ON INVARIANT CCC σ-IDEALS ON $2^{\mathbb{N}}$

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Abstract. We study structural properties of the collection of all σ-ideals in the σ-algebra of Borel subsets of the Cantor group $2^{\mathbb{N}}$, especially those which satisfy the countable chain condition (ccc) and are translation invariant. We prove that the latter collection contains an uncountable family of pairwise orthogonal members and, as a consequence, a strictly decreasing sequence of length ω_1.

We also make some observations related to the σ-ideal I_{ccc} on $2^{\mathbb{N}}$, consisting of all Borel sets which belong to every translation invariant ccc σ-ideal on $2^{\mathbb{N}}$. In particular, improving earlier results of Recław, Kraszewski and Komjáth, we show that:

- every subset of $2^{\mathbb{N}}$ of cardinality less than $\aleph_0$-$\mathfrak{s}$ can be covered by a set from I_{ccc},
- there exists a set $C \in I_{ccc}$ such that every countable subset Y of $2^{\mathbb{N}}$ is contained in a translate of C.

1. Introduction

Given an uncountable Polish space X by a *σ-ideal* in the σ-algebra $\mathscr{B}(X)$ of Borel subsets of X (shortly: a *σ-ideal on X*) we mean a nonempty family $I \subseteq \mathscr{B}(X)$ which is closed under taking Borel subsets and countable unions. Throughout the paper we assume that I contains all singletons. Note, however, that we do not assume that $I \neq \mathscr{B}(X)$. If the latter is the case, then I is referred to as a *proper* σ-ideal.

We say that a σ-ideal I on X is *ccc* if there is no uncountable family of pairwise disjoint Borel subsets of X outside I. In particular, $\mathscr{B}(X)$ is the largest ccc σ-ideal in $\mathscr{B}(X)$.

The aim of this note is to point out that looking at structural properties of the collection of all ccc σ-ideals on X leads to either generalisations or at least new and easier proofs of some known results of Recław [14], Kraszewski [10] and Komjáth [8].

The collection of all σ-ideals in $\mathscr{B}(X)$ equipped with the ordering of inclusion is a lattice with the operations of join and meet given by $I_0 \wedge I_1 = I_0 \cap I_1$ and $I_0 \vee I_1 = \{A_0 \cup A_1 : A_0 \in I_0,\ A_1 \in I_1\}$.

We say that the σ-ideals I_0 and I_1 on X are *orthogonal* (or *singular to each other*, cf. [4]), in symbols: $I_0 \perp I_1$, if $I_0 \vee I_1 = \mathscr{B}(X)$ or, equivalently, if $X = A_0 \cup A_1$ where $A_0 \in I_0$, $A_1 \in I_1$ (and with no loss

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of generality $A_0 \cap A_1 = \emptyset$). In particular, $\mathscr{B}(X) \perp I$ for every σ-ideal I on X.

Given a σ-ideal J in $\mathscr{B}(X)$ and $A \in \mathscr{B}(X)$ we let

$$J|A = \{C \in \mathscr{B}(X) : C \cap A \in J\}.$$

Clearly, $J|A$ is also a σ-ideal on X, $J \subseteq J|A$ and $J|A \neq \mathscr{B}(X)$ if and only if $A \notin J$.

Finally, we let $J^* = \{X \setminus A : A \in J\}$ and $J^+ = \mathscr{B}(X) \setminus J$.

In Section 2 we either recall or prove some of the basic structural properties of the collection of all σ-ideals on an arbitrary uncountable Polish space X, related to the notions above. In particular, we recall an ideal version of the Lebesgue decomposition theorem (see Theorem 2.4), due to Capek [1] and derive two consequences from it (see Theorems 2.5 and 2.7). We also prove (see Theorem 2.8) that given an uncountable collection of pairwise orthogonal proper σ-ideals on X every ccc σ-ideal is orthogonal to at least one of them; in particular, their intersection is not ccc.

In Section 3 we shift our attention to invariant σ-ideals on $2^{\mathbb{N}}$ (where $2^{\mathbb{N}}$ is considered with the coordinatewise addition modulo 2 and referred to as the *Cantor group*). In general, a σ-ideal I on a Polish abelian group $(G, +)$ is *translation invariant* (shortly: *invariant*), if

$$\forall x \in G \ \forall A \subseteq G \ (A \in I \ \Rightarrow \ x + A \in I).$$

We prove (see Theorem 3.1) that there exists an uncountable collection of pairwise orthogonal ccc invariant σ-ideals on $2^{\mathbb{N}}$ and derive some consequences from this fact. In particular, together with Theorem 2.8 mentioned above, it immediately implies that the intersection of all invariant ccc σ-ideals on $2^{\mathbb{N}}$, denoted by I_{ccc}, is not itself ccc, the result proved earlier in [17].

