AN EXAMPLE OF A CAPACITY FOR WHICH ALL POSITIVE BOREL SETS ARE THICK

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Abstract. On the Cantor cube $\{0,1\}^{\mathbb{N}}$ with the standard product topology we construct a finite Choquet capacity with respect to the family of all compact sets such that every compact set of positive capacity contains continuum many pairwise disjoint compact subsets of positive capacity.

Let E be any set and $\mathcal{E}$ be any lattice (a family closed under finite unions and finite intersections) of subsets of E containing the empty set. A *Choquet $\mathcal{E}$-capacity* on E is any function $c : P(E) \to [-\infty, \infty]$ ($P(E)$ denotes here the family of all subsets of E) such that the following three conditions hold (cf. [1]):

(i) $A \subseteq B \subseteq E$ implies $c(A) \le c(B)$;
(ii) if $A_1 \subseteq A_2 \subseteq \ldots$ is any ascending sequence of subsets of E, then $\lim_{n\to\infty} c(A_n) = c(\bigcup_{n=1}^{\infty} A_n)$;
(iii) if $E_1 \supseteq E_2 \supseteq \ldots$ is any descending sequence of subsets from $\mathcal{E}$, then $\lim_{n\to\infty} c(E_n) = c(\bigcap_{n=1}^{\infty} E_n)$.

Capacities play an important role in the theory of general stochastic processes and the main tool for their applications is celebrated Choquet's capacitability theorem that says that for any set B from the σ-δ-lattice $\hat{\mathcal{E}}$ (a σ-δ-lattice is a family closed under countable unions and countable intersections) generated by the family $\mathcal{E}$ the capacity of B can be approximated from below by capacities of subsets of B which are elements of $\mathcal{E}$ (see [1], Theorem 31, Chap. 1).

An element B of the σ-δ-lattice $\hat{\mathcal{E}}$ generated by $\mathcal{E}$ is called *thick* (with respect to a capacity c) if it contains uncountably many pairwise disjoint elements from $\hat{\mathcal{E}}$ of positive capacity c (cf. [1], Definition D9, Chap. 2).

The above definition of a capacity is quite general, and such generality is not always required. Capacities considered on a topological space X are usually non-negative $\mathcal{K}(X)$-capacities, where $\mathcal{K}(X)$ is the family of all compact subsets of X (cf. [3], [2]). For example, this is the case in Kechris-Louveau-Woodin's paper [4] where, moreover, capacities are considered on compact metrizable spaces (then $\hat{\mathcal{E}} = \mathcal{B}(X)$ is the family of all Borel subsets of X). In this setup the authors mention the property (and call it a *strange* one) that every compact set of positive capacity is thick (cf. [4], the remarks following Corollary 8).

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In this note we construct an example of a $\mathcal{K}(X)$-capacity, where X is the Cantor cube such that all Borel sets of positive capacity are thick (Here $\mathcal{E} = \mathcal{K}(X)$, and $\hat{\mathcal{E}} = \mathcal{B}(X)$ is the family of all Borel subsets of X).

Let X be the Cantor cube, i.e. $X = \{0,1\}^{\mathbb{N}}$. Let η denote the measure defined on $\{0,1\}$ by $\eta(\{0\}) = \eta(\{1\}) = \frac{1}{2}$. Let $\Lambda_1, \Lambda_2, \dots$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N}$. Let $\Lambda_i = \{n_{i,j} : j \in \mathbb{N}\}$, where $j < k$ implies $n_{i,j} < n_{i,k}$. We will now define a family of perfect subsets of X, $\{C(\xi^{(1)}, \dots, \xi^{(n)}) : \xi^{(1)}, \dots, \xi^{(n)} \in \{0,1\}^{\mathbb{N}}, n \in \mathbb{N}\}$. Let $C(\emptyset) = X$. For $\xi^{(1)}, \dots, \xi^{(n)} \in \{0,1\}^{\mathbb{N}}$ we define $C(\xi^{(1)}, \dots, \xi^{(n)})$ as

$$C(\xi^{(1)}, \dots, \xi^{(n)}) = \prod_{k \in \mathbb{N}} D_k,$$

where $D_k = \{\xi^{(i)}_j\}$ if $k = n_{i,j} \in \Lambda_i$, $i \leq n$, and $D_k = \{0,1\}$ if $k \notin \bigcup_{i \leq n} \Lambda_i$. Let

