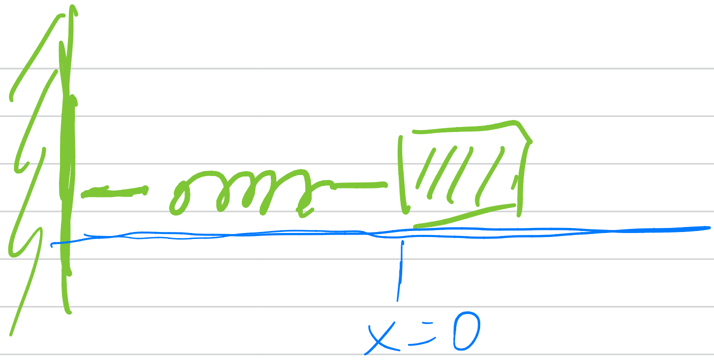
$$m\ddot{x} = -\alpha x$$



$$D\sin(\omega t + \omega_0),$$

$$x = A\cos\omega t + B\sin\omega t,$$

gdzie $\omega = \sqrt{\alpha/m}$, $A, B \in \mathbb{R}$

$$\ddot{x} + \frac{\alpha}{m}x = 0$$

$$\lambda^2 + \frac{\alpha}{m} = 0$$

$$\lambda = \pm i\sqrt{\frac{\alpha}{m}}$$

$$\lambda = \pm i\omega$$

$$x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t$$

$$m\ddot{x} = -\alpha x + \delta \cos \lambda t \quad / : m$$

1) $\lambda \neq \omega$ $\qquad \omega = \left(\frac{\alpha}{m}\right)^{1/2}$

2) $\lambda = \omega$.

$$\ddot{x} + \omega^2 x = \frac{\delta}{m} \cos \lambda t$$

$$(D + i\omega)(D - i\omega\, x) = \frac{\delta}{m} \cos \lambda t.$$

ad 1. $D + i\omega$ i $D - i\omega$ są izomorfizmami na przestrzeni rozpiętej przez $e^{i\lambda t}$, $e^{-i\lambda t}$

czyli rozwiązanie będzie postaci

$$\underbrace{A \cos \omega t + B \sin \omega t}_{\text{wyznaczone przez war. pocz.}} + \underbrace{C \cos \lambda t + D \sin \lambda t}_{\text{zależą od } \frac{\delta}{m}}$$

$$(D+i\omega)(D-i\omega\, x) = \frac{\delta}{m}\cos \lambda t.$$

$$\lambda = \omega.$$

$$(D+i\omega)(D-i\omega) = \frac{\delta}{2m} e^{i\omega t} + \frac{\delta}{2m} e^{-i\omega t}$$

$$(D-i\omega)^{-1} \frac{\delta}{2m} e^{i\omega t} = \underbrace{\underline{\underline{A_1}}}_{\text{zależy od } \frac{\delta}{2m}}\, t\, e^{i\omega t} + \underbrace{A_2}_{\text{dowolne}}\, e^{i\omega t}$$

$$(D-i\omega)\left(\underbrace{A_1\, t\, e^{i\omega t}}_{\frac{\delta}{2m}} + \underbrace{A_2\, e^{i\omega t}}_{0}\right) =$$

$$(D-(-i\omega))^{-1}\left(A_1\, t\, e^{i\omega t} + A_2\, e^{i\omega t}\right) =$$

$$A_1'\, t e^{i\omega t} + A_2'\, e^{i\omega t}$$

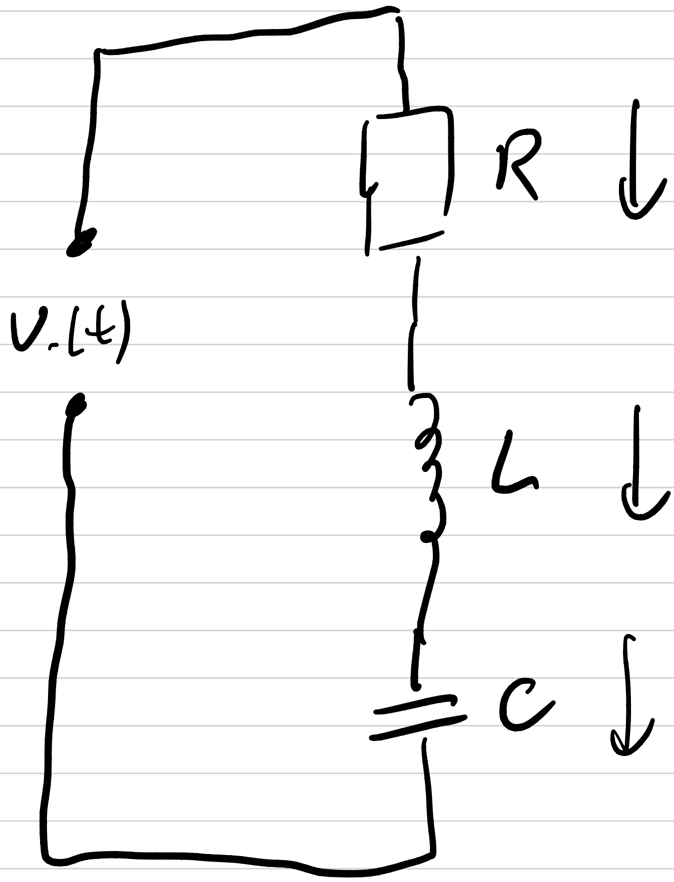
$$x(t) = A_1' t e^{i\omega t} + A_2' e^{i\omega t} +$$

