

# Geometry of log-concave Ensembles of random matrices and approximate reconstruction <sup>\*</sup>

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## Abstract

We study the Restricted Isometry Property of a random matrix  $\Gamma$  with independent isotropic log-concave rows. To this end, we introduce a parameter  $\Gamma_{k,m}$  that controls uniformly the operator norm of sub-matrices with  $k$  rows and  $m$  columns. This parameter is estimated by means of new tail estimates of order statistics and deviation inequalities for norms of projections of an isotropic log-concave vector.

## Résumé

On étudie la propriété d'isométrie restreinte d'une matrice aléatoire  $\Gamma$  dont les lignes sont des vecteurs aléatoires indépendants isotropes log-concave. Pour cela on introduit un paramètre  $\Gamma_{k,m}$  qui contrôle uniformément les normes d'opérateurs des sous-matrices de  $k$  lignes et  $m$  colonnes. Ce paramètre est estimé à l'aide de nouvelles inégalités de queue des statistiques d'ordre et d'inégalités de déviation des normes de projections d'un vecteur aléatoire log-concave.

**Introduction.** Let  $T \subset \mathbb{R}^N$  and  $\Gamma$  be an  $n \times N$  matrix. Consider the problem of reconstructing any vector  $x \in T$  from the data  $\Gamma x \in \mathbb{R}^n$ , with a fast algorithm. Clearly one needs some a priori hypothesis on the subset  $T$  and of course, the matrix  $\Gamma$  should be suitably chosen. The common and useful hypothesis is that  $T$  consists of sparse vectors, that is vectors with short support. In that setting, Compressed Sensing provides a way of reconstructing the original signal

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$x$  from its compression  $\Gamma x$  with  $n \ll N$  by the so-called  $\ell_1$ -minimization method. The problem of reconstruction can be reformulated after D. Donoho [10] in a language of high dimensional geometry, namely, in terms of neighborliness of polytopes obtained by taking the convex hull of the columns of  $\Gamma$ . In this spirit, the sensing matrix is described by its columns. From another point of view, the matrix  $\Gamma$  may be also determined by measurements, e.g. by its rows.

Let  $0 \leq m \leq N$ . Denote by  $U_m$  the subset of unit vectors in  $\mathbb{R}^N$ , which are  $m$ -sparse, i.e. have at most  $m$  non-zero coordinates. The natural scalar product, the Euclidean norm and the unit sphere are denoted by  $\langle \cdot, \cdot \rangle$ ,  $|\cdot|$  and  $S^{N-1}$ . We also denote by the same notation  $|\cdot|$  the cardinality of a set. For any  $x = (x_i) \in \mathbb{R}^n$  we let  $\|x\|_\infty = \max_i |x_i|$ . By  $C, C_1, c$ , etc. we will denote absolute positive constants.

Let  $\delta_m = \delta_m(\Gamma) = \sup_{x \in U_m} |\Gamma x|^2 - \mathbb{E}|\Gamma x|^2|$  be the Restricted Isometry Property (RIP) parameter of order  $m$ . This concept was introduced by E. Candes and T. Tao in [9] and its important feature is that if  $\delta_{2m}$  is appropriately small then every  $m$ -sparse vector  $x$  can be reconstructed from its compression  $\Gamma x$  by the  $\ell_1$ -minimization method. The goal now is to check this property for certain models of matrices.

The articles [1, 2, 3, 4, 5] considered random matrices with independent *columns*, and investigated high dimensional geometric properties of the convex hull of the columns and the RIP for various models of matrices, including the log-concave Ensemble build with independent isotropic log-concave columns. It was shown that various properties of random vectors can be efficiently studied via operator norms and the parameter  $\Gamma_{n,m}$  recalled below. In order to control this parameter an efficient technique of chaining was developed in [3] and [4].

In [14], the authors studied the RIP and more generally the parameter  $\delta_T = \sup_{x \in T} |\Gamma x|^2 - \mathbb{E}|\Gamma x|^2|$  for random matrices with independent isotropic subgaussian *rows*. It is natural to ask whether random matrices with independent isotropic log-concave *rows* also have the RIP.

Fix integers  $n, N \geq 1$ . Let  $Y_1, \dots, Y_n$  be independent random vectors in  $\mathbb{R}^N$  and let  $\Gamma$  be the  $n \times N$  random matrix with rows  $Y_i$ . Let  $T \subset S^{N-1}$  and  $1 \leq k \leq n$  and define the parameter  $\Gamma_k(T)$  by

$$\Gamma_k(T)^2 = \sup_{y \in T} \sup_{\substack{I \subset \{1, \dots, n\} \\ |I|=k}} \sum_{i \in I} |\langle Y_i, y \rangle|^2. \quad (1)$$

We also denote  $\Gamma_{k,m} = \Gamma_k(U_m)$ . The role of this parameter with respect to the RIP is revealed by the following lemma which reduces a concentration inequality to a deviation inequality.