$$\nu_n = \prod_{k \notin \bigcup_{i \leq n} \Lambda_i} \eta.$$

Let $A \subseteq C(\xi^{(1)}, \dots, \xi^{(n)})$. Then A may be identified with the set

$$A = \left(\prod_{i \leq n, j \in \mathbb{N}} \{\xi^{(i)}_j\} \right) \times \pi_{X_n}(A),$$

where

$$X_n = \prod_{k \notin \bigcup_{i \leq n} \Lambda_i} \{0,1\} \quad \text{and} \quad \pi_{X_n} : X \to X_n \text{ is the projection of } X \text{ onto } X_n.$$

For $A \subseteq X$, let

$$c(A) = \sup \left\{ \frac{1}{n} \nu_n^*(\pi_{X_n}(A \cap C(\xi^{(1)}, \dots, \xi^{(n)}))) : \xi^{(1)}, \dots, \xi^{(n)} \in \{0,1\}^{\mathbb{N}}, n \in \mathbb{N} \right\}, \tag{1}$$

where ν_n^* denotes the outer measure related to ν_n.

Recall that $\mathcal{K}(X)$ denotes the family of all compact subsets of X.

Theorem 1. *The function $c : P(X) \to [0,1]$ is a non-negative Choquet $\mathcal{K}(X)$-capacity and if B is a Borel subset of X with $c(B) > 0$, then B contains continuum many pairwise disjoint Borel subsets of positive capacity.*

Proof. First, we show that c is a $\mathcal{K}(X)$-Choquet capacity.

It is obvious that $A \subseteq B \subseteq X$ implies $c(A) \leq c(B)$.

Let us now assume that $K_1 \supseteq K_2 \supseteq \dots$ is a sequence of compact subsets of X. As the function c is monotone we have

$$c\left(\bigcap_{n=1}^{\infty} K_n \right) \leq \lim_{n \to \infty} c(K_n).$$

If $\lim_{n\to\infty} c(K_n) = 0$ we have $c(\bigcap_{n=1}^{\infty} K_n) = \lim_{n\to\infty} c(K_n)$.
If $\lim_{n\to\infty} c(K_n) > 0$, then there exists $m \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{m}\nu_m(\pi_{X_m}(K_n \cap C(\xi^{(1,n)}, \ldots, \xi^{(m,n)}))) = \lim_{n\to\infty} c(K_n), \tag{2}$$

for some $\xi^{(1,n)}, \ldots, \xi^{(m,n)} \in \{0,1\}^{\mathbb{N}}$. Passing, if necessary, to a subsequence, we can assume that

$$\lim_{n\to\infty} (\xi^{(1,n)}, \ldots, \xi^{(m,n)}) = (\xi^{(1)}, \ldots, \xi^{(m)}). \tag{3}$$

By (3) and the compactness of the sets K_n, $C(\xi^{(1)}, \ldots, \xi^{(m)})$ and the sets $C(\xi^{(1,n)}, \ldots, \xi^{(m,n)})$, $n \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} \pi_{X_m}(K_n \cap C(\xi^{(1,n)}, \ldots, \xi^{(m,n)})) \subseteq \pi_{X_m}\left(\bigcap_{n=1}^{\infty} K_n \cap C(\xi^{(1)}, \ldots, \xi^{(m)})\right).$$

By the Fatou Lemma and the above inclusion

$$\lim_{n\to\infty} \nu_m(\pi_{X_m}(K_n \cap C(\xi^{(1,n)}, \ldots, \xi^{(m,n)}))) =$$

$$= \lim_{n\to\infty} \int_{X_m} \mathbb{1}_{\pi_{X_m}(K_n \cap C(\xi^{(1,n)}, \ldots, \xi^{(m,n)}))} d\nu_m \leq$$

$$\leq \int_{X_m} \limsup_{n\to\infty} \mathbb{1}_{\pi_{X_m}(K_n \cap C(\xi^{(1,n)}, \ldots, \xi^{(m,n)}))} d\nu_m =$$

$$= \nu_m(\limsup_{n\to\infty} \pi_{X_m}(K_n \cap C(\xi^{(1,n)}, \ldots, \xi^{(m,n)}))) \leq$$

$$\leq \nu_m\left(\pi_{X_m}\left(\bigcap_{n=1}^{\infty} K_n \cap C(\xi^{(1)}, \ldots, \xi^{(m)})\right)\right).$$

