Analytic Tableau Systems for Propositional Bimodal Logics of Knowledge and Belief *

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Abstract. We give sound and complete analytic tableau systems for the propositional bimodal logics $\mathbb{KB}$, $\mathbb{KB}_{\text{-}C}$, $\mathbb{KB}_{\text{-}5}$, and $\mathbb{KB}_{\text{-}5C}$. These logics have two universal modal operators $\mathbb{K}$ and $\mathbb{B}$, where $\mathbb{K}$ stands for knowing and $\mathbb{B}$ stands for believing. The logic $\mathbb{KB}$ is a combination of the modal logic $S5$ (for $\mathbb{K}$) and $KD45$ (for $\mathbb{B}$) with the interaction axioms $I : \mathbb{K}\,\phi \to \mathbb{B}\,\phi$ and $C : \mathbb{B}\,\phi \to \mathbb{K}\,\mathbb{B}\,\phi$. The logics $\mathbb{KB}_{\text{-}C}$, $\mathbb{KB}_{\text{-}5}$, $\mathbb{KB}_{\text{-}5C}$ are obtained from $\mathbb{KB}$ respectively by deleting the axiom C (for $\mathbb{KB}_{\text{-}C}$), the axioms 5 (for $\mathbb{KB}_{\text{-}5}$), and both of the axioms C and 5 (for $\mathbb{KB}_{\text{-}5C}$). As analytic sequent-like tableau systems, our calculi give simple decision procedures for reasoning about both knowledge and belief in the mentioned logics.

1 Introduction

Modal logics can be used to reason about knowledge and belief. Between basic modal logics, the logic $S5$ (resp. $KD45$) is the most suitable for reasoning about knowledge (resp. belief). Both of these logics have the axioms $4 : \Box\phi \to \Box\Box\phi$ and $5 : \neg\Box\phi \to \Box\neg\Box\phi$, which mean that knowledge and belief satisfy positive and negative introspection. Furthermore, the logic $S5$ has the axiom $T : \Box\phi \to \phi$, which means that knowledge is veridical, while $KD45$ has the axiom $D : \Box\phi \to \neg\Box\neg\phi$, which means that belief is consistent. These logics can be combined to reason about both knowledge and belief. To distinguish the modalities, we use $\mathbb{K}$ and $\mathbb{B}$ as the modal operators standing respectively for knowing and believing. As interaction axioms between $\mathbb{K}$ and $\mathbb{B}$, one can adopt $I : \mathbb{K}\,\phi \to \mathbb{B}\,\phi$ and $C : \mathbb{B}\,\phi \to \mathbb{K}\,\mathbb{B}\,\phi$. The axiom[1] I means that knowledge is understood to be stronger than belief, and the axiom C means that one is conscious about one's beliefs. This combined logic, denoted by $\mathbb{KB}$, is discussed in [18] by Hoek and Meyer.

Analytic tableau systems have been developed for all 15 basic modal logics, i.e. the ones obtained from the logic K by adding any combination of the

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[1] The axioms I and C are named by us.

axioms D, T, B, 4, 5 (see the works by Fitting [6], Goré [10], Massacci [14], and ours [15]). As some recent works devoted to developing tableau systems for modal logics about knowledge or belief, there are the works by Fitting et al [7], Rosati [17], Baldoni et al [2], and Wooldridge et al [20]. Besides, tableau algorithms have been also developed for description logics, which are a family of knowledge representation formalisms (see the overview by Baader and Sattler [1]). The mentioned works [7, 17, 2, 20], however, do not suit to reasoning about both knowledge and belief: the multimodal logics studied in [7, 20] have modalities of the same type, the grammar logics considered in [2] have modalities of different types but do not contain the axioms D and 5.

In this paper, basing on known tableau systems for the monomodal logics $S5$, $KD45$, $S4$, and $KD4$, we develop analytic sequent-like tableau systems for the logic $\mathbb{KB}$ and three other bimodal logics $\mathbb{KB}_{-5}$, $\mathbb{KB}_{-C}$, and $\mathbb{KB}_{-5C}$. These latter logics are obtained from $\mathbb{KB}$ by deleting the axioms 5 – for $\mathbb{KB}_{-5}$, the axiom C – for $\mathbb{KB}_{-C}$, and both of the axioms 5 and C – for $\mathbb{KB}_{-5C}$. We choose these logics because $\mathbb{KB}$ is a popular modal logic for reasoning about both knowledge and belief, $\mathbb{KB}_{-5}$ eliminates some undesirable properties of $\mathbb{KB}$ (e.g. $\mathbb{B}\,\mathbb{K}\,\phi \to \phi$), and $\mathbb{KB}_{-C}$, $\mathbb{KB}_{-5C}$ are interesting from the theoretical point of view as combinations of the component logics with the only interaction axiom I. Our tableau systems are sound and complete. As analytic sequent-like tableau systems, they give simple set-based backtracking decision procedures for the considered logics. Using the systems, we show that the satisfiability problem in $\mathbb{KB}_{-5}$ and $\mathbb{KB}_{-5C}$ is PSPACE-complete, in $\mathbb{KB}$ and $\mathbb{KB}_{-C}$ is NP-complete.

In [3], Beckert and Gabbay developed a method for fibring tableaux. They described how to uniformly construct a sound and complete tableau calculus for the combined logic from "well-behaved" calculi for the component logics. In this paper, we do not apply the method as there are interaction axioms between $\mathbb{K}$ and $\mathbb{B}$, and the calculi considered in [3] are labeled tableau systems. The combination of modal logics has gained a lot of attention in the past years (see e.g. [12, 19, 5, 8, 13, 9, 4]). The logics considered in this paper are fusions of the component logics (with some interaction axioms). In the way of [4], the logic $\mathbb{KB}_{-C}$ is denoted by $S5 \oplus_{\subseteq} KD45$, and $\mathbb{KB}_{-5C}$ by $S4 \oplus_{\subseteq} KD4$.

The formulation of our systems is based on the work by Goré [10]. We use a similar technique to prove completeness of the systems. To show completeness of a calculus $\mathcal{C}L$ we give an algorithm that, given a finite $\mathcal{C}L$-consistent formula set X, constructs an L-model graph that satisfies every one of its formulae at the corresponding world.

2 Preliminaries

2.1 Definitions for Bimodal Logics

A modal formula, hereafter simply called a *formula*, is defined by the following rules: any primitive proposition p is a formula, $\perp$ is a formula (which stands for falsity), and if ϕ and ψ are formulae then so are $\neg\phi$, $\phi \wedge \psi$, $\mathbb{K}\,\phi$, and $\mathbb{B}\,\phi$. The modal operator $\mathbb{K}$ stands for knowing and $\mathbb{B}$ for believing. We write $\phi \vee \psi$ and $\phi \rightarrow \psi$ to denote the shortened forms of $\neg(\neg\phi \wedge \neg\psi)$ and $\neg(\phi \wedge \neg\psi)$, respectively.

We use small letters p and q to denote primitive propositions, Greek letters like ϕ and ψ to denote formulae, and block letters like X, Y, Z to denote formula sets. A *classical literal* is either a primitive proposition or negation of a primitive proposition (i.e. $\neg p$ for some p).

A (bimodal) Kripke *frame* is a tuple $\langle W, \tau, R, S\rangle$, where W is a nonempty set of possible worlds, $\tau \in W$ is the actual world, R and S are binary relations on W called accessibility relations. If $R(w, u)$ (resp. $S(w, u)$) holds, then we say that the world u is accessible (or reachable) from u via R (resp. S).

A (bimodal) Kripke *model* is a tuple $\langle W, \tau, R, S, h\rangle$, where $\langle W, \tau, R, S\rangle$ is a Kripke frame and h is a function that maps each world of W to a set of primitive propositions.

