ON ABSOLUTELY BAIRE NONMEASURABLE FUNCTIONS

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Abstract. We shall show that under Martin's Axiom there exist absolutely Baire nonmeasurable additive functions. This provides a Baire category counterpart of an analogous measure-theoretic result of A. B. Kharazishvili.

1. Introduction

Following Kharazishvili [5], given an uncountable set X we shall call a function $f : X \to \mathbb{R}$ *absolutely nonmeasurable* if it is not measurable with respect to (the completion of) any non-zero σ-finite nonatomic measure defined on an arbitrary σ-algebra of subsets of X containing all singletons. Kharazishvili [5, Example 1] (see also [6, Chapter 11]) proved that under Martin's Axiom (**MA**) there exists an absolutely nonmeasurable function $f : \mathbb{R} \to \mathbb{R}$ which is moreover injective and additive, i.e., it is an injective homomorphism of the additive group $\mathbb{R}$ into itself.

The aim of this note is to provide a Baire category counterpart of this result.

We consider only Hausdorff topological spaces.

Given an uncountable set X, we say that a function $f : X \to \mathbb{R}$ is *absolutely Baire nonmeasurable* if it is not Baire measurable with respect to any second countable topology τ on X such that X has no isolated points and is not meager in the topology τ.

The main result of this note is the following theorem.

Theorem 1.1. *Assuming* **MA***, there exists an absolutely Baire nonmeasurable injective additive function* $f : \mathbb{R} \to \mathbb{R}$.

In Section 2 we shall present a proof of this theorem. In Section 3 we shall indicate how to modify it so that to obtain functions with additional strange properties. We shall also make some comments about properties of arbitrary absolutely Baire nonmeasurable functions and discuss questions concerning their characterization and existence.

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2. A proof of Theorem 3.1

Our approach is analogous to that in [5, Example 1], the main difference being that we shall use universally meager sets, a Baire category counterpart of universally null sets employed by Kharazishvili in [5].

Let us recall, cf. [10], that a set $A \subseteq \mathbb{R}$ is *universally null* if it has (outer) measure zero for every σ-finite nonatomic measure defined on the σ-algebra of (relative) Borel subsets of A. On the other hand, one of several equivalent definitions of a *universally meager* set, see [16, Theorem 2.1], states that it is such a set $A \subseteq \mathbb{R}$ which is meager in every second countable topology τ on A such that A has no isolated points and all Borel subsets of A (in the topology inherited from the original topology on $\mathbb{R}$) have the Baire Property in the topology τ. It is well-known that there exist uncountable universally meager sets (cf. [16]).

A key step in our proof is the following observation.

Claim 2.1. *If there exists a universally meager set of size continuum, then there is also one with the structure of a vector space over the field of rationals* $\mathbb{Q}$*.*

To prove the claim, we shall follow closely an argument of Recław (cf. [12, the proof of Theorem 4]). Let $Z \subseteq \mathbb{R}$ be a perfect set linearly independent over $\mathbb{Q}$, whose existence is guaranteed by a theorem of von Neumann [15] (cf. [4, 19.2]). Since there exists a universally meager set in $\mathbb{R}$ of size continuum and the image of a universally meager set under any Borel isomorphism is universally meager, let us fix a universally meager set $X \subseteq Z$ with $|X| = \mathfrak{c}$. Since the product of two universally meager sets is universally meager, cf. [16, Theorem 2.2], for any natural number $n > 0$ the set X^n is universally meager and so is the set

$$X^{(n)} = \{(x_1, \ldots, x_n) \in X^n : \ x_i < x_j \quad \text{for} \quad i < j \leq n\}.$$

Its image under the function

$$f_\tau(x_1, \ldots, x_n) = \sum_{i=1}^{n} q_i x_i,$$

where $\tau = (q_1, \ldots, q_n)$ is any sequence of non-zero rationals, is also universally meager, f_τ being continuous and injective on $Z^{(n)}$.

Finally, if A is the vector space over $\mathbb{Q}$ spanned by X, then it is universally meager as the (countable) union of all sets of the form $f_\tau(X^{(n)})$ and $\{0\}$. This completes the proof of the claim.