We observe (see Proposition 3.6) that sets of the form $[f] = \{x \in 2^{\mathbb{N}} : f \subseteq x\}$, where f is a function from an infinite subset of $\mathbb{N}$ to $\{0, 1\}$, belong to I_{ccc}. This observation, combined with a result of Cichoń and Kraszewski [3], emposes a lower bound on the cardinal invariant $\mathrm{non}(I_{ccc})$, the smallest cardinality of a subset of $2^{\mathbb{N}}$ not covered by a set from I_{ccc}, improving an earlier result of Kraszewski [10]. We also prove that there exists a set $C \in I_{ccc}$ such that every countable subset Y of $2^{\mathbb{N}}$ is contained in a translate of C improving an earlier result of Komjáth [8].

2. Preliminaries

Throughout this section X is an uncountable Polish space.

The following fact was observed by several people (see [1] and [4]).

Proposition 2.1. Let I and J be σ-ideals in $\mathscr{B}(X)$ and assume that J is ccc. Then the following are equivalent:

(1) $J \subseteq I$,
(2) There is a set $A \in I^*$ such that $I = J|A$.

Remark 2.2. Apparently, the usefulness of Proposition 2.1 has not been fully recognized. To see it in action, let us give a simple proof of the following fact, particular cases of which were earlier showed using different methods (Woodin for the meager σ-ideal, Kechris-Miller for the null σ-ideal (see [9, Remark 10.3]); a related result for a broader class of σ-ideals is proved in [7, Claim 4.2.3] by forcing methods):

Given a ccc σ-ideal I in $\mathscr{B}(X)$ and a countable group G of Borel automorphisms of X, there is a Borel set $A \in I^*$ such that if $B \subseteq A$, $gB \subseteq A$ and $B \in I$, then $gB \in I$ (we may additionally require that $\bigcup_{g\in G} gA = X$).

To prove it, just let $J = \{B \in \mathscr{B}(X) : \forall g \in G\ gB \in I\}$. Then since $J \subseteq I$ and J is clearly ccc, there is a set $A \in I^*$ such that $I = J|A$. It is easy to see that A is a desired set (to make A satisfy the additional condition, replace it by $A \cup (X \setminus \bigcup_{g\in G} gA)$).

As a corollary of Proposition 2.1 we get a useful characterization of orthogonality of σ-ideals.

Proposition 2.3. Let I_0 and I_1 be σ-ideals in $\mathscr{B}(X)$ and additionally assume that I_1 is ccc. Then the following are equivalent:

(1) $I_0 \perp I_1$,
(2) $I_0 \subseteq I_1|C$ for no $C \in I_1^+$,
(3) If, moreover, I_0 is ccc, then $I_0|A = I_1|A$ for no $A \in I_0^+ \cap I_1^+$.

Proof. The implication (1) $\Rightarrow$ (2) is clear.

To see that (1) $\Rightarrow$ (3), assume $\neg$(3) and let $I = I_0|A = I_1|A$ for a certain $A \in I_0^+ \cap I_1^+$. But then $I_0, I_1 \subseteq I \neq \mathscr{B}(X)$, contradicting (1).

To show that (2) $\Rightarrow$ (1), assume $\neg$(1) and let $I = I_0 \vee I_1 \neq \mathscr{B}(X)$. Then, by Proposition 2.1, there is $C \in I_1^+$ such that $I = I_1|C$ hence $I_0 \subseteq I_1|C$, contradicting (2).

Finally, to prove that (3) $\Rightarrow$ (1), assume $\neg$(1) and let $I = I_0 \vee I_1$. Then $I \neq \mathscr{B}(X)$, so by Proposition 2.1, there are sets $A_0,\ A_1 \in I^*$ such that $I = I_0|A_0 = I_1|A_1$. Letting $A = A_0 \cap A_1$ we have $A \in I^*$ (hence, since $I \neq \mathscr{B}(X)$, $A \in I_0^+ \cap I_1^+$) and $I_0|A = I|A = I_1|A$, which contradicts (3). □

The following ideal version of the Lebesgue decomposition theorem is a consequence (at least as far as the existence of decompositions is concerned) of a more general result due to Capek [1] (see also [4]). For the sake of completeness, we present a sketch of its proof in our framework.