Given some Kripke model $M = \langle W, \tau, R, S, h\rangle$ and some $w \in W$, the satisfaction relation $M, w \vDash \phi$ is defined recursively as follows.

$$
\begin{array}{ll}
M, w \nvDash \bot & \\
M, w \vDash p & \text{iff } \ p \in h(w); \\
M, w \vDash \neg\phi & \text{iff } \ M, w \nvDash \phi; \\
M, w \vDash \phi \wedge \psi & \text{iff } \ M, w \vDash \phi \text{ and } M, w \vDash \psi; \\
M, w \vDash \mathbb{K}\,\phi & \text{iff } \ \text{for all } u \in W \text{ such that } R(w, u),\ M, u \vDash \phi; \\
M, w \vDash \mathbb{B}\,\phi & \text{iff } \ \text{for all } u \in W \text{ such that } S(w, u),\ M, u \vDash \phi.
\end{array}
$$

We say that M *satisfies* ϕ *at* w, and ϕ is *satisfied at* w *in* M iff $M, w \vDash \phi$. If $M, \tau \vDash \phi$ then we say that M *satisfies* ϕ and ϕ is *satisfied in* M. If M satisfies ϕ then we also call M a *model of* ϕ.

The axiom schemata mentioned in the introduction are mirrored by certain properties of the accessibility relations. The axioms T, D, 4 and 5 correspond respectively to reflexivity (of R), seriality (of S), transitivity and euclideaness[2] (of R and S). The axiom $I : \mathbb{K}\,\phi \rightarrow \mathbb{B}\,\phi$ corresponds to the property $S \subseteq R$, while $C : \mathbb{B}\,\phi \rightarrow \mathbb{K}\,\mathbb{B}\,\phi$ corresponds to the condition $\forall x, y, z\ R(x, y) \wedge S(y, z) \rightarrow S(x, z)$. The classes of admissible interpretations for the logics $\mathbb{KB}_{-5C}$, $\mathbb{KB}_{-5}$, $\mathbb{KB}_{-C}$, and $\mathbb{KB}$ are specified by the frame restrictions given in Table 1. For L being one of these logics, we refer to the restrictions as *L-frame restrictions.*

We call a model M an *L-model* if the accessibility relations of M satisfy all L-frame restrictions. We say that ϕ is *L-satisfiable* if there exists an L-model of ϕ. A formula ϕ is *L-valid* if it is satisfied in every L-model.

The logics $\mathbb{KB}_{-5C}$, $\mathbb{KB}_{-5}$, $\mathbb{KB}_{-C}$, and $\mathbb{KB}$ are axiomatized by the standard axioms for the classical propositional logic, the *modus ponens* inference rule, plus the axioms mentioned in the Introduction, the K-axioms: $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$, and the necessitation rules: $\vdash \phi \Longrightarrow \vdash \Box\phi$, where $\Box$ can be either $\mathbb{K}$ or $\mathbb{B}$. It can be shown that for L being one of the considered logics, a formula is provable by the axiomatization system of L iff it is L-valid (see [18] for $\mathbb{KB}$). In this paper we use only the fact that all axioms of L are L-valid.

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[2] i.e. $\forall x, y, z\ R(x, y) \wedge R(x, z) \rightarrow R(y, z)$, and similarly for S

Logic	Frame Restrictions
$\mathbb{KB}_{\text{-}5C}$	R is reflexive and transitive, S is serial and transitive, $S \subseteq R$
$\mathbb{KB}_{\text{-}5}$	the conditions as for $\mathbb{KB}_{\text{-}5C}$ plus $\forall x, y, z\ R(x, y) \wedge S(y, z) \rightarrow S(x, z)$
$\mathbb{KB}_{\text{-}C}$	R is reflexive, transitive, and euclidean, S is serial, transitive, and euclidean, $S \subseteq R$
$\mathbb{KB}$	the conditions as for $\mathbb{KB}_{\text{-}C}$ plus $\forall x, y, z\ R(x, y) \wedge S(y, z) \rightarrow S(x, z)$

Table 1. Frame restrictions for the logics $\mathbb{KB}_{\text{-}5C}$, $\mathbb{KB}_{\text{-}5}$, $\mathbb{KB}_{\text{-}C}$, and $\mathbb{KB}$.

2.2 Definitions for Tableau Systems

Our tableau formulation is adopted from the work by Goré [10], which in turn is related to the ones by Hintikka [11] and Rautenberg [16]. A number of terms and notations used in this work are borrowed from Goré [10].

A *tableau rule* δ consists of a numerator N above the line and a (finite) list of denominators D_1, D_2, ..., D_k (below the line) separated by vertical bars.

$$\frac{N}{D_1 \mid D_2 \mid \ \dots \ \mid D_k}$$

The numerator is a finite formula set and so is each denominator. As we shall see later, each rule is read downwards as "if the numerator is L-satisfiable, then so is one of the denominators". The numerator of each tableau rule contains one or more distinguished formulae called the *principal formulae*.

A *tableau system* (or *calculus*) $\mathcal{C}L$ is a finite set of tableau rules.

A $\mathcal{C}L$-tableau for X is a tree with root X whose nodes carry finite formula sets. A tableau rule with numerator N is applicable to a node carrying a set Y if Y is an instance of N. The steps for extending a tableau are:

- choose a leaf node n carrying Y where n is not an end node (defined below), and choose a rule δ which is applicable to n;
- if δ has k denominators then create k successors nodes for n, with successor i carrying an appropriate instance of denominator D_i;
- all with the proviso that if a successor s carries a set Z and Z has already appeared on the branch from the root to s then s is an *end node*.

Let Δ be a set of tableau rules. We say that Y is *obtainable from* X *by applications of rules from* Δ if there exists a tableau for X which uses only rules from Δ and has a node that carries Y.

A branch in a tableau is *closed* if it ends with $\perp$. A tableau is *closed* if every its branch is closed. A tableau is *open* if it is not closed. A finite formula set X

is said to be $\mathcal{C}L$-*consistent* if every $\mathcal{C}L$-tableau for X is open. If there is a closed $\mathcal{C}L$-tableau for X then we say that X is $\mathcal{C}L$-*inconsistent*.

A tableau system $\mathcal{C}L$ is said to be *sound* if for any finite formula set X, if X is L-satisfiable then X is $\mathcal{C}L$-consistent. A tableau system $\mathcal{C}L$ is said to be *complete* if for any finite formula set X, if X is $\mathcal{C}L$-consistent then X is L-satisfiable.

Let δ be one of the rules of $\mathcal{C}L$. We say that δ is sound wrt. L if for any instance δ' of δ, if the numerator of δ' is L-satisfiable then so is one of the denominators of δ'. It is clear that if $\mathcal{C}L$ contains only rules sound wrt. L then $\mathcal{C}L$ is sound.

3 Tableau Systems for the Logics $\mathbb{KB}_{\mathbf{5}C}$, $\mathbb{KB}_{\mathbf{5}}$, $\mathbb{KB}_{C}$, $\mathbb{KB}$

By a sequence $\phi_1; \phi_2; \ldots; \phi_n$ we denote the set $\{\phi_1, \phi_2, \ldots, \phi_n\}$. If X and Y are formula sets and ϕ is a formula, then by $X; Y$ (resp. $X; \phi$) we denote the set $X \cup Y$ (resp. $X \cup \{\phi\}$). We use $\mathbb{K}\, X$ to denote the set $\{\mathbb{K}\,\phi \mid \phi \in X\}$. The sets $\mathbb{B}\, X$ and $\neg X$ are defined similarly. If X is a formula set, then by (X) we denote the formula being the conjunction of formulae of X. Note that $\neg\mathbb{K}\,\neg(X)$ is different to $\neg\mathbb{K}\,\neg X$. In Table 2, we give tableau rules and calculi for the logics $\mathbb{KB}_{5C}$, $\mathbb{KB}_5$, $\mathbb{KB}_C$, and $\mathbb{KB}$.

