A CHARACTERIZATION OF THE MEAGER IDEAL

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Abstract. We give a classical proof of the theorem stating that the σ-ideal of meager sets is the unique σ-ideal on a Polish group, generated by closed sets which is invariant under translations and ergodic.

1. Introduction

The σ-ideal $\mathscr{M}$ of meager subsets of $\mathbb{R}$ has the following remarkable properties:

- $\mathscr{M}$ is generated by closed sets,
- $\mathscr{M}$ satisfies the countable chain condition (ccc),
- $\mathscr{M}$ is invariant under translations,
- $\mathscr{M}$ is $\mathbb{Q}$-ergodic, i.e., every $\mathbb{Q}$-invariant Borel subset of $\mathbb{R}$ is either meager or comeager.

These properties are interrelated and conjunctions of some of them characterize $\mathscr{M}$.

Balcerzak and Rogowska [1] and, independently (using a different method), Recław and Zakrzewski [4] proved that if a σ-ideal $\mathcal{I}$ on a Polish space X is generated by closed sets and ccc, then it is Borel isomorphic to $\mathscr{M}$. Both proofs are based on a deep theorem by Kechris and Solecki [3, Theorem 3] which provides a characterization of those σ-ideals on Polish spaces which are generated by closed sets and fulfil ccc. As a corollary, Kechris and Solecki [3] also showed that the σ-ideal of meager sets on a Polish group is the unique σ-ideal generated by closed sets which is invariant under translations and ccc.

Zapletal (see [7]) in turn proved that if a σ-ideal on $\mathbb{R}$ (respectively, on a Polish space X) is generated by closed sets and $\mathbb{Q}$-ergodic (respectively, ergodic; see Section 2 for a general definition of ergodicity), then it is ccc.

Combining the last two statements we arrive at the following characterization of the σ-ideal of meager sets on Polish groups.

Theorem 1.1. The σ-ideal of meager sets on a Polish group G is the unique σ-ideal on G which is generated by closed sets, invariant under translations by elements of G and ergodic.

2010 *Mathematics Subject Classification.* 03E15, 54H05.

Key words and phrases. Polish group, σ-ideal, meager sets.

This research was partially supported by MNiSW Grant Nr N N201 543638.

The original Zapletal's proof of the fact that ergodicity of a σ-ideal which is generated by closed sets implies countable chain condition used forcing (see [5, Lemma 1.3] or [6, Lemma 5.4.2]). The aim of this note is to give a "classical" proof of this result.

2. Preliminaries

Throughout the paper X (more precisely: (X,τ)) is an uncountable Polish (i.e., a separable, completely metrizable) topological space. The σ-algebra of Borel subsets of X is denoted by $\mathscr{B}(X)$.

By a *σ-ideal* $\mathcal{I}$ on X we understand a collection of subsets of X, closed under countable unions and such that for any $A \in \mathcal{I}$, all subsets of A are in $\mathcal{I}$. Throughout the paper we assume that $X \notin \mathcal{I}$ and $\mathcal{I}$ contains all singletons.

We say that a σ-ideal $\mathcal{I}$ on X is *generated by closed sets* if there is a family $\mathcal{F} \subseteq \mathcal{I}$ consisting of sets closed in X such that each element of $\mathcal{I}$ can be covered by countably many elements of $\mathcal{F}$.

Given a σ-ideal $\mathcal{I}$ on X we shall use the following notation and terminology:

- $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$,
- $A_1,\ A_2 \in \mathscr{B}(X) \setminus \mathcal{I}$ are *almost disjoint* if $A_1 \cap A_2 \in \mathcal{I}$,
- A family $\mathcal{A} \subseteq \mathscr{B}(X) \setminus \mathcal{I}$ is almost disjoint if it consists of pairwise almost disjoint sets.