Theorem 2.4. Let I_0 and I_1 be σ-ideals in $\mathscr{B}(X)$.

If I_1 is ccc, then there is a partition of $X = A \cup B$ into disjoint Borel subsets A and B such that $I_0 \subseteq I_1|A$ and $I_0 \perp I_1|B$. Moreover, such a partition is unique mod I_1 in the following sense: if $X = A_1 \cup B_1$ is another such a partition, then $A \triangle A_1 \in I_1$ (hence also $B \triangle B_1 \in I_1$).

Proof. In order to prove the existence of a desired partition, consider three cases:

Case 1. The σ-ideals I_0 and I_1 are comparable. First assume that $I_0 \subseteq I_1$. Then $A = X$ and $B = \emptyset$ satisfy the requirements. Next assume that $I_1 \subseteq I_0$. Then, by Proposition 2.1, there is a set $A \in I_0^*$ such that $I_0 = I_1|A$. Taking $B = X \setminus A$ we have $B \in I_0$, $A \in I_1|B$ and $X = B \cup A$ which shows that $I_0 \perp I_1|B$. Hence A and B satisfy the requirements.

Case 2. The σ-ideals I_0 and I_1 are orthogonal. Then $I_0 \subseteq \mathscr{B}(X) = I_1|\emptyset$ and $I_0 \perp I_1 = I_1|X$ so $A = \emptyset$ and $B = X$ clearly work.

Case 3. The σ-ideals I_0 and I_1 are neither comparable nor orthogonal. Consider a maximal collection $\mathcal{R}$ of pairwise disjoint Borel sets $C \in I_1^+$ such that $I_0 \subseteq I_1|C$. By Proposition 2.3 and the assumption that $I_0 \not\perp I_1$, the family $\mathcal{R}$ is nonempty. By the ccc property of I_1, it is countable.

Let $A = \bigcup \mathcal{R}$ and $B = X \setminus A$. Then, by the countability of $\mathcal{R}$, we have $A,\ B \in \mathscr{B}(X)$ and $I_0 \subseteq I_1|A$. On the other hand, by the maximality of $\mathcal{R}$, $I_0 \subseteq I_1|C$ for no $C \subseteq B$, $C \in I_1^+$. By Proposition 2.3, this implies that $I_0 \perp I_1|B$.

Thus the existence of a desired partition is ensured.

To prove its essential uniqueness, let $X = A \cup B = A_1 \cup B_1$ be two partitions of X into disjoint Borel subsets such that $I_0 \subseteq I_1|A \cap I_1|A_1$, $I_0 \perp I_1|B$ and $I_0 \perp I_1|B_1$. Then we have

$$I_0 \subseteq I_1|A_1 \subseteq I_1|(B \cap A_1) \quad \text{and} \quad I_1|B \subseteq I_1|(B \cap A_1)$$

which implies that $A_1 \setminus A = B \cap A_1 \in I_1$ since otherwise $I_0 \vee I_1|B \neq \mathscr{B}(X)$ – a contradiction with $I_0 \perp I_1|B$. Analogically, $A \setminus A_1 \in I_1$. □

When both σ-ideals are ccc Theorem 2.4 admits the following strengthening (cf. Remark 2.6).

Theorem 2.5. If I_0 and I_1 are ccc σ-ideals in $\mathscr{B}(X)$, then there is a partition of $X = A \cup B$ into disjoint Borel subsets A and B such that $I_0|A = I_1|A$ and $I_0|B \perp I_1|B$. Moreover, such a partition is unique mod $I = I_0 \cap I_1$ in the following sense: if $X = A_1 \cup B_1$ is another such a partition, then $A \triangle A_1 \in I$ (hence also $B \triangle B_1 \in I$).

Proof. Since I_1 is ccc, there is, by Theorem 2.4, a partition $X = A' \cup B'$ into Borel sets such that $I_0 \subseteq I_1|A'$ and $I_0 \perp I_1|B'$.

Let $J = I_0$ and $I = I_1|A'$. Since I_0 is ccc, there is, by Proposition 2.1, a set $E \in I^*$ such that $I = J|E$.

Let $A = A' \cap E$ and $B = X \setminus A$. We claim that A and B satisfy the requirements.

As $I_1|A' = I = J|E = I_0|E$ we have $I_0|A = I_1|A$.