**Lemma 1** *Let  $Y_1, \dots, Y_n$  be independent isotropic random vectors in  $\mathbb{R}^N$ . Let  $T \subset S^{N-1}$  be a finite set. Let  $0 < \theta < 1$  and  $B \geq 1$ . Then with probability at least  $1 - |T| \exp(-3\theta^2 n / 8B^2)$  one has*

$$\sup_{y \in T} \left| \frac{1}{n} \sum_{i=1}^n (|\langle Y_i, y \rangle|^2 - \mathbb{E}|\langle Y_i, y \rangle|^2) \right| \leq \theta + \frac{1}{n} (\Gamma_k(T)^2 + \mathbb{E}\Gamma_k(T)^2),$$

where  $k \leq n$  is the largest integer satisfying  $k \leq (\Gamma_k(T)/B)^2$ .

In this note we focus on the compressed sensing setting where  $T$  is the set of sparse vectors. Lemma 1 shows that after a suitable discretization, checking the RIP reduces to estimating  $\Gamma_{k,m}$ . The idea of such an approach, when  $k = n$ , originated from the work of J. Bourgain [8] on the empirical covariance matrix. It was developed in [3] and [5] (with  $T = U_m$ ), where the estimate of  $\Gamma_{n,m}$  played a central role for solving the Kannan-Lovász-Simonovits conjecture related to complexity of computing high-dimensional volumes ([11]); and it was studied in [13], where  $\Gamma_k(T)$  was estimated by means of Talagrand  $\gamma$ -functionals.

Using Lemma 1 it can be shown (cf., [5] for a similar argument) that if  $0 < \theta < 1$ ,  $B \geq 1$ , and  $m \leq N$  satisfies  $m \log(CN/m) \leq 3\theta^2 n/16B^2$ , then with probability at least  $1 - \exp(-3\theta^2 n/16B^2)$  one has

$$\delta_m(\Gamma/\sqrt{n}) = \sup_{y \in U_m} \left| \frac{1}{n} \sum_{i=1}^n (|\langle Y_i, y \rangle|^2 - \mathbb{E}|\langle Y_i, y \rangle|^2) \right| \leq C\theta + \frac{C}{n} (\Gamma_{k,m}^2 + \mathbb{E}\Gamma_{k,m}^2), \quad (2)$$

where  $k \leq n$  is the largest integer satisfying  $k \leq (\Gamma_{k,m}/B)^2$  (note that  $k$  is a random variable).

We consider the log-concave Ensemble of  $n \times N$  matrices with independent isotropic log-concave rows. Recall that a random vector is isotropic log-concave if it is centered, its covariance matrix is the identity and its distribution has a log-concave density. Our goal is to bound  $\Gamma_{k,m}$  for this Ensemble. This leads to questions that require a deeper understanding of some geometric parameters of log-concave measures, such as tail estimates for order statistics and deviation inequalities for norms of projections. Proofs and related results will be presented in [6].

**Main results.** Our main result, Theorem 6, provides upper estimates for  $\Gamma_{k,m}$  valid with large probability for matrices from the log-concave Ensemble. To achieve this we need some intermediate steps also of a major importance. For a random vector  $X$  and  $p > 0$ , we define the following natural parameter

$$\sigma_X(p) = \sup_{t \in S^{N-1}} (\mathbb{E}|\langle t, X \rangle|^p)^{1/p}.$$

It is known that  $\sigma_X(p) \leq p$  for isotropic log-concave  $X$  and  $p \geq 2$ . Paouris' Theorem ([15]) states

$$(\mathbb{E}|X|^p)^{1/p} \leq C((\mathbb{E}|X|^2)^{1/2} + \sigma_X(p)). \quad (3)$$

It is a consequence of Theorem 8.2 combined with Lemma 3.9 in [15], note that Lemma 3.9 holds not only for convex bodies but for log-concave measures as well.

We extend the Paouris Theorem to the following bound on deviations of norm of projections of an isotropic log-concave vector, uniform over all coordinate projections  $P_I$  of a fixed rank.