$$+ A_3' t e^{-i\omega t} + A_4' e^{i\omega t} =$$

$$= B_1 t \cos \omega t + B_2 t \sin \omega t +$$

$$+ B_3 \cos \omega t + B_4 \sin \omega t$$

B_1, B_2, B_3, B_4 stałe



$$RI(t)$$

$$L \cdot \frac{dI}{dt}$$

$$V_C = \frac{1}{C}\int_0^t I(\tau)\,d\tau.$$

$$v(t) = V_R + V_I + V_C =$$

$$= RI(t) + L\dot{I}(t) + \frac{1}{C}\int_0^t I(\tau)\,d\tau.$$

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = \dot{v}(t)$$

uproszczenie $R=0$.

$$\ddot{I} + \frac{1}{LC}I = \dot{v}(t)$$

$$\ddot{I} + \frac{1}{LC} I = \dot{v}(t)$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad \dot{v}(t) = \alpha \cos \lambda t.$$

albo $\omega \neq \lambda$ złożenie dwóch drgań.

$$\omega = \lambda \Rightarrow$$

$$I(t) = A t \cos \omega t + \dots$$

Two body problem

zagadnienie dwóch ciał.

$(0,0)$ (x,y)

$$\dot r = \frac{1}{r}(x\dot x + y\dot y) \qquad r = \sqrt{x^2+y^2}$$

$$\dot\varphi = \frac{1}{r^2}(x\dot y - y\dot x) \qquad \varphi = \operatorname{arc\,tg} \frac{y}{x}$$

Przypuśćmy, że siła przyciągania jest zależna od promienia.

$$\ddot x = -F(r)x$$

$$\ddot y = -F(r)y$$

Lemat 1: $\ddot{\varphi} = -2\frac{\dot{r}}{r}\dot{\varphi}$.

Dowód:

$$\ddot{\varphi} = \frac{d}{dt}\left(\frac{1}{r^2}(x\dot{y} - y\dot{x})\right) =$$

$$= -\frac{2\dot{r}}{r^3}(x\dot{y} - y\dot{x}) + \frac{1}{r^2}(\dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x}) =$$

$$= -2\frac{\dot{r}}{r}\dot{\varphi} + \frac{1}{r^2}(x\ddot{y} - y\ddot{x}) =$$

$$= \left\{\begin{array}{l} \ddot{x} = -F(r)x \\ \ddot{y} = -F(r)y \end{array}\right\} = -2\frac{\dot{r}}{r}\dot{\varphi}.$$

Lemat 2: $r^2\dot\varphi$ nie zależy od t.

$M = r^2\dot\varphi$ jest całką pierwszą

Dowód:

$$\frac{d}{dt}\left(r^2\dot\varphi\right) = 2r\dot r\dot\varphi + r^2\ddot\varphi =$$

$$= \left\{ \ddot\varphi = -2\frac{\dot r}{r}\dot\varphi \right\} = 2r\dot r\dot\varphi +$$

$$- 2\dot r r\dot\varphi = 0.$$

$r^2\dot\varphi = M$ nie zależy od t.

Lemat 3: $\ddot{r} = \frac{a^2}{r^3} - F(r)r$.

Dowód!

$$\ddot{r} = -\frac{\dot{r}}{r^2}(x\dot{x}+y\dot{y}) +$$

$$+ \frac{1}{r}(x\ddot{x}+\dot{x}\dot{x}+y\ddot{y}+\dot{y}\dot{y}) =$$

$$= \frac{-\dot{r}^2}{r} + \frac{1}{r}\left(-F(r)(x^2+y^2)\right) + \frac{\dot{x}^2+\dot{y}^2}{r} =$$

$$= \frac{-\dot{r}^2+\dot{x}^2+\dot{y}^2}{r} - rF(r) = \dots$$

$$\dot{x}^2+\dot{y}^2-\dot{r}^2 = \dot{x}^2+\dot{y}^2-\frac{1}{r^2}(\dot{x}x+y\dot{y})^2 =$$

$$= \frac{1}{r^2}\Big(\underline{\dot{x}^2x^2}+\dot{x}^2y^2+\dot{y}^2x^2+\underline{\dot{y}^2y^2} +$$

