have

$$\mathbb{P}\big(Z\in\lambda K+(1-\lambda)L\big)\ge\mathbb{P}(Z\in K)^\lambda\mathbb{P}(Z\in L)^{1-\lambda}.$$

Log-concave vectors are a natural generalization of the class of uniform distributions over convex bodies and they are widely investigated in convex geometry and high dimensional probability (see the monographs [4, 10]). By the result of Borell [9] we know that log-concave vectors with nondegenerate covariance matrix are exactly the vectors with a log-concave density, i.e., with a density whose logarithm is a concave function with values in $[-\infty,\infty)$.

A random vector Z in $\mathbb{R}^k$ is called unconditional, if for every choice of signs $\eta\in\{-1,1\}^k$ the vectors Z and $(\eta_iZ_i)_{i\le k}$ are equally distributed (or, equivalently, that Z and $(\varepsilon_iZ_i)_{i\le k}$ are equally distributed, where $\varepsilon_1,\ldots,\varepsilon_k$ are iid symmetric Bernoulli variables independent of Z). A random vector is called isotropic if it is centered and its covariance matrix is the identity.

[19, Theorem 2] yields that for every bounded nonempty set U in $\mathbb{R}^k$ (see Lemma 8 below for a standard reduction to the case of symmetric index sets) and every k-dimensional unconditional isotropic log-concave random vector Y,

$$\mathbb{E}\sup_{u\in U}\sum_{i=1}^{k} u_i Y_i \lesssim \mathbb{E}\sup_{u\in U}\sum_{i=1}^{k} u_i E_i, \tag{2}$$

where $E_1, E_2, \dots, E_k$ are independent symmetric exponential r.v.'s (i.e., iid Weibull r.v.'s with shape parameter $r = 1$). Hence, Theorem 1 yields the following Chevet-type bound for isotropic unconditional log-concave random matrices.

Corollary 3. *Let $E_1, E_2, \dots$ be iid symmetric exponential r.v.'s, and let $Y = (Y_{i,j})_{1\le i\le m, 1\le j\le n}$ be a random matrix with isotropic unconditional log-concave distribution on $\mathbb{R}^{mn}$. Then for every bounded nonempty sets $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ we have*

$$\mathbb{E}\sup_{s\in S, t\in T}\sum_{i\le m, j\le n} Y_{i,j}s_i t_j \lesssim \sup_{s\in S}\|s\|_\infty \mathbb{E}\sup_{t\in T}\sum_{j=1}^{n} E_j t_j + \sup_{s\in S}\|s\|_2 \mathbb{E}\sup_{t\in T}\sum_{j=1}^{n} g_j t_j$$
$$+ \sup_{t\in T}\|t\|_\infty \mathbb{E}\sup_{s\in S}\sum_{i=1}^{m} E_i s_i + \sup_{t\in T}\|t\|_2 \mathbb{E}\sup_{s\in S}\sum_{i=1}^{m} g_i s_i.$$

Let us stress that estimate (2) is no longer true for general isotropic log-concave vectors as [1, Theorem 5.1] shows. We do not know whether there exists a counterpart of Corollary 3 for arbitrary isotropic log-concave random matrices.

The next subsection reveals how our Chevet-type inequalities imply precise bounds for norms of random matrices.

1.2. **Norms of random matrices.** Initially motivated by mathematical physics, the theory of random matrices [3, 29] is now used in many areas of mathematics. A great effort was made to understand the asymptotic behaviour of the edge of the spectrum of random matrices with independent entries. In particular, numerous bounds on their spectral norm (i.e., the largest singular value) were derived. The seminal result of Seginer [27] states that in the iid case the expectation of the spectral norm is of the same order as the expectation of the maximum Euclidean norm of rows and columns of a given random matrix. We know from [25] that the same is true for the structured Gaussian matrices $G_A = (a_{i,j}g_{i,j})_{i\le m, j\le n}$, where $g_{i,j}$'s are iid standard Gaussian r.v.'s, and $(a_{i,j})_{i,j}$ is a deterministic matrix encoding the covariance structure of G_A. Although in the structured Gaussian case we still assume that the entries are independent, obtaining optimal bounds in this case was much more challenging than in the non-structured case. Upper bounds for the spectral norm of some Gaussian random matrices with dependent entries were obtained very recently in [5].

In this note we are interested in bounding more general operator norms of random matrices. For $\rho \in [1,\infty)$ by $\|x\|_\rho = (\sum_i |x_i|^\rho)^{1/\rho}$, we denote the ℓ_ρ-norm of a vector x. A similar notation, $\|S\|_\rho = (\mathbb{E}|S|^\rho)^{1/\rho}$ is used for the L_ρ-norm of a random variable S. For $\rho = \infty$ we write $\|x\|_\infty := \max_i |x_i|$. By B_ρ^k we denote the unit ball in $(\mathbb{R}^k, \|\cdot\|_\rho)$. For an $m\times n$ matrix $X = (X_{i,j})_{i\le m, j\le n}$ we denote by

$$\|X\|_{\ell_p^n\to\ell_q^m} = \sup_{t\in B_p^n}\|Xt\|_q = \sup_{t\in B_p^n, s\in B_{q^*}^m} s^T X t = \sup_{t\in B_p^n, s\in B_{q^*}^m}\sum_{i\le m, j\le n} X_{i,j}s_i t_j$$

its operator norm from ℓ_p^n to ℓ_q^m. In particular, $\|X\|_{\ell_2^n\to\ell_2^m}$ is the spectral norm of X. When $(p,q)\ne(2,2)$, the moment method used to upper bound the operator norm cannot be employed. This is one of the reasons why upper bounds for $\mathbb{E}\|X\|_{\ell_p^n\to\ell_q^m}$ are known only in some special cases, and most of them are optimal only up to logarithmic factors. Before we move to a brief survey of these results, let us note that bounds for $\mathbb{E}\|X\|_{\ell_p^n\to\ell_q^m}$ yield both tail bounds for $\|X\|_{\ell_p^n\to\ell_q^m}$ and bounds for $(\mathbb{E}\|X\|^\rho_{\ell_p^n\to\ell_q^m})^{1/\rho}$ for every $\rho\geq 1$, provided that the entries of X satisfy a mild regularity assumption; see [2, Proposition 1.16] for more details.

Chevet's inequality together with, say, Remark 21 below easily yields the following two-sided estimate for $\ell_p^n\to\ell_q^m$ norms of iid Gaussian matrices for every $p,q\in[1,\infty]$,

(3)
$$\mathbb{E}\big\|(g_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m}\sim\begin{cases} m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\leq 2,\\ \sqrt{p^*\wedge\operatorname{Log} n}\,n^{1/p^*}m^{1/q-1/2}+m^{1/q}, & q\leq 2\leq p^*,\\ n^{1/p^*}+\sqrt{q\wedge\operatorname{Log} m}\,m^{1/q}n^{1/p^*-1/2}, & p^*\leq 2\leq q,\\ \sqrt{p^*\wedge\operatorname{Log} n}\,n^{1/p^*}+\sqrt{q\wedge\operatorname{Log} m}\,m^{1/q}, & 2\leq q,p^*\end{cases}$$
$$\sim\sqrt{p^*\wedge\operatorname{Log} n}\,m^{(1/q-1/2)\vee 0}n^{1/p^*}+\sqrt{q\wedge\operatorname{Log} m}\,n^{(1/p^*-1/2)\vee 0}m^{1/q},$$

where to simplify the notation we define

$$\operatorname{Log} n=\max\{1,\ln n\}.$$

If the entries $X_{i,j}$ are bounded and centered, then it is known that

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m,j\leq n}\big\|_{\ell_p^n\to\ell_q^m}\lesssim_{p,q}\begin{cases} m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\leq 2,\\ m^{1/q-1/2}n^{1/p^*}+m^{1/q}, & q\leq 2\leq p^*,\\ n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*\leq 2\leq q,\\ n^{1/p^*}+m^{1/q}, & 2\leq p^*,q.\end{cases}\tag{4}$$

This was proven in [7] in the case $p=2\leq q$ and may be easily extrapolated to the whole range $1\leq p,q\leq\infty$ (see [6, 11], cf., [2, Remark 4.2]). Moreover, in the case of matrices with iid symmetric Bernoulli r.v.'s inequality (4) may be reversed. In [26, Lemma 172] it was shown that in the square case (i.e., when $m=n$) estimate (4) holds with a constant non depending on p and q. The two-sided estimate for rectangular Bernoulli matrices is more complicated – we capture the correct dependence of the underlying constants on p and q in an upcoming article [22]. As for the Gaussian random matrices, also the Bernoulli *structured* case is much more difficult to deal with, even when $p=q=2$. Nevertheless, in this case an upper bound optimal up to $\log\log$ factor is known in the case of circulant matrices due to [23, Theorem 1.3].