**Theorem 2** *Let  $m \leq N$  and  $X$  be an isotropic log-concave vector in  $\mathbb{R}^N$ . Then for every  $t \geq 1$  one has*

$$\mathbb{P} \left( \sup_{\substack{I \subset \{1, \dots, N\} \\ |I|=m}} |P_I X| \geq Ct\sqrt{m} \log \left( \frac{eN}{m} \right) \right) \leq \exp \left( -t \frac{\sqrt{m}}{\sqrt{\log(em)}} \log \left( \frac{eN}{m} \right) \right).$$

This theorem is sharp up to  $\sqrt{\log(em)}$  in the probability estimate as the case of a vector with independent exponential coordinates shows. Actually our further applications require a stronger result in which the bound for probability is improved by involving the parameter  $\sigma_X$  and its inverse  $\sigma_X^{-1}$ , namely

**Theorem 3** *Let  $m \leq N$  and  $X$  be an isotropic log-concave vector in  $\mathbb{R}^N$ . Then for any  $t \geq 1$ ,*

$$\mathbb{P} \left( \sup_{\substack{I \subset \{1, \dots, N\} \\ |I|=m}} |P_I X| \geq Ct\sqrt{m} \log \left( \frac{eN}{m} \right) \right) \leq \exp \left( -\sigma_X^{-1} \left( \frac{t\sqrt{m} \log \left( \frac{eN}{m} \right)}{\sqrt{\log(em/m_0)}} \right) \right),$$

where  $m_0 = m_0(X, t) = \sup \{k \leq m : k \log(eN/k) \leq \sigma_X^{-1}(t\sqrt{m} \log(eN/m))\}$ .

Theorem 3 is based on tail estimates for order statistics of isotropic log-concave vectors. By  $(X^*(i))_i$  we denote the non-increasing rearrangement of  $(|X(i)|)_i$ . Combining (3) with methods of [12] we obtain

**Theorem 4** *Let  $X$  be an  $N$ -dimensional isotropic log-concave vector. Then for every  $t \geq C \log(eN/\ell)$ ,*

$$\mathbb{P}(X^*(\ell) \geq t) \leq \exp(-\sigma_X^{-1}(C^{-1}t\sqrt{\ell})).$$

Introduction of the parameter  $\sigma_X$  enables us to obtain new inequalities for convolutions of log-concave measures. Let  $X_1, \dots, X_n$  be independent isotropic log-concave random vectors in  $\mathbb{R}^N$ . We will consider weighted sums of the vectors  $X_i$  of the form  $Y = \sum_{i=1}^n x_i X_i$ , where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Bernstein's inequality and  $\psi_1$  estimate for isotropic log-concave random vectors give  $\sigma_Y(p) \leq C(\sqrt{p}|x| + p\|x\|_\infty)$  for  $p \geq 1$ . Together with Theorem 3 this yields the following

**Corollary 5** *Assume that  $|x| \leq 1$  and  $1 \geq b \geq \max(\|x\|_\infty, 1/\sqrt{m})$ . Then for any  $t \geq 1$ ,*

$$\mathbb{P} \left( \sup_{\substack{I \subset \{1, \dots, N\} \\ |I|=m}} |P_I Y| \geq Ct\sqrt{m} \log \left( \frac{eN}{m} \right) \right) \leq \exp \left( -\frac{t\sqrt{m} \log \left( \frac{eN}{m} \right)}{b\sqrt{\log(e^2 b^2 m)}} \right).$$

We now pass to bounds on deviation of  $\Gamma_{k,m}$ . To get a slightly simplified formula we assume that  $N \geq n$ .

**Theorem 6** *Let  $1 \leq n \leq N$ , and let  $\Gamma$  be an  $n \times N$  random matrix with independent isotropic log-concave rows. For any integers  $k \leq n$ ,  $m \leq N$  and any  $t \geq 1$ , we have*

$$\mathbb{P}(\Gamma_{k,m} \geq Ct\lambda) \leq \exp(-t\lambda/\sqrt{\log(3m)}),$$

where  $\lambda = \sqrt{\log \log(3m)}\sqrt{m} \log(eN/m) + \sqrt{k} \log(en/k)$ .

The threshold value  $\lambda$  in Theorem 6 is optimal, up to the factor of  $\sqrt{\log \log(3m)}$ . Assuming additionally unconditionality of the distributions of the rows, we can remove this factor and get a sharp estimate ([7]).

The proof of the above theorem is composed of two parts, depending on the relation between  $k$  and the quantity  $k' = \inf\{\ell \geq 1: m \log(eN/m) \leq \ell \log(en/\ell)\}$ . First, we adjust the chaining argument from [3] to reduce the problem to the case  $k \leq k'$ . This step also involves Theorem 2. Next, we use Corollary 5 combined with another chaining to complete the argument.

Theorem 6 together with (2) allows us to prove the RIP result for matrices  $\Gamma$  with independent isotropic log-concave rows. The result is optimal, up to the factor  $\log \log 3m$ , as shown in [4]. As for Theorem 6, assuming unconditionality of the distributions of the rows, we can remove this factor ([7]).

**Theorem 7** *Let  $0 < \theta < 1$ ,  $1 \leq n \leq N$ . Let  $\Gamma$  be an  $n \times N$  random matrix with independent isotropic log-concave rows. There exists  $c(\theta) > 0$  such that  $\delta_m(\Gamma/\sqrt{n}) \leq \theta$  with overwhelming probability whenever*

$$m \log^2(2N/m) \log \log 3m \leq c(\theta)n.$$

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