Since $I_0 \perp I_1|B'$, there is a partition $X = C \cup D$ into Borel sets such that $C \in I_0$ and $D \cap B' \in I_1$. Since obviously, $C \cap B \in I_0$, to prove that $I_0|B \perp I_1|B$, it suffices to show that $D \cap B \in I_1$. To that end note that $B = B' \cup (A' \setminus E)$ hence

$$D \cap B \subseteq (D \cap B') \cup (A' \setminus E)$$

and we are done since $E \in I^*$ just means that $A' \setminus E \in I_1$.

This completes the proof of the existence of a desired partition.

To prove its essential uniqueness, let $X = A \cup B = A_1 \cup B_1$ be two partitions of X into disjoint Borel subsets such that $I_0|A = I_1|A$, $I_0|A_1 = I_1|A_1$, $I_0|B \perp I_1|B$ and $I_0|B_1 \perp I_1|B_1$.

Then $I_0 \subseteq I_1|A \cap I_1|A_1$. It is also easy to see that $I_0 \perp I_1|B$ and $I_0 \perp I_1|B_1$ (see Remark 2.6). Consequently, by the uniqueness part of Theorem 2.4, $A \triangle A_1 \in I_1$. Likewise, by symmetry, $A \triangle A_1 \in I_0$.

□

Remark 2.6. To see that Theorem 2.5 is indeed a strengthening of Theorem 2.4, note that $I_0|A = I_1|A$ clearly implies $I_0 \subseteq I_1|A$, whereas $I_0|B \perp I_1|B$ is in fact equivalent to $I_0 \perp I_1|B$ for arbitrary (i.e., not necessary ccc) σ-ideals I_0 and I_1 on X. For assuming that $I_0|B \perp I_1|B$ let $X = M \cup N$ where $M,\ N \in \mathscr{B}(X)$ are such that $M \cap B \in I_0$ and $N \cap B \in I_1$. Then

$$X = (M \cap B) \cup \big((N \cap B) \cup (X \setminus B)\big) \qquad (*)$$

where $M \cap B \in I_0$ and $\big((N \cap B) \cup (X \setminus B)\big) \in I_1|B$. Consequently, $(*)$ implies that $I_0 \perp I_1|B$. The other direction is obvious.

Theorems 2.4 and 2.5 have the following interesting consequence in the case when the σ-ideals are invariant (cf. Remark 3.7).

Theorem 2.7. Let I_0 and I_1 be invariant σ-ideals on a Polish group G.

(1) If I_1 is ccc, then there exists a (unique mod I_1) partition of G into disjoint Borel subsets A and B such that $I_0 \subseteq I_1|A$, $I_0 \perp I_1|B$ and the sets A and B are I_1-almost invariant in the following sense: if $g \in G$, then $gA \triangle A \in I_1$ (hence also $gB \triangle B \in I_1$).

(2) If I_0 and I_1 are ccc, then there exists a (unique mod $I_0 \cap I_1$) partition of X into disjoint Borel subsets A and B such that $I_0|A = I_1|A$, $I_0|B \perp I_1|B$ and the sets A and B are $(I_0 \cap I_1)$-almost invariant.

Proof. To prove part (1), let $G = A \cup B$ be a partition whose existence is guaranteed by Theorem 2.4, i.e., $I_0 \subseteq I_1|A$ and $I_0 \perp I_1|B$. Fix $g \in G$, let $A_1 = gA$, $B_1 = gB$ and note that, due to invariance of I_0 and I_1, $G = A_1 \cup B_1$ is another partition of G such that $I_0 \subseteq I_1|A_1$

and $I_0 \perp I_1|B_1$. The uniqueness property of such partitions gives the desired conclusion.

Part (2) is proved analogically with the help of Theorem 2.5.

□

The last result of this section will turn out to be especially important in the sequel.

Theorem 2.8. Let $\langle I_\alpha : \alpha < \omega_1\rangle$ be a sequence of pairwise orthogonal proper σ-ideals on X.

Then for every ccc σ-ideal J on X there is $\alpha < \omega_1$ such that $J \perp I_\alpha$. Consequently, the σ-ideal $I = \bigcap_{\alpha<\omega_1} I_\alpha$ is not ccc.