Following Goré [10], we categorize each rule either as a *static rule* or as a *transitional rule*. The intuition behind this sorting is that in the static rules, the numerator and the denominators represent the same world (in the same model), whereas in the transitional rules, the numerator and the denominator represent different worlds (in the same model). In our systems thinning is built into the transitional rules, whereas in [10] Goré uses an explicit thinning rule.

The meaning of the static rules $(\bot)$, $(\neg)$, $(\wedge)$, $(\vee)$, (T_K), and (D_B) is quite clear. For intuition and soundness of the transitional rules, see the proof of Lemma 1. The transitional rule (4_K) is often used in tableau systems for the monomodal logics $K4$, $KD4$, and $S4$, while the transitional rule (45_K) is often used in tableau systems for the monomodal logics $K45$, $KD45$, and $S5$ (see [10]).

The rule $(4I_B)$ is just a simple modification of the rule (4_K) for dealing with the modal operator $\mathbb{B}$. It uses the axiom $I : \mathbb{K}\,\phi \to \mathbb{B}\,\phi$. The rule $(4C_K)$ differs from $(4I_B)$ in the point that Z does not occur in the denominator. The reason is that the modal operator $\mathbb{B}$ is weaker than $\mathbb{K}$. In the transitional rule $(4C_K)$, the presence of $\mathbb{B}\, Z$ in the denominator is due to the axiom $C : \mathbb{B}\,\psi \to \mathbb{K}\,\mathbb{B}\,\psi$. Similar things can be said about the transitional rules $(45I_B)$ and $(45C_K)$ in comparison with the rule (45_K).

The static rule (T'_K) plays a similar role as the rule (π) in Fitting's tableau system $\mathcal{C}S5\pi$ for the logic $S5$ (see [6, 10]). As we will see, we may apply this rule only once, before applying transitional rules. The rule (T''_K) is similar to (T'_K), but it cannot be replaced by (T'_K) for $\mathcal{C}\mathbb{KB}_C$ because that in the rule (45_K) formulae of the form $\mathbb{B}\,\psi$ or $\neg\mathbb{B}\,\psi$ are not transferred from the numerator to the denominator as in the rule $(45C_K)$ of $\mathcal{C}\mathbb{KB}$. The presence of $\neg\mathbb{K}\,\neg(Y; \mathbb{B}\, U; \neg\mathbb{B}\, V)$ in the denominator of (T''_K) guarantees that we can reconsider the content of the current world in the future when having more information.

$$(\bot)\ \frac{X;\phi;\neg\phi}{\bot} \qquad (\neg)\ \frac{X;\neg\neg\phi}{X;\phi} \qquad (\wedge)\ \frac{X;\phi\wedge\psi}{X;\phi;\psi} \qquad (\vee)\ \frac{X;\neg(\phi\wedge\psi)}{X;\neg\phi \mid X;\neg\psi}$$

$$(T_K)\ \frac{X;\mathbb{K}\,\phi}{X;\mathbb{K}\,\phi;\phi} \qquad (D_B)\ \frac{X}{X;\neg\mathbb{B}\,\bot}$$

$$(T'_K)\ \frac{X;Y}{X;Y;\neg\mathbb{K}\,\neg(Y)} \qquad (T''_K)\ \frac{X;Y;\mathbb{B}\,U;\neg\mathbb{B}\,V}{X;Y;\mathbb{B}\,U;\neg\mathbb{B}\,V;\neg\mathbb{K}\,\neg(Y;\mathbb{B}\,U;\neg\mathbb{B}\,V)}$$

where in the rules (T'_K) and (T''_K), Y is a set of classical literals

$$(4I_B)\ \frac{X;\mathbb{K}\,Y;\mathbb{B}\,Z;\neg\mathbb{B}\,\phi}{\mathbb{K}\,Y;Y;\mathbb{B}\,Z;Z;\neg\phi} \qquad (45I_B)\ \frac{X;\mathbb{K}\,Y;\mathbb{B}\,Z;\neg\mathbb{K}\,U;\neg\mathbb{B}\,V;\neg\mathbb{B}\,\phi}{\mathbb{K}\,Y;Y;\mathbb{B}\,Z;Z;\neg\mathbb{K}\,U;\neg\mathbb{B}\,V;\neg\mathbb{B}\,\phi;\neg\phi}$$

$$(4_K)\ \frac{X;\mathbb{K}\,Y;\neg\mathbb{K}\,\phi}{\mathbb{K}\,Y;Y;\neg\phi} \qquad (45_K)\ \frac{X;\mathbb{K}\,Y;\neg\mathbb{K}\,U;\neg\mathbb{K}\,\phi}{\mathbb{K}\,Y;Y;\neg\mathbb{K}\,U;\neg\mathbb{K}\,\phi;\neg\phi}$$

$$(4C_K)\ \frac{X;\mathbb{K}\,Y;\mathbb{B}\,Z;\neg\mathbb{K}\,\phi}{\mathbb{K}\,Y;Y;\mathbb{B}\,Z;\neg\phi} \qquad (45C_K)\ \frac{X;\mathbb{K}\,Y;\mathbb{B}\,Z;\neg\mathbb{K}\,U;\neg\mathbb{B}\,V;\neg\mathbb{K}\,\phi}{\mathbb{K}\,Y;Y;\mathbb{B}\,Z;\neg\mathbb{K}\,U;\neg\mathbb{B}\,V;\neg\mathbb{K}\,\phi;\neg\phi}$$

Calculus	Static Rules	Transitional Rules
$\mathcal{C}\mathbb{KB}_{\text{-}5C}$	$(\bot),(\neg),(\wedge),(\vee),(T_K),(D_B)$	$(4I_B),(4_K)$
$\mathcal{C}\mathbb{KB}_{\text{-}5}$	$(\bot),(\neg),(\wedge),(\vee),(T_K),(D_B)$	$(4I_B),(4C_K)$
$\mathcal{C}\mathbb{KB}_{\text{-}C}$	$(\bot),(\neg),(\wedge),(\vee),(T_K),(T''_K),(D_B)$	$(45I_B),(45_K)$
$\mathcal{C}\mathbb{KB}$	$(\bot),(\neg),(\wedge),(\vee),(T_K),(T'_K),(D_B)$	$(45I_B),(45C_K)$

Table 2. Tableau systems for the logics $\mathbb{KB}_{\text{-}5C}$, $\mathbb{KB}_{\text{-}5}$, $\mathbb{KB}_{\text{-}C}$, and $\mathbb{KB}$.

Example 1. Suppose that we want to prove $(\mathbb{K}\,p \wedge (\mathbb{B}\,p \to \mathbb{K}\,q)) \to q$ in $\mathbb{KB}$, where $\to$ has the usual meaning. Denote the formula by ϕ. By Lemma 1 given below, $\mathcal{C}\mathbb{KB}$ is sound, hence it suffices to show that there exists a closed $\mathcal{C}\mathbb{KB}$-tableau for $\neg\phi = (\neg q;\mathbb{K}\,p;\neg(\mathbb{B}\,p\wedge\neg\mathbb{K}\,q))$. Here is such a tableau:

$$(\vee)\ \cfrac{\neg q;\mathbb{K}\,p;\neg(\mathbb{B}\,p\wedge\neg\mathbb{K}\,q)}{(45I_B)\ \cfrac{\neg q;\mathbb{K}\,p;\neg\mathbb{B}\,p}{(\bot)\ \cfrac{\neg q;\mathbb{K}\,p;p;\neg\mathbb{B}\,p;\neg p}{\bot}} \ \Big|\ (\neg)\ \cfrac{\neg q;\mathbb{K}\,p;\neg\neg\mathbb{K}\,q}{(T_K)\ \cfrac{\neg q;\mathbb{K}\,p;\mathbb{K}\,q}{(\bot)\ \cfrac{\neg q;\mathbb{K}\,p;\mathbb{K}\,q;q}{\bot}}}}$$

We write $Sf(\phi)$ to denote the set of all subformulae of ϕ, and by $Sf(X)$ we denote the set $\bigcup_{\phi\in X} Sf(\phi)$. By $\widetilde{X}$ we denote the set $Sf(X)\cup\neg Sf(X)\cup\{\bot\}$. By $Ss(X)$ we denote the set of formulae of the form (Y) with $Y\subseteq X$.