Having verified the claim, we shall follow Kharazishvili's construction of an absolutely nonmeasurable injective additive function, cf. [5].

First, let us recall that under **MA** there exists a universally meager set of size continuum. One way to see this is to appeal to a result of Grzegorek [2] and [3] who proved the existence of a universally meager subset of $\mathbb{R}$ of size equal to the smallest cardinality of a non-meager set in $\mathbb{R}$ (which under **MA** equals $\mathfrak{c}$, see [14, T057])).

Using the claim, we can fix a universally meager set A of size $\mathfrak{c}$ having the structure of a vector space over $\mathbb{Q}$.

Let $f : \mathbb{R} \to A$ be the isomorphism of the respective vector spaces over $\mathbb{Q}$. The function f is clearly injective and additive. We shall prove that it is absolutely Baire nonmeasurable.

Striving for a contradiction, we assume that f is Baire measurable with respect to a topology τ on $\mathbb{R}$ which makes $\mathbb{R}$ a second countable topological space X with no isolated points and not meager in the topology τ. By passing to a dense G_δ-set G in X such that $f|G$ is continuous and then deleting from G, if necessary, first the (countable) union of all meager basic open sets in G and next possibly some countably many points, we obtain a Baire space $Y \subseteq X$ (recall that this means that $Y \neq \emptyset$ and no non-empty relatively open subset of Y is meager in Y) with no isolated points in the topology inherited from X. Moreover, the map $f|Y : Y \to A$ is continuous.

A contradiction is now reached by appealing to the following property of universally meager sets (see [17, Lemma 2.1]).

Lemma 2.2. *If $A \subseteq \mathbb{R}$ is universally meager, then for every second countable Baire space Y and continuous map $f : Y \to \mathbb{R}$, if the fibers $f^{-1}(x)$ of all $x \in A$ are meager in Y, then the preimage of A is meager in Y.*

This completes the proof of Theorem 3.1. □

3. Additional comments

3.1. **Absolutely Baire nonmeasurable functions with additional properties.** If **MA** holds, then there exists a subset of $\mathbb{R}$ of size $\mathfrak{c}$ which is both universally meager and universally null. By a theorem of Plewik [11], an example of such a set (which lives in the space of all subsets of natural numbers with the topology of the Cantor space but is easily transferable to $\mathbb{R}$ by a Borel isomorphism) is a tower of length $\mathfrak{c}$. Since the classes of universally null and universally meager sets are closed under finite products and Borel isomorphisms, the proof of Theorem 3.1 actually provides an example of an injective additive function $f : \mathbb{R} \to \mathbb{R}$ which is both absolutely Baire nonmeasurable and absolutely nonmeasurable.

Recall that $f : \mathbb{R} \to \mathbb{R}$ is called a *Sierpiński-Zygmund function* (or shortly: an SZ-function) if no restriction of f to a set of size $\mathfrak{c}$ is continuous (the first example of such a function was given by Sierpiński and Zygmund in [13]). Kharazishvili [7, Theorem 3] proved that if $A \subseteq \mathbb{R}$ is a vector space over $\mathbb{Q}$ of size $\mathfrak{c}$, then there is an injective and additive SZ-function $f : \mathbb{R} \to A$. Combining this with the preceding remarks which allowed us to assume, under **MA**, that A is, moreover, both universally meager and universally null, we obtain the following result.

Theorem 3.1. *Assuming* **MA***, there exists an injective additive absolutely nonmeasurable and absolutely Baire nonmeasurable SZ-function acting from* $\mathbb{R}$ *to* $\mathbb{R}$.

Let us, however, note that an analogous refinement of the proof of [7, Theorem 2] shows that under **MA** there also exists an injective additive absolutely Baire nonmeasurable and absolutely nonmeasurable function which is constant on a set of size $\mathfrak{c}$.

3.2. **An "SZ-like" property of absolutely Baire nonmeasurable functions.** Though, as we pointed out in the preceding paragraph, an absolutely Baire nonmeasurable function may (e.g., under **MA**) not be an SZ-function, it always enjoys an "SZ-like" property.