Proof. Suppose that J is not orthogonal to any of the I_α's. Using Proposition 2.3, for each $\alpha < \omega_1$ find $A_\alpha \in J^+$ such that $I_\alpha \subseteq J|A_\alpha$. J being ccc, there are $\alpha \neq \beta$ with $A_\alpha \cap A_\beta \in J^+$; let $A = A_\alpha \cap A_\beta$. Then $I_\alpha \vee I_\beta \subseteq J|A \neq \mathscr{B}(X)$ and we obtain a contradiction with the assumption that $I_\alpha \perp I_\beta$. □

3. Invariant ccc σ-ideals on $2^{\mathbb{N}}$

Let $\mathcal{N}$ and $\mathscr{M}$ denote as usual the σ-ideals of null and, respectively, of meager Borel subsets of $2^{\mathbb{N}}$. More generally, if Y is a nonempty subset of $\mathbb{N}$ and $G = (2^{\mathbb{N}})^Y$ is a product of countably many copies of the group $2^{\mathbb{N}}$, then $\mathcal{N}(G)$ and $\mathscr{M}(G)$ are the σ-ideals consisting of those Borel subsets of G which are, respectively, null (with respect to the product of ordinary measures on $2^{\mathbb{N}}$) and meager (with respect to the product topology on G). Note that if we identify G with $2^{\mathbb{N}}$ using the canonical topological group isomorphism then, up to this identification, $\mathcal{N}(G) = \mathcal{N}$ and $\mathscr{M}(G) = \mathscr{M}$.

Recall that $\mathcal{N} \otimes \mathscr{M}$, the Fubini product of $\mathcal{N}$ and $\mathscr{M}$, is the σ-ideal on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ consisting of Borel sets $B \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\{x \in 2^{\mathbb{N}} : B_x \notin \mathscr{M}\} \in \mathcal{N}$. By a theorem of Gavalec [6] (see also [5] for a more general result), the σ-ideal $\mathcal{N} \otimes \mathscr{M}$ is ccc.

Theorem 3.1. There is a collection $\langle I_\alpha : \alpha < \mathfrak{c}\rangle$ of continuum many ccc pairwise orthogonal invariant proper σ-ideals on $2^{\mathbb{N}}$. Moreover, each σ-ideal I_α is essentially equal to $\mathcal{N} \otimes \mathscr{M}$. More precisely, for every α there is a topological group isomorphism ϕ_α between $2^{\mathbb{N}}$ and $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that for every $B \in \mathscr{B}(2^{\mathbb{N}})$

$$B \in I_\alpha \iff \phi_\alpha[B] \in \mathcal{N} \otimes \mathscr{M}.$$

Proof. We closely follow the proof of Theorem 2.1 from [17] based on ideas of Solecki [16]. For every non-constant sequence $y \in 2^{\mathbb{N}}$ let $Z_0^y = \{n \in \mathbb{N} : y(n) = 0\}$ and $Z_1^y = \{n \in \mathbb{N} : y(n) = 1\}$. The idea is to identify $2^{\mathbb{N}}$ with the product group $G = G_0^y \times G_1^y$ where $G_i^y = (2^{\mathbb{N}})^{Z_i^y}$ for $i \in \{0, 1\}$ and to let $I_y = \mathcal{N}(G_0^y) \otimes \mathscr{M}(G_1^y)$.

More precisely, we fix canonical topological group isomorphisms:

(1) ψ between $2^{\mathbb{N}}$ and $(2^{\mathbb{N}})^{\mathbb{N}}$ defined by
$$x \mapsto \big\langle \langle x(p(n,m) : m \in \mathbb{N}\rangle :\ n \in \mathbb{N}\big\rangle,$$
where p is a fixed bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$,

(2) ψ_y between $(2^{\mathbb{N}})^{\mathbb{N}}$ and $G_0^y \times G_1^y$ defined by
$$\overline{x} \mapsto \langle \overline{x}|Z_0^y, \overline{x}|Z_1^y\rangle.$$

Then let $\phi_y = \psi_y \circ \psi$ and define
$$I_y = \{B \in \mathscr{B}(2^{\mathbb{N}}) :\ \phi_y[B] \in \mathcal{N}(G_0^y) \otimes \mathscr{M}(G_1^y)\}.$$

Clearly, each σ-ideal I_y is essentially equal to $\mathcal{N} \otimes \mathscr{M}$. It remains to be proved that the σ-ideals I_y are pairwise orthogonal.

To that end we fix a partition of $2^{\mathbb{N}}$ into a meager F_σ-set M of measure 1 and a dense G_δ-set N of measure 0.