The case of structured Gaussian matrices in the range $p\leq 2\leq q$ was investigated in [17]; in this case

$$\mathbb{E}\|G_A\|_{\ell_p^n\to\ell_q^m}\sim_{p,q}\max_{j\leq n}\|(a_{i,j})_{i=1}^m\|_q+(\operatorname{Log} m)^{1/q}\Big(\max_{i\leq m}\|(a_{i,j})_{j=1}^n\|_{p^*}+\mathbb{E}\max_{i\leq m,j\leq n}|a_{i,j}g_{i,j}|\Big).$$

Since in the range $p\leq 2\leq q$ we have

$$\|(a_{i,j})_{i=1}^m\|_q+\|(a_{i,j})_{j=1}^n\|_{p^*}+\mathbb{E}\max_{i\leq m,j\leq n}|a_{i,j}g_{i,j}|\sim_{p,q}\mathbb{E}\max_{i\leq m}\|(a_{i,j}g_{i,j})_j\|_{p^*}+\mathbb{E}\max_{j\leq n}\|(a_{i,j}g_{i,j})_i\|_q$$

(see [2, Remark 1.1]), it seems natural to expect, that, as in the case $p=q=2$,

$$\mathbb{E}\|G_A\|_{\ell_p^n\to\ell_q^m}\overset{?}{\lesssim}_{p,q}\mathbb{E}\max_{i\leq m}\|(a_{i,j}g_{i,j})_j\|_{p^*}+\mathbb{E}\max_{j\leq n}\|(a_{i,j}g_{i,j})_i\|_q.$$

However, this bound fails outside the range $p\leq 2\leq q$ (see [2, Remark 1.1]) and, as discussed in [2], a more reasonable guess is that

$$\mathbb{E}\|G_A\colon\ell_p^n\to\ell_q^m\|\overset{?}{\lesssim}_{p,q}D_1+D_2+D_3,\tag{5}$$

where

$$D_1:=\|(a_{i,j}^2)_{i\leq m,j\leq n}\colon\ell_{p/2}^n\to\ell_{q/2}^m\|^{1/2},\qquad b_j:=\|(a_{i,j})_{i\leq m}\|_{2q/(2-q)},$$
$$D_2:=\|(a_{j,i}^2)_{j\leq n,i\leq m}\colon\ell_{q^*/2}^m\to\ell_{p^*/2}^n\|^{1/2},\qquad d_i:=\|(a_{i,j})_{j\leq n}\|_{2p/(p-2)},$$

$$D_3 = \begin{cases} \mathbb{E}\max_{i\le m, j\le n} |a_{i,j} g_{i,j}| & \text{if } p \le 2 \le q, \\ \max_{j\le n} \sqrt{\ln(j+1)} b_j^* & \text{if } p \le q \le 2, \\ \max_{i\le m} \sqrt{\ln(i+1)} d_i^* & \text{if } 2 \le p \le q, \\ 0 & \text{if } q < p, \end{cases}$$

and $(c_i^*)_{i=1}^k$ is the nonincreasing rearrangement of $(|c_i|)_{i=1}^k$. It is known by [2, (1.13) and Corollary 1.4] that (5) holds up to logarithmic terms.

It seems that proving the correct asymptotic bound for the operator norm from ℓ_p to ℓ_q of a structured Gaussian is a challenge. All the more, there is currently no hope of getting two-sided bounds in a general case of the structured matrices $(a_{i,j}X_{i,j})_{i\le m, j\le n}$ for a wider class of iid random variables $X_{i,j}$. Therefore, in this paper we restrict ourselves to a special class of variance structures $(a_{i,j})_{i\le m,j\le n}$: the tensor structure. In other words, we assume that the structure has a tensor form $a_{i,j} = a_i b_j$ for some $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$. In this case Chevet-type bounds stated in Theorem 1 allow us to provide two-sided bounds — with constants independent of p and q — for exponential, Gaussian, and, more general, Weibull tensor structured random matrices. Since these bounds have quite complicated forms we postpone the exact formulations to Section 3. Let us only announce here two corollaries from these bounds. The first one is an affirmative answer to conjecture (5) in the case when $(a_{i,j})$ has a tensor form (see Corollary 10 below). In Section 3 we state also a counterpart of this conjecture for weighted Weibull matrices and verify it in the tensor case. Moreover, using (2) and a bound for $\mathbb{E}\|(a_i b_j E_{i,j})\|_{\ell_p^n \to \ell_q^m}$ we provide a two-sided bound for weighted unconditional isotropic log-concave random matrices $(a_{i,j}Y_{i,j})$ in the tensor case $a_{i,j} = a_i b_j$ (see Corollary 12 below). We do not know whether a similar bound holds without the unconditionality assumption.

Let us now move to another application of Theorem 1. The authors of [1] used their Chevet-type bound to provide upper bounds for maximal spectral norms of $k \times l$ submatrices of unconditional isotropic log-concave random matrices (which turn out to be sharp in the case of independent exponential entries). Our improved Chevet-type inequality allows us to extend this result and derive two-sided bounds for maximal $\ell_p \to \ell_q$ norms of submatrices. Let us first formulate the result for Weibull matrices. By l_p^J we denote the space $\{(x_j)_{j\in J} \colon \sum_{j\in J} |x_j|^p \le 1\}$ equipped with the norm $\|x\|_p \coloneqq (\sum_{j\in J} |x_j|^p)^{1/p}$.

Theorem 4. *Let $r \in [1, 2]$ and $(X_{i,j})_{i\le m,j\le n}$ be independent, centered, ψ_r random variables with constant σ. Then for any $1 \le k \le m$, $1 \le l \le n$ and $p, q \in [1, \infty]$,*

$$\begin{aligned}\mathbb{E}\sup_{I,J} \big\|(X_{i,j})_{i\in I, j\in J}\big\|_{\ell_p^J \to \ell_q^I} \lesssim \sigma\Big(& k^{(1/q-1/r)\vee 0} l^{1/p^*} \Big(\mathrm{Log}\Big(\frac{n}{l}\Big) \vee (p^* \wedge \mathrm{Log}\, l)\Big)^{1/r} \\ & + k^{(1/q-1/2)\vee 0} l^{1/p^*} \Big(\mathrm{Log}\Big(\frac{n}{l}\Big) \vee (p^* \wedge \mathrm{Log}\, l)\Big)^{1/2} \\ & + l^{(1/p^*-1/r)\vee 0} k^{1/q} \Big(\mathrm{Log}\Big(\frac{m}{k}\Big) \vee (q \wedge \mathrm{Log}\, k)\Big)^{1/r} \\ & + l^{(1/p^*-1/2)\vee 0} k^{1/q} \Big(\mathrm{Log}\Big(\frac{m}{k}\Big) \vee (q \wedge \mathrm{Log}\, k)\Big)^{1/2}\Big),\end{aligned}$$

where the supremum runs over all sets $I \subset [m]$, $J \subset [n]$ such that $|I| = k$ and $|J| = l$. Moreover, the above bound may be reversed if $(X_{i,j})_{i\le m,j\le n}$ are iid symmetric Weibull r.v.'s with parameter r.

Theorem 4 applied with $r = 1$, and (2) yield the following corollary.

Corollary 5. *Let $(Y_{i,j})_{i\le m,j\le n}$ be isotropic log-concave unconditional matrix. Then for any $1\le k\le m$, $1\le l\le n$ and $p,q\in[1,\infty]$,*

$$\begin{aligned}\mathbb{E}\sup_{I,J}\big\|(Y_{i,j})_{i\in I,j\in J}\big\|_{\ell_p^J\to\ell_q^I} \lesssim\, & l^{1/p^*}\Big(\mathrm{Log}\Big(\frac{n}{l}\Big)\vee(p^*\wedge\mathrm{Log}\, l)\Big)\\ &+k^{(1/q-1/2)\vee 0}l^{1/p^*}\Big(\mathrm{Log}\Big(\frac{n}{l}\Big)\vee(p^*\wedge\mathrm{Log}\, l)\Big)^{1/2}\\ &+k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee(q\wedge\mathrm{Log}\, k)\Big)\\ &+l^{(1/p^*-1/2)\vee 0}k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee(q\wedge\mathrm{Log}\, k)\Big)^{1/2},\end{aligned}$$

where the supremum runs over all sets $I\subset[m]$, $J\subset[n]$ such that $|I|=k$ and $|J|=l$. Moreover, the above bound may be reversed if $(Y_{i,j})_{i\le m,j\le n}$ are iid symmetric exponential r.v.'s.