$$-\big(\underline{x^2\dot{x}^2}+2x\dot{x}y\dot{y}+\underline{y^2\dot{y}^2}\big)\Big) =$$

$$= \frac{1}{r^2}(x\dot{y} - y\dot{x})^2 = r^2\dot{\varphi}^2$$

$$= r\dot{\varphi}^2 - F(r)\,r. \quad =$$

ale: $r^2\dot{\varphi} = M.$

$$= \frac{M^2}{r^3} - F(r)\,r.$$

Co to jest $F(r)$? $\qquad F(r) = \frac{\alpha}{r^3}$

$$\ddot{x} = -F(r)\,x$$

$$\ddot{y} = -F(r)\,y$$

$$\ddot{r} = \frac{M^2}{r^3} - \frac{\alpha}{r^2} = \left(-\frac{1}{2}\frac{M^2}{r^2} + \frac{\alpha}{r}\right)'$$

$$\ddot{r} = \frac{M^2}{r^3} - \frac{k}{r^2} = \left(-\frac{1}{2}\frac{M}{r^2} + \frac{k}{r}\right)'$$

ma całkę pierwszą.

$$E = \frac{1}{2}\dot{r}^2 - \frac{k}{r} + \frac{M}{2r^2}.$$

wtedy $\frac{d}{dt}E(t) = 0$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 - \frac{k}{r} + \frac{M}{2r^2}\right) =$$

$$= \dot{r}\ddot{r} + \frac{k}{r^2}\dot{r} - \frac{M}{r^3}\dot{r} =$$

$$= \dot{r}\left(\ddot{r} + \frac{k}{r^2} - \frac{M}{r^3}\right) = 0$$

$$\dot{r} = \sqrt{2E - \frac{M}{r^2} + \frac{2k}{r}}$$

$$\frac{d}{d\varphi}\left(\frac{1}{r}\right) = \frac{dt}{d\varphi} \cdot \frac{d\left(\frac{1}{r}\right)}{dt} =$$

$$= \frac{1}{\dot\varphi} \cdot \left(-\frac{\dot r}{r^2}\right) = \frac{-\dot r}{r^2 \dot\varphi} = \frac{-\dot r}{M}.$$

$$\dot r = -M \frac{d}{d\varphi}\left(\frac{1}{r}\right). \qquad u = \frac{1}{r}$$

$$\dot r = \sqrt{2E - \frac{M}{r^2} + \frac{2\alpha}{r}}$$

$$-M \frac{d}{d\varphi} u = \sqrt{2E - Mu + 2\alpha u}$$

$$\frac{du}{d\varphi} = \frac{-1}{M}\sqrt{2E - Mu^2 + 2\alpha u}.$$

$$\frac{du}{d\varphi} = \frac{-1}{M}\sqrt{2E - Mu^2 + 2ku}.$$

$$\int \frac{1}{\sqrt{2E - Mu^2 + 2ku}}\, du = -\frac{1}{M}\varphi + C.$$

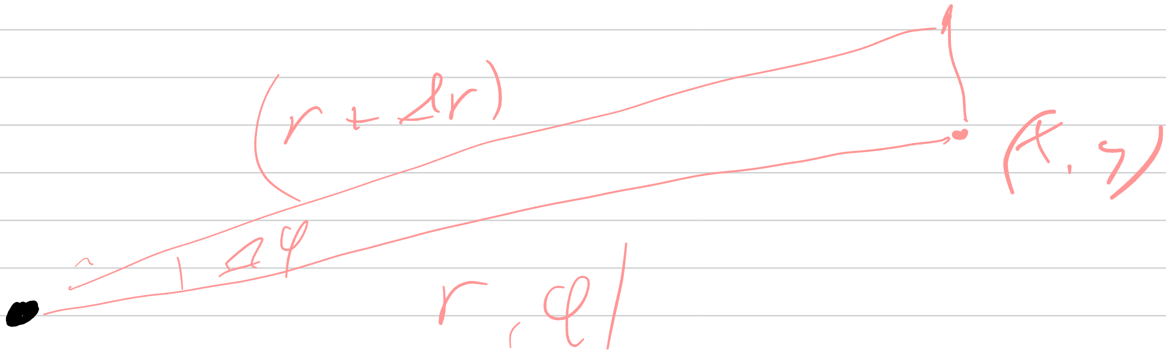
$$\downarrow$$

$$A \arcsin(Bu + D)$$

$$u = \alpha \sin(\beta\varphi + \gamma) + \delta \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

$$r(\varphi) = \frac{1}{\alpha \sin(\beta\varphi + \gamma) + \delta}$$

można wyznaczyć $\alpha, \beta, \gamma, \delta$ w zależności od E, M, k.



$$\frac{1}{2} r^2 d\varphi + \ldots$$

$$\frac{1}{2} M$$