A tableau system $\mathcal{C}L$ has the *analytic superformula* property iff to every finite set X we can assign a finite set $X^*_{\mathcal{C}L}$ such that $X^*_{\mathcal{C}L}$ contains all formulae that may appear in any tableau for X. Our systems have the analytic superformula property, with $X^*_{\mathcal{C}L} = \widetilde{X}$ for $L\in\{\mathbb{KB}_{\text{-}5C},\mathbb{KB}_{\text{-}5}\}$, and $X^*_{\mathcal{C}L} = \widetilde{X}\cup\neg\mathbb{K}\,\neg Ss(\widetilde{X})\cup\neg\neg Ss(\widetilde{X})$ for $L\in\{\mathbb{KB}_{\text{-}C},\mathbb{KB}\}$.

Lemma 1. *The calculi* $\mathbb{KB}_{5C}$, $\mathbb{KB}_{5}$, $\mathbb{KB}_{C}$, *and* $\mathbb{KB}$ *are sound.*

Proof. We show that $\mathcal{C}L$ contains only rules sound wrt. L, where L is $\mathbb{KB}_{5C}$, $\mathbb{KB}_{5}$, $\mathbb{KB}_{C}$, or $\mathbb{KB}$. Suppose that the numerator of the considered rule is satisfied at a world w in a model $M = \langle W, \tau, R, S, h\rangle$. It can be shown that if the considered rule is static, then one of the denominators is satisfied at w, else the only denominator is satisfied at some world reachable from w (via R or S). Nontrivial cases are when the considered rule is one of the transitional rules.

Consider the rule (4_K) of $\mathcal{C}\mathbb{KB}_{5C}$. We have $M, w \models X; \mathbb{K}\,Y; \neg\mathbb{K}\,\phi$. Since $M, w \models \neg\mathbb{K}\,\phi$, there exists u such that $R(w, u)$ and $M, u \models \neg\phi$. Since R is transitive and $M, w \models \mathbb{K}\,Y$, we have $M, u \models \mathbb{K}\,Y; Y$. Hence the denominator of the rule is satisfied at u.

Consider the rule $(4I_B)$ of $\mathcal{C}\mathbb{KB}_{5C}$ and $\mathcal{C}\mathbb{KB}_{5}$. We have $M, w \models X; \mathbb{K}\,Y; \mathbb{B}\,Z; \neg\mathbb{B}\,\phi$. Since $M, w \models \neg\mathbb{B}\,\phi$, there exists u such that $S(w, u)$ and $M, u \models \neg\phi$. We have $M, u \models Y; Z$. Since $M, w \models \mathbb{K}\,Y; \mathbb{B}\,Z$, by the axioms 4, we have $M, w \models \mathbb{K}\,\mathbb{K}\,Y; \mathbb{B}\,\mathbb{B}\,Z$, and by the axiom I, $M, w \models \mathbb{B}\,\mathbb{K}\,Y; \mathbb{B}\,\mathbb{B}\,Z$. Hence $M, u \models \mathbb{K}\,Y; \mathbb{B}\,Z$, and the denominator of the rule is satisfied at u.

Consider the rule $(4C_K)$ of $\mathcal{C}\mathbb{KB}_{5}$. We have $M, w \models X; \mathbb{K}\,Y; \mathbb{B}\,Z; \neg\mathbb{K}\,\phi$. Since $M, w \models \neg\mathbb{K}\,\phi$, there exists u such that $R(w, u)$ and $M, u \models \neg\phi$. Since R is transitive and $M, w \models \mathbb{K}\,Y$, it follows that $M, u \models \mathbb{K}\,Y; Y$. For any world v such that $S(u, v)$, by the property $\forall x, y, z\; R(x, y) \wedge S(y, z) \to S(x, z)$, we have $S(w, v)$ (since $R(w, u)$), hence $M, v \models Z$ (since $M, w \models \mathbb{B}\,Z$). Therefore $M, u \models \mathbb{B}\,Z$, and the denominator of the rule is satisfied at u.

Consider the rule (45_K) of $\mathcal{C}\mathbb{KB}_{C}$. We have $M, w \models X; \mathbb{K}\,Y; \neg\mathbb{K}\,U; \neg\mathbb{K}\,\phi$. Since $M, w \models \neg\mathbb{K}\,\phi$, there exists u such that $R(w, u)$ and $M, u \models \neg\phi$. Similarly as for the rule (4_K), we have $M, u \models \mathbb{K}\,Y; Y$. Since $M, w \models \neg\mathbb{K}\,U; \neg\mathbb{K}\,\phi$, by the axiom 5, we have $M, w \models \mathbb{K}\,\neg\mathbb{K}\,U; \mathbb{K}\,\neg\mathbb{K}\,\phi$. Hence $M, u \models \neg\mathbb{K}\,U; \neg\mathbb{K}\,\phi$. Therefore the denominator of the rule is satisfied at u.

Consider the rule $(45I_B)$ of $\mathcal{C}\mathbb{KB}_{C}$ and $\mathcal{C}\mathbb{KB}$. We have $M, w \models X; \mathbb{K}\,Y; \mathbb{B}\,Z; \neg\mathbb{K}\,U; \neg\mathbb{B}\,V; \neg\mathbb{B}\,\phi$. Since $M, w \models \neg\mathbb{B}\,\phi$, there exists u such that $S(w, u)$ and $M, u \models \neg\phi$. Analogously as for the rule $(4I_B)$, we derive $M, u \models \mathbb{K}\,Y; Y; \mathbb{B}\,Z; Z$. Since $M, w \models \neg\mathbb{K}\,U; \neg\mathbb{B}\,V; \neg\mathbb{B}\,\phi$, by the axioms 5, we have $M, w \models \mathbb{K}\,\neg\mathbb{K}\,U; \mathbb{B}\,\neg\mathbb{B}\,V; \mathbb{B}\,\neg\mathbb{B}\,\phi$. By the axiom I, it follows that $M, w \models \mathbb{B}\,\neg\mathbb{K}\,U$. Hence $M, u \models \neg\mathbb{K}\,U; \neg\mathbb{B}\,V; \neg\mathbb{B}\,\phi$, and the denominator of the rule is satisfied at u.

Finally, consider the rule $(45C_K)$ of $\mathcal{C}\mathbb{KB}$. We have $M, w \models X; \mathbb{K}\,Y; \mathbb{B}\,Z; \neg\mathbb{K}\,U; \neg\mathbb{B}\,V; \neg\mathbb{K}\,\phi$. Since $M, w \models \neg\mathbb{K}\,\phi$, there exists u such that $R(w, u)$ and $M, u \models \neg\phi$. Analogously as for the rule $(45I_B)$, we derive $M, u \models \mathbb{K}\,Y; Y; \neg\mathbb{K}\,U; \neg\mathbb{K}\,\phi$. Since $M, w \models \mathbb{B}\,Z$, similarly as for the rule $(4C_K)$, we have $M, u \models \mathbb{B}\,Z$. Since $M, w \models \neg\mathbb{B}\,V$, by the axiom T, $M, w \models \neg\mathbb{K}\,\mathbb{B}\,V$. Hence, by the axiom 5, $M, w \models \mathbb{K}\,\neg\mathbb{K}\,\mathbb{B}\,V$, and by the axiom I, $M, w \models \mathbb{B}\,\neg\mathbb{K}\,\mathbb{B}\,V$. Hence $M, u \models \neg\mathbb{K}\,\mathbb{B}\,V$, and by the axiom C, $M, u \models \neg\mathbb{B}\,V$. Therefore the denominator of the rule is satisfied at u.