Applying Theorem 4 with $k=m$ and $l=n$ we derive the following bound which extends (3) to the case of Weibull matrices (this also follows from Theorem 11 from Section 3 applied with $a_i=b_j=1$).

Corollary 6. *Let $(X_{i,j})_{i\le m,i\le n}$ be iid symmetric Weibull r.v.'s with parameter $r\in[1,2]$. Then for every $1\le p,q\le\infty$,*

$$\begin{aligned}&\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}\\ &\sim\begin{cases} m^{1/q-1/2}n^{1/p^*}+n^{1/p^*-1/2}m^{1/q}, & p^*,q\le 2,\\ (p^*\wedge\mathrm{Log}\, n)^{1/r}n^{1/p^*}m^{(1/q-1/r)\vee 0}+\sqrt{p^*\wedge\mathrm{Log}\, n}\, n^{1/p^*}m^{1/q-1/2}+m^{1/q}, & q\le 2\le p^*,\\ n^{1/p^*}+(q\wedge\mathrm{Log}\, m)^{1/r}m^{1/q}n^{(1/p^*-1/r)\vee 0}+\sqrt{q\wedge\mathrm{Log}\, m}\, m^{1/q}n^{1/p^*-1/2}, & p^*\le 2\le q,\\ (p^*\wedge\mathrm{Log}\, n)^{1/r}n^{1/p^*}+(q\wedge\mathrm{Log}\, m)^{1/r}m^{1/q}, & 2\le p^*,q\end{cases}\\ &\sim(p^*\wedge\mathrm{Log}\, n)^{1/r}m^{(1/q-1/r)\vee 0}n^{1/p^*}+\sqrt{p^*\wedge\mathrm{Log}\, n}\, m^{(1/q-1/2)\vee 0}n^{1/p^*}\\ &\qquad+(q\wedge\mathrm{Log}\, m)^{1/r}n^{(1/p^*-1/r)\vee 0}m^{1/q}+\sqrt{q\wedge\mathrm{Log}\, m}\, n^{(1/p^*-1/2)\vee 0}m^{1/q}.\end{aligned}$$

In particular, if $n=m$ then

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n}\sim\begin{cases} n^{1/q+1/p^*-1/2}, & p^*,q\le 2,\\ (p^*\wedge q\wedge\mathrm{Log}\, n)^{1/r}n^{1/(p^*\wedge q)}, & p^*\vee q\ge 2.\end{cases}$$

Lemma 15 below and the bound $\|X_{i,j}\|_\rho=\big(\Gamma(\rho/r+1)\big)^{1/\rho}\sim(\rho/r)^{1/r}\sim\rho^{1/r}$ imply that the estimates in Corollary 6 are equivalent to

$$\mathbb{E}\big\|(X_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}\sim m^{1/q}\sup_{t\in B_p^n}\Big\|\sum_{j=1}^n t_jX_{1,j}\Big\|_{q\wedge\mathrm{Log}\, m}+n^{1/p^*}\sup_{s\in B_{q^*}^m}\Big\|\sum_{i=1}^m s_iX_{i,1}\Big\|_{p^*\wedge\mathrm{Log}\, n}\tag{6}$$

and, in the square case, to

$$\mathbb{E}\big\|(X_{i,j})_{i,j=1}^n\big\|_{\ell_p^n\to\ell_q^n}\sim\begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & p^*,q\le 2,\\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge\mathrm{Log}\, n}, & p^*\vee q\ge 2.\end{cases}\tag{7}$$

In the upcoming work [22] we show that (6) and (7) hold for a wider class of centered iid random matrices satisfying the following mild regularity assumption: there exists $\alpha\ge 1$ such that for every $\rho\ge 1$,

$$\|X_{i,j}\|_{2\rho}\le\alpha\|X_{i,j}\|_\rho;$$

this class contains, e.g., all log-concave random matrices with iid entries and iid Weibull random variables with shape parameter $r\in(0,\infty]$.

The rest of this paper is organized as follows. Section 2 contains the proof of Theorem 1, Corollary 2, and inequality (2). In Section 3 we formulate and prove bounds for norms of random matrices in the tensor structured case. Finally, Section 4 contains the proof of Theorem 4.

2. Proofs of Chevet-type bounds

In this section we show how to derive Chevet-type bounds. Then we move to the proofs of Corollary 2 and inequality (2).

Proof of Theorem 1. The second estimate follows by Chevet's inequality. The proof of the first upper bound is a modification of the proof of Theorem 3.1 from [1].

Let us briefly recall the notation from [1]. For a metric space (U, d) and $\rho > 0$ let

$$\gamma_\rho(U,d) := \inf_{(U_l)_{l=0}^\infty} \sup_{u\in U} \sum_{l=0}^\infty 2^{l/\rho} d(u, U_l),$$

where the infimum is taken over all admissible sequences of sets, i.e., all sequences $(U_l)_{l=0}^\infty$ of subsets of U, such that $|U_0| = 1$, and $|U_l| \leq 2^{2^l}$ for $l \geq 1$. Let d_ρ be the ℓ_ρ-metric in the appropriate dimension. Since $r \in [1, 2]$, by the result of Talagrand [28] (one may also use the more general [24, Theorem 2.4] to see more explicitely that two-sided bounds hold with constants independent of parameter r) for every nonempty $U \subset \mathbb{R}^k$,

$$\mathbb{E} \sup_{u\in U} \sum_{i\leq k} u_i g_i \sim \gamma_2(U, d_2) \quad \text{and} \quad \mathbb{E} \sup_{u\in U} \sum_{i=1}^k u_i X_i \sim \gamma_r(U, d_{r^*}) + \gamma_2(U, d_2). \tag{8}$$

For nonempty sets $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ let

$$S \otimes T = \{s \otimes t \colon s \in S, t \in T\},$$

where $s \otimes t := (s_i t_j)_{i\leq m, j\leq n}$ belongs to the space of real $m \times n$ matrices, which we identify with $\mathbb{R}^{mn}$. Now we will prove that

$$\gamma_r(S \otimes T, d_{r^*}) \sim \sup_{t\in T} \|t\|_{r^*} \gamma_r(S, d_{r^*}) + \sup_{s\in S} \|s\|_{r^*} \gamma_r(T, d_{r^*}). \tag{9}$$

Let $S_l \subset S$ and $T_l \subset T$, $l = 0, 1, \ldots$ be admissible sequences of sets. Set $T_{-1} := T_0$, $S_{-1} := S_0$ and define $U_l := S_{l-1} \otimes T_{l-1}$. Then $(U_l)_{l\geq 0}$ is an admissible sequence of subsets of $S \otimes T$.

Note that for all $s', s'' \in S$, and $t', t'' \in T$ we have

$$\begin{aligned} d_{r^*}(s' \otimes t', s'' \otimes t'') &= \|s' \otimes t' - s'' \otimes t''\|_{r^*} \leq \|s' \otimes (t' - t'')\|_{r^*} + \|(s' - s'') \otimes t''\|_{r^*} \\ &= \|s'\|_{r^*} \|t' - t''\|_{r^*} + \|t''\|_{r^*} \|s' - s''\|_{r^*} \\ &\leq \sup_{s\in S} \|s\|_{r^*} d_{r^*}(t', t'') + \sup_{t\in T} \|t\|_{r^*} d_{r^*}(s', s''). \end{aligned}$$

Therefore

$$\begin{aligned} \gamma_r(S \otimes T, d_{r^*}) &\leq \sup_{s\in S, t\in T} \sum_{l=0}^\infty 2^{l/r} d_{r^*}(s \otimes t, U_l) \\ &\leq \sup_{s\in S} \|s\|_{r^*} \sup_{t\in T} \sum_{l=0}^\infty 2^{l/r} d_{r^*}(t, T_{l-1}) + \sup_{t\in T} \|t\|_{r^*} \sup_{s\in S} \sum_{l=0}^\infty 2^{l/r} d_{r^*}(s, S_{l-1}). \end{aligned}$$

Taking the infimum over all admissible sequences $(S_l)_{l\geq 0}$ and $(T_l)_{l\geq 0}$ we get the upper bound (9).