Proposition 3.2. *If a function* $f : \mathbb{R} \to \mathbb{R}$ *is absolutely Baire nonmeasurable, then no restriction of* f *to an uncountable Borel set is a Borel mapping.*

Proof. Striving for a contradiction suppose that B is a Borel set in $\mathbb{R}$ such that $f|B$ is a Borel mapping. Shrinking B, if necessary, we can assume that $|\mathbb{R} \setminus B| = \mathfrak{c}$. Let $\phi : \mathbb{R} \to \overline{B(0,1)}$ be a bijection between $\mathbb{R}$ and the closed unit ball in $\mathbb{R}^2$ such that $\phi|B : B \to B(0,1)$ is a Borel isomorphism between B and the open ball $B(0,1)$. With the help of ϕ we can turn $\mathbb{R}$ into a topological space X homeomorphic to $\overline{B(0,1)}$, with ϕ being the witnessing homeomorphism. Note that $f|B$ viewed as a mapping from B, treated as a subspace of X, remains Borel and its domain B is comeager in X. It follows that $f : X \to \mathbb{R}$ has the Baire Property, contradicting the assumption that f was absolutely Baire nonmeasurable. □

With the help of the Luzin theorem on restricting measurable functions to continuous ones (see [4, 17.12]) we obtain the following.

Corollary 3.3. *Every absolutely Baire nonmeasurable function is not measurable with respect to (the completion of) any non-zero* σ*-finite nonatomic Borel measure on* $\mathbb{R}$.

3.3. **A characterization of absolutely Baire nonmeasurable functions.** Let us recall that by a result of Kharazishvili and Kirtadze [8], a function acting from $\mathbb{R}$ to $\mathbb{R}$ is absolutely nonmeasurable if and only if its range is universally null and all its fibers are at most countable.

The analogous characterization of absolutely Baire nonmeasurable functions in terms of the cardinalities of their fibers and universally meager sets is somewhat more complicated.

Proposition 3.4. *Let κ be the minimal cardinality of a second countable topological space with no isolated points and not meager in itself.*

A function acting from $\mathbb{R}$ to $\mathbb{R}$ is absolutely Baire nonmeasurable if and only if its range is universally meager and all its fibers have cardinalities less than κ.

Proof. Let us fix a function $f : \mathbb{R} \to \mathbb{R}$.

First let us assume that the range of f is universally meager and all its fibers have cardinalities less than κ. Let us suppose that f is not absolutely Baire nonmeasurable, i.e., f is Baire measurable with respect to a topology τ on $\mathbb{R}$ which makes $\mathbb{R}$ a second countable topological space X with no isolated points which is not meager in the topology τ. But then, all the fibers of f having size less than κ and hence being meager in any second countable space with no isolated points, a contradiction is reached by appealing to Lemma 2.2 in exactly the same way as in the last part of the proof of Theorem 3.1.

To prove the converse implication, we shall use the following simple observation.

Claim 3.5. *For any infinite cardinal $\lambda \leq \mathfrak{c}$ there exists a separable metrizable space with no isolated points and meager in itself of size λ.*

Indeed, if $\lambda = \aleph_0$, then we can just take $Z = \mathbb{Q}$. Assuming that $\aleph_0 < \lambda \leq \mathfrak{c}$ let X_1 and X_2 be two dense-in-itself subsets of $\mathbb{R}$ such that X_1 is a universally meager (hence meager in itself) set of cardinality $\aleph_1$ and X_2 has cardinality λ. Then the product space $X_1 \times X_2$ has the required properties.

We shall now consider two cases.

Case 1. $|f^{-1}(a_0)| \geq \kappa$ for a certain $a_0 \in f(\mathbb{R})$.

Then let $Y \subseteq f^{-1}(a_0)$ be such that $|Y| = \kappa$ and $|\mathbb{R} \setminus Y| \geq \aleph_0$. We can define a space X as the direct sum of Y equipped with a topology that makes it a second countable topological space with no isolated points which is not meager in itself and $\mathbb{R} \setminus Y$ equipped with a separable metrizable topology which makes it a topological space with no isolated points and meager in itself (cf. Claim 3.5). It is evident that the space X is second countable, has no isolated points and is not meager in itself. The set Y being comeager in X, f is Baire measurable as a function acting from the space X to $\mathbb{R}$.