4 Completeness of the Calculi

We prove completeness of our calculi via model graphs in a similar way as Rautenberg [16] and Goré [10] do for their systems. Let L denote one of the logics $\mathbb{KB}_{5C}$, $\mathbb{KB}_{5}$, $\mathbb{KB}_{C}$, $\mathbb{KB}$; and $\mathcal{C}L$ denote the corresponding calculus. To show completeness of $\mathcal{C}L$ we give an algorithm that, given a finite $\mathcal{C}L$-consistent formula set X, constructs an L-model graph (defined in Section 4.2) for X that satisfies every one of its formulae at the corresponding world.

4.1 Saturation

In the rules $(\neg)$, $(\wedge)$, $(\vee)$, the principal formula does not occur in the denominators. For δ being one of these rules, let δ' denote the rule obtained from δ by adding the principal formula to each of the denominators. Let $\mathcal{SC}L$ denote the set of static rules of $\mathcal{C}L$ with $(\neg)$, $(\wedge)$, $(\vee)$ replaced by $(\neg')$, $(\wedge')$, $(\vee')$. Note that for any rule of $\mathcal{SC}L$ except $(\bot)$, the numerator is included in each of the denominators.

For X being a finite $\mathcal{C}L$-consistent formula set, a formula set Y is called a *$\mathcal{C}L$-saturation of X* if Y is a maximal $\mathcal{C}L$-consistent set obtainable from X by applications of the rules of $\mathcal{SC}L$.

A set X is *closed wrt. a tableau rule* if, whenever the rule is applicable to X, one of the corresponding instances of the denominators is equal to X.

As stated by the following lemma, $\mathcal{C}L$-saturations have the same nature as "downward saturated sets" defined in the works by Hintikka [11] and Goré [10].

Lemma 2. *Let X be a finite $\mathcal{C}L$-consistent formula set and Y a $\mathcal{C}L$-saturation of X. Then $X \subseteq Y \subseteq X^*_{\mathcal{C}L}$ and Y is closed wrt. the rules of $\mathcal{SC}L$. Furthermore, there is an effective procedure that, given a finite $\mathcal{C}L$-consistent formula set X, constructs some $\mathcal{C}L$-saturation of X.*

Proof. It is clear that $X \subseteq Y \subseteq X^*_{\mathcal{C}L}$. Observe that if a rule of $\mathcal{SC}L$ is applicable to Y, then one of the corresponding instances of the denominators is $\mathcal{C}L$-consistent. In fact, in a sequence of applications of the rules of $\mathcal{SC}L$, the second application of a rule with the same principal formula does not change the formula set; and when applying a transitional rule, formulae that may be principal formulae of static rules are ignored. Since Y is a $\mathcal{C}L$-saturation (of X), Y is closed wrt. the rules of $\mathcal{SC}L$.

We construct a $\mathcal{C}L$-saturation of X as follows: let $Y = X$; while there is some rule δ of $\mathcal{SC}L$ applicable to Y with the property that one of the corresponding instances of the denominators, denoted by Z, is $\mathcal{C}L$-consistent and strictly contains Y, set $Y = Z$.

At each iteration, $Y \subset Z \subseteq X^*_{\mathcal{C}L}$. Hence the above process always terminates. It is clear that the resulting set Y is a $\mathcal{C}L$-saturation of X.

4.2 Proving Completeness Via Model Graphs

A *model graph* is a tuple $\langle W, \tau, R, S, H\rangle$, where $\langle W, \tau, R, S\rangle$ is a Kripke frame, and H is a function that maps each world to a formula set. We sometimes treat model graphs as models with $h(w)$ being $H(w)$ restricted to the set of primitive propositions. A model graph is called an *L-model graph* if its frame satisfies all L-frame restrictions. A model graph $\langle W, \tau, R, S, H\rangle$ is said to be *saturated* if it satisfies the following conditions for every $w \in W$:

- if $\neg\neg\phi \in H(w)$, then $\phi \in H(w)$;
- if $\phi \wedge \psi \in H(w)$, then $\{\phi, \psi\} \subseteq H(w)$;
- if $\neg(\phi \wedge \psi) \in H(w)$, then $\neg\phi \in H(w)$ or $\neg\psi \in H(w)$;
- if $\mathbb{K}\,\phi \in H(w)$, then for every u such that $R(w, u)$, $\phi \in H(u)$;
- if $\neg\mathbb{K}\,\phi \in H(w)$, then there exists $u \in W$ such that $R(w, u)$ and $\neg\phi \in H(u)$;
- if $\mathbb{B}\,\phi \in H(w)$, then for every u such that $S(w, u)$, $\phi \in H(u)$;
- if $\neg\mathbb{B}\,\phi \in H(w)$, then there exists $u \in W$ such that $S(w, u)$ and $\neg\phi \in H(u)$.

A saturated model graph is said to be *consistent* if every one of its worlds does not contain $\perp$ nor a pair of formulae ϕ and $\neg\phi$. We use the term "model graph" merely to denote a data structure, while in the definition by Rautenberg's model graphs are required to be saturated and consistent.

Lemma 3. *If $M = \langle W, \tau, R, S, H\rangle$ is a consistent saturated model graph, then M is a model of $H(\tau)$.*

This lemma can be proved by induction on the construction of ϕ that if $\phi \in H(w)$ then $M, w \vDash \phi$.

Given a finite $\mathcal{C}L$-consistent set X, as an L-model for X we construct a consistent saturated L-model graph $M = \langle W, \tau, R, S, H\rangle$ such that $H(\tau) \to X$ is L-valid. If for every $w \in W$, $H(w)$ is a $\mathcal{C}L$-saturation of some set, then the first three conditions of being a *saturated* model graph are satisfied. The constructions given in the next subsections differ by frame properties of the constructed model graphs: for both $\mathbb{KB}_C$ and $\mathbb{KB}$, $R = W \times W$; and additionally for $\mathbb{KB}$, $S = W \times W_B$ for some $W_B \subset W$, $\tau \notin W_B$.

4.3 Completeness of $\mathcal{C}\mathbb{KB}_{5C}$ and $\mathcal{C}\mathbb{KB}_5$

In this subsection, let L denote either $\mathbb{KB}_{5C}$ or $\mathbb{KB}_5$. In the following algorithm, the worlds of the constructed model graph are marked either as *unresolved* or as *resolved*.

Algorithm 1
Input: a finite $\mathcal{C}L$-consistent set X of formulae.
Output: an L-model graph $M = \langle W, \tau, R, S, H\rangle$ satisfying X.