To establish the lower bound in (9) it is enough to observe that

$$\gamma_r(S\otimes T,d_{r^*})\geq \max\Big\{\sup_{t\in T}\gamma_r\big(S\otimes\{t\},d_{r^*}\big),\ \sup_{s\in S}\gamma_r\big(\{s\}\otimes T,d_{r^*}\big)\Big\}$$
$$=\max\Big\{\sup_{t\in T}\|t\|_{r^*}\gamma_r(S,d_{r^*}),\ \sup_{s\in S}\|s\|_{r^*}\gamma_r(T,d_{r^*})\Big\}.$$

Bounds (8) and (9) imply

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\leq m,j\leq n}X_{i,j}s_it_j\sim\gamma_2(S\otimes T,d_2)+\gamma_r(S\otimes T,d_{r^*}) \tag{10}$$
$$\sim\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\leq m,j\leq n}g_{i,j}s_it_j+\sup_{t\in T}\|t\|_{r^*}\gamma_r(S,d_{r^*})+\sup_{s\in S}\|s\|_{r^*}\gamma_r(T,d_{r^*}).$$

Moreover, Chevet's inequality and (8) yield

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\leq m,j\leq n}g_{i,j}s_it_j\gtrsim\sup_{t\in T}\|t\|_2\gamma_2(S,d_2)+\sup_{s\in S}\|s\|_2\gamma_2(T,d_2) \tag{11}$$
$$\geq\sup_{t\in T}\|t\|_{r^*}\gamma_2(S,d_2)+\sup_{s\in S}\|s\|_{r^*}\gamma_2(T,d_2).$$

The first asserted inequality follows by applying (10), (11) and (8). □

Corollary 2 immediately follows by a symmetrization, Theorem 1 and the following standard lemma.

Lemma 7. *Let $X_{i,j}$'s be iid symmetric Weibull r.v.'s with parameter $r\in[1,2]$, and $Y_{i,j}$'s be independent symmetric ψ_r random variables with constant σ. Then for every bounded nonempty sets $S\subset\mathbb{R}^m$ and $T\subset\mathbb{R}^n$ we have*

$$\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\leq m,j\leq n}Y_{i,j}s_it_j\leq 2\sigma\mathbb{E}\sup_{s\in S,t\in T}\sum_{i\leq m,j\leq n}X_{i,j}s_it_j.$$

Proof. The ψ_r assumption gives $\mathbb{P}(|Y_{i,j}|\geq t)\leq 2\mathbb{P}(|\sigma X_{i,j}|\geq t)$. Let $(\delta_{i,j})_{i\leq m,j\leq n}$ be iid r.v.'s independent of all the others, such that $\mathbb{P}(\delta_i=1)=1/2=1-\mathbb{P}(\delta_i=0)$. Then $\mathbb{P}(|\delta_{i,j}Y_{i,j}|\geq t)\leq\mathbb{P}(|\sigma X_{i,j}|\geq t)$ for every $t\geq 0$, so we may find such a representation of $(X_{i,j},Y_{i,j},\delta_{i,j})_{i\leq m,j\leq n}$, that $\sigma|X_{i,j}|\geq|\delta_{i,j}Y_{i,j}|$ a.s. Let $(\varepsilon_{i,j})_{i\leq m,j\leq n}$ be a matrix with iid symmetric ±1 entries (Rademachers) independent of all the others. Then the contraction principle and Jensen's inequality imply

$$\sigma\mathbb{E}\sup_{s\in S,t\in T}\sum_{i,j}X_{i,j}s_it_j=\mathbb{E}\sup_{s\in S,t\in T}\sum_{i,j}\varepsilon_{i,j}|\sigma X_{i,j}|s_it_j\geq\mathbb{E}\sup_{s\in S,t\in T}\sum_{i,j}\varepsilon_{i,j}|\delta_{i,j}Y_{i,j}|s_it_j$$
$$=\mathbb{E}\sup_{s\in S,t\in T}\sum_{i,j}\delta_{i,j}Y_{i,j}s_it_j\geq\mathbb{E}\sup_{s\in S,t\in T}\sum_{i,j}Y_{i,j}\mathbb{E}\delta_{i,j}s_it_j$$
$$=\frac{1}{2}\mathbb{E}\sup_{s\in S,t\in T}\sum_{i,j}Y_{i,j}s_it_j.$$
□

Lemma 8. *Formula* (2) *holds for every bounded nonempty set $U\subset\mathbb{R}^k$ and every k-dimensional unconditional isotropic log-concave random vector Y.*

Proof. [19, Theorem 2] states that for every norm $\|\cdot\|$ on $\mathbb{R}^k$, $\mathbb{E}\|Y\|\leq C\mathbb{E}\|E\|$, where $E=(E_1,\ldots,E_k)$. In other words, (2) holds for bounded symmetric sets U.

Now, let U be arbitrary. Take any point $v\in U$. Since $\mathbb{E}\sum_{i=1}^k v_iY_i=0$ we have

$$\mathbb{E}\sup_{u\in U}\sum_{i=1}^k u_iY_i=\mathbb{E}\sup_{u\in U-v}\sum_{i=1}^k u_iY_i\leq\mathbb{E}\sup_{u\in U-v}\Big|\sum_{i=1}^k u_iY_i\Big|\leq C\mathbb{E}\sup_{u\in U-v}\Big|\sum_{i=1}^k u_iE_i\Big|,$$

where the last inequality follows by (2) applied to the symmetric set $(U-v)\cup(v-U)$. On the other hand, the distribution of E is symmetric, $0\in U-v$, and $\mathbb{E}\sum_{i=1}^k v_iE_i=0$, so

$$\mathbb{E}\sup_{u\in U-v}\Big|\sum_{i=1}^k u_iE_i\Big| \le \mathbb{E}\sup_{u\in U-v}\Big(\sum_{i=1}^k u_iE_i\Big)\vee 0+\mathbb{E}\sup_{u\in U-v}\Big(-\sum_{i=1}^k u_iE_i\Big)\vee 0$$
$$=2\mathbb{E}\sup_{u\in U-v}\Big(\sum_{i=1}^k u_iE_i\Big)\vee 0=2\mathbb{E}\sup_{u\in U-v}\sum_{i=1}^k u_iE_i=2\mathbb{E}\sup_{u\in U}\sum_{i=1}^k u_iE_i. \qquad \square$$

3. Matrices $(a_ib_jX_{ij})$

In this section we shall consider matrices of the form $(a_ib_jX_{i,j})_{i\le m,j\le n}$. Before presenting our results we need to introduce some notation.

By $(c_i^*)_{i=1}^k$ we will denote the nonincreasing rearrangement of $(|c_i|)_{i=1}^k$. For $\rho\ge 1$ we set

$$\varphi_\rho(t)=\begin{cases}\exp(2-2t^{-\rho}), & t>0.\\ 0, & t=0,\end{cases}$$

and define

$$\|(c_i)_{i\le k}\|_{\varphi_\rho}:=\inf\Big\{t>0\colon \sum_{i=1}^k\varphi_\rho(|c_i|/t)\le 1\Big\}.$$

The function φ_ρ is not convex on $\mathbb{R}_+$. However, it is increasing and convex on $[0,1]$ and $\varphi_\rho(1)=1$. So we may find a convex function $\tilde\varphi_\rho$ on $[0,\infty)$ such that $\varphi_\rho=\tilde\varphi_\rho$ on $[0,1]$. Then clearly $\|\cdot\|_{\varphi_\rho}=\|\cdot\|_{\tilde\varphi_\rho}$. Thus $\|\cdot\|_{\varphi_\rho}$ is an Orlicz norm.

Let us first present the bound in the Gaussian case.