This, taking into account (1) and (2), gives

$$\lim_{n\to\infty} c(K_n) \leq c\left(\bigcap_{n=1}^{\infty} K_n\right).$$

Thus finally,

$$\lim_{n\to\infty} c(K_n) = c\left(\bigcap_{n=1}^{\infty} K_n\right).$$

Now let $A_1 \subseteq A_2 \subseteq \ldots$ be any subsets of X. We will prove that

$$\lim_{n\to\infty} c(A_n) = c\left(\bigcup_{n=1}^{\infty} A_n\right). \tag{4}$$

If the right-hand side of (4) is equal to zero, then (4) holds by monotonicity of c. Assume that $c(\bigcup_{n=1}^{\infty} A_n) = a > 0$. Fix $\varepsilon > 0$. Then

$$\frac{1}{m}\nu_m^*\left(\pi_{X_m}\left(\bigcup_{n=1}^{\infty} A_n \cap C(\xi^{(1)},\dots,\xi^{(m)})\right)\right) > a - \varepsilon$$

for some $\xi^{(1)},\dots,\xi^{(m)} \in \{0,1\}^{\mathbb{N}}$, and $m \in \mathbb{N}$. By the continuity of the outer measure ν_m^* under ascending limits we have

$$\lim_{n\to\infty} \nu_m^*(\pi_{X_m}(A_n \cap C(\xi^{(1)},\dots,\xi^{(m)}))) =$$

$$= \nu_m^*\left(\bigcup_{n=1}^{\infty} \pi_{X_m}(A_n \cap C(\xi^{(1)},\dots,\xi^{(m)}))\right) =$$

$$= \nu_m^*\left(\pi_{X_m}\left(\bigcup_{n=1}^{\infty} A_n \cap C(\xi^{(1)},\dots,\xi^{(m)})\right)\right).$$

But this shows that

$$\lim_{n\to\infty} c(A_n) > a - \varepsilon,$$

and (4) follows.

Now let B be any Borel subset of X such that $c(B) > 0$. Hence

$$\nu_n(\pi_{X_n}(B \cap C(\xi^{(1)},\dots,\xi^{(n)}))) > 0,$$

where

$$X_n = \prod_{k \notin \bigcup_{i\le n} \Lambda_i} \{0,1\},$$

for some $\xi^{(1)},\dots,\xi^{(n)} \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. We have

$$\nu_n = \nu_{n+1} \times \prod_{k\in\Lambda_{n+1}} \eta$$

and

$$\nu_n(\pi_{X_n}(B \cap C(\xi^{(1)},\dots,\xi^{(n)}))) =$$

$$= \int_{\prod_{k\in\Lambda_{n+1}}\{0,1\}} \nu_{n+1}(\pi_{X_{n+1}}(B \cap C(\xi^{(1)},\dots,\xi^{(n)},\xi)))d\left(\prod_{k\in\Lambda_{n+1}} \eta\right)(\xi).$$

Thus the set of those $\xi \in \prod_{k\in\Lambda_{n+1}}\{0,1\}$ for which

$$\nu_{n+1}(\pi_{X_{n+1}}(B \cap C(\xi^{(1)},\dots,\xi^{(n)},\xi))) > 0$$

has positive measure. Hence, also the set of those $\xi \in \prod_{k\in\Lambda_{n+1}}\{0,1\}$ for which

$$c(B \cap C(\xi^{(1)},\dots,\xi^{(n)},\xi)) > 0,$$

must have positive measure and thus it must have cardinality $\mathfrak{c}$. □

Using similar methods we can construct the following more general example.

Assume now that $X_1, X_2, \dots$ are compact metrizable perfect spaces and, for each $n \in \mathbb{N}$, μ_n is a probability Borel measure on X_n which vanishes on points.