Now let y and y' be two different non-constant elements of $2^{\mathbb{N}}$. Let $n \in \mathbb{N}$ be such that $y(n) \neq y'(n)$, say $y(n) = 0$ and $y'(n) = 1$. Let $B_0 = \{\overline{x} \in (2^{\mathbb{N}})^{\mathbb{N}} : \overline{x}(n) \in N\}$ and $B_1 = \{\overline{x} \in (2^{\mathbb{N}})^{\mathbb{N}} : \overline{x}(n) \in M\}$. Clearly, $B_0 \cup B_1 = (2^{\mathbb{N}})^{\mathbb{N}}$ hence $\psi_y[B_0 \cup B_1] = G_0^y \times G_1^y$ and $\psi_{y'}[B_0 \cup B_1] = G_0^{y'} \times G_1^{y'}$. Moreover,
$$\psi_y[B_0] = \{\overline{z} \in G_0^y : \overline{z}(n) \in N\} \times G_1^y \in \mathcal{N}(G_0^y) \otimes \mathscr{M}(G_1^y)$$
and
$$\psi_{y'}[B_1] = G_0^{y'} \times \{\overline{z} \in G_1^{y'} : \overline{z}(n) \in M\} \in \mathcal{N}(G_0^{y'}) \otimes \mathscr{M}(G_1^{y'}).$$
Finally, letting $A_0 = \psi^{-1}[B_0]$ and $A_1 = \psi^{-1}[B_1]$ we have $A_0 \in I_y$, $A_1 \in I_{y'}$ and $A_0 \cup A_1 = 2^{\mathbb{N}}$ which proves that $I_y \perp I_{y'}$.

□

Corollary 3.2.

(1) There exists a strictly decreasing sequence of length ω_1 consisting of invariant ccc σ-ideals on $2^{\mathbb{N}}$.

(2) If $\langle J_\alpha : \alpha < \omega_1\rangle$ is a strictly decreasing sequence of invariant ccc σ-ideals on $2^{\mathbb{N}}$, then its intersection is not ccc. Consequently, there is no strictly decreasing sequence of any length $\alpha > \omega_1$ consisting of invariant ccc σ-ideals on $2^{\mathbb{N}}$.

Proof. To prove part (1), let $\langle I_\alpha : \alpha < \omega_1\rangle$ be a sequence of ccc pairwise orthogonal invariant proper σ-ideals on $2^{\mathbb{N}}$ whose existence is guaranteed by Theorem 3.1. We define a sequence $\langle J_\alpha : \alpha < \omega_1\rangle$ by letting
$$J_\alpha = \bigcap_{\beta<\alpha} I_\beta, \quad \text{for} \quad \alpha < \omega_1.$$
Clearly, each σ-ideal J_α is ccc as the intersection of countably many ccc σ-ideals. But, by Theorem 2.8, the σ-ideal $\bigcap_{\beta<\omega_1} I_\beta$ is not ccc. It follows that the sequence $\langle J_\alpha : \alpha < \omega_1\rangle$ contains a strictly decreasing subsequence of length ω_1, completing the proof of part (1) (as a matter

of fact, it is not difficult to prove directly that the sequence $\langle J_\alpha : \alpha < \omega_1\rangle$ is strictly decreasing itself).

To prove part (2), let

$$I = \bigcap_{\alpha<\omega_1} J_\alpha$$

and suppose that I is ccc.

Using Proposition 2.1, for each $\alpha < \omega_1$ find $A_\alpha \in J^*_\alpha$ such that $J_\alpha = I|A_\alpha$. Note that $\beta < \alpha < \omega_1$ implies that $A_\beta \setminus A_\alpha \in I$. Indeed, otherwise, letting $B = A_\beta \setminus A_\alpha$ we have $B \in I|A_\alpha \setminus I|A_\beta$ contradicting the fact that $J_\alpha \subseteq J_\beta$. Now, I being ccc, there must be $\alpha < \omega_1$ such that $A_\alpha \triangle A_{\alpha+1} \in I$. But then $J_\alpha = J_{\alpha+1}$ and we have reached a contradiction. □

Let I_{ccc} be the σ-ideal on $2^{\mathbb{N}}$ consisting of Borel sets which belong to every invariant ccc σ-ideal on $2^{\mathbb{N}}$ (see [17], where this σ-ideal was introduced and studied).

Theorems 3.1 and 2.8 immediately imply the following two corollaries, first of which was proved in a stronger form in [17].

Corollary 3.3. I_{ccc} is not ccc.