Theorem 9. *For every $1\le p,q\le\infty$ and deterministic sequences $(a_i)_{i\le m}$, $(b_j)_{j\le m}$,*

$$\mathbb{E}\big\|(a_ib_jg_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}$$
$$\sim\begin{cases}\|a\|_{\frac{2q^*}{q^*-2}}\|b\|_{p^*}+\|a\|_q\|b\|_{\frac{2p}{p-2}}, & p^*,q<2,\\ \|a\|_{\frac{2q^*}{q^*-2}}\big(\|(b_j^*)_{j\le e^{p^*}}\|_{\varphi_2}+\sqrt{p^*}\|(b_j^*)_{j>e^{p^*}}\|_{p^*}\big)+\|a\|_q\|b\|_\infty, & q<2\le p^*,\\ \|a\|_\infty\|b\|_{p^*}+\big(\|(a_i^*)_{i\le e^q}\|_{\varphi_2}+\sqrt{q}\|(a_i^*)_{i>e^q}\|_q\big)\|b\|_{\frac{2p}{p-2}}, & p^*<2\le q,\\ \|a\|_\infty\big(\|(b_j^*)_{j\le e^{p^*}}\|_{\varphi_2}+\sqrt{p^*}\|(b_j^*)_{j>e^{p^*}}\|_{p^*}\big) & \\ \quad+\big(\|(a_i^*)_{i\le e^q}\|_{\varphi_2}+\sqrt{q}\|(a_i^*)_{i>e^q}\|_q\big)\|b\|_\infty, & 2\le p^*,q.\end{cases}$$

Before we move to the Weibull case, let us see how Theorem 9 implies conjecture (5) for the tensor structured Gaussian matrices.

Corollary 10. *Assume that there exists $a\in\mathbb{R}^m$ and $b\in\mathbb{R}^n$ such that $a_{ij}=a_ib_j$ for every $i\le m,j\le n$. Then conjecture (5) holds.*

Proof. If $p^*=\infty$ or $q=\infty$, then (5) is satisfied for an arbitrary matrix $(a_{i,j})_{i,j}$ by [17, Remark 1.4], [2, Proposition 1.8 and Corollary 1.11]

In the case $p^*,q<\infty$ we shall show that

$$D_1+D_2\lesssim\mathbb{E}\big\|(a_ib_jg_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}\lesssim\sqrt{q}D_1+\sqrt{p^*}D_2. \tag{12}$$

The lower bound follows by [2, Proposition 5.1 and Corollary 5.2].

To establish the upper bound let us first compute D_1 and D_2 in the case $a_{i,j} = a_i b_j$. If $p > 2$, then $2(p/2)^* = 2p/(p-2)$, so for every $p \in [1,\infty]$,

$$D_1 = \sup_{t \in B^n_{p/2}} \left(\sum_{i=1}^m |a_i|^q \Big| \sum_{j=1}^n b_j^2 t_j \Big|^{q/2} \right)^{1/q} = \|a\|_q \sup_{t \in B^n_{p/2}} \Big| \sum_{j=1}^n b_j^2 t_j \Big|^{1/2} = \|a\|_q \begin{cases} \|b\|_{2p/(p-2)} & p^* < 2, \\ \|b\|_\infty & p^* \geq 2, \end{cases}$$

and dually

$$D_2 = \|b\|_{p^*} \begin{cases} \|a\|_{2q^*/(q^*-2)} & q < 2, \\ \|a\|_\infty & q \geq 2. \end{cases}$$

Moreover,

$$\|(b_j^*)_{j \leq e^{p^*}}\|_{\varphi_2} + \sqrt{p^*}\|(b_j^*)_{j > e^{p^*}}\|_{p^*} \leq 2\sqrt{p^*} b_1^* + \sqrt{p^*}\|(b_j^*)_{j > e^{p^*}}\|_{p^*} \leq 3\sqrt{p^*}\|(b_j^*)_j\|_{p^*},$$

and similarly

$$\|(a_i^*)_{i \leq e^q}\|_{\varphi_2} + \sqrt{q}\|(a_i^*)_{i > e^q}\|_q \lesssim \sqrt{q}\|(a_i^*)_i\|_q.$$

Hence, Theorem 9 yields the upper bound in (12). □

In the Weibull case we get the following bound.

Theorem 11. *Let $(X_{i,j})_{i \leq m, j \leq n}$ be iid symmetric Weibull r.v.'s with parameter $r \in [1,2]$. Then for every $1 \leq p, q \leq \infty$ and deterministic sequences $a = (a_i)_{i \leq m}$ and $b = (b_j)_{j \leq n}$,*

$$\mathbb{E}\big\|(a_i b_j X_{i,j})_{i \leq m, j \leq n}\big\|_{\ell_p^n \to \ell_q^m}$$

$$\sim \begin{cases} \|a\|_{\frac{2q^*}{q^*-2}} \|b\|_{p^*} + \|a\|_q \|b\|_{\frac{2p}{p-2}}, & p^*, q < 2, \\ \|a\|_{\frac{2q^*}{q^*-2}} \big(\|(b_j^*)_{j \leq e^{p^*}}\|_{\varphi_2} + \sqrt{p^*}\|(b_j^*)_{j > e^{p^*}}\|_{p^*}\big) & \\ \quad + \|a\|_{\frac{r^* q^*}{q^* - r^*}} \big(\|(b_j^*)_{j \leq e^{p^*}}\|_{\varphi_r} + (p^*)^{1/r}\|(b_j^*)_{j > e^{p^*}}\|_{p^*}\big) + \|a\|_q \|b\|_\infty, & q < r, 2 \leq p^*, \\ \|a\|_{\frac{2q^*}{q^*-2}} \big(\|(b_j^*)_{j \leq e^{p^*}}\|_{\varphi_2} + \sqrt{p^*}\|(b_j^*)_{j > e^{p^*}}\|_{p^*}\big) & \\ \quad + \|a\|_\infty \big(\|(b_j^*)_{j \leq e^{p^*}}\|_{\varphi_r} + (p^*)^{1/r}\|(b_j^*)_{j > e^{p^*}}\|_{p^*}\big) + \|a\|_q \|b\|_\infty, & r \leq q < 2 \leq p^*, \\ \|a\|_\infty \|b\|_{p^*} + \big(\|(a_i^*)_{i \leq e^q}\|_{\varphi_2} + \sqrt{q}\|(a_i^*)_{i > e^q}\|_q\big)\|b\|_{\frac{2p}{p-2}} & \\ \quad + \big(\|(a_i^*)_{i \leq e^q}\|_{\varphi_r} + q^{1/r}\|(a_i^*)_{i > e^q}\|_q\big)\|b\|_{\frac{r^* p}{p - r^*}}, & p^* < r, 2 \leq q, \\ \|a\|_\infty \|b\|_{p^*} + \big(\|(a_i^*)_{i \leq e^q}\|_{\varphi_2} + \sqrt{q}\|(a_i^*)_{i > e^q}\|_q\big)\|b\|_{\frac{2p}{p-2}} & \\ \quad + \big(\|(a_i^*)_{i \leq e^q}\|_{\varphi_r} + q^{1/r}\|(a_i^*)_{i > e^q}\|_q\big)\|b\|_\infty, & r \leq p^* < 2 \leq q, \\ \|a\|_\infty \big(\|(b_j^*)_{j \leq e^{p^*}}\|_{\varphi_r} + (p^*)^{1/r}\|(b_j^*)_{j > e^{p^*}}\|_{p^*}\big) & \\ \quad + \big(\|(a_i^*)_{i \leq e^q}\|_{\varphi_r} + q^{1/r}\|(a_i^*)_{i > e^q}\|_q\big)\|b\|_\infty, & 2 \leq p^*, q. \end{cases}$$

Corollary 12. *Suppose that $r \in [1,2]$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, and $(X_{i,j})_{i \leq m, j \leq n}$ is a random matrix with independent ψ_r entries with constant σ such that $\mathbb{E}|X_{ij}| \geq \gamma$. Then*

$$\gamma(D_1 + D_2) \lesssim \mathbb{E}\big\|(a_i b_j X_{i,j})_{i \leq m, j \leq n}\big\|_{\ell_p^n \to \ell_q^m} \lesssim \sigma\big(q^{1/r} D_1 + (p^*)^{1/r} D_2\big). \tag{13}$$

Moreover, if $(X_{i,j})_{i \leq m, j \leq n}$ is an isotropic log-concave unconditional random matrix, then two-sided estimate (13) *holds with $r = \sigma = \gamma = 1$.*

Proof. The lower bound follows by the proof of [2, Proposition 5.1] (which in fact shows that the assertion of [2, Proposition 5.1] holds for unconditional random matrices whose entries satisfy $\mathbb{E}|X_{ij}| \geq c$). To derive the upper bound we proceed similarly as in the proof of Corollary 10 using Theorem 11 (instead of Theorem 9) and then apply Lemma 7 — or inequality (2) in the log-concave case. □

Corollary 12 suggests, that in a non-tensor case it makes sense to pose the following counterpart of conjecture (5).