Let $X = \prod_{i=1}^{n} X_i$. Given a set $A \subseteq X$ and a finite sequence $(x_1, \dots, x_n) \in \prod_{i=1}^{n} X_i$, we set

$$A_{x_1,\dots,x_n} = \left\{ (z_{n+1}, z_{n+2}, \dots) \in \prod_{i=n+1}^{\infty} X_i : \ (x_1, \dots, x_n, z_{n+1}, z_{n+2}, \dots) \in A \right\}.$$

Let $\mathcal{B}^{(n)}$ denote the σ-algebra of Borel sets in $\prod_{i=n+1}^{\infty} X_i$ and let ν_n be the product of the measures $\mu_{n+1}, \mu_{n+2}, \dots$. Then ν_n is a Borel probability measure defined on $\mathcal{B}^{(n)}$ and we may identify ν_n with the product of μ_{n+1} and ν_{n+1}. Let ν_n^* be the outer measure associated with ν_n.

Theorem 2. *The function $c : P(X) \to [0,1]$ defined by letting*

$$c(A) = \sup \left\{ \frac{1}{n} \nu_n^*(A_{x_1,\dots,x_n}) : \ \forall i \le n \ x_i \in X_i, \ n = 1, 2, \dots \right\}$$

is a non-negative Choquet $\mathcal{K}(X)$-capacity and if B is a Borel subset of X with $c(B) > 0$, then B contains continuum many pairwise disjoint Borel subsets of positive capacity.

Sketch of the proof. First, we show that c is a $\mathcal{K}(X)$-Choquet capacity.

It is obvious that $A \subseteq B \subseteq X$ implies $c(A) \le c(B)$.

Let us now assume that $K_1 \supseteq K_2 \supseteq \dots$ are compact subsets of X and $\lim_{n\to\infty} c(K_n) > 0$. Passing to a subsequence, if necessary, we can assume that there exist $m \in \mathbb{N}$ and for each $k \le m$ sequences $(x_{k,n})_{n=1}^{\infty}$ of elements of X_k such that

$$\lim_{n\to\infty} \frac{1}{m} \nu_m((K_n)_{x_{1,n},\dots,x_{m,n}}) = \lim_{n\to\infty} c(K_n),$$

where, moreover, by compactness of X_k, every sequence $(x_{k,n})_{n=1}^{\infty}$ can be assumed to be convergent:

$$\lim_{n\to\infty} x_{k,n} = x_k \text{ for each } k \le m.$$

One can easily verify that

$$\bigcap_{k=1}^{\infty} \bigcup_{i \ge k} (K_i)_{x_{1,i},\dots,x_{m,i}} \subseteq \bigcap_{n=1}^{\infty} (K_n)_{x_1,\dots,x_m}.$$

The rest of the proof is very similar to the proof of the previous theorem. □

Quite often capacities are two-valued. Let us notice that such a capacity can be easily constructed on the space $X = \omega_1 \times [0,1]$, where ω_1 is equipped with the discrete topology (clearly, the space X is metrizable but not separable). Indeed, it is enough to set $c(A) = 1$ if $\{\alpha : \ A \cap (\{\alpha\} \times [0,1]) \ne \emptyset\}$ is uncountable and $c(A) = 0$ otherwise. Notice that each compact set has capacity zero but there are many Borel sets of capacity 1. Moreover, each of them can be partitioned into continuum many Borel sets of capacity 1.

On the other hand, if X is a compact (or even just σ-compact) metrizable space, then no $\mathcal{K}(X)$-capacity with the property that each compact set of positive capacity is thick is two-valued. Indeed, with the help of such a capacity one could construct a transfinite strictly descending sequence of compact sets of length ω_1, contradicting the separability of X.

References

[1] C. Dellacherie, Capacités et Processus Stochastiques, Springer-Verlag, Berlin, Heidelberg, New York 1972.
[2] C. Dellacherie, D. Feyel, G. Mokobodzki, Intégrales de capacités fortement sous-additive, Séminaire de Probabilités 16, 8–28, Lecture Notes in Mathematics 920, Springer 1982.
[3] A.S. Kechris, Classical Descriptive Set Theory, Springer-Verlag, Berlin, Heidelberg, New York 1994.
[4] A.S. Kechris, A. Louveau, W. H. Woodin, The structure of σ-ideals of compact sets, Transactions of the American Mathematical Society, 301 (1987), 263–288.

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