Corollary 3.4. If J is an arbitrary invariant ccc σ-ideal on $2^{\mathbb{N}}$, then there is an invariant ccc σ-ideal I on $2^{\mathbb{N}}$, essentially equal (cf. Theorem 3.1) to $\mathcal{N} \otimes \mathcal{M}$, such that $J \perp I$.

Recall that if I is a proper σ-ideal on $2^{\mathbb{N}}$ then $\mathrm{non}(I)$ is the smallest cardinality of a subset of $2^{\mathbb{N}}$ not covered by a set from I and $\mathrm{cov}^*(I)$ is the smallest number of translates of a fixed set from I required to cover $2^{\mathbb{N}}$ (consistently, $\mathrm{cov}^*(\mathcal{M})$ might be bigger than the minimal cardinality of a covering of $2^{\mathbb{N}}$ by sets from $\mathcal{M}$, see [13]).

The following corollary was earlier proved (in an even stronger form) by Recław [14] using a different method.

Corollary 3.5. If J is an arbitrary proper invariant ccc σ-ideal on $2^{\mathbb{N}}$, then

$$\mathrm{non}(J) \geq \min(\mathrm{cov}^*(\mathcal{M}), \mathrm{cov}^*(\mathcal{N})).$$

Proof. Observe that

$$\mathrm{cov}^*(\mathcal{N} \otimes \mathcal{M}) = \min(\mathrm{cov}^*(\mathcal{M}), \mathrm{cov}^*(\mathcal{N}))$$

and recall a result due to Rothberger [15] stating that if J and I are invariant σ-ideals on $2^{\mathbb{N}}$, $J \perp I$ and J is proper, then $\mathrm{non}(J) \geq \mathrm{cov}^*(I)$. It is now enough to appeal to Corollary 3.5. □

Let $\mathbb{S}_2$ be the σ-ideal on $2^{\mathbb{N}}$, generated by sets of the form

$$[f] = \{x \in 2^{\mathbb{N}} : f \subseteq x\},$$

where f is a function from an infinite subset of $\mathbb{N}$ to $\{0,1\}$ (see Cichoń and Kraszewski [3], where this σ-ideal was introduced and studied).

Kraszewski [10] realised that Recław's argument from [14] actually establishes the inequality $\mathrm{non}(J) \geq \min(\mathrm{non}(I), \mathrm{cov}^*(I))$ for arbitrary proper invariant σ-ideals J and I on $2^{\mathbb{N}}$ provided J is ccc. Applying this to $I = \mathbb{S}_2$, Kraszewski noted that if J is as above then $\mathrm{non}(J) \geq \aleph_0\text{-}\mathfrak{s}$, the *countably splitting number*, i.e., the minimal cardinality of a collection $\mathscr{S} \subseteq [\mathbb{N}]^{\aleph_0}$ such that (see [12])

$$\forall \mathcal{A} \in [[\mathbb{N}]^{\aleph_0}]^{\aleph_0}\ \exists S \in \mathscr{S}\ \forall A \in \mathcal{A}\ \big(|A \cap S| = |A \setminus S| = \aleph_0\big).$$

The point is that, by the results of Cichoń and Kraszewski (see [3] and [11])

$$\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s} \leq \mathrm{cov}^*(\mathbb{S}_2) = \mathfrak{c}.$$

Taking this into account we may obtain a strengthening of Kraszewski's result quoted above by the following simple reasoning which avoids the use of Recław's argument from [14].

Proposition 3.6.

$$\mathbb{S}_2 \subseteq I_{ccc}.$$

Consequently, $\mathrm{non}(I_{ccc}) \geq \aleph_0\text{-}\mathfrak{s}$, so MA implies that $\mathrm{non}(I_{ccc}) = \mathfrak{c}$

Proof. Let f be a function from an infinite subset A of $\mathbb{N}$ to $\{0,1\}$. It suffices to show that $[f] \in I_{ccc}$. This, however, follows immediately from the fact that there are $\mathfrak{c}$ many pairwise disjoint translations of $[f]$ of the form $t + [f]$ where $t \in 2^{\mathbb{N}}$ is such that $t(n) = 0$ for every $n \notin A$. □

Remark 3.7. The above quoted result of Kraszewski [10] stating that the inequality

(1) $\mathrm{non}(J) \geq \min(\mathrm{non}(I), \mathrm{cov}^*(I))$

holds for arbitrary proper invariant σ-ideals J and I on $2^{\mathbb{N}}$, provided J is ccc, follows easily from Theorem 2.7.