A σ-ideal $\mathcal{I}$ on X is:

- *ccc* if it satisfies the countable chain condition, i.e., if there is no uncountable almost disjoint family $\mathcal{A} \subseteq \mathscr{B}(X) \setminus \mathcal{I}$,
- *ergodic* if there is a countable Borel equivalence relation R on X such every set $B \in \mathscr{B}(X)$ which is the union of a family of R-equivalence classes is either in $\mathcal{I}$ or in $\mathcal{I}^*$,
- *invariant* (under translations) if $(X, \cdot)$ is a Polish group and $x \cdot A \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and $x \in X$.

3. Classical proofs of Zapletal's results

We start with the following lemma which in the forcing terminology is closely related to the fact that "forcing with a σ-ideal generated by closed sets does not collapse $\aleph_1$" (cf. [5] and [7]).

Lemma 3.1 (Main Lemma)**.** Assume that $\mathcal{I}$ is a σ-ideal on X generated by closed sets. Let $\langle \mathcal{A}_n : n \in \mathbb{N}\rangle$ be a sequence of maximal almost disjoint subfamilies of $\mathscr{B}(X)\setminus\mathcal{I}$. Then there exists a set $E \in \mathscr{B}(X)\setminus\mathcal{I}$ such that for every $n \in \mathbb{N}$ we have

$$|\{A \in \mathcal{A}_n :\ E \cap A \notin \mathcal{I}\}| \leq \aleph_0.$$

Proof. For each $n \in \mathbb{N}$, using the maximality of $\mathcal{A}_n$, fix a function $\psi_n : \mathscr{B}(X) \to \mathscr{B}(X)$ such that

$$B \in \mathscr{B}(X) \setminus \mathcal{I} \Rightarrow \big(\psi_n(B) \in \mathcal{A}_n \wedge B \cap \psi_n(B) \notin \mathcal{I}\big)$$

and let $\varphi_n : \mathscr{B}(X) \to \mathscr{B}(X)$ be the function defined by

$$\varphi_n(B) = B \cap \psi_n(B) \quad \text{for} \quad B \in \mathscr{B}(X).$$

Note that if $B \in \mathscr{B}(X) \setminus \mathcal{I}$, then $\varphi_n(B) \notin \mathcal{I}$ and $\psi_n(B)$ is the only $A \in \mathcal{A}_n$ such that $\varphi_n(B) \cap A \notin \mathcal{I}$.

Recall that τ is the topology of X and let (U_n) be a countable basis of (X, τ).

Sublemma 3.2. There exist a field $\mathscr{C}$ of subsets of X, a Polish topology $\bar{\tau}$ extending τ and a countable base $\mathscr{V}$ of $\bar{\tau}$ satisfying the following conditions:

(1) $\mathscr{C} \subseteq \mathscr{B}(X)$,
(2) $\mathscr{C}$ is countable,
(3) $\varphi_n(B) \in \mathscr{C}$ for every $B \in \mathscr{C}$ and $n \in \mathbb{N}$,
(4) $\mathscr{C} \subseteq \bar{\tau}$,
(5) $\mathscr{V} \subseteq \mathscr{C}$.

Proof of Sublemma 3.2. We construct inductively fields $\mathscr{C}_n$ of subsets of X and Polish topologies τ_n on X with associated countable bases $\mathscr{V}_n$, $n \in \mathbb{N}$, so that:

- $n > 0$ implies τ_n is zero-dimensional,
- $\mathscr{V}_n \subseteq \mathscr{C}_n$,
- $\mathscr{C}_n \subseteq clop(X, \tau_n)$, the field of clopen subsets of (X, τ_n),
- $n < m$ implies $\mathscr{C}_n \subseteq \mathscr{C}_m$,
- $n < m$ implies $\tau_n \subseteq \tau_m$,
- $\mathscr{C}_n$ is countable,
- $B \in \mathscr{C}_n$ and $m \in \mathbb{N}$ implies $\varphi_m(B) \in \mathscr{C}_{n+1}$,

Let $\mathscr{C}_0$ be the field of subsets of X generated by $\{U_k : k \in \mathbb{N}\}$ and $\mathscr{V}_0 = \{U_k : k \in \mathbb{N}\}$.