Case 2. The set $f(\mathbb{R})$ is not universally meager.

Let us fix a set $A \subseteq f(\mathbb{R})$ such that $|f(\mathbb{R}) \setminus A| = \aleph_0$ and a second countable topology τ on A such that A has no isolated points, is not meager in τ and all Borel subsets of A (in the topology inherited from the original topology on $\mathbb{R}$) have the Baire Property in the topology τ.

Let S be a selector of the collection of fibers $\{f^{-1}(a) : a \in A\}$. Then $f|S$ is a bijection from S onto A and $|\mathbb{R} \setminus S| \geq \aleph_0$, the set $f^{-1}(f(\mathbb{R}) \setminus A)$

being infinite and disjoint from S. Let us now define a space X as the direct sum of S equipped essentially with the topology τ (transferred to it by $f|S$) and $\mathbb{R} \setminus S$ equipped with a separable metrizable topology which makes it a topological space with no isolated points and meager in itself (cf. Claim 3.5). Again it is evident that the space X is second countable, has no isolated points and is not meager in itself. It is now easy to see that f is Baire measurable as a function acting from the space X to $\mathbb{R}$. Indeed, if $U \subseteq \mathbb{R}$ is open in $\mathbb{R}$, then $U \cap A$ has the Baire Property in the topology τ hence $(f|S)^{-1}(U)$ has the Baire Property in the space S and so in the space X as well. But S being comeager in X, the set $f^{-1}(U)$ differs from $(f|S)^{-1}(U)$ by a meager set in X, so it has the Baire Property in X as well.

□

To indicate the symmetry between the above characterizations of the two notions of nonmeasurability, related to measure and category, respectively, let us note that $\aleph_1$ is the smallest cardinality of the domain of a non-zero, σ-finite, nonatomic measure defined on a arbitrary σ-algebra of subsets of this domain, containing all its singletons. On the other hand, however, the cardinal κ from the formulation of Proposition 3.4 can be bigger than $\aleph_1$. In particular, it is well-known that under **MA** it is equal $\mathfrak{c}$ (see [14, T057]).

A direct consequence of Proposition 3.4 is the following equivalence.

Corollary 3.6. *There exists an absolutely Baire nonmeasurable function acting from $\mathbb{R}$ to $\mathbb{R}$ if and only if there exists a universally meager set of size $\mathfrak{c}$.*

Proof. Only "the only if" part requires an argument. So assume that no universally meager set has size $\mathfrak{c}$. Let κ be the cardinal from the formulation of Proposition 3.4. By a result of Grzegorek [2] and [3], it follows that there exists a non-meager subset of $\mathbb{R}$ of size less than $\mathfrak{c}$, which readily implies that $\kappa < \mathfrak{c}$. Now, if $f : \mathbb{R} \to \mathbb{R}$, then either the range of f is not universally meager or else its cardinality is less than $\mathfrak{c}$ but then the cardinalities of its fibers cannot be jointly bounded by κ. By Proposition 3.4, in either case the function f is not absolutely Baire nonmeasurable.

□

Since by a result of Miller [9], the statement that there exists a universally meager set of size $\mathfrak{c}$ is consistently false, it is consistent with **ZFC** that absolutely Baire nonmeasurable functions acting from $\mathbb{R}$ to $\mathbb{R}$ do not exist.

As noted by Kharazishvili [7], another natural assumption which immediately refutes the existence of absolutely nonmeasurable functions acting from $\mathbb{R}$ to $\mathbb{R}$ is that $\mathfrak{c}$ is a real-valued measurable cardinal (indeed, any real-valued function defined on $\mathbb{R}$ is measurable with respect to a σ-finite nonatomic measure defined on the σ-algebra of all subsets of $\mathbb{R}$). An attempt to refute the existence of absolutely Baire nonmeasurable functions using a symmetric argument is, however, bound to fail, since by a theorem of Fremlin [1, 7E], there is no second countable topological space X with no isolated points such that X is not meager but all subsets of X have the Baire Property.

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