1. Let $W = \{\tau\}$, $R_0 = S_0 = \emptyset$, and $H(\tau)$ be a $\mathcal{C}L$-saturation of X. Mark τ as unresolved.

2. While there are unresolved worlds, take one, denoted by w, and do the following:
   (a) For every formula $\neg\mathbb{B}\,\phi$ in $H(w)$:
      i. Let Y be the result of the application of the rule $(4I_B)$ to $H(w)$, i.e.
$$Y = \{\neg\phi\} \cup \{\psi \mid \mathbb{K}\,\psi \in H(w) \text{ or } \mathbb{B}\,\psi \in H(w) \text{ or } \psi \text{ is of the form } \mathbb{K}\,\zeta \text{ or } \mathbb{B}\,\zeta\}.$$
      and let Z be a $\mathcal{C}L$-saturation of Y.
      ii. If there exists a world $u \in W$ such that $H(u) = Z$, then add the pair (w, u) to S_0. Otherwise, add a new world w_ϕ with content Z to W, mark it as unresolved, and add the pair (w, w_ϕ) to S_0.
   (b) For every formula $\neg\mathbb{K}\,\phi$ in $H(w)$:
      i. Let δ be the rule (4_K) if $L = \mathbb{KB}_{5C}$, or $(4C_K)$ if $L = \mathbb{KB}_5$. Let Y be the result of the application of δ to $H(w)$, i.e.
$$Y = \{\neg\phi\} \cup \{\psi \mid \mathbb{K}\,\psi \in H(w) \text{ or } \psi \text{ is of the form } \mathbb{K}\,\zeta\} \cup \{\mathbb{B}\,\zeta \mid \mathbb{B}\,\zeta \in H(w) \text{ and } L = \mathbb{KB}_5\}$$
      and let Z be a $\mathcal{C}L$-saturation of Y.
      ii. If there exists a world $u \in W$ such that $H(u) = Z$, then add the pair (w, u) to R_0. Otherwise, add a new world w_ϕ with content Z to W, mark it as unresolved, and add the pair (w, w_ϕ) to R_0.
   (c) Mark w as resolved.
3. Let R and S be respectively the least extensions of R_0 and S_0 such that $\langle W, \tau, R, S\rangle$ is an L-frame.

This algorithm always terminates because H is one-to-one, and for any world w, $H(w) \subseteq X^*_{\mathcal{C}L}$.

Lemma 4. *Let X be a finite $\mathcal{C}L$-consistent formula set and $M = \langle W, \tau, R, S, H\rangle$ be the model graph constructed by the above algorithm for X. Then M is a consistent saturated L-model graph satisfying X.*

Proof. It is clear that M is an L-model graph and for any $w \in W$, $H(w)$ is $\mathcal{C}L$-consistent. We now show that M is a saturated model graph. For this aim it is sufficient to show that for every $w \in W$,

– if $\mathbb{K}\,\phi \in H(w)$, then for every u such that $R(w, u)$, $\phi \in H(u)$;
– if $\mathbb{B}\,\phi \in H(w)$, then for every u such that $S(w, u)$, $\phi \in H(u)$.

The first assertion follows from the observation that for any $x \in W$, if $\mathbb{K}\,\phi \in H(x)$ then $\phi \in H(x)$ (by the rule (T_K)) and for any $y \in W$ such that $R_0(x, y)$ or $S_0(x, y)$, $\{\mathbb{K}\,\phi, \phi\} \subseteq H(y)$.

Consider the second assertion. Note that for any $x, y \in W$, if $\mathbb{B}\,\phi \in H(x)$ and $S_0(x, y)$ then $\{\mathbb{B}\,\phi, \phi\} \subseteq H(y)$. This implies the second assertion for the case $L = \mathbb{KB}_{5C}$. The assertion also holds for the case $L = \mathbb{KB}_5$ because that for any $x, y \in W$ with $\mathbb{B}\,\phi \in H(x)$, if $S_0(x, y)$ then $\{\mathbb{B}\,\phi, \phi\} \subseteq H(y)$, if $R_0(x, y)$ then $\mathbb{B}\,\phi \in H(y)$.

Since $X \subseteq H(\tau)$, by Lemma 3, M satisfies X.

The following theorem follows from Lemmas 1 and 4.

Theorem 1. *The calculi $\mathcal{C}\mathbb{KB}_{-5C}$ and $\mathcal{C}\mathbb{KB}_{-5}$ are sound and complete.*

Let L still denote $\mathbb{KB}_{-5C}$ or $\mathbb{KB}_{-5}$. By the above theorem, a formula set X is L-satisfiable iff every $\mathcal{C}L$-tableau for X has an unclosed branch. Thus checking L-satisfiability can be reduced to searching an and-or tree. It can be shown that the depths of $\mathcal{C}L$-tableaux for X as well as the sizes of formula sets carried by nodes of the search tree are bounded by a polynomial in the size of X. Hence the satisfiability problem in $\mathbb{KB}_{-5C}$ and $\mathbb{KB}_{-5}$ is decidable in PSPACE, and then is PSPACE-complete (since the satisfiability problem in $S4$ is PSPACE-complete).

4.4 Completeness of $\mathcal{C}\mathbb{KB}$

Let $X_\#$ denote the set $\{\phi \in X \mid \phi \text{ is of the form } \mathbb{K}\,\psi, \mathbb{B}\,\psi, \neg\mathbb{K}\,\psi, \text{ or } \neg\mathbb{B}\,\psi\}$. Note that if Y is obtainable from X by applications of the rules of $\mathcal{C}\mathbb{KB}$, then $X_\# \subseteq Y_\#$. Given a finite $\mathcal{C}\mathbb{KB}$-consistent set X of formulae, in order to construct a $\mathbb{KB}$-model graph for X, we first repeatedly apply the rules of $\mathcal{C}\mathbb{KB}$ to X until obtaining a set Y such that for any set Y' obtainable from Y by applications of the rules of $\mathcal{C}\mathbb{KB}$, $Y'_\# = Y_\#$.

For δ being one of the transitional rules of $\mathcal{C}\mathbb{KB}$, X being a formula set, and $\phi \in X$ being an instance of the principal formula of δ, let $apply(\delta, X, \phi)$ denote the result of the application of δ to X with ϕ being the principal formula, i.e.

$$apply((45I_B), X, \neg\mathbb{B}\,\phi) = \{\neg\phi\} \cup X_\# \cup \{\psi \mid \mathbb{K}\,\psi \in X \text{ or } \mathbb{B}\,\psi \in X\}$$
$$apply((45C_K), X, \neg\mathbb{K}\,\phi) = \{\neg\phi\} \cup X_\# \cup \{\psi \mid \mathbb{K}\,\psi \in X\}$$

Algorithm 2
Input: a finite $\mathcal{C}\mathbb{KB}$-consistent set X of formulae.
Output: a $\mathbb{KB}$-model graph $M = \langle W, \tau, R, S, H\rangle$ satisfying X.

1. Let X_1 be a $\mathcal{C}\mathbb{KB}$-saturation of X, and let $Y = X_1$.
2. While there exist $\neg\mathbb{B}\,\phi$ (resp. $\neg\mathbb{K}\,\phi$) from Y and a $\mathcal{C}\mathbb{KB}$-saturation Y' of the set $apply((45I_B), Y, \neg\mathbb{B}\,\phi)$ (resp. $apply((45C_K), Y, \neg\mathbb{K}\,\phi)$) with the property that $Y_\# \subset Y'_\#$, let $Y = Y'$.
3. Let $X_2 = Y$ and $W_K = W_B = \emptyset$.
4. Let Z be the set of all classical literals from X_1. We have $\neg\mathbb{K}\,\neg(Z) \in X_1$, and hence $\neg\mathbb{K}\,\neg(Z) \in X_2$. Let $H(\tau)$ be a $\mathcal{C}\mathbb{KB}$-saturation of the set $apply((45C_K), X_2, \neg\mathbb{K}\,\neg(Z))$.
5. For every $\neg\mathbb{B}\,\phi \in X_2$, add a new world w to W_B with $H(w)$ being a $\mathcal{C}\mathbb{KB}$-saturation of the set $apply((45I_B), X_2, \neg\mathbb{B}\,\phi)$.
6. For every $\neg\mathbb{K}\,\phi \in X_2$, add a new world w to W_K with $H(w)$ being a $\mathcal{C}\mathbb{KB}$-saturation of the set $apply((45C_K), X_2, \neg\mathbb{K}\,\phi)$.
7. Let $W = \{\tau\} \cup W_K \cup W_B$, $R = W \times W$ and $S = W \times W_B$.