If $\mathscr{C}_n$, τ_n and $\mathscr{V}_n$ have been defined, let

$$\mathscr{R}_{n+1} = \mathscr{C}_n \cup \bigcup_{m \in \mathbb{N}} \varphi_m[\mathscr{C}_n]$$

and extend τ_n to a Polish zero-dimensional topology τ_{n+1} on X such that $\mathscr{R}_{n+1} \subseteq clop(X, \tau_{n+1})$. Then let $\mathscr{V}_{n+1}$ be a countable base of τ_{n+1} consisting of sets clopen in τ_{n+1}. Finally, let $\mathscr{C}_{n+1}$ be the field of subsets of X generated by $\mathscr{R}_{n+1} \cup \mathscr{V}_{n+1}$. This completes the construction.

Now let $\mathscr{C} = \bigcup_{n \in \mathbb{N}} \mathscr{C}_n$ and let $\bar{\tau}$ be the topology generated by $\bigcup_{n \in \mathbb{N}} \tau_n$.

The topology $\bar{\tau}$ is Polish and finite intersections of elements of $\bigcup_{n \in \mathbb{N}} \mathscr{V}_n$ form a countable base $\mathscr{V}$ of $\bar{\tau}$ (cf. [2, Lemma 13.3]). $\mathscr{C}$ being closed under finite intersections, we have $\mathscr{V} \subseteq \mathscr{C}$.

It is easy to see that $\mathscr{C}$, $\bar{\tau}$ and $\mathscr{V}$ satisfy conditions (1)–(5) above which completes the proof of Sublemma 3.2.

□

Continuing the proof of Main Lemma enumerate $\mathscr{V}$ as $\{V_k : k \in \mathbb{N}\}$ and let
$$D = X \setminus \bigcup \{V_k : k \in \mathbb{N} \text{ and } V_k \in \mathcal{I}\}.$$

Note that

- $D \in \mathcal{I}^*$,
- D is closed in $\bar{\tau}$, so uncountable Polish in the relative topology; *in the rest of the proof all topological notions concerning subsets of D will refer, unless stated otherwise, to this topology*,
- no nonempty open subset of D is in $\mathcal{I}$,
- if P is closed in D and $P \in \mathcal{I}$, then P is nowhere dense in D.

For every $n \in \mathbb{N}$ let
$$\mathcal{O}_n = D \cap \bigcup_{k \in \mathbb{N}} \varphi_n(V_k).$$

We claim that each $\mathcal{O}_n$ is open and dense in D.

To see that $\mathcal{O}_n$ is open, use (5), (3) and (4).

To prove that $\mathcal{O}_n$ is dense in D, take a basic open subset od D of the form $V_k \cap D \neq \emptyset$.

Then $V_k \in \mathscr{C} \setminus \mathcal{I}$ hence $\varphi_n(V_k) \in \mathscr{C} \setminus \mathcal{I}$.

Consequently, $\varphi_n(V_k)$ being a member of $\mathscr{C}$ is $\bar{\tau}$-open and $\varphi_n(V_k) \cap D \neq \emptyset$ since $D \in \mathcal{I}^*$.

But $\varphi_n(V_k) \subseteq V_k$ and $D \cap \varphi_n(V_k) \subseteq \mathcal{O}_n$ which implies that
$$(V_k \cap D) \cap \mathcal{O}_n \supseteq V_k \cap (D \cap \varphi_n(V_k)) = \varphi_n(V_k) \cap D \neq \emptyset,$$
completing the proof that $\mathcal{O}_n$ is dense in D.