Conjecture 13. *Assume that* $r \in [1,2]$, $(a_{i,j})_{i\le m,j\le n}$ *is a deterministic* $m\times n$ *matrix, and* $(X_{i,j})_{i\le m,j\le n}$ *be iid symmetric Weibull r.v.'s with parameter* $r\in[1,2]$. *Let*

$$D_{3,r} = \begin{cases} \mathbb{E}\max_{i\le m,j\le n}|a_{i,j}X_{i,j}| & \text{if } p\le 2\le q,\\ \max_{j\le n} b_j^* \ln^{1/r}(j+1) & \text{if } p\le q\le 2,\\ \max_{i\le m} d_i^* \ln^{1/r}(i+1) & \text{if } 2\le p\le q,\\ 0 & \text{if } q<p. \end{cases}$$

Is it true that

$$\mathbb{E}\|(a_{i,j}X_{i,j})_{i,j}\colon \ell_p^n\to\ell_q^m\| \sim_{p,q} D_1+D_2+D_{3,r} \quad ?$$

Remark 14. Using similar methods as in the proofs of [2, Propositions 1.8 and 1.10], one may show that Conjecture 13 holds whenever $p\in\{1,\infty\}$ or $q\in\{1,\infty\}$. Moreover, it follows by [25, Theorem 4.4] and a counterpart of [2, equation (1.11)] for iid Weibull r.v.'s that Conjecture 13 holds in the case $p=2=q$.

Now we provide the following lemma yielding the equivalence between (6) and the assertion of Corollary 6.

Lemma 15. *Let* $X_1,X_2,\dots,X_k$ *be iid symmetric Weibull r.v.'s with parameter* $r\in[1,2]$ *and let* $\rho_1,\rho_2\in[1,\infty)$. *Then*

$$\sup_{t\in B^k_{\rho_1}}\Big\|\sum_{i=1}^k t_iX_i\Big\|_{\rho_2} \sim \begin{cases} \rho_2^{1/r}, & 1\le\rho_1\le 2,\\ \rho_2^{1/r}+\sqrt{\rho_2}k^{1/2-1/\rho_1}, & 2\le\rho_1\le r^*,\\ \rho_2^{1/r}k^{1/r^*-1/\rho_1}+\sqrt{\rho_2}k^{1/2-1/\rho_1}, & \rho_1\ge r^* \end{cases}$$
$$\sim \rho_2^{1/r}k^{\left(\frac{1}{\rho_1^*}-\frac{1}{r}\right)\vee 0}+\sqrt{\rho_2}k^{\left(\frac{1}{\rho_1^*}-\frac{1}{2}\right)\vee 0}.$$

Proof. The Gluskin–Kwapień inequality [15] easily implies that

$$\Big\|\sum_{i=1}^k t_iX_i\Big\|_{\rho_2} \sim \rho_2^{1/r}\|t\|_{r^*}+\rho_2^{1/2}\|t\|_2. \tag{14}$$

Hence it is enough to observe that

$$\sup_{t\in B^k_{\rho_1}}\|t\|_\rho = \begin{cases} 1, & \rho_1\le\rho,\\ k^{1/\rho-1/\rho_1}, & \rho_1\ge\rho. \end{cases} \qquad \square$$

Before proving Theorems 9 and 11 we need to formulate several technical results. Hölder's inequality yields the following simple lemma.

Lemma 16. *Let* $1\le\rho_1,\rho_2\le\infty$ *and* $c=(c_i)\in\mathbb{R}^k$. *Then*

$$\sup_{t\in B^k_{\rho_1}}\|(c_it_i)\|_{\rho_2} = \begin{cases} \|c\|_\infty, & \rho_1\le\rho_2,\\ \|c\|_{\rho_1\rho_2/(\rho_1-\rho_2)}, & \rho_2<\rho_1<\infty,\\ \|c\|_{\rho_2}, & \rho_1=\infty. \end{cases}$$

The next result is a two-sided bound for the ℓ_ρ-norms of a weighted sequence of independent Weibull r.v.'s. Much more general two-sided estimates for the Orlicz norms of weighted vectors with iid coordinates were obtained in [16]. However, the formula stated therein is quite involved and not easy to decrypt in the case of ℓ_ρ-norms. Therefore we give an alternative proof in our special setting, providing a form of the two-sided estimate which is more handy for the purpose of proving Theorems 9 and 11.

Proposition 17. *Let $(X_i)_{i\le k}$ be iid symmetric Weibull r.v.'s with parameter $r\in[1,2]$. Then for every $1\le\rho\le\infty$ and every sequence $c=(c_i)_{i=1}^k$ we have*

$$\mathbb{E}\big\|(c_iX_i)_{i=1}^k\big\|_\rho \sim \|(c_i^*)_{i\le e^\rho}\|_{\varphi_r}+\rho^{1/r}\|(c_i^*)_{i>e^\rho}\|_\rho, \tag{15}$$

where $\varphi_r(x)=\exp(2-2x^{-r})$ and $(c_i^*)_{i\le k}$ is the nonincreasing rearrangement of $(|c_i|)_{i\le k}$.

Remark 18. For $\rho\le 2$ we have $\mathbb{E}\|(c_iX_i)\|_\rho\sim\|c\|_\rho$. It is not hard to deduce this from (15). Alternatively one may use the Khintchine–Kahane-type inequality $\mathbb{E}\|(c_iX_i)\|_\rho\sim(\mathbb{E}\|(c_iX_i)\|_\rho^\rho)^{1/\rho}$.

Proof of Proposition 17. First we show that for every $1\le l\le k$,

$$\mathbb{E}\big\|(c_iX_i)_{i\le l}\big\|_\infty\sim\|(c_i)_{i\le l}\|_{\varphi_r}. \tag{16}$$

Let $t=2^{1/r}\|(c_i)_{i\le l}\|_{\varphi_r}$. Then

$$\sum_{i=1}^l\mathbb{P}(|c_iX_i|\ge t)=\sum_{i=1}^l e^{-2}\varphi_r\big(2^{1/r}|c_i|/t\big)=e^{-2}.$$

This and the independence of X_i's imply

$$\mathbb{E}\big\|(c_iX_i)_{i=1}^l\big\|_\infty\ge t\mathbb{P}(\max_{i\le l}|c_iX_i|\ge t)\gtrsim t.$$

Moreover, for $u\ge1$,

$$\begin{aligned}\mathbb{P}(\max_{i\le l}|c_iX_i|\ge ut)&\le\sum_{i=1}^l\mathbb{P}(|c_iX_i|\ge ut)=\sum_{i=1}^l e^{-2}\varphi_r\big(2^{1/r}|c_i|/(tu)\big)\\&\le\sum_{i=1}^l e^{-2u^r}\varphi_r\big(2^{1/r}|c_i|/t\big)=e^{-2u^r},\end{aligned}$$

where the second inequality follows since $(xu)^r\ge x^r+u^r-1$ for $x,u\ge1$. Thus integration by parts yields $\mathbb{E}\|(c_iX_i)_{i\le l}\|_\infty\lesssim t$ and (16) follows.

To establish (15) for $\rho\in[1,\infty)$ we may and will assume that $c_1\ge c_2\ge\ldots\ge c_k\ge0$. Then $c_k^*=c_k$.

We have by (16) applied with $l=\lfloor e^\rho\rfloor\wedge k$,

$$\mathbb{E}\|(c_iX_i)_{i\le e^\rho}\|_\rho\sim\mathbb{E}\|(c_iX_i)_{i\le e^\rho}\|_\infty\sim\|(c_i)_{i\le e^\rho}\|_{\varphi_r}.$$

Moreover,

$$\mathbb{E}\|(c_iX_i)_{i>e^\rho}\|_\rho\le\big(\mathbb{E}\|(c_iX_i)_{i>e^\rho}\|_\rho^\rho\big)^{1/\rho}=\|X_1\|_\rho\|(c_i)_{i>e^\rho}\|_\rho\sim\rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho.$$

Therefore the upper bound in (15) easily follows.

Now we will show the lower bound. By (16) we have

$$\mathbb{E}\|(c_iX_i)_{i=1}^k\|_\rho\ge\mathbb{E}\|(c_iX_i)_{i\le e^\rho}\|_\infty\sim\|(c_i)_{i\le e^\rho}\|_{\varphi_r}, \tag{17}$$

so it is enough to show that

$$\mathbb{E}\|(c_iX_i)_{i=1}^k\|_\rho\gtrsim\rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho. \tag{18}$$

Observe that

$$\mathbb{E}\|(c_iX_i)_{i=1}^k\|_\rho=\mathbb{E}\|(c_i|X_i|)_{i=1}^k\|_\rho\ge\|(c_i\mathbb{E}|X_i|)_{i=1}^k\|_\rho=\mathbb{E}|X_1|\|c\|_\rho\gtrsim\|c\|_\rho.$$

In particular, (18) holds for $\rho\le2$.