Indeed, let $2^{\mathbb{N}} = A \cup B$ be a partition into disjoint Borel subsets A and B such that:

(2) $I \subseteq J|A$,
(3) $I \perp J|B$,
(4) $t \in 2^{\mathbb{N}}$ implies $(t + B) \triangle B \in J$ (cf. Theorem 2.7).

In order to prove (1), take $Z \subseteq 2^{\mathbb{N}}$ with $|Z| < \min(\mathrm{non}(I), \mathrm{cov}^*(I))$. Our aim is to show that Z is contained in a set from J or, equivalently, covered by countably many sets from J.

Since $|Z| < \mathrm{non}(I)$, Z is contained in a set $E \in I$. Then by (2),

(5) $Z \cap A \subseteq E \cap A \in J$.

Next by (3), there is a partition $2^{\mathbb{N}} = C \cup D$ into disjoint Borel subsets C and D such that $C \in I$ and $D \in J|B$.

It follows that

(6) $Z \cap B \cap D \subseteq E \cap B \cap D \in J$,

so by (5) and (6), it remains to be proved that $Z \cap B \cap C$ is contained in a set from J.

To see the latter, note that since $|Z| < \mathrm{cov}^*(I)$ and $D \in I^*$, there is $t \in 2^{\mathbb{N}}$ such that $t + Z \subseteq D$. Consequently,

(6) $t + (Z \cap B \cap C) \subseteq \big((t + B) \setminus B\big) \cup (D \cap B)$.

Then by (4), $(t + B) \setminus B \in J$ and $D \in J|B$ means that $D \cap B \in J$. Hence $t+(Z\cap B\cap C)$ is covered by a set from J, so due to the invariance of J, the same is true for $Z \cap B \cap C$, completing the proof.

Recall (cf. [2]) that for a cardinal κ a subset C of a group G ($2^{\mathbb{N}}$ or $\mathbb{R}$) is a *κ-covering* if every subset Y of G of size κ is contained in a translate of C. Marczewski proved that there exists a measure zero first category ω-covering. An example of such a set is given by Komjáth in [8]). Komjáth also proved there that under MA for every $\kappa < \mathfrak{c}$ there exists a measure zero first category κ-covering in $\mathbb{R}$. As another immediate corollary of Theorem 3.1 we obtain the following strengthening of these results in the case of $2^{\mathbb{N}}$.

Corollary 3.8.

(1) If J is an arbitrary invariant ccc σ-ideal on $2^{\mathbb{N}}$, then there exists an ω-covering in $2^{\mathbb{N}}$ which belongs to J.
(2) Under MA if J is an arbitrary invariant ccc σ-ideal on $2^{\mathbb{N}}$, then for every $\kappa < \mathfrak{c}$ there exists a κ-covering in $2^{\mathbb{N}}$ which belongs to J.

Proof. Note that a set $C \subseteq 2^{\mathbb{N}}$ is a κ-covering if and only if $2^{\mathbb{N}}$ cannot be covered by κ-many translates of its complement. It follows that there exists an ω-covering in a σ-ideal J if and only if J is orthogonal to a proper invariant σ-ideal. By Corollary 3.4, the latter condition is true for every invariant ccc σ-ideal J on $2^{\mathbb{N}}$ which proves (1).

Analogically, there exist a κ-covering in a σ-ideal J if and only if J is orthogonal to a proper invariant σ-ideal I with $\mathrm{cov}^*(I) > \kappa$. But under MA we have $\mathrm{cov}^*(\mathscr{N} \otimes \mathscr{M}) = \mathfrak{c}$ hence again by Corollary 3.4, (2) follows. □

In fact, the existence of an ω-covering in every invariant ccc σ-ideal may be further strengthened as follows.

Corollary 3.9. There exist an ω-covering in $2^{\mathbb{N}}$ which belongs to I_{ccc}.

Proof. We closely follow the idea behind the example of an ω-covering presented by Komjáth in [8]). Partition $\mathbb{N}$ into infinitely many pairwise disjoint infinite sets H_n and for every $n \in \mathbb{N}$ let f_n be the function with the domain H_n and of constant value 0. Let $A = \bigcup_{n\in\mathbb{N}} [f_n]$.

Then the fact that A is an ω-covering is proved exactly as in [8]. But $A \in \mathbb{S}_2$, so by Proposition 3.6, $A \in I_{ccc}$. □

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