Finally, let
$$E = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n.$$

To complete the proof of Main Lemma it suffices to prove the following

Claim

(6) $E \notin \mathcal{I}$,
(7) $\forall n \quad \{A \in \mathcal{A}_n : \ E \cap A \notin \mathcal{I}\} \subseteq \{\psi_n(V_k) : k \in \mathbb{N}\}$.

To prove (6), we shall use the fact that $\mathcal{I}$ is generated by closed sets. So let (D_n) be a sequence of τ-closed sets from $\mathcal{I}$. Our aim is to show that
$$E \not\subseteq \bigcup_{n \in \mathbb{N}} D_n.$$

Note that:

- $E = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n$ is a dense G_δ subset od D.
- Each D_n being τ-closed is also closed in $\bar{\tau}$, so $D_n \cap D$ is closed nowhere dense in D.

By the Baire category theorem, we are done.

To prove (7), recall that for each n:

- if $B \in \mathscr{B}(X) \setminus \mathcal{I}$, then $\psi_n(B)$ is the only $A \in \mathcal{A}_n$ such that $\varphi_n(B) \cap A \notin \mathcal{I}$.
- $E = \bigcap_{m\in\mathbb{N}} \mathcal{O}_m \subseteq \mathcal{O}_n = D \cap \bigcup_{k\in\mathbb{N}} \varphi_n(V_k) \subseteq \bigcup_k \varphi_n(V_k)$.

Fix n and let $A \in \mathcal{A}_n$ be such that $E \cap A \notin \mathcal{I}$. Then there is k with $\varphi_n(V_k) \cap A \notin \mathcal{I}$. But the only $A \in \mathcal{A}_n$ with this property is $A = \psi_n(V_k)$ which shows (7) and completes the proof of Main Lemma.

□

With the help of Main Lemma we are now ready to finish our proof of Zapletal's theorem (cf. [5, Lemma 1.3] and [6, Lemma 5.4.2]).

Theorem 3.3 (Zapletal)**.** If a σ-ideal $\mathcal{I}$ on X is generated by closed sets and ergodic, then $\mathcal{I}$ is ccc.

Proof. Recall that ergodicity of $\mathcal{I}$ means that there is a countable Borel equivalence relation R on X such that every set $B \in \mathscr{B}(X)$ which is the union of a family of R-equivalence classes is either in $\mathcal{I}$ or in $\mathcal{I}^*$.

By the Feldman–Moore theorem, R is the orbit equivalence relation for a certain countable group $G = \{g_n : n \in \mathbb{N}\}$ of Borel automorphisms of X.

So, ergodicity of $\mathcal{I}$ means that

$$B \in \mathscr{B}(X) \setminus \mathcal{I} \;\Rightarrow\; \bigcup_n g_n B \in \mathcal{I}^*.$$

Suppose that $\mathcal{I}$ is *not* ccc and let $\{A_\alpha : \alpha < \omega_1\}$ be a disjoint family of sets in $\mathscr{B}(X) \setminus \mathcal{I}$.

For each n let

$$\mathcal{A}_n = \{g_n A_\alpha : \alpha < \omega_1\} \setminus \mathcal{I}.$$

$\mathcal{A}_n$ is a disjoint (perhaps empty) collection of sets in $\mathscr{B}(X) \setminus \mathcal{I}$ hence by Main Lemma, there is $E \in \mathscr{B}(X) \setminus \mathcal{I}$ such that

$$\forall n \quad |\{\alpha < \omega_1 :\ E \cap g_n A_\alpha \notin \mathcal{I}\}| \leq \aleph_0. \tag{$*$}$$

On the other hand, by ergodicity, for every $\alpha < \omega_1$ there is $n \in \mathbb{N}$ such that

$$E \cap g_n A_\alpha \notin \mathcal{I},$$

so there is a single $n \in \mathbb{N}$ with

$$|\{\alpha < \omega_1 :\ E \cap g_n A_\alpha \notin \mathcal{I}\}| = \aleph_1,$$

contradicting $(*)$.

□

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