It is clear that this algorithm always terminates.

Lemma 5. *Let X be a finite $\mathcal{C}\mathbb{KB}$-consistent formula set and $M = \langle W, \tau, R, S, H\rangle$ be the model graph constructed by the above algorithm for X. Then M is a consistent saturated $\mathbb{KB}$-model graph satisfying X.*

Proof. Clearly, M is a $\mathbb{KB}$-model graph and for every $w \in W$, $H(w)$ is $\mathcal{C}\mathbb{KB}$-consistent. We show that M is a saturated model graph. Since for $w \in W$, $H(w)$ is a $\mathcal{C}\mathbb{KB}$-saturation of some set, it suffices to show that

- if $\mathbb{K}\,\phi \in H(w)$, then for every $u \in W$, $\phi \in H(u)$;
- if $\neg\mathbb{K}\,\phi \in H(w)$, then there exists $u \in W$ such that $\neg\phi \in H(u)$;
- if $\mathbb{B}\,\phi \in H(w)$, then for every $u \in W_B$, $\phi \in H(u)$;
- if $\neg\mathbb{B}\,\phi \in H(w)$, then there exists $u \in W_B$ such that $\neg\phi \in H(u)$.

These assertions hold because for any $w \in W$, $H(w)_{\#} = X_{2\#}$.

We now show that $H(\tau) \to X$ is $\mathbb{KB}$-valid. We have $Z \cup X_{1\#} \subseteq H(\tau)$. Hence $X_1 - H(\tau)$ is a subset of $X_1 - (Z \cup X_{1\#})$, which contains only formulae of the form $\neg\neg\phi$, $\phi\wedge\psi$, or $\neg(\phi\wedge\psi)$. Since X_1 is a $\mathcal{C}\mathbb{KB}$-saturation (of X), it follows that $H(\tau) \to X_1$ is $\mathbb{KB}$-valid. By Lemma 3, M satisfies $H(\tau)$, hence it also satisfies X_1 and X.

The following theorem follows from Lemmas 1 and 5.

Theorem 2. *The calculus $\mathcal{C}\mathbb{KB}$ is sound and complete.*

If the language does not contain the modal operator $\mathbb{B}$, then we have the monomodal logic $S5$. As a corollary of Theorem 2, the rules $(\bot)$, $(\neg)$, $(\wedge)$, $(\vee)$, (T_K), (T'_K), and (45_K) form a sound and complete *cut-free* tableau system for the logic $S5$. The system has the analytic superformula property and can give a decision procedure for $S5$.

When constructing a $\mathbb{KB}$-model graph for a $\mathcal{C}\mathbb{KB}$-consistent formula set, we may use the rule (T'_K) only once, for the actual world after saturating its content. In fact, suppose that we modify the definition of $\mathcal{C}\mathbb{KB}$-saturation by excluding the rule (T'_K) from the set $\mathcal{SC}\mathbb{KB}$ and replacing step 1 of Algorithm 2 by: "Let X' be a $\mathcal{C}\mathbb{KB}$-saturation of X, and let $Y = X_1 = X' \cup \neg\mathbb{K}\,\neg(Z)$, where $Z = \{\phi \in X' \mid \phi$ is a classical literal$\}$". Then the proof of Lemma 5 can still be used for the modified algorithm.

With the above mentioned modifications, computing a $\mathcal{C}\mathbb{KB}$-saturation of a $\mathcal{C}\mathbb{KB}$-consistent set U is done in nondeterministic polynomial time in the size of U. In the modified algorithm, the formula sets X_1, Y, X_2 have size bounded by a polynomial in the size of X. Hence the modified algorithm terminates in nondeterministic polynomial time in the size of X. A nondeterministic execution of the modified algorithm gives a correct solution if the resulting model graph is consistent. Therefore the satisfiability problem in $\mathbb{KB}$ is NP-complete (the lower bound is taken as the complexity of the satisfiability problem in $S5$).

4.5 Completeness of $\mathcal{C}\mathbb{KB}_C$

Let $X_{\#K}$ denote the set $\{\phi \in X \mid \phi$ is of the form $\mathbb{K}\,\psi$ or $\neg\mathbb{K}\,\psi\}$. Note that if Y is obtainable from X by applications of the rules of $\mathcal{C}\mathbb{KB}_C$, then $X_{\#K} \subseteq Y_{\#K}$. Despite that $X_{\#K}$ does not contain formulae of the form $\mathbb{B}\,\phi$ or $\neg\mathbb{B}\,\phi$, it contains information about such formulae because of the rule (T''_K). Similarly as for the case of $\mathcal{C}\mathbb{KB}$, to construct a $\mathbb{KB}_C$-model graph for a $\mathcal{C}\mathbb{KB}_C$-consistent set X of formulae we first repeatedly apply the rules of $\mathcal{C}\mathbb{KB}_C$ to X until obtaining a set Y such that for any Y' obtainable from Y by applications of the rules of $\mathcal{C}\mathbb{KB}_C$, ${Y'}_{\#K} = Y_{\#K}$.

We also use $X_{\#B}$ to denote the set $\{\phi \in X \mid \phi \text{ is of the form } \mathbb{B}\,\psi \text{ or } \neg\mathbb{B}\,\psi\}$.

For $\neg\mathbb{B}\,\phi \in X$, let $apply((45I_B), X, \neg\mathbb{B}\,\phi)$ be defined as in the previous subsection. For $\neg\mathbb{K}\,\phi \in X$, let $apply((45_K), X, \neg\mathbb{K}\,\phi)$ be the result of the application of the rule (45_K) to X with $\neg\mathbb{K}\,\phi$ being the principal formula, i.e. $apply((45_K), X, \neg\mathbb{K}\,\phi) = \{\neg\phi\} \cup X_{\#K} \cup \{\psi \mid \mathbb{K}\,\psi \in X\}$.

Algorithm 3
Input: a finite $\mathcal{C}\mathbb{KB}_C$-consistent set X of formulae.
Output: a $\mathbb{KB}_C$-model graph $M = \langle W, \tau, R, S, H\rangle$ satisfying X.