Let C_1 be a suitably chosen constant (to be fixed later). If $\rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho\le C_1\|(c_i)_{i\le e^\rho}\|_{\varphi_r}$, then (17) yields (18). Thus we may assume that $\rho>2$ and $\rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho>C_1\|(c_i)_{i\le e^\rho}\|_{\varphi_r}$.

The variables X_i have log-concave tails, hence [18, Theorem 1] yields

$$\mathbb{E}\|(c_iX_i)_{i>e^\rho}\|_\rho \geq \frac{1}{C}\big(\mathbb{E}\|(c_iX_i)_{i>e^\rho}\|_\rho^\rho\big)^{1/\rho} - \sup_{t\in B^k_{\rho^*}}\Big\|\sum_{i>e^\rho} t_ic_iX_i\Big\|_\rho.$$

We have

$$\big(\mathbb{E}\|(c_iX_i)_{i>e^\rho}\|_\rho^\rho\big)^{1/\rho} \sim \rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho.$$

Inequality (14) and Lemma 16 (applied with $\rho_1 = \rho^* \leq 2$ and $\rho_2 \in \{2, r^*\}$) yield

$$\begin{aligned}\sup_{t\in B^n_{\rho^*}}\Big\|\sum_{i>e^\rho} t_ic_iX_i\Big\|_\rho &\lesssim (\rho^{1/r}+\rho^{1/2})\max_{i>e^\rho}|c_i| \leq 2\rho^{1/r}c_{\lceil e^\rho\rceil} \leq 2\rho^{1/r}\|(c_i)_{i\leq e^\rho}\|_{\varphi_r}\varphi_r^{-1}\Big(\frac{1}{\lfloor e^\rho\rfloor}\Big)\\ &\lesssim \|(c_i)_{i\leq e^\rho}\|_{\varphi_r}.\end{aligned}$$

Therefore

$$\mathbb{E}\|(c_iX_i)\|_\rho \geq \frac{1}{C_2}\rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho - C_3\|(c_i)_{i\leq e^\rho}\|_{\varphi_r} \geq \Big(\frac{1}{C_2}-\frac{C_3}{C_1}\Big)\rho^{1/r}\|(c_i)_{i>e^\rho}\|_\rho.$$

So to get (18) and conclude the proof it is enough to choose $C_1 = 2C_2C_3$. □

We shall also use the following lemma which is standard, but we prove it for the sake of completeness.

Lemma 19. *Let $(X_i)_{i=1}^k$ be iid Weibull r.v.'s with parameter 2. Then for any norm $\|\cdot\|$ on $\mathbb{R}^k$ we have*

$$\mathbb{E}\|(X_i)_{i=1}^k\| \sim \mathbb{E}\|(g_i)_{i=1}^k\|.$$

Moreover, if $(Y_i)_{i=1}^k$ are iid Weibull r.v.'s with parameter $r \in [1,2]$, then for any norm $\|\cdot\|$ on $\mathbb{R}^k$ we have

$$\mathbb{E}\|(Y_i)_{i=1}^k\| \gtrsim \mathbb{E}\|(g_i)_{i=1}^k\|. \tag{19}$$

Proof. We have $\mathbb{P}(|g_i| \geq t) \leq e^{-t^2/2} = \mathbb{P}(|\sqrt{2}X_i| \geq t)$. Thus we may find such a representation of X_i's and g_i's that $|g_i| \leq |\sqrt{2}X_i|$ a.s. Let $(\varepsilon_i)_{i=1}^k$ be a sequence of iid symmetric ±1 r.v.'s (Rademachers) independent of $(X_i)_{i=1}^k$ and $(g_i)_{i=1}^k$. Then the contraction principle implies

$$\mathbb{E}\|(g_i)_{i=1}^k\| = \mathbb{E}\|(\varepsilon_i|g_i|)_{i=1}^k\| \leq \mathbb{E}\|(\varepsilon_i|\sqrt{2}X_i|)_{i=1}^k\| = \sqrt{2}\mathbb{E}\|(X_i)_{i=1}^k\|.$$

To justify the opposite inequality observe that there exists $c > 0$ (one may take $c = 1/4$) such that for all $t \geq 0$, $\mathbb{P}(|g_i| \geq t) \geq ce^{-t^2}$ and proceed similarly as in the proof of Lemma 7.

Using the inequality $e^{-t^2/2} \leq Ce^{-t^r/2}$ for $t \geq 0$ (one may take $C = \sqrt{e}$) and proceeding in a similar way as above we may prove (19). □

Proposition 17, Remark 18 and Lemma 19 yield the following bound for ℓ_ρ-norms of a Gaussian sequence.

Corollary 20. *For every $1 \leq \rho \leq \infty$ and every sequence $c = (c_i)_{i\leq k}$ we have*

$$\mathbb{E}\big\|(c_ig_i)_{i=1}^k\big\|_\rho \sim \|(c_i^*)_{i\leq e^\rho}\|_{\varphi_2} + \sqrt{\rho}\|(c_i^*)_{i>e^\rho}\|_\rho.$$

In particular, for $\rho \leq 2$ we have

$$\mathbb{E}\big\|(c_ig_i)_{i=1}^k\big\|_\rho \sim \|c\|_\rho.$$

Remark 21. In the case $c_i = 1$ Corollary 20 yields the well known bound

$$\mathbb{E}\big\|(g_i)_{i=1}^k\big\|_\rho \sim \begin{cases}\rho^{1/2}k^{1/\rho}, & 1 \leq \rho \leq \operatorname{Log} k,\\ (\operatorname{Log} k)^{1/2}, & \rho \geq \operatorname{Log} k\end{cases} \sim (\rho \wedge \operatorname{Log} k)^{1/2}k^{1/\rho}.$$

Proof of Theorem 9. Chevet's inequality (1), appplied with $S=\{(a_is_i)\colon s\in B_{q^*}^m\}$ and $T=\{(b_jt_j)\colon t\in B_p^n\}$, yields

$$\mathbb{E}\big\|(a_ib_jg_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}\sim\sup_{s\in B_{q^*}^m}\|(a_is_i)\|_2\mathbb{E}\|(b_jg_j)\|_{p^*}+\sup_{t\in B_p^n}\|(b_jt_j)\|_2\mathbb{E}\|(a_ig_i)\|_q.$$

Lemma 16 and Corollary 20 yield the assertion. □

Proof of Theorem 11. Theorem 1, appplied with $S=\{(a_is_i)\colon s\in B_{q^*}^m\}$ and $T=\{(b_jt_j)\colon t\in B_p^n\}$, yields

$$\begin{aligned}\mathbb{E}\big\|(a_ib_jX_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}\sim&\,\mathbb{E}\big\|(a_ib_jg_{i,j})_{i\le m,j\le n}\big\|_{\ell_p^n\to\ell_q^m}\\&+\sup_{s\in B_{q^*}^m}\|(a_is_i)\|_{r^*}\mathbb{E}\|(b_jX_j)\|_{p^*}+\sup_{t\in B_p^n}\|(b_jt_j)\|_{r^*}\mathbb{E}\|(a_iX_i)\|_q.\end{aligned}$$

To get the assertion we use Theorem 9, Lemma 16, Proposition 17, and Remark 18 together with the following observations:

- for $q<r\le 2$ we have $\|a\|_{2q^*/(q^*-2)}\ge\|a\|_{r^*q^*/(q^*-r^*)}$, and for $p^*<r\le 2$ we have $\|b\|_{2p/(p-2)}\ge\|b\|_{r^*p/(p-r^*)}$,
- $\mathbb{E}\|(a_iX_i)_{i=1}^m\|_q\gtrsim\mathbb{E}\|(a_ig_i)_{i=1}^m\|_q$ and $\mathbb{E}\|(b_jX_j)_{j=1}^n\|_{p^*}\gtrsim\mathbb{E}\|(b_jg_j)_{j=1}^n\|_{p^*}$, which follows by inequality (19). □

4. Operator norms of submatrices

In this section we prove Theorem 4 about the norms of submatrices. To prove it we shall use Theorem 1 and Corollary 2. Thus, we need to estimate

$$\mathbb{E}\sup_{|I|=k}\Big(\sum_{i\in I}|X_i|^q\Big)^{1/q}=\mathbb{E}\Big(\sum_{i=1}^k(X_i^*)^q\Big)^{1/q},$$

where $(X_1^*,X_2^*,\dots,X_m^*)$ denotes the non-increasing rearrangement of $(|X_1|,\dots,|X_m|)$. This is done in the next two technical lemmas.