1. Let X_1 be a $\mathcal{C}\mathbb{KB}_C$-saturation of X, and let $Y = X_1$.
2. Repeat the following until $Y_{\#K} = Y_{0\#K}$ (after executing step 2d).
   (a) Let $Y_0 = Y$.
   (b) If there exist $\neg\mathbb{B}\,\phi \in Y$ and a set Z being a $\mathcal{C}\mathbb{KB}_C$-saturation of the set $apply((45I_B), Y, \neg\mathbb{B}\,\phi)$ such that $Y_{\#K} \subset Z_{\#K}$, then let $Y = Z$.
   (c) If there exist $\neg\mathbb{K}\,\phi \in Y$ and a set Z being a $\mathcal{C}\mathbb{KB}_C$-saturation of the set $apply((45_K), Y, \neg\mathbb{K}\,\phi)$ such that $Y_{\#K} \subset Z_{\#K}$, then let $Y = Z$.
   (d) If there exist formula sets Z_1, Z_2, and formulae $\neg\mathbb{K}\,\neg(U) \in Y$, $\neg\mathbb{B}\,\phi \in Z_1$ such that:
      – Z_1 is a $\mathcal{C}\mathbb{KB}_C$-saturation of $apply((45_K), Y, \neg\mathbb{K}\,\neg(U))$,
      – Z_2 is a $\mathcal{C}\mathbb{KB}_C$-saturation of $apply((45I_B), Z_1, \neg\mathbb{B}\,\phi)$, and
      – $Y_{\#K} \subset Z_{2\#K}$,
      then let $Y = Z_2$.
3. Let X_2 be the set obtained from Y by deleting every formula $\neg\mathbb{K}\,\neg(U)$ such that there exists $\neg\mathbb{K}\,\neg(U') \in Y$ with $U' \supset U$.
4. For each formula $\neg\mathbb{K}\,\neg(U) \in X_2$, create a new world w with $H(w)$ being a $\mathcal{C}\mathbb{KB}_C$-saturation of $apply((45_K), X_2, \neg\mathbb{K}\,\neg(U))$. Let W_K be the set of such worlds.
5. Let $W = W_K$ and $S_0 = \emptyset$.
6. For each $w \in W_K$ do the following:
   (a) Let $U = H(w)_{\#B}$.
   (b) Let $\neg\mathbb{K}\,\neg(U_w)$ be a formula of X_2 such that $U_w \supseteq U$ and there is no $\neg\mathbb{K}\,\neg(U') \in X_2$ with ${U'}_{\#B} \supset U_{w\#B}$.
   (c) Let Z_w be a $\mathcal{C}\mathbb{KB}_C$-saturation of $apply((45_K), X_2, \neg\mathbb{K}\,\neg(U_w))$.
   (d) For each formula $\neg\mathbb{B}\,\phi \in Z_w$, create a new world u with $H(u)$ being a $\mathcal{C}\mathbb{KB}_C$-saturation of $apply((45I_B), Z_w, \neg\mathbb{B}\,\phi)$, add u to W and the pair (w, u) to S_0.
7. Let τ be a world of W_K such that $H(\tau)$ contains all classical literals and formulae of the form $\mathbb{B}\,\psi$ or $\neg\mathbb{B}\,\psi$ that belong to X_1.
8. Let $R = W \times W$ and S be the euclidean closure of S_0.

Due to the rule (T''_K), the existences of $\neg\mathbb{K}\,\neg(U_w)$ at step 6b and τ at step 7 are guaranteed. It is clear that this algorithm always terminates.

Lemma 6. *Let X be a finite $\mathcal{C}\mathbb{KB}_{C}$-consistent formula set and $M = \langle W, \tau, R, S, H\rangle$ be the model graph constructed by the above algorithm for X. Then M is a consistent saturated $\mathbb{KB}_{C}$-model graph satisfying X.*

Proof. It is easily seen that S is serial (because of the rule (D_B)), transitive and euclidean. Hence M is a $\mathbb{KB}_{C}$-model graph. M is consistent because $H(w)$ is $\mathcal{C}\mathbb{KB}_{C}$-consistent for every $w \in W$.

We now show that M is a saturated model graph. Since for every $w \in W$, $H(w)$ is $\mathcal{C}\mathbb{KB}_{C}$-saturation of some set, it is sufficient to show that, for every $w \in W$,

- if $\mathbb{K}\,\phi \in H(w)$, then for every $u \in W$, $\phi \in H(u)$;
- if $\neg\mathbb{K}\,\phi \in H(w)$, then there exists $u \in W$ such that $\neg\phi \in H(u)$;
- if $\mathbb{B}\,\phi \in H(w)$, then for every u such that $S(w,u)$, $\phi \in H(u)$;
- if $\neg\mathbb{B}\,\phi \in H(w)$, then there exists $u \in W$ such that $S(w,u)$ and $\neg\phi \in H(u)$.

The first two of the above assertions hold because that if ψ is of the form $\mathbb{K}\,\phi$ or $\neg\mathbb{K}\,\phi$, and $\psi \in H(w)$ for some w, then $\psi \in X_{2\#K}$ (because of the step 2).

For the third assertion, suppose that $S(w,u)$ holds and $\mathbb{B}\,\phi \in H(w)$. If $w \in W_K$ and $S_0(w,u)$, then $\phi \in H(u)$ follows from $\mathbb{B}\,\phi \in H(w)$. If $w \notin W_K$ then there exists $v \in W_K$ such that $S_0(v,w)$ and $S_0(v,u)$. By the step 6b and the rules $(45I_B)$, (T''_K), we have $U_{v\#B} = H(w)_{\#B}$. This implies that $\mathbb{B}\,\phi \in U_v$, and hence $\mathbb{B}\,\phi \in Z_v$ and $\phi \in u$. Therefore the third assertion holds.

For the fourth assertion, suppose that $\neg\mathbb{B}\,\phi \in H(w)$. If $w \in W_K$, then there exists u such that $S_0(w,u)$ and $\neg\phi \in H(u)$. Suppose that $w \notin W_K$. There exists $v \in W_K$ such that $S_0(v,w)$. Since $\neg\mathbb{B}\,\phi \in H(w)$, similarly as for the case of the third assertion, we derive that $\neg\mathbb{B}\,\phi \in U_v$. It follows that $\neg\mathbb{B}\,\phi \in Z_v$ and there exists u such that $S_0(v,u)$ and $\neg\phi \in H(u)$. It is clear that $S(w,u)$ holds.

We now show that M satisfies X. Since M is a saturated model graph, by Lemma 3, $M \models H(\tau)$. Let V be the set of formulae of X_1 that are of the form $\neg\neg\phi$, $\phi \wedge \psi$, or $\neg(\phi \wedge \psi)$. We have $(X_1 - V) \subseteq H(\tau)$. Hence $M \models (X_1 - V)$. Since X_1 is a $\mathcal{C}\mathbb{KB}_{C}$-saturation of some set, it follows that $M \models X_1$. Therefore $M \models X$. This completes the proof.

Theorem 3. *The calculus $\mathcal{C}\mathbb{KB}_{C}$ is sound and complete.*

This theorem follows from Lemmas 1 and 6. Now consider the complexity of the satisfiability problem in $\mathbb{KB}_{C}$. We first adopt some restrictions on the use of the rule (T''_K). Assume that when applying the rule (T''_K), a formula of the form $\neg\mathbb{K}\,\neg(U)$ is created in the denominator only for U being the largest set satisfying the condition of the rule. Also assume that when computing a $\mathcal{C}\mathbb{KB}_{C}$-saturation of some set, the rule (T''_K) is applied only once as the last rule, and if in the resulting set there are $\neg\mathbb{K}\,\neg(U)$ and $\neg\mathbb{K}\,\neg(U')$ with $U \subset U'$ then the formula $\neg\mathbb{K}\,\neg(U)$ is deleted from the set. Consider Algorithm 3 under such restrictions: computing a $\mathcal{C}\mathbb{KB}_{C}$-saturation of a set U is done in nondeterministic polynomial time in the size of U, and the sets X_1, Y, X_2 have size bounded by a polynomial in the size of X. Hence the new algorithm terminates in nondeterministic polynomial time in the size of X. Analogously as for $\mathbb{KB}$, we conclude that the satisfiability problem in $\mathbb{KB}_{C}$ is NP-complete.

5 Conclusions

We have given analytic sequent-like tableau systems for the propositional bimodal logics $\mathbb{KB}$, $\mathbb{KB}_C$, $\mathbb{KB}_5$, and $\mathbb{KB}_{5C}$. They are sound and complete and can be used to reason about both knowledge and belief. Using the systems and their completeness proofs, we have shown that the satisfiability problem in $\mathbb{KB}_5$ and $\mathbb{KB}_{5C}$ is PSPACE-complete, in $\mathbb{KB}$ and $\mathbb{KB}_C$ is NP-complete. Our tableau systems can be used to prove the Craig interpolation lemma for the considered logics, as done in our work [15] for other logics.

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