Lemma 22. *Let $X_1,\dots,X_m$ be iid symmetric Weibull r.v.'s with shape parameter $r\in[1,2]$. Then for every $q\ge 1$ and $1\le k\le m$ we have*

$$\Big(\mathbb{E}\sum_{i=1}^k(X_i^*)^q\Big)^{1/q}\sim k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee q\Big)^{1/r}.$$

Proof. By, say, [21, Theorem 3.2], we get

$$\mathbb{E}\sum_{i=1}^k(X_i^*)^q\sim kt_*,\quad\text{where}\quad t_*:=\inf\Big\{t>0\colon\mathbb{E}|X_1|^qI_{\{|X_1|^q>t\}}\le t\frac{k}{m}\Big\}.$$

Let $t_1:=(2q+\ln(\frac{m}{k}))^{q/r}$. Then

$$\begin{aligned}\mathbb{E}|X_1|^qI_{\{|X_1|^q>t_1\}}&\le\sum_{l=0}^\infty e^{(l+1)q}t_1\mathbb{P}(|X_1|>e^lt_1^{1/q})=t_1\sum_{l=0}^\infty e^{(l+1)q}e^{-e^{lr}t_1^{r/q}}\\&\le t_1e^q\sum_{l=0}^\infty e^{lq}e^{-(1+lr)(2q+\ln(\frac{m}{k}))}\le t_1e^{-q}\frac{k}{m}\sum_{l=0}^\infty e^{-(2r-1)ql}<t_1\frac{k}{m}.\end{aligned}$$

Thus, $t_*^{1/q}\le t_1^{1/q}\sim(\mathrm{Log}(\frac{m}{k})\vee q)^{1/r}$.

Let $t_2 := \log^{q/r}(m/k)$ and $t_3 = \frac{1}{2}\mathbb{E}|X_1|^q = \frac{1}{2}\Gamma(\frac{q}{r}+1)$. We have

$$\mathbb{E}|X_1|^q I_{\{|X_1|^q>t_2\}} > t_2\mathbb{P}(|X_1|>t_2^{1/q}) = t_2 e^{-t_2^{r/q}} = t_2\frac{k}{m},$$

and

$$\mathbb{E}|X_1|^q I_{\{|X_1|^q>t_3\}} = \mathbb{E}|X_1|^q - \mathbb{E}|X_1|^q I_{\{|X_1|^q\le t_3\}} > \frac{1}{2}\mathbb{E}|X_1|^q \ge t_3\frac{k}{m}.$$

Therefore, $t_*^{1/q} \ge (t_2\vee t_3)^{1/q} \sim (\mathrm{Log}(\frac{m}{k})\vee q)^{1/r}$. □

Lemma 23. *Let $X_1,\ldots,X_m$ be iid symmetric Weibull r.v.'s with shape parameter $r\in[1,2]$. Then for $q\ge 1$, $1\le k\le m$ we have*

$$\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \sim \begin{cases} \mathrm{Log}^{1/r} m & q\ge \mathrm{Log}\, k \\ k^{1/q}\big(\mathrm{Log}(\frac{m}{k})\vee q\big)^{1/r} & q<\mathrm{Log}\, k\end{cases} \sim k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee (q\wedge \mathrm{Log}\, k)\Big)^{1/r}.$$

Proof. If $q\ge \mathrm{Log}\, k$, then

$$\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \sim \mathbb{E}\max_{i\le m}|X_i| \sim \mathrm{Log}^{1/r} m,$$

where the last (standard) bound follows e.g. by Lemma 22 applied with $k=q=1$.

If $q\in[1,2]$ then Khinchine-Kahane-type inequality (cf., [20, Corollary 1.4]) and Lemma 22 yield

$$\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \sim \Big(\mathbb{E}\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \sim k^{1/q}\,\mathrm{Log}^{1/r}\Big(\frac{m}{k}\Big).$$

From now on assume that $2\le q\le \mathrm{Log}\, k$. By Lemma 22

$$\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \le \Big(\mathbb{E}\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \sim k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee q\Big)^{1/r}.$$

Variables X_i have log-concave tails, so by [18, Theorem 1],

$$\begin{aligned}\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} &= \mathbb{E}\sup_{|I|=k}\sup_{s\in B_{q^*}^I}\sum_{i\in I} s_iX_i \\ &\ge \frac{1}{C_1}\Big\|\sup_{|I|=k}\sup_{s\in B_{q^*}^I}\sum_{i\in I} s_iX_i\Big\|_q - \sup_{|I|=k}\sup_{s\in B_{q^*}^I}\Big\|\sum_{i\in I} s_iX_i\Big\|_q.\end{aligned}$$

Since $q^*\le 2\le r^*$, (14) implies that for any $I\subset[m]$ and any $s\in B_{q^*}^I$ we have ,

$$\Big\|\sum_{i\in I} s_iX_i\Big\|_q \lesssim q^{1/r}\|s\|_{r^*} + q^{1/2}\|s\|_2 \le q^{1/r}+q^{1/2}\le 2q^{1/r}.$$

This together with Lemma 22 yields

$$\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q} \ge \frac{1}{C_1C_2}k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee q\Big)^{1/r} - C_3q^{1/r}.$$

Thus if $k\ge (2C_1C_2C_3)^q$ we get $\mathbb{E}(\sum_{i=1}^k (X_i^*)^q)^{1/q} \ge \frac{1}{2C_1C_2}k^{1/q}(\mathrm{Log}(\frac{m}{k})\vee q)^{1/r}$. If $k\le (2C_1C_2C_3)^q$ then $k^{1/q}\sim 1$ and $\mathbb{E}(\sum_{i=1}^k (X_i^*)^q)^{1/q} \ge (\mathbb{E}(X_1^*)^q)^{1/q} \sim (\mathrm{Log}\, m\vee q)^{1/r}$. □

Proof of Theorem 4. We use Theorem 1 and Corollary 2 with

$$S=\bigcup_I B^I_{q*},\quad T=\bigcup_J B^J_p,$$

where B^I_{q*} is the unit ball in the space $\ell^I_{q^*}$, and the sums run over, respectively, all sets $I\subset[m]$ and $J\subset[n]$ such that $|I|=k$ and $|J|=l$. We only need to estimate the quantities on the right-hand side of the two-sided bounds from Theorem 1 and Corollary 2. We have for $\rho\in\{2,r^*\}$,

$$\sup_{s\in S}\|s\|_\rho=k^{(1/q-1/\rho^*)\vee 0},\quad \sup_{t\in T}\|t\|_\rho=l^{(1/p^*-1/\rho^*)\vee 0}.$$

Lemmas 23 and 19 yield

$$\mathbb{E}\sup_{s\in S}\sum_{i=1}^m s_iX_i=\mathbb{E}\Big(\sum_{i=1}^k (X_i^*)^q\Big)^{1/q}\sim k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee(q\wedge \mathrm{Log}\,k)\Big)^{1/r},$$

$$\mathbb{E}\sup_{s\in S}\sum_{i=1}^m s_ig_i=\mathbb{E}\Big(\sum_{i=1}^k (g_i^*)^q\Big)^{1/q}\sim k^{1/q}\Big(\mathrm{Log}\Big(\frac{m}{k}\Big)\vee(q\wedge \mathrm{Log}\,k)\Big)^{1/2}.$$

Similarily,

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_iX_i=\mathbb{E}\Big(\sum_{i=1}^l (X_i^*)^{p^*}\Big)^{1/p^*}\sim l^{1/p^*}\Big(\mathrm{Log}\Big(\frac{n}{l}\Big)\vee(p^*\wedge \mathrm{Log}\,l)\Big)^{1/r},$$

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^n t_ig_i=\mathbb{E}\Big(\sum_{i=1}^l (g_i^*)^{p^*}\Big)^{1/p^*}\sim l^{1/p^*}\Big(\mathrm{Log}\Big(\frac{n}{l}\Big)\vee(p^*\wedge \mathrm{Log}\,l)\Big)^{1/2}